MATHEMATICS

CLASS XI

R.D. SHARMA

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MATHEMATICS

CLASS XI

Based on the latest revised syllabus prescribed by CBSE for Class XI under 10+2 Pattern of Senior School Certificate Examination

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DHANPAT RAI PUBLICATIONS (P) LTD.
22, Ansari Road, Daryaganj, New Delhi-110002
Ph.: 2327 4073, 2324 657 [illegible] E-Mail: [illegible]@gmail.com

Price : ₹ 495.00

Always ask the bookseller to put his stamp on the first page of this book.

ISBN: 978-93-83182-95-4

First Edition: 1998, Reprints: 1999-01, Second Revised Edition: 2002, Reprints: 2003-05, Third Revised Edition: 2006, Reprint: 2007, Fourth Revised Edition: 2008, Reprints: 2009-12, Fifth Revised Edition: 2013, Sixth Revised Edition : 2014, Reprint : 2015, Seventh Revised Edition: 2016

Published by Ish Kapur, for Dhanpat Rai Publications (P) Ltd.,
Printed at Deepak Pustak Bandhnalaya, New Delhi.

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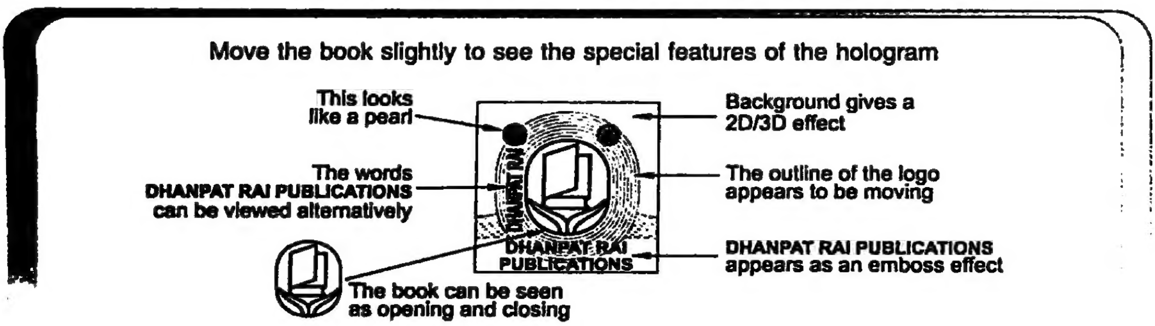

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Dear Teachers & Students

In the present edition, ***Illustrative Examples*** *and* ***Exercises*** *given at the end of every section/sub-section in each chapter have been arranged in the increasing order of difficulty level and have been categorized into two levels, namely,* ***Level-1*** *and* ***Level-2****.*

Main highlights of the present edition are:

— ***The entire text has been re-written.***

— ***New illustrative examples have been added and alternative solutions (Aliter) of many of them have been provided.***

— ***Level-wise categorization of illustrative examples and exercises.***

I am grateful to all the teachers who have given their valuable suggestions for the improvement of the quality of the book.

Please send your valuable suggestions, feedback or queries through e-mail to **ish.dhanpat@gmail.com**

Any suggestion/criticism to enhance the quality of the book will be gratefully acknowledged.

With my Best Wishes

Dr. R.D. SHARMA

CONTENTS

MATHEMATICS - XI

CHAPTER 1

SETS

1.1 SETS

It is a well known fict that any attempt to define a set has always led mathematicians to unsurmountable difficulties. For example, suppose one defines the term set as *"a well defined collection of objects"*. One may then ask what is meant by a collection. If one answers that a collection is an aggregte of objects or things. What is then an aggregate? Perhaps then one may define that an aggregee is a class of things. What is then a class? Now, one may define a class as a collection. In this mnner question after question, since our language is finite, we find that after some time we vill have to use some words which have already been questioned. The definition thus becoms circular and worthless. Thus, mathematicians realized that there must be some undefined (r primitive) terms. In this chapter, we start with two undefined (or primitive) terms — "ement" and "set". We assume that the word "set" is synonymous with the words "collectioi", "aggregate", "class" and is comprised of elements. The words "element", "object", "member" are synonymous.

If a is an element of a st A, then we write $a \in A$ and say a belongs to A or a is in A or a is a member of A. If a does not belog to A, then we write $a \notin A$. It is assumed here that if A is any set and a is any element, then eithr $a \in A$ or $a \notin A$ and the two possibilities are mutually exclusive. Thus, one cannot say "consier the set A of some positive integers", because it is not sure whether $3 \in A$ or $3 \notin A$.

Throughout this chapr we shall denote sets by capital alphabets e.g. A, B, C, X, Y, Z etc. and the elements by the smalllphabets e.g. a, b, c, x, y, z etc.

The following are son illustrations of sets:

ILLUSTRATION 1 *Thellection of vowels in English alphabets. This set contains five elements, namely,* a, e, i, o, u.

ILLUSTRATION 2 *Thollection of first five prime natural numbers is a set containing the elements* 2, 3, 5, 7, 11.

ILLUSTRATION 3 *Thollection of all States in the Indian Union is a set.*

ILLUSTRATION 4 *Thollection of past presidents of the Indian union is a set.*

ILLUSTRATION 5 *Thollection of cricketers in the world who were out for 99 runs in a test match is a set.*

ILLUSTRATION 6 *Thollection of good cricket players of India is not a set, since the term "good player is vague and it is not w defined".*

Similarly, collection good teachers in a school is not a set. However, the collection of all teachers in a school is set.

In this chapter we whave frequent interaction with some sets, so we reserve some letters for these sets as listed bew:

N : for the set oftural numbers.

Z : for the set oftegers.

Z^+ : for the set of positive integers.

Q : for the set of all rational numbers.
Q^+ : for the set of all positive rational numbers.
R : for the set of all real numbers.
R^+ : for the set of all positive real numbers.
C : for the set of all complex numbers.

EXERCISE 1.1

LEVEL-1

1. What is the difference between a collection and a set? Give reasons to support your answer?
2. Which of the following collections are sets? Justify your answer:
   (i) A collection of all natural numbers less than 50.
   (ii) The collection of good hockey players in India.
   (iii) The collection of all girls in your class.
   (iv) The collection of most talented writers of India. [NCERT]
   (v) The collection of difficult topics in Mathematics.
   (vi) The collection of all months of a year beginning with the letter J [NCERT]
   (vii) A collection of novels written by Munshi Prem Chand. [NCERT]
   (viii) The collection of all questions in this chapter. [NCERT]
   (ix) A collection of most dangerous animals of the world. [NCERT]
   (x) The collection of prime integers.
3. If $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then insert the appropriate symol $\in$ or $\notin$ in each of the following blank spaces:
   (i) 4 ..... A (ii) -4 ...... A (iii) 12 .. A
   (iv) 9 .... A (v) 0 ..... A (vi) -2 .. A

ANSWERS

1. Every set is a collection but a collection is not necessarily a set. Onlywell defined collections are sets. For example, group of good cricket players is a collectiooput it is not a set.
2. (i), (iii), (vi), (vii), (viii), (x)
3. (i) $\in$ (ii) $\notin$ (iii) $\notin$ (iv) $\in$ (v) $\in$ (vi) $\notin$

HINTS TO NCERT & SELECTED PROBLEMS

2. (iv) The collection of most talented writers of India is not a seas there is no specific criterion to determine whether a writer is talented or not.
   (vi) The collection of all months of a year beginning with the ;ter J is a set given by {January, June July}.
   (vii) The collection of novels written by Munshi Prem Chand is a set because one can determine whether a novel is written by him or not.
   (viii) The collection of all questions in this chapter is a set because a question is given one can easily decide whether it is a question of this chapter or t.
   (ix) The collection of most dangerous animals of the world is notset because there is no criterion to determine whether an animal is most dangerous not.

1.2 DESCRIPTION OF A SET

A set is often described in the following two forms. One can make us any one of these two ways according to his (her) convenience.

(i) Roster form or Tabular form (ii) Set-builder form

Let us now discuss these forms.

1.2.1 ROSTER FORM

In this form a set is described by listing elements, separated by commas, within braces { }.

ILLUSTRATION 1 *The set of vowels of English Alphabet may be described as* $\{a, e, i, o, u\}$.

ILLUSTRATION 2 *The set of even natural numbers can be described as* $\{2, 4, 6, \ldots\}$. *Here the dots stand for 'and so on'.*

ILLUSTRATION 3 *If A is the set of all prime numbers less than 11, then* $A = \{2, 3, 5, 7\}$.

NOTE *The order in which the elements are written in a set makes no difference. Thus,* $\{a, e, i, o, u\}$ *and* $\{e, a, i, o, u\}$ *denote the same set. Also, the repetition of an element has no effect. For example,* $\{1, 2, 3, 2\}$ *is the same set as* $\{1, 2, 3\}$.

1.2.2 SET-BUILDER FORM

In this form, a set is described by a characterizing property $P(x)$ of its elements x. In such a case the set is described by $\{x : P(x) \text{ holds}\}$ or, $\{x \mid P(x) \text{ holds}\}$, which is read as 'the set of all x such that $P(x)$ holds'. The symbol '$|$' or ':' is read as 'such that'.

In other words, in order to describe a set, a variable x (say) (to denote each element of the set) is written inside the braces and then after putting a colon the common property $P(x)$ possessed by each element of the set is written within the braces.

ILLUSTRATION 4 *The set E of all even natural numbers can be written as*

$$E = \{x : x \text{ is a natural number and } x = 2n \text{ for } n \in N\}$$

or, $E = \{x : x \in N, x = 2n, n \in N\}$ or, $E = \{x \in N : x = 2n, n \in N\}$

ILLUSTRATION 5 *The set* $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ *can be written as* $A = \{x \in N : x \leq 8\}$.

ILLUSTRATION 6 *The set of all real numbers greater than –1 and less than 1 can be described as* $\{x \in R : -1 < x < 1\}$.

ILLUSTRATION 7 *The set* $A = \{0, 1, 4, 9, 16, \ldots\}$ *can be written as* $A = \{x^2 : \in Z\}$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON DESCRIBING OR REPRESENTING SETS IN TABULAR FORM OR ROSTER FORM**

EXAMPLE 1 *Describe the following sets in Roster form:*

(i) *The set of all letters in the word 'MATHEMATICS'*

(ii) *The set of all letters in the word 'ALGEBRA'*

(iii) *The set of all vowels in the word 'EQUATION'*

(iv) *The set of all natural numbers less than 7.*

(v) *The set of squares of integers.*

SOLUTION (i) We observe that distinct letters in the word 'MATHEMATICS' are:

$$M, A, T, H, E, I, C, S$$

Since the order in which the elements of a set are written is immaterial and the repetition of elements has no effect. So, required set can be described as follows:

$$\{M, A, T, H, E, I, C, S\}$$

(ii) We find that the word 'ALGEBRA' has following distinct letters: A, L, G, E, B, R

Hence, required set can be described in Roster form as follows: $\{A, L, G, E, B, R\}$

(iii) Clearly, word 'EQUATION' has following vowels: A, E, I, O, U

So, required set can be described as follows: $\{A, E, I, O, U\}$

(iv) Natural numbers less than 7 are: 1, 2, 3, 4, 5, 6.

Hence, required set can be described as follows: {1, 2, 3, 4, 5, 6}.

(v) Since square of a negative integer is same as the square of its absolute value. Therefore, squares of integers are 0, 1, 4, 9, 16, 25, ... Hence, required set is {0, 1, 4, 9, 16, ...}.

Type II ON DESCRIBING OR REPRESENTING SETS IN SET-BUILDER FORM

EXAMPLE 2 *Describe the following sets in set-builder form:*

(i) *The set of all letters in the word 'PROBABILITY'.*

(ii) *The set of reciprocals of natural numbers.* (iii) *The set of all odd natural numbers.*

(iv) *The set of all even natural numbers.*

SOLUTION (i) Given set in set-builder form can be described as follows:

{x : x is a letter in the word 'PROBABILITY'}

(ii) Given set can be described in set-builder form as follows:

{x : x is reciprocal of a natural number} or, $\left\{x : x = \frac{1}{n}, \in N\right\}$ or, $\left\{\frac{1}{n} : \in N\right\}$

(iii) An odd natural number can be written in the form $(2n-1)$. So, given set can be described as follows $\{x : x = 2n-1, n \in N\}$ or, $\{2n-1 : n \in N\}$.

(iv) An even natural number can be written as $2n$, where $n \in N$. Therefore, set of all even natural numbers can be written in the form $\{x : x = 2n, n \in N\}$ or, $\{2n : n \in N\}$

EXAMPLE 3 *Write the set of all integers whose cube is an even integer.*

SOLUTION We know that the cube of an even integer is also an even integer. Hence, the required set is the set of all even integers which can also be written in the set-builder form as $\{2n : n \in Z\}$.

EXAMPLE 4 *Write the set of all real numbers which cannot be written as the quotient of two integers in the set-builder form.*

SOLUTION We know that all rational numbers are expressible as the quotient of two integers. Therefore, the required set is the set of all irrational numbers which can be written as

{x : x is real and irrational} or, $\{x : x \in R$ but $x \notin Q\}$.

Type III ON DESCRIBING A SET IN ROSTER FORM WHEN IT IS GIVEN IN SET-BUILDER FORM

EXAMPLE 5 *Describe each of the following sets in Roster form*

(i) {x : *x is a positive integer and a divisor of 9*} (ii) $\{x : x \in Z$ *and* $|x| \le 2\}$

(iii) {x : *x is a letter of the word 'PROPORTION'*} (iv) $\left\{x : x = \frac{n}{n^2+1} \text{ and } 1 \le n \le 3, \text{ where } n \in N\right\}$

SOLUTION (i) Since x is a positive integer and a divisor of 9. So, x can take values 1, 3, 9.

$\therefore$ {x : x is a positive integer and a divisor of 9} = {1, 3, 9}

(ii) We find that x is an integer satisfying $|x| \le 2$.

and, $|x| = 0, 1, 2 \Rightarrow x = 0, \pm 1, \pm 2$

So, x can take values $-2, -1, 0, 1, 2$.

$\therefore$ $\{x : x \in Z \text{ and } |x| \le 2\} = \{-2, -1, 0, 1, 2\}$

(iii) We find that distinct letters in the word 'PROPORTION' are P, R, O, T, N, I. So, x can be P, R, O, T, I, N.

Hence, {x : x is a letter in the word '*PROPORTION*'} = $\{P, R, O, T, I, N\}$

(iv) We have,

$x = \frac{n}{n^2+1}$ where $n \in N$ and $1 \le n \le 3$.

$\therefore \quad x = \frac{n}{n^2+1}$, where $n = 1, 2, 3.$

$\Rightarrow \quad x = \frac{1}{1^2+1}, \frac{2}{2^2+1}, \frac{3}{3^2+1}$

$\Rightarrow \quad x = \frac{1}{2}, \frac{2}{5}, \frac{3}{10}$

Hence, $\left\{x : x = \frac{n}{n^2+1} \text{ and } 1 \le n \le 3, \text{ where } n \in N\right\} = \left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}\right\}$

EXAMPLE 6 *Write the set of all vowels in English alphabet which precede s.*

SOLUTION The vowels in English alphabet which precede s are a, e, i, o. So, the set $A = \{a, e, i, o\}$ is the set of all vowels in English alphabet which precede s.

EXAMPLE 7 *Write the set $A = \{x : x \in Z, x^2 < 20\}$ in the roster form.*

SOLUTION We observe that the integers whose squares are less than 20 are: $0, \pm 1, \pm 2, \pm 3, \pm 4$. Therefore, the set A in roster form is $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

EXAMPLE 8 *Match each of the set on the left described in the roster form with the same set on the right described in the set-builder form.* [NCERT]

(i) $\{P, R, I, N, C, A, L\}$	(a) $\{x : x$ *is a positive integer and is a divisor of* $18\}$
(ii) $\{0\}$	(b) $\{x : x$ *is an integer and* $x^2 - 9 = 0\}$
(iii) $\{1, 2, 3, 6, 9, 18\}$	(c) $\{x : x$ *is an integer and* $x + 1 = 1\}$
(iv) $\{-3, 3\}$	(d) $\{x : x$ *is a letter of the word 'PRINCIPAL'*$\}$

SOLUTION (i) Clearly, $\{P, R, I, N, C, A, L\} = \{P, R, I, N, C, I, P, A, L\}$
$= \{x : x \text{ is a letter of the word 'PRINCIPAL'}\}$

Hence, (i) matches with (d).

(ii) $\{0\} = \{x : x \text{ is an integer equal to zero}\} = \{x : x \text{ is an integer and } x + 1 = 1\}$

Hence, (ii) matches with (c).

(iii) $\{1, 2, 3, 6, 9, 18\}$ = Set of all positive divisors of 18
$= \{x : x \text{ is a positive integer and is a divisor of } 18\}$

Hence, (iii) matches with (a).

(iv) Clearly, $\{-3, 3\} = \{x : x \text{ is an integer and } x^2 - 9 = 0\}$. Hence, (iv) matches with (b).

Type IV ON DESCRIBING A SET IN ROSTER FORM WHEN IT IS GIVEN IN SET-BUILDER FORM

EXAMPLE 9 *Write the set* $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}\right\}$ *in the set-builder form.* [NCERT]

SOLUTION We observe that each element in the given set has the denominator one more than the numerator. Also, the numerator begins from 1 and do not exceed 9. Hence, in the set-builder form the given set can be written as $\left\{x : x = \frac{n}{n+1}, n \in N, n \le 9\right\}$.

EXAMPLE 10 *Write the set* $X = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right\}$ *in the set -builder form.*

SOLUTION We observe that the elements of set X are the reciprocals of the squares of all natural numbers. So, the set X in set builder form is $X = \left\{\frac{1}{n^2} : n \in N\right\}$.

EXAMPLE 11 *Write the following sets in Roster form:*

(i) $A=\{a_n : n \in N, a_{n+1}=3a_n \text{ and } a_1 = 1\}$ (ii) $B=\{a_n : n \in N, a_{n+2}=a_{n+1}+a_n, a_1=a_2=1\}$

SOLUTION (i) We have, $a_1=1$ and $a_{n+1}=3a_n$ for all $n \in N$

Putting $n=1$ in $a_{n+1} = 3a_n$, we get

$$a_2=3a_1=3\times 1 = 3 \qquad [\because a_1=1]$$

Putting $n = 2$ in $a_{n+1} = 3a_n$, we get

$$a_3=3a_2=3\times 3=3^2 \qquad [\because a_2=3]$$

Putting $n=3$ in $a_{n+1}=3a_n$, we get

$$a_4=3a_3=3\times 3^2=3^3 \qquad [\because a_3=3]$$

Similarly, we obtain

$$a_5=3a_4=3\times 3^3=3^4,\ a_6=3a_5=3\times 3^4=3^5 \text{ and so on.}$$

Hence, $A=\{a_1, a_2, a_3, a_4, a_5, a_6, ...\} = \{1, 3, 3^2, 3^3, 3^4, 3^5, ...\}$

(ii) We have, $a_1 = 1, a_2 = 1$ and $a_{n+2} = a_{n+1}+a_n$.

Putting $n = 1, 2, 3, 4, ...$ in $a_{n+2}=a_{n+1}+a_n$, we get

$$a_3=a_2+a_1=1+1=2;\ a_4 = a_3+a_2 = 2+1 = 3;\ a_5 = a_4+a_3 = 3+2 = 5;$$

$$a_6 = a_5+a_4 = 5+3 = 8 \text{ and so on.}$$

Hence, $B = \{a_1, a_2, a_3, a_4, a_5, a_6, ...\} = \{1, 1, 2, 3, 5, 8, ...\}$

EXERCISE 1.2

LEVEL-1

1. Describe the following sets in Roster form:
   (i) $\{x : x$ is a letter before e in the English alphabet$\}$.
   (ii) $\{x \in N : x^2 < 25\}$.
   (iii) $\{x \in N : x$ is a prime number, $10 < x < 20\}$.
   (iv) $\{x \in N : x = 2n, n \in N\}$.
   (v) $\{x \in R : x > x\}$.
   (vi) $\{x : x$ is a prime number which is a divisor of $60\}$.
   (vii) $\{x : x$ is a two digit number such that the sum of its digits is $8\}$.
   (viii) The set of all letters in the word 'Trigonometry'.
   (ix) The set of all letters in the word 'Better'.
2. Describe the following sets in set-builder form:
   (i) $A=\{1, 2, 3, 4, 5, 6\}$ (ii) $B=\{1, 1/2, 1/3, 1/4, 1/4, ...\}$
   (iii) $C=\{0, 3, 6, 9, 12, ...\}$ (iv) $D=\{10, 11, 12, 13, 14, 15\}$
   (v) $E=\{0\}$ (vi) $\{1, 4, 9, 16, \dots, 100\}$
   (vii) $\{2, 4, 6, 8, ...\}$ (viii) $\{5, 25, 125, 625\}$
3. List all the elements of the following sets:
   (i) $A=\{x : x^2 \le 10, x \in Z\}$ (ii) $B = \left\{x : x=\frac{1}{2n-1}, 1 \le n \le 5\right\}$
   (iii) $C=\left\{x : x \text{ is an integer}, -\frac{1}{2} < x < \frac{9}{2}\right\}$
   (iv) $D=\{x : x$ is a vowel in the word "EQUATION"$\}$
   (v) $E=\{x : x$ is a month of a year not having 31 days$\}$
   (vi) $F=\{x : x$ is a letter of the word "MISSISSIPPI"$\}$

4. Match each of the sets on the left in the roster form with the same set on the right described in the set-builder form:

(i) $\{A, R, L, E\}$	(i) $\{x : x + 5 = 5, x \in Z\}$
(ii) $\{5, -5\}$	(ii) $\{x : x$ is a prime natural number and a divisor of 10 $\}$
(iii) $\{0\}$	(iii) $\{x : x$ is a letter of the word "RAJASTHAN" $\}$
(iv) $\{1, 2, 5, 10\}$	(iv) $\{x : x$ is a natural number and divisor of 10$\}$
(v) $\{A, H, J, R, S, T, N\}$	(v) $\{x : x^2 - 25 = 0\}$
(vi) $\{2, 5\}$	(vi) $\{x : x$ is a letter of the word "APPLE"$\}$

5. Write the set of all vowels in the English alphabet which precede q.
6. Write the set of all positive integers whose cube is odd.
7. Write the set $\left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}, \frac{6}{37}, \frac{7}{50}\right\}$ in the set-builder form.

ANSWERS

1. (i) $\{a, b, c, d\}$ (ii) $\{1, 2, 3, 4\}$ (iii) $\{11, 13, 17, 19\}$ (iv) $\{2, 4, 6, 8, ...\}$
(v) ϕ (vi) $\{2, 3, 5\}$ (vii) $\{17, 26, 35, 44, 53, 62, 71, 80\}$
(viii) $\{T, R, I, G, O, N, M, E, Y\}$ (ix) $\{B, E, T, R\}$
2. (i) $\{x : x \in N, x < 7\}$ (ii) $\{x : x = 1/n, x \in N\}$ (iii) $\{x : x = 3n, n \in Z^+\}$
(iv) $\{x : x \in N, 9 < x < 16\}$ (v) $\{x : x = 0\}$ (vi) $\{x^2 : x \in N, 1 \le x \le 10\}$
(vii) $\{x : x = 2n, n \in N\}$ (viii) $\{5^n : n \in N, 1 \le n \le 4\}$
3. (i) $A = \{0, \pm 1, \pm 2, \pm 3\}$ (ii) $B = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\right\}$
(iii) $C = \{0, 1, 2, 3, 4\}$ (iv) $D = \{A, E, I, O, U\}$
(v) $E = \{$Feb., April, June, Sept., November$\}$ (vi) $F = \{M, I, S, P\}$
4. (i)→ (vi) ; (ii)→ (v); (iii) → (i) ; (iv) → (iv); (v) → (iii); (vi) → (ii)
5. $\{a, e, i, o\}$ 6. $\{2n + 1 : n \in Z, n > 0\}$ 7. $\left\{\frac{n}{n^2 + 1} : n \in N, n \le 7\right\}$

1.3 TYPES OF SETS

EMPTY SET *A set is said to be empty or null or void set if it has no element and it is denoted by ϕ. In Roster method, ϕ is denoted by $\{\ \}$.*

It follows from this definition that a set A is an empty set if the statement $x \in A$ is not true for any x.

ILLUSTRATION 1 $\{x \in R : x^2 = -2\} = \phi$.

ILLUSTRATION 2 $\{x \in N : 5 < x < 6\} = \phi$.

ILLUSTRATION 3 *The set A given by A = $\{x$: x is an even prime number greater than 2$\}$ is an empty set because 2 is the only even prime number.*

A set consisting of at least one element is called a non-empty or non-void set.

NOTE *If A and B are any two empty sets, then $x \in A$ iff (if and only if) $x \in B$ is satisfied because there is no element x in either A or B to which the condition may be applied. Thus, A = B. Hence, there is only one empty set and we denote it by ϕ. Therefore, article 'the' is used before empty set.*

SINGLETON SET *A set consisting of a single element is called a singleton set.*

ILLUSTRATION 4 *The set {5} is a singleton set.*

ILLUSTRATION 5 *The set* $\{x : x \in N \text{ and } x^2 = 9\}$ *is a singleton set equal to* $\{3\}$.

FINITE SET *A set is called a finite set if it is either void set or its elements can be listed (counted, labelled) by natural numbers 1, 2, 3, ... and the process of listing terminates at a certain natural number n (say).*

CARDINAL NUMBER OF A FINITE SET *The number n in the above definition is called the cardinal number or order of a finite set A and is denoted by* $n(A)$.

INFINITE SET *A set whose elements cannot be listed by the natural numbers 1, 2, 3, ..., n, for any natural number n is called an infinite set.*

ILLUSTRATION 6 *Each one of the following sets is a finite set:*

(i) *Set of even natural numbers less than 100.* (ii) *Set of soldiers in Indian army.*
(iii) *Set of even prime natural numbers.* (iv) *Set of all persons on the earth.*

ILLUSTRATION 7 *Each one of the following sets is an infinite set:*

(i) *Set of all points in a plane.* (ii) *Set of all lines in a plane.* (iii) $\{x \in R : 0 < x < 1\}$.

EQUIVALENT SETS *Two finite sets A and B are equivalent if their cardinal numbers are same. i.e.* $n(A) = n(B)$.

EQUAL SETS *Two sets A and B are said to be equal if every element of A is a member of B, and every element of B is a member of A.*

If sets A and B are equal, we write $A = B$ and $A \neq B$ when A and B are not equal.

If $A = \{1, 2, 5, 6\}$ and $B = \{5, 6, 2, 1\}$. Then $A = B$, because each element of A is an element of B and vice-versa. Note that the elements of a set may be listed in any order.

It follows from the above definition and the definition of equivalent sets that equal sets are equivalent but equivalent sets need not be equal.

For example, $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ are equivalent sets but not equal sets.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON IDENTIFYING WHETHER GIVEN SET IS EMPTY OR NOT**

EXAMPLE 1 *Which of the following sets are empty sets?*

(i) $A = \{x : x^2 - 3 = 0 \text{ and } x \text{ is rational}\}$ (ii) $B = \{x : x \text{ is an even prime number}\}$

(iii) $C = \{x : 4 < x < 5, x \in N\}$ (iv) $D = \{x : x^2 = 25, \text{and } x \text{ is an odd integer}\}$

SOLUTION (i) We know that there is no rational number whose square is 3. So, $x^2 - 3 = 0$ is not satisfied by any rational number. Hence, A is an empty set.

(ii) We know that 2 is the only even prime number. Therefore, $B = \{2\}$. So, B is not an empty set.

(iii) Since there is no natural number between 4 and 5. So, C is an empty set.

(iv) Since $x = 5, -5$ satisfy $x^2 = 25$ and ± 5 are odd integers. Therefore, $D = \{-5, 5\}$. Thus, D is a non-empty set.

Type II **ON EQUAL SETS**

EXAMPLE 2 *Find the pairs of equal sets, from the following sets, if any, giving reasons:*

$A = \{0\}$, $B = \{x : x > 15 \text{ and } x < 5\}$, $C = \{x : x - 5 = 0\}$, $D = \{x : x^2 = 25\}$

$E = \{x : x \text{ is an integral positive root of the equation } x^2 - 2x - 15 = 0\}$. [NCERT]

SOLUTION We have.

$$A = \{0\},\ B = \{x : x > 15 \text{ and } x < 5\} = \phi,\ C = \{x : x - 5 = 0\} = \{5\},$$
$$D = \{x : x^2 = 25\} = \{-5, 5\}, \text{ and, } E = \{5\}.$$

Clearly, $C = E$.

EXAMPLE 3 *Which of the following pairs of sets are equal ? Justify your answer.*

(i) $A = \{x : x \text{ is a letter in the word "LOYAL"}\},\ B = \{x : x \text{ is a letter of the word "ALLOY"}\}$

(ii) $A = \{x : x \in Z \text{ and } x^2 \le 8\},\ B = \{x : x \in R \text{ and } x^2 - 4x + 3 = 0\}$

SOLUTION (i) We have,

$$A = \{L, O, Y, A, L\} = \{L, O, Y, A\} \text{ and, } B = \{A, L, L, O, Y\} = \{L, O, Y, A\}$$

Clearly, $A = B$.

(ii) $A = \{x : x \in Z \text{ and } x^2 \le 8\} = \{-2, -1, 0, 1, 2\}$ and, $B = \{x : x \in R \text{ and } x^2 - 4x + 3 = 0\} = \{1, 3\}$.

We observe that $0 \in A$ but $0 \notin B$. So, $A \ne B$.

Type III ON FINITE AND INFINITE SETS

EXAMPLE 4 *State which of the following sets are finite and which are infinite:*

(i) $A = \{x : x \in Z \text{ and } x^2 - 5x + 6 = 0\}$ (ii) $B = \{x : x \in Z \text{ and } x^2 \text{ is even}\}$

(iii) $C = \{x : x \in Z \text{ and } x^2 = 36\}$ (iv) $D = \{x : x \in Z \text{ and } x > -10\}$

SOLUTION (i) $A = \{x : x \in Z \text{ and } x^2 - 5x + 6 = 0\} = \{2, 3\}$

So, A is a finite set

(ii) $B = \{x : x \in Z \text{ and } x^2 \text{ is even}\} = \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$

Clearly, B is an infinite set.

(iii) $C = \{x : x \in Z \text{ and } x^2 = 36\} = \{6, -6\}$

Clearly, A is a finite set.

(iv) $D = \{x : x \in Z \text{ and } x > -10\} = \{-9, -8, -7, \dots\}$

Clearly, D is an infinite set.

EXERCISE 1.3

LEVEL-1

1. Which of the following are examples of empty set ?
   (i) Set of all even natural numbers divisible by 5.
   (ii) Set of all even prime numbers.
   (iii) $\{x : x^2 - 2 = 0 \text{ and } x \text{ is rational}\}$.
   (iv) $\{x : x \text{ is a natural number}, x < 8 \text{ and simultaneously } x > 12\}$.
   (v) $\{x : x \text{ is a point common to any two parallel lines}\}$.
2. Which of the following sets are finite and which are infinite ?
   (i) Set of concentric circles in a plane. (ii) Set of letters of the English Alphabets.
   (iii) $\{x \in N : x > 5\}$ (iv) $\{x \in N : x < 200\}$
   (v) $\{x \in Z : x < 5\}$ (vi) $\{x \in R : 0 < x < 1\}$.
3. Which of the following sets are equal ?
   (i) $A = \{1, 2, 3\}$ (ii) $B = \{x \in R : x^2 - 2x + 1 = 0\}$
   (iii) $C = \{1, 2, 2, 3\}$ (iv) $D = \{x \in R : x^3 - 6x^2 + 11x - 6 = 0\}$.

4. Are the following sets equal ?
   $A = \{x : x$ is a letter in the word reap$\}$, $B = \{x : x$ is a letter in the word paper$\}$,
   $C = \{x : x$ is a letter in the word rope$\}$.
5. From the sets given below, pair the equivalent sets:
   $A = \{1, 2, 3\}$, $B = \{t, p, q, r, s\}$, $C = \{\alpha, \beta, \gamma\}$, $D = \{a, e, i, o, u\}$.
6. Are the following pairs of sets equal ? Give reasons.
   (i) $A = \{2, 3\}$, $B = \{x : x$ is a solution of $x^2 + 5x + 6 = 0\}$
   (ii) $A = \{x : x$ is a letter of the word "WOLF" $\}$,
   $B = \{x : x$ is a letter of the word "FOLLOW" $\}$ [NCERT]
7. From the sets given below, select equal sets and equivalent sets.
   $A = \{0, [illegible]\}$, $B = \{1, 2, 3, 4\}$, $C = \{4, 8, 12\}$, $D = \{3, 1, 2, 4\}$, $E = \{1, 0\}$, $F = \{8, 4, 12\}$
   $G = \{1, 5, 7, 11\}$, $H = \{a, b\}$.
8. Which of the following sets are equal?
   $A = [illegible]$, $B = \{1, 2\}$, $C = \{3, 1\}$,
   $D = \{x : x \in N$, x is odd, $x < 5\}$, $E = \{1, 2, 1, 1\}$, $F = \{1, 1, 3\}$.
9. Show that the set of letters needed to spell "CATARACT" and the set of letters needed to spell "TRACT" are equal. [NCERT]

ANSWERS

1. (iii), (iv), (v) 2. (i) Infinite (ii) finite (iii) Infinite (iv) Finite
   (v) Infinite (vi) Infinite.
3. $A = C = D$ 4. No 5. $A, C ; B, D$ 6. (i) No (ii) Yes
7. Equal sets : $B = D, C = F$ Equivalent sets : $A, E, H ; B, D, G ; C, F$
8. $A = B = E$, $C = D = F$

HINTS TO NCERT & SELECTED PROBLEMS

6. (ii) We have,
   $A = \{x : x$ is a letter of the word "WOLF"$\} = \{W, O, L, F\}$
   $B = \{x : x$ is a letter of the word "FOLLOW"$\} = \{W, O, L, F\}$
   Clearly, $A = B$.
9. $A =$ Set of letters of the word "CATARACT" $= \{A, C, R, T\}$
   $B =$ Set of letters of the word "TRACT" $= \{A, C, R, T\}$
   Clearly, $A = B$.

1.4 SUBSETS

SUBSETS *Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B.*
If A is a subset of B, we write $A \subseteq B$, which is read as "A is a subset of B" or "A is contained in B".
Thus, $A \subseteq B$ iff

$$a \in A \Rightarrow a \in B.$$

The symbol "$\Rightarrow$" stands for "implies".
If A is a subset of B, we say that B contains A or, B is a super set of A and we write $B \supset A$.
If A is not a subset of B, we write $A \not\subseteq B$.
Obviously, every set is a subset of itself and the empty set is subset of every set. A subset A of a set B is called a *proper subset* of B if $A \neq B$ and we write $A \subset B$. In such a case, we also say that B is a super set of A. An improper subset is a subset containing every element of the original set. A proper subset contains some but not all of the elements of the original set. The empty set is a proper subset of [illegible] given set.

Thus, if A is a proper subset of B, then there exists an element $x \in B$ such that $x \notin A$.

It follows immediately from this definition and the definition of equal sets that two sets A and B are equal iff $A \subseteq B$ and $B \subseteq A$.

Thus, whenever it is to be proved that two sets A and B are equal, we must prove that $A \subseteq B$ and $B \subseteq A$.

ILLUSTRATION 1 *Clearly* $\{1\} \subseteq \{1, 2, 3\}$, *but* $\{1, 4\} \not\subseteq \{1, 2, 3\}$.

ILLUSTRATION 2 *Clearly,* $N \subset Z \subset Q \subset R \subset C$, *where* N, Z, Q, R *and* C *have their usual meanings.*

ILLUSTRATION 3 *If A is the set of all divisors of 68 and B is the set of all prime divisors of 68, then B is the subset of A and we write $B \subset A$.*

1.4.1 SOME RESULTS ON SUBSETS

THEOREM 1 *Every set is a subset of itself.*

PROOF Let A be any set. Then, each element of A is clearly in A itself. Hence, $A \subseteq A$.

THEOREM 2 *The empty set is a subset of every set.*

PROOF Let A be any set and ϕ be the empty set. In order to show that $\phi \subseteq A$, we must show that every element of ϕ is an element of A also. But, ϕ contains no element. So, every element of ϕ is in A. Hence, $\phi \subset A$.

THEOREM 3 *The total number of subsets of a finite set containing n elements is 2^n.*

PROOF Let A be a finite set containing n elements. Let $0 \le r \le n$. Consider those subsets of A that have r elements each. We know that the number of ways in which r elements can be chosen out of n elements is ${}^nC_r$. Therefore, the number of subsets of A having r elements each is ${}^nC_r$. Hence, the total number of subsets of A is

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = (1+1)^n = 2^n \qquad \text{[Using binomial theorem]}$$

ILLUSTRATION 1 *Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. Find the values of m and n.*

[NCERT EXEMPLAR]

SOLUTION Let A and B be two sets having m and n elements respectively. Then,

Number of subsets of set $A = 2^m$, Number of subsets of set $B = 2^n$.

It is given that,

$$2^m - 2^n = 56$$
$$\Rightarrow \quad 2^n(2^{m-n} - 1) = 2^3(2^3 - 1)$$
$$\Rightarrow \quad n = 3 \text{ and } m - n = 3$$
$$\Rightarrow \quad n = 3 \text{ and } m = 6.$$

ILLUSTRATION 2 *If* $X = \{4^n - 3n - 1 : n \in N\}$ *and* $Y = \{9(n-1) : n \in N\}$, *prove that* $X \subset Y$.

SOLUTION Let $x_n = 4^n - 3n - 1, n \in N$. Then, $x_1 = 4 - 3 - 1 = 0$.

And, for any $n \ge 2$, we have

$$x_n = 4^n - 3n - 1 = (1+3)^n - 3n - 1$$
$$\Rightarrow \quad x_n = {}^nC_0 + {}^nC_1(3) + {}^nC_2(3^2) + {}^nC_3(3^3) + \dots + {}^nC_n(3^n) - 3n - 1$$

[Using Binomial Theorem]

$$\Rightarrow \quad x_n = 1 + 3n + {}^nC_2(3^2) + {}^nC_3(3^3) + \dots + {}^nC_n(3^n) - 3n - 1 \qquad [\because {}^nC_0 = 1, {}^nC_1 = n]$$
$$\Rightarrow \quad x_n = 3^2\left\{{}^nC_2 + {}^nC_3(3) + {}^nC_4(3^2) + \dots + {}^nC_n(3^{n-2})\right\}$$
$$\Rightarrow \quad x_n = 9\left\{{}^nC_2 + {}^nC_3(3) + {}^nC_4(3^2) + \dots + {}^nC_n(3^{n-2})\right\}.$$

$\Rightarrow \qquad x_n$ is some positive integral multiple of 9 for all $n \geq 2$.

Thus, X consists of all those positive integral multiples of 9 which are of the form $9\left\{{}^nC_2 + 3 \times {}^nC_3 + 3^2 \times {}^nC_4 + \ldots + 3^{n-2} \times {}^nC_n\right\}$ together with 0.

Clearly, $Y = \{9(n-1) : n \in N\}$ consists of all integral multiples of 9 together with 0.

Hence, $X \subset Y$.

1.4.2 SUBSETS OF THE SET *R* OF REAL NUMBERS

Following sets are important subsets of the set R of all real numbers:

(i) The set of all natural numbers $N = \{1, 2, 3, 4, 5, 6, \ldots\}$

(ii) The set of all integers $Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

(iii) The set of all rational numbers $Q = \left\{x : x = \frac{m}{n}, m, n \in Z, n \neq 0\right\}$.

(iv) The set of all irrational numbers. It is denoted by T.

Thus, $T = \{x : x \in R \text{ and } x \notin Q\}$.

Clearly, $N \subset Z \subset Q \subset R, T \subset R$ and $N \not\subset T$.

1.4.3 INTERVALS AS SUBSETS OF *R*

On real line various types of infinite subsets are designated as intervals as defined below:

CLOSED INTERVAL *Let a and b be two given real numbers such that $a < b$. Then, the set of all real numbers x such that $a \leq x \leq b$ is called a closed interval and is denoted by $[a, b]$.*

Thus, $[a, b] = \{x \in R : a \leq x \leq b\}$.

On the real line, $[a, b]$ may be graphed as shown in Fig. 1.1

Fig. 1.1

For example, $[-1, 2] = \{x \in R : -1 \leq x \leq 2\}$ is the set of all real numbers lying between -1 and 2 including the end points. Clearly, it is an infinite subset of R.

OPEN INTERVAL *If a and b are two real numbers such that $a < b$, then the set of all real numbers x satisfying $a < x < b$ is called an open interval and is denoted by (a, b) or $]a, b[$.*

Thus, $(a, b) = \{x \in R : a < x < b\}$

On the real line, (a, b) may be graphed as shown in Fig. 1.2.

Fig. 1.2

Here, encircling a and b means that a and b are not included in the set.

For example, $(1, 2) = \{x \in R : 1 < x < 2\}$ is the set of all real numbers lying between 1 and 2 excluding the end-points 1 and 2. This is an infinite subset of R.

SEMI-OPEN OR SEMI-CLOSED INTERVAL *If a and b are two real numbers such that $a < b$, then the sets $(a, b] = \{x \in R : a < x \leq b\}$ and $[a, b) = \{x \in R : a \leq x < b\}$ are known as semi-open or semi-closed intervals. $(a, b]$ and $[a, b)$ are also denoted by $]a, b]$ and $[a, b[$ respectively.*

On real line these sets may be graphed as shown in Figs. 1.3 and 1.4 respectively.

Fig. 1.3

Fig. 1.4

The number $b - a$ is called the length of any of the intervals (a, b), $[a, b]$, $[a, b)$ and $(a, b]$.

These notations provide an alternative way of designating the subsets of the set R of all real numbers. For example, the interval $[0, \infty)$ denotes the set R^+ of all non-negative real numbers, while the interval $(-\infty, 0)$ denotes the set R^- of all negative real numbers. The interval $(-\infty, \infty)$ denotes the set R of all real numbers.

1.5 UNIVERSAL SET

In any discussion in set theory, there always happens to be a set that contains all sets under consideration i.e. it is a super set of each of the given sets. Such a set is called the universal set and is denoted by U.

Thus, a set that contains all sets in a given context is called the universal set.

ILLUSTRATION 1 *When we study two dimensional coordinate geometry, then the set of all points in xy-plane is the universal set.*

ILLUSTRATION 2 *When we are using sets containing natural numbers, then N is the universal set.*

ILLUSTRATION 3 *If $A = \{1, 2, 3\}$, $B = \{2, 4, 5, 6\}$ and $C = \{1, 3, 5, 7\}$, then $U = \{1, 2, 3, 4, 5, 6, 7\}$ can be taken as the universal set.*

ILLUSTRATION 4 *When we are using intervals on real line, the set R of real numbers is taken as the universal set.*

1.6 POWER SET

POWER SET *Let A be a set. Then the collection or family of all subsets of A is called the power set of A and is denoted by P (A).*

That is, $P(A) = \{S : S \subset A\}$.

Since the empty set and the set A itself are subsets of A and are therefore elements of $P(A)$. Thus, the power set of a given set is always non-empty.

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$. Then, the subsets of A are :* ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$. *Hence, $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.*

ILLUSTRATION 2 *If A is the void set ϕ, then P(A) has just one element ϕ i.e. $P(\phi) = \{\phi\}$.*

ILLUSTRATION 3 *Show that $n\,\{P\,[P\,(P\,(\phi)]\} = 4$.*

SOLUTION We have ,

$$P(\phi) = \{\phi\}$$

$$\therefore \quad P(P(\phi)) = \{\phi, \{\phi\}\}$$

$$\Rightarrow \quad P\,[P\,(P(\phi))] = \{\,\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\,\}.$$

Hence, $P\,[P\,(P\,(\phi))]$ consists of 4 elements i.e. $n\,\{P\,[\,P\,(P\,(\phi)\,)]\,\} = 4$.

REMARK *We know that a set having n elements has 2^n subsets. Therefore, if A is a finite set having n elements, then P(A) has 2^n elements.*

ILLUSTRATION 4 *If $A = \{a, \{b\}\}$, find P (A).*

SOLUTION Let $B = \{b\}$. Then, $A = \{a, B\}$.

$$\therefore \quad P(A) = \{\,\phi, \{a\}, \{B\}, \{a, B\}\} = \{\phi, \{a\}, \{\{b\}\}, \{a, \{b\}\}\}.$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Consider the following sets: ϕ, $A = \{1, 2\}$, $B = \{1, 4, 8\}$, $C = \{1, 2, 4, 6, 8\}$. Insert the correct symbol $\subset$ or $\not\subset$ between each of the following pair of sets:*

(i) $\phi \ldots B$ (ii) $A \ldots B$ (iii) $A \ldots C$ (iv) $B \ldots C$

SOLUTION (i) Since null set is subset of every set. Therefore, $\phi \subset B$.

(ii) Clearly, $2 \in A$ but $2 \notin B$. So, $A \not\subset B$.

(iii) Since all elements of set A are in C and $A \neq C$. So, $A \subset C$.

(iv) Clearly, all elements of set B are in set C and $B \neq C$. So, $B \subset C$.

EXAMPLE 2 *Let $A = \{a, b, c, d\}$, $B = \{a, b, c\}$ and $C = \{b, d\}$. Find all sets X such that:*

(i) $X \subset B$ *and* $X \subset C$ (ii) $X \subset A$ *and* $X \not\subset B$.

SOLUTION (i) We have,

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, ...\}$, $P(B) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

and, $P(C) = \{\phi, \{b\}, \{d\}, \{b, d\}\}$

Now, $X \subset B$ and $X \subset C$

$\Rightarrow$ $X \in P(B)$ and $X \in P(C)$

$\Rightarrow$ $X \in \{\phi, \{b\}\}$

$\Rightarrow$ $X = \phi, \{b\}$

(ii) We have,

$X \subset A$ and $X \not\subset B$

$\Rightarrow$ X is a subset of A but X is not a subset of B

$\Rightarrow$ $X \in P(A)$ but $X \notin P(B)$

$\Rightarrow$ $X = \{d\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c, d\}$.

EXAMPLE 3 *Let A, B and C be three sets. If $A \in B$ and $B \subset C$, is it true that $A \subset C$? If not give an example.*

SOLUTION Consider the following sets: $A = \{a\}$, $B = \{\{a\}, b\}$ and $C = \{\{a\}, b, c\}$.

Clearly, $A \in B$ and $B \subset C$. But, $A \not\subset C$ as $a \in A$ but $a \notin C$. Thus, the given statement is not true.

EXAMPLE 4 *Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3\}$ and $C = \{2, 4\}$. Find all sets X satisfying each pair of conditions:*

(i) $X \subset B$ and $X \not\subset C$ (ii) $X \subset B, X \neq B$ and $X \not\subset C$ (iii) $X \subset A, X \subset B$ and $X \subset C$.

SOLUTION (i) We have,

$X \subset B$ and $X \not\subset C$

$\Rightarrow$ X is a subset of B but X is not a subset of C

$\Rightarrow$ $X \in P(B)$ but $x \notin P(C)$

$\Rightarrow$ $X = \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

(ii) We have,

$X \subset B, X \neq B$ and $X \not\subset C$

$\Rightarrow$ X is a subset of B other than B itself and X is not a subset of C

$\Rightarrow$ $X \in P(B), X \notin P(C)$ and $X \neq B$

$\Rightarrow$ $X = \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$

(iii) We have,

$X \subset A, X \subset B$ and $X \subset C$

$\Rightarrow$ $X \in P(A), X \in P(B)$ and $X \in P(C)$

$\Rightarrow$ X is a subset of A, B and C

$\Rightarrow$ $X = \phi, \{2\}$.

EXAMPLE 5 *Let B be a subset of a set A and let $P(A : B) = \{X \in P(A) : X \supset B\}$.*

(i) *Show that: $P(A : \phi) = P(A)$*

(ii) *If $A = \{a, b, c, d\}$ and $B = \{a, b\}$. List all the members of the set $P(A : B)$.*

SOLUTION (i) We have,

$P(A : B) = \{X \in P(A) : X \supset B\} = \{X \in P(A) : B \subset X\}$

$=$ Set of all those subsets of A which contain B

$\therefore \quad P(A:\phi)$ = Set of all those subsets of A which contain ϕ

= Set of all subsets of set $A = P(A)$.

(ii) If $A = \{a, b, c, d\}$ and $B = \{a, b\}$. Then,

$P(A:B)$ = Set of all those subsets of set A which contain B

$= \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}$

EXAMPLE 6 *Prove that $A \subset \phi$ implies $A = \phi$.*

SOLUTION We know that two sets A and B are equal iff $A \subset B$ and $B \subset A$. Also, we know that

$\phi \subset A$

and, $\quad A \subset \phi$ [Given]

$\therefore \quad A = \phi$

EXAMPLE 7 *In each of the following, determine whether the statement is true or false. If it is true, prove it. If it is false, give an example.*

(i) If $x \in A$ and $A \in B$, then $x \in B$ (ii) *If $A \subset B$ and $B \in C$, then $A \in C$*

(iii) *If $A \subset B$ and $B \subset C$, then $A \subset C$* (iv) *If $A \not\subset B$ and $B \not\subset C$, then $A \not\subset C$*

(v) *If $x \in A$ and $A \not\subset B$, then $x \in B$* (vi) *If $A \subset B$ and $x \notin B$, then $x \notin A$*

SOLUTION (i) False:

Consider sets $A = \{1\}$ and, $B = \{\{1\}, 2\}$.

Clearly $1 \in A$ and $A \in B$, but $1 \notin B$. So, $x \in A$ and $A \in B$ need not imply that $x \in B$.

(ii) False:

Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \{\{1, 2\}, 3\}$. Then, we observe that $A \subset B$ and $B \in C$ but $A \notin C$.

Thus, $A \subset B$ and $B \in C$ need not imply that $A \in C$.

(iii) True:

Let $x \in A$. Then,

$A \subset B \Rightarrow x \in B \Rightarrow x \in C$ $[\because B \subset C]$

Thus, $x \in A \Rightarrow x \in C$ for all $x \in A$. So, $A \subset C$.

Hence, $A \subset B$ and $B \subset C \Rightarrow A \subset C$.

(iv) False:

Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 2, 5\}$. Then, $A \not\subset B$ and $B \not\subset C$. But, $A \subset C$.

Thus, $A \not\subset B$ and $B \not\subset C$ need not imply that $A \not\subset C$.

(v) False:

Let $A = \{1, 2\}$ and $B = \{2, 3, 4, 5\}$. Then, we observe that $1 \in A$ and $A \not\subset B$, but $1 \notin B$.

Thus, $x \in A$ and $A \not\subset B$ need not imply that $x \in B$.

(vi) True:

Let $A \subset B$. Then, we observe that

$x \in A \Rightarrow x \in B \Leftrightarrow x \notin B \Rightarrow x \notin A$.

EXAMPLE 8 *Write the following subsets of R as intervals:*

(i) $\{x : x \in R, -4 < x \leq 6\}$ (ii) $\{x : x \in R, -12 < x < -10\}$

(iii) $\{x : x \in R, 0 \leq x < 7\}$ (iv) $\{x : x \in R, 3 \leq x \leq 4\}$.

Also, find the length of each interval.

SOLUTION (i) $\{x : x \in R, -4 < x \leq 6\} = (-4, 6]$. Length $= 6 - (-4) = 10$

(ii) $\{x : x \in R, -12 < x < -10\} = (-12, -10)$. Length $= -10 - (-12) = 2$

(iii) $\{x : x \in R, 0 \leq x < 7\} = [0, 7)$. Length $= 7 - 0 = 7$

(iv) $\{x : x \in R, 3 \leq x \leq 4\} = [3, 4]$. Length $= 4 - 3 = 1$

EXAMPLE 9 *Write the following intervals in the set-builder form:*

(i) $(-7, 0)$ (ii) $[6, 12]$ (iii) $(6, 12]$ (iv) $[-20, 3)$

SOLUTION (i) $(-7, 0) = \{x : x \in R$ and $-7 < x < 0\}$

(ii) $[6, 12] = \{x : x \in R$ and $6 \leq x \leq 12\}$

(iii) $(6, 12] = \{x : x \in R \text{ and } 6 < x \leq 12\}$

(iv) $[-20, 3) = \{x : x \in R \text{ and } -20 \leq x < 3\}$

EXAMPLE 10 *Let* $A = \{1, 2, \{3, 4\}, 5\}$. *Which of the following statements are incorrect and why?*

(i) $\{3, 4\} \subset A$ (ii) $\{3, 4\} \in A$ (iii) $\{\{3, 4\}\} \subset A$ (iv) $1 \in A$

(v) $1 \subset A$ (vi) $\{1, 2, 5\} \subset A$ (vii) $\{1, 2, 5\} \in A$ (viii) $\{1, 2, 3\} \subset A$

(ix) $\phi \in A$ (x) $\phi \subset A$ (xi) $\{\phi\} \subset A$

SOLUTION $\{3, 4\}$ is an element of set A. Therefore, $\{3, 4\} \in A$ is correct and $\{3, 4\} \subset A$ is incorrect.

So, (i) is incorrect and (ii) is correct. As $\{3, 4\}$ is an element of set A. Therefore, $\{\{3, 4\}\}$ is a set containing element $\{3, 4\}$ which belongs to A. So, $\{\{3, 4\}\} \subset A$.

Hence, (iii) is correct.

Since 1 is an element of set A. So, $1 \in A$ is correct and $1 \subset A$ is incorrect.

So, (iv) is correct and (v) is incorrect.

Since 1, 2, 5 are elements of set A. Therefore, $\{1, 2, 5\}$ is a subset of set A.

Hence, (vi) is correct and (vii) is incorrect.

As 3 is not an element of set A. So, $\{1, 2, 3\} \subset A$ is incorrect. The null set is subset of every set.

So, $\phi \subset A$ is correct and $\phi \in A$ is incorrect. Hence, (ix) is incorrect and (x) is correct.

As $\phi \subset A$ but $\{\phi\}$ is not a subset of A. So, (xi) is incorrect.

EXERCISE 1.4

LEVEL-1

1. Which of the following statements are true ? Give reason to support your answer.
   (i) For any two sets A and B either $A \subseteq B$ or $B \subseteq A$.
   (ii) Every subset of an infinite set is infinite.
   (iii) Every subset of a finite set is finite.
   (iv) Every set has a proper subset.
   (v) $\{a, b, a, b, a, b, \ldots\}$ is an infinite set.
   (vi) $\{a, b, c\}$ and $\{1, 2, 3\}$ are equivalent sets.
   (vii) A set can have infinitely many subsets.
2. State whether the following statements are true or false:
   (i) $1 \in \{1, 2, 3\}$ (ii) $a \subset \{b, c, a\}$ (iii) $\{a\} \in \{a, b, c\}$
   (iv) $\{a, b\} = \{a, a, b, b, a\}$ (v) The set $\{x : x + 8 = 8\}$ is the null set.
3. Decide among the following sets, which are subsets of which:
   $A = \{x : x \text{ satisfies } x^2 - 8x + 12 = 0\}$, $B = \{2, 4, 6\}$, $C = \{2, 4, 6, 8, \ldots\}$, $D = \{6\}$.
4. Write which of the following statements are true ? Justify your answer.
   (i) The set of all integers is contained in the set of all rational numbers.
   (ii) The set of all crows is contained in the set of all birds.
   (iii) The set of all rectangles is contained in the set of all squares.
   (iv) The set of all real numbers is contained in the set of all complex numbers.
   (v) The sets $P = \{a\}$ and $B = \{\{a\}\}$ are equal.
   (vi) The sets $A = \{x : x$ is a letter of the word "LITTLE"$\}$ and, $B = \{x : x$ is a letter of the word "TITLE"$\}$ are equal.
5. Which of the following statements are correct? Write a correct form of each of the incorrect statements.
   (i) $a \subset \{a, b, c\}$ (ii) $\{a\} \in \{a, b, c\}$ (iii) $a \in \{\{a\}, b\}$

(iv) $\{a\} \subset \{\{a\}, b\}$ (v) $\{b, c\} \subset \{a, \{b, c\}\}$ (vi) $\{a, b\} \subset \{a, \{b, c\}\}$
(vii) $\phi \in \{a, b\}$ (viii) $\phi \subset \{a, b, c\}$ (ix) $\{x : x + 3 = 3\} = \phi$

6. Let $A = \{a, b, \{c, d\}, e\}$. Which of the following statements are false and why?
(i) $\{c. d\} \subset A$ (ii) $\{c, d\} \in A$ (iii) $\{\{c, d\}\} \subset A$
(iv) $a \in A$ (v) $a \subset A$ (vi) $\{a, b, e\} \subset A$
(vii) $\{a, b, e\} \in A$ (viii) $\{a, b, c\} \subset A$ (ix) $\phi \in A$
(x) $\{\phi\} \subset A$

7. Let $A = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$. Determine which of the following is true or false:
(i) $1 \in A$ (ii) $\{1, 2, 3\} \subset A$ (iii) $\{6, 7, 8\} \in A$
(iv) $\{\{4, 5\}\} \subset A$ (v) $\phi \in A$ (vi) $\phi \subset A$

8. Let $A = \{\phi, \{\phi\}, 1, \{1, \phi\}, 2\}$. Which of the following are true?
(i) $\phi \in A$ (ii) $\{\phi\} \in A$ (iii) $\{1\} \in A$
(iv) $\{2, \phi\} \subset A$ (v) $2 \subset A$ (vi) $\{2, \{1\}\} \not\subset A$
(vii) $\{\{2\}, \{1\}\} \not\subset A$ (viii) $\{\phi, \{\phi\}, \{1, \phi\}\} \subset A$ (ix) $\{\{\phi\}\} \subset A$.

9. Write down all possible subsets of each of the following sets:
(i) $\{a\}$ (ii) $\{0, 1\}$ (iii) $\{a, b, c\}$
(iv) $\{1, \{1\}\}$ (v) $\{\phi\}$

10. Write down all possible proper subsets each of the following sets:
(i) $\{1, 2\}$ (ii) $\{1, 2, 3\}$ (iii) $\{1\}$

11. What is the total number of proper subsets of a set consisting of n elements?

12. If A is any set, prove that: $A \subseteq \phi \Leftrightarrow A = \phi$.

13. Prove that: $A \subseteq B$, $B \subseteq C$ and $C \subseteq A \Rightarrow A = C$.

14. How many elements has $P(A)$, if $A = \phi$?

15. What universal set (s) would you propose for each of the following:
(i) The set of right triangles. (ii) The set of isosceles triangles.

LEVEL-2

16. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49(n-1) : n \in N\}$, then prove that $X \subseteq Y$.

ANSWERS

1. (i) F, $A = \{1, 2, 3\}$, $B = \{a, b\}$ (ii) F, $A = \{1, 2\}$ is a finite subset of N.
(iii) T (iv) F, ϕ does not have a proper subset
(v) F, Given set $= \{a, b\}$ (vi) T (vii) F

2. (i) T (ii) F (iii) F (iv) T (v) F 3. $D \subset A \subset B \subset C$

4. (i) T (ii) T (iii) F (iv) T (v) F (vi) T

5. (i) $a \in \{a, b, c\}$ (ii) $\{a\} \subset \{a, b, c\}$ (iii) $\{a\} \in \{\{a\}, b\}$
(iv) $\{\{a\}\} \subset \{\{a\}, b\}$ (v) $\{b, c\} \in \{a, \{b, c\}\}$ (vi) $\{a, b\} \not\subset \{a, \{b, c\}\}$
(vii) $\phi \subset \{a, b\}$ (viii) $\phi \subset \{a, b, c\}$ (ix) $\{x : x + 3 = 3\} \neq \phi$

6. (i) F (ii) T (iii) T (iv) T (v) F (vi) T
(vii) F (viii) F (ix) F (x) F

7. (i) F (ii) F (iii) T (iv) T (v) F (vi) T

8. (i) T (ii) T (iii) F (iv) T (v) F (vi) T (vii) T
(viii) T (ix) T

9. (i) $\phi, \{a\}$ (ii) $\phi, \{0\}, \{1\}, \{0, 1\}$ (iii) $\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$
(iv) $\phi, \{1\}, \{\{1\}\}, \{1, \{1\}\}$ (v) $\phi, \{\phi\}$.

10. (i) $\phi, \{1\}, \{2\}$ (ii) $\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}$ (iii) ϕ

11. $2^n - 1$ 14. 1

15. (i) The setof all triangles in a plane. (ii) The set of all triangles in a plane.

HINTS TO SELECTED PROBLEM

16. Let $x_n = 8^n - 7n - 1 = (1+7)^n - 7n - 1 = {}^nC_2\, 7^2 + {}^nC_3\, 7^3 + \dots + {}^nC_n\, 7^n$

$= 49\,({}^nC_2 + {}^nC_3\, 7 + \dots + {}^nC_n\, 7^{n-2})$ for $n \geq 2$.

For $n = 1$, $x_n = 0$. Thus, X contains all positive integral multiples of 49 of the form $49\,k_n$, where $k_n = {}^nC_2 + {}^nC_3\,(7) + {}^nC_4\,(7^2) + \dots + {}^nC_n\,(7^{n-2})$.

Also, Y contains all positive integral multiples of 49 including zero. Thus, $X \subseteq Y$.

1.7 VENN DIAGRAMS

Sometimes pictures are very helpful in our thinking. First of all a Swiss mathematician Euler gave an idea to represent a set by the points in a closed curve. Later on British mathematician John-Venn (1834-1883) brought this idea to practice. That is why the diagrams drawn to represent sets are called *Venn-Euler diagrams* or simply Venn-diagrams. In Venn-diagrams the universal set U is represented by points within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle. If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B as shown in Fig. 1.5 (i). If A and B are not equal but they have some common elements, then to represent A and B we draw two intersecting circles. (See Fig. 1.5 (ii)). Two disjoint sets are represented by two non-intersecting circles. (See Fig. 1.5 (iii))

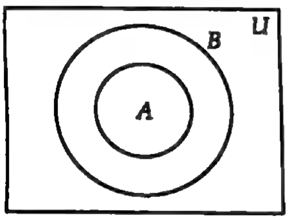

Fig. 1.5 (i)

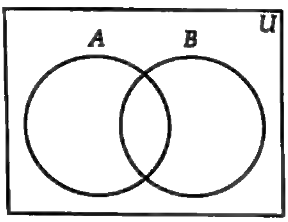

Fig. 1.5 (ii)

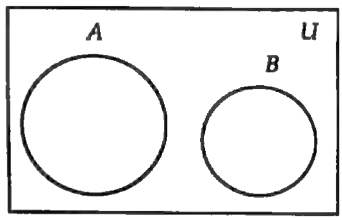

Fig. 1.5 (iii)

1.8 OPERATIONS ON SETS

In this section, we shall introduce some operations on sets to construct new sets from given ones.

UNION OF SETS *Let A and B be two sets. The union of A and B is the set of all those elements which belong either to A or to B or to both A and B.*

We shall use the notation $A \cup B$ (read as "A union B") to denote the union of A and B.

Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Clearly, $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$.

And, $x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B$.

In Fig. 1.6 the shaded part represents $A \cup B$. It is evident from the definition that $A \subseteq A \cup B$, $B \subseteq A \cup B$.

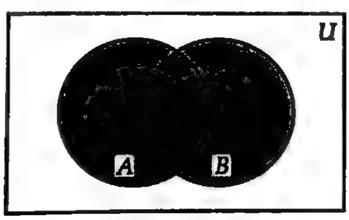

Fig. 1.6

If A and B are two sets such that $A \subset B$, then $A \cup B = B$. Also, $A \cup B = A$, if $B \subset A$.

ILLUSTRATION 1 *If $A = \{1, 2, 3\}$ and $B = \{1, 3, 5, 7\}$, then $A \cup B = \{1, 2, 3, 5, 7\}$.*

ILLUSTRATION 2 *If $A = \{x : x = 2n + 1, n \in Z\}$ and $B = \{x : x = 2n, n \in Z\}$, then*

$$A \cup B = \{x : x \text{ is an odd integer}\} \cup \{x : x \text{ is an even integer}\} = \{x : x \text{ is an integer}\} = Z.$$

NOTE *If A_1, A_2, ..., A_n is a finite family of sets, then their union is denoted by $\bigcup_{i=1}^{n} A_i$ or, $A_1 \cup A_2 \cup A_3 \ldots \cup A_n$.*

ILLUSTRATION 3 *Let $A = \{1, 2, 3\}$, $B = \{3, 5\}$, $C = \{4, 7, 8\}$. Then, $A \cup B \cup C = \{1, 2, 3, 4, 5, 7, 8\}$*

INTERSECTION OF SETS *Let A and B be two sets. The intersection of A and B is the set of all those elements that belong to both A and B.*

The intersection of A and B is denoted by $A \cap B$ (read as "A intersection B").

Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Clearly, $x \in A \cap B \iff x \in A$ and $x \cap B$

In Fig. 1.7 the shaded region represents $A \cap B$. Evidently, $A \cap B \subseteq A$, $A \cap B \subseteq B$.

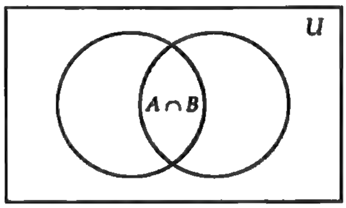

Fig. 1.7

If A and B are two sets, then $A \cap B = A$, if $A \subset B$ and $A \cap B = B$, if $B \subset A$.

NOTE *If $A_1, A_2, \ldots, A_n$ is a finite family of sets, then their intersection is denoted by*

$$\bigcap_{i=1}^{n} A_i \text{ or, } A_1 \cap A_2 \cap \ldots \cap A_n.$$

ILLUSTRATION 4 If $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 9, 12\}$, then $A \cap B = \{1, 3\}$.

ILLUSTRATION 5 If $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{2, 4, 6, 8, 10\}$ *and* $C = \{4, 6, 7, 8, 9, 10, 11\}$, *then* $A \cap B = \{2, 4, 6\}$. *Therefore,* $A \cap B \cap C = \{4, 6\}$.

ILLUSTRATION 6 *If $A = \{x : x = 2n, n \in Z\}$ and $B = \{x : x = 3n, n \in Z\}$, then*

$$\begin{aligned} A \cap B &= \{x : x = 2n, n \in Z\} \cap \{x : x = 3n, n \in Z\} \\ &= \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\} \cap \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\} \\ &= \{\ldots, -6, 0, 6, 12, \ldots\} = \{x : x = 6n, n \in Z\}. \end{aligned}$$

ILLUSTRATION 7 *If $A = \{x : x = 3n, n \in Z\}$ and $B = \{x : x = 4n, n \in Z\}$, then find $A \cap B$.*

SOLUTION Clearly,

$x \in A \cap B$

$\iff$ $x = 3n$ and $x = 4n, n \in Z$

$\iff$ x is a multiple of 3 and x is a multiple of 4

$\iff$ x is a multiple of 3 and 4 both

$\iff$ x is a multiple of 12.

$\iff$ $x = 12n$, $n \in Z$

Hence, $A \cap B = \{x : x = 12n, n \in Z\}$.

DISJOINT SETS *Two sets A and B are said to be disjoint, if $A \cap B = \phi$.*

If $A \cap B \neq \phi$, then A and B are said to be intersecting or overlapping sets.

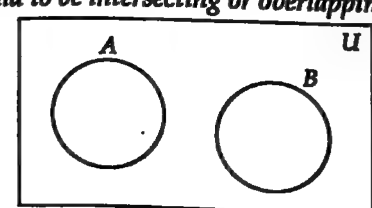

Fig. 1.8

ILLUSTRATION 8 *If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{7, 8, 9, 10, 11\}$ and $C = \{6, 8, 10, 12, 14\}$, then A and B are disjoint sets, while A and C are intersecting sets.*

DIFFERENCE OF SETS *Let A and B be two sets. The difference of A and B, written as $A - B$, is the set of all those elements of A which do not belong to B.*

Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$

or, $A - B = \{x \in A : x \notin B\}$.

Clearly, $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$.

In Fig. 1.9, the shaded part represents $A - B$.

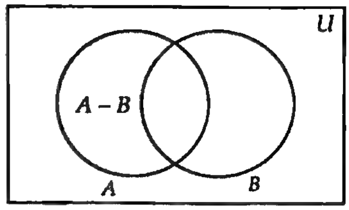

Fig. 1.9

Similarly, the difference $B - A$ is the set of all those elements of B that do not belong to A i.e. $B - A = \{x \in B : x \notin A\}$.

In Fig. 1.10, the shaded part represents $B - A$.

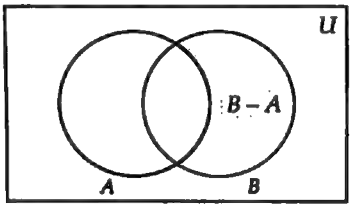

Fig. 1.10

ILLUSTRATION 9 *If $A = \{2, 3, 4, 5, 6, 7\}$ and $B = \{3, 5, 7, 9, 11, 13\}$, then $A - B = \{2, 4, 6\}$ and $B - A = \{9, 11, 13\}$.*

SYMMETRIC DIFFERENCE OF TWO SETS *Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.*

Thus, $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$.

The shaded part in Fig. 1.11 represents $A \Delta B$.

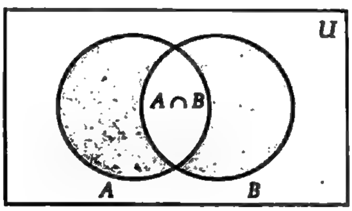

Fig. 1.11

ILLUSTRATION 10 *If $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 3, 5, 6, 7, 8, 9\}$, then $A - B = \{2, 4\}$, $B - A = \{9\}$*

$\therefore \quad A \Delta B = \{2, 4, 9\}$.

ILLUSTRATION 11 *If $A = \{x \in R : 0 < x < 3\}$ and $B = \{x \in R : 1 \leq x \leq 5\}$, then*

$A - B = \{x \in R : 0 < x < 1\}$, $B - A = \{x \in R : 3 \leq x \leq 3\}$

and, $A \Delta B = \{x \in R : 0 < x < 1\} \cap \{x \in R : 3 \leq x \leq 5\} = \{x \in R : 0 < x < 1 \text{ or } 3 \leq x \leq 5\}$.

COMPLEMENT OF A SET *Let U be the universal set and let A be a set such that $A \subset U$. Then, the complement of A with respect to U is denoted by A' or A^C or $U - A$ and is defined the set of all those elements of U which are not in A.*

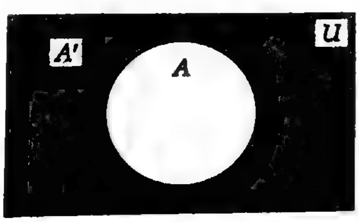

Fig. 1.12

Thus $A' = \{x \in U : x \notin A\}$. Clearly, $x \in A' \Leftrightarrow x \notin A$.

ILLUSTRATION 12 *Let the set of natural numbers $N = \{1, 2, 3, 4, ...\}$ be the universal set and let $A = \{2, 4, 6, 8, ...\}$. Then $A' = \{1, 3, 5, ...\}$.*

ILLUSTRATION 13 *If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{1, 3, 5, 7, 9\}$, then $A' = \{2, 4, 6, 8\}$.*

Following results are direct consequences of the definition of the complement of a set:

(i) $U' = \{x \in U : x \notin U\} = \phi$

(ii) $\phi' = \{x \in U : x \notin \phi\} = U$

(iii) $(A')' = \{x \in U : x \notin A'\} = \{x \in U : x \in A\} = A$

(iv) $A \cup A' = \{x \in U : x \in A\} \cup \{x \in U : x \notin A\} = U$

(v) $A \cap A' = \{x \in U : x \in A\} \cap \{x \in U : x \notin A\} = \phi$.

EXERCISE 1.5

LEVEL-1

1. If A and B are two sets such that $A \subset B$, then Find:
   (i) $A \cap B$ (ii) $A \cup B$
2. If $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, $C = \{7, 8, 9, 10, 11\}$ and $D = \{10, 11, 12, 13, 14\}$. Find:
   (i) $A \cup B$ (ii) $A \cup C$ (iii) $B \cup C$
   (iv) $B \cup D$ (v) $A \cup B \cup C$ (vi) $A \cup B \cup D$
   (vii) $B \cup C \cup D$ (viii) $A \cap (B \cup C)$ (ix) $(A \cap B) \cap (B \cap C)$
   (x) $(A \cup D) \cap (B \cup C)$.
3. Let $A = \{x : x \in N\}$, $B = \{x : x = 2n, n \in N\}$, $C = \{x : x = 2n - 1, n \in N\}$ and, $D = \{x : x$ is a prime natural number $\}$. Find:
   (i) $A \cap B$ (ii) $A \cap C$ (iii) $A \cap D$
   (iv) $B \cap C$ (v) $B \cap D$ (vi) $C \cap D$
4. Let $A = \{3, 6, 12, 15, 18, 21\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$ and $D = \{5, 10, 15, 20\}$. Find:
   (i) $A - B$ (ii) $A - C$ (iii) $A - D$
   (iv) $B - A$ (v) $C - A$ (vi) $D - A$
   (vii) $B - C$ (viii) $B - D$
5. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. Find:
   (i) A' (ii) B' (iii) $(A \cap C)'$
   (iv) $(A \cup B)'$ (v) $(A')'$ (vi) $(B - C)'$
6. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{2, 3, 5, 7\}$. Verify that:
   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

ANSWERS

1. (i) A (ii) B
2. (i) $\{1, 2, 3, 4, 5, 6, 7, 8\}$ (ii) $\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11\}$
   (iii) $\{4, 5, 6, 7, 8, 9, 10, 11\}$ (iv) $\{4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}$
   (v) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ (vi) $\{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}$
   (vii) $\{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ (viii) $\{4, 5\}$ (ix) ϕ (x) $\{4, 5, 10, 11\}$
3. (i) B (ii) C (iii) D

(iv) ϕ (v) $\{2\}$ (vi) $D-\{2\}$

4. (i) $\{3, 6, 15, 18, 21\}$ (ii) $\{3, 15, 18, 21\}$ (iii) $\{3, 6, 12, 18, 21\}$
(iv) $\{4, 8, 16, 20\}$ (v) $\{2, 4, 8, 10, 14, 16\}$ (vi) $\{5, 10, 20\}$
(vii) $\{20\}$ (viii) $\{4, 8, 12, 16\}$

5. (i) $\{5, 6, 7, 8, 9\}$ (ii) $\{1, 3, 5, 7, 9\}$ (iii) $\{1, 2, 5, 6, 7, 8, 9\}$
(iv) $\{5, 7, 9\}$ (v) A (vi) $\{1, 3, 4, 5, 6, 7, 9\}$

1.9 LAWS OF ALGEBRA OF SETS

In this section, we shall state and prove some fundamental laws of algebra of sets.

THEOREM 1 *(Idempotent Laws) For any set A,*

(i) $A \cup A = A$ (ii) $A \cap A = A$.

PROOF (i) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$

(ii) $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A$.

THEOREM 2 *(identity Laws) For any set A,*

(i) $A \cup \phi = A$ (ii) $A \cap U = A$.

i.e. ϕ and U are identity elements for union and intersection respectively.

PROOF (i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$

(ii) $A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A$

THEOREM 3 *(Commutative Laws) For any two sets A and B*

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

i.e. union and intersection are commutative.

PROOF Recall that two sets X and Y are equal iff $X \subseteq Y$ and $Y \subseteq X$. Also, $X \subseteq Y$ if every element of X belongs to Y.

(i) Let x be an arbitrary element of $A \cup B$. Then,

$$x \in A \cup B \Rightarrow x \in A \text{ or } x \in B \Rightarrow x \in B \text{ or } x \in A \Rightarrow x \in B \cup A$$

$\therefore$ $A \cup B \subseteq B \cup A$.

Similarly, $B \cup A \subseteq A \cup B$.

Hence, $A \cup B = B \cup A$.

THEOREM 4 *(Associative Laws) If A, B and C are any three sets, then*

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$ [NCERT EXEMPLAR]

i.e. union and intersection are associative.

PROOF (i) Let x be an arbitrary element of $(A \cup B) \cup C$. Then,

$x \in (A \cup B) \cup C$

$\Rightarrow$ $x \in (A \cup B)$ or $x \in C$

$\Rightarrow$ $(x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow$ $x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cup C)$

$\Rightarrow$ $x \in A \cup (B \cup C)$

$\therefore$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

Similarly, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

Hence, $(A \cup B) \cup C = A \cup (B \cup C)$.

(ii) Let x be an arbitrary element of $A \cap (B \cap C)$. Then,

$x \in A \cap (B \cap C)$

$\Rightarrow$ $x \in A$ and $x \in (B \cap C)$

$\Rightarrow \quad x \in A$ and $(x \in B$ and $x \in C)$

$\Rightarrow \quad (x \in A$ and $x \in B)$ and $x \in C$

$\Rightarrow \quad x \in (A \cap B)$ and $x \in C$

$\Rightarrow \quad x \in (A \cap B) \cap C$

$\therefore \quad A \cap (B \cap C) \subseteq (A \cap B) \cap C.$

Similarly, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

Hence, $A \cap (B \cap C) = (A \cap B) \cap C$.

THEOREM 5 *(Distributive Laws) If A, B and C are any three sets, then*

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i.e. union and intersection are distributive over intersection and union respectively.

[NCERT EXEMPLAR]

PROOF (i) Let x be an arbitrary element of $A \cup (B \cap C)$. Then,

$x \in A \cup (B \cap C)$

$\Rightarrow \quad x \in A$ or $x \in (B \cap C)$

$\Rightarrow \quad x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow \quad (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$ [$\because$ 'or' is distributive over 'and']

$\Rightarrow \quad x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow \quad x \in ((A \cup B) \cap (A \cup C))$

$\therefore \quad A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Similarly, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(ii) Let x be an arbitrary element of $A \cap (B \cup C)$. Then,

$x \in A \cap (B \cup C)$

$\Rightarrow \quad x \in A$ and $x \in (B \cup C)$

$\Rightarrow \quad x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow \quad (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow \quad x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow \quad x \in (A \cap B) \cup (A \cap C)$

$\therefore \quad A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Similarly, $(A \cup B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

THEOREM 6 *(De-Morgan's Laws) If A and B are any two sets, then*

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

PROOF (i) Let x be an arbitrary element of $(A \cup B)'$. Then,

$x \in (A \cup B)'$

$\Rightarrow \quad x \notin (A \cup B)$

$\Rightarrow \quad x \notin A$ and $\notin B$

$\Rightarrow \quad x \in A'$ and $x \in B'$

$\Rightarrow \quad x \in A' \cap B'$.

$\therefore \quad (A \cap B)' \subseteq A' \cap B'$.

Again, let y be an arbitrary element of $A' \cap B'$. Then,

$y \in A' \cap B'$

$\Rightarrow \quad y \in A'$ and $y \in B'$

$\Rightarrow \quad y \notin A$ and $y \notin B$

$\Rightarrow \quad y \notin A \cup B.$

$\Rightarrow \quad y \in (A \cup B)'$

$\therefore \quad A' \cap B' \subseteq (A \cup B)'$

Hence, $(A \cup B)' = A' \cap B'$.

(ii) Let x be an arbitrary element of $(A \cap B)'$. Then,

$$x \in (A \cap B)'$$
$$\Rightarrow \quad x \notin (A \cap B)$$
$$\Rightarrow \quad x \notin A \text{ or } x \notin B$$
$$\Rightarrow \quad x \in A' \text{ or } x \in B'$$
$$\Rightarrow \quad x \in A' \cup B'$$
$$\Rightarrow \quad (A \cap B)' \subseteq A' \cup B'.$$

Again, let y be an arbitrary element of $A' \cup B'$. Then,

$$y \in A' \cup B'$$
$$\Rightarrow \quad y \in A' \text{ or } y \in B'$$
$$\Rightarrow \quad y \notin A \text{ or } y \notin B$$
$$\Rightarrow \quad y \notin (A \cap B)$$
$$\Rightarrow \quad y \in (A \cap B)'$$
$$\therefore \quad A' \cup B' \subseteq (A \cap B)'.$$

Hence, $(A \cap B)' = A' \cup B'$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If $a \in N$ such that $aN = \{ax : x \in N\}$. Describe the set $3N \cap 7N$.*

SOLUTION We have, $aN = \{ax : x \in N\}$

$\therefore \quad 3N = \{3x : x \in N\} = \{3, 6, 9, 12, \ldots,\}$ and, $7N = \{7x : x \in N\} = \{7, 14, 21, 28, \ldots\}$

Hence, $3N \cap 7N = \{21, 42, \ldots\} = \{21x : x \in N\} = 21N$.

EXAMPLE 2 *If $A = \{1, 3, 5, 7, 11, 13, 15, 17\}$, $B = \{2, 4, 6, \ldots, 18\}$ and N is the universal set, then find $A' \cup ((A \cup B) \cap B')$.*

SOLUTION Clearly, $(A \cup B) \cap B' = A$ [$\because$ A, B are disjoint sets]

$\therefore \quad A' \cup ((A \cup B) \cap B') = A' \cup A = N$.

EXAMPLE 3 *For any natural number a, we define $aN = \{ax : x \in N\}$. If b, c, $d \in N$ such that $bN \cup cN = dN$, then prove that d is the l.c.m. of b and c.*

SOLUTION We have,

$bN = \{bx : x \in N\}$ = The set of positive integral multiples of b

$cN = \{cx : x \in N\}$ = The set of positive integral multiples of c

$\therefore \quad bN \cap cN$ = The set of positive integral multiples of b and c both.

$\Rightarrow \quad bN \cap cN = \{kx : x \in N\}$, where k is the l.c.m. of b and c.

Hence, d = l.c.m. of b and c.

LEVEL-2

EXAMPLE 4 *Suppose $A_1, A_2, \ldots A_{30}$ are thirty sets each with five elements and $B_1, B_2, \ldots, B_n$ are n sets each with three elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$. Assume that each element of S belongs to exactly ten of the A_i's and exactly 9 of B_j's. Find n.*

SOLUTION Since each A_i has 5 elements and each element of S belongs to exactly 10 of A_i's.

$$\therefore \quad S = \bigcup_{i=1}^{30} A_i \Rightarrow n(S) = \frac{1}{10} \sum_{i=1}^{30} n(A_i) = \frac{1}{10}(5 \times 30) = 15 \qquad \ldots(i)$$

Again, each B_j has 3 elements and each element of S belongs to exactly 9 of B_j's

$$\therefore \quad S = \bigcup_{j=1}^{n} B_j \Rightarrow n(S) = \frac{1}{9}\sum_{j=1}^{n} n(B_j) = \frac{1}{9}(3n) = \frac{n}{3} \quad \text{...(ii)}$$

From (i) and (ii), we get $: 15 = \frac{n}{3} \Rightarrow n = 45$.

EXAMPLE 5 *For any two sets A and B, prove that $A \cup B = A \cap B \Leftrightarrow A = B$.*

SOLUTION First let $A = B$. Then,

$$A \cup B = A \text{ and } A \cap B = A \Rightarrow A \cup B = A \cap B$$

Thus, $A = B \Rightarrow A \cup B = A \cap B$ ...(i)

Conversely, let $A \cup B = A \cap B$. Then, we have to prove that $A = B$. For this, let

$x \in A \Rightarrow x \in A \cup B$

$\Rightarrow x \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$

$\therefore$ $A \subset B$ ...(ii)

Now, let

$y \in B \Rightarrow y \in A \cup B$

$\Rightarrow y \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow y \in A$ and $y \in B$

$\Rightarrow y \in A$

$\therefore$ $B \subset A$ ...(iii)

From (ii) and (iii), we get $A = B$.

Thus, $A \cup B = A \cap B \Rightarrow A = B$ ...(iv)

From (i) and (iv), we obtain

$$A \cup B = A \cap B \Leftrightarrow A = B.$$

EXAMPLE 6 *Let A, B and C be the sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.*

SOLUTION We have,

$A \cup B = A \cup C$

$\Rightarrow$ $(A \cup B) \cap C = (A \cup C) \cap C$

$\Rightarrow$ $(A \cap C) \cup (B \cap C) = C$ $[\because (A \cup C) \cap C = C]$

$\Rightarrow$ $(A \cap B) \cup (B \cap C) = C$ $[\because A \cap C = A \cap B]$ ...(i)

Again, $A \cup B = A \cup C$

$\Rightarrow$ $(A \cup B) \cap B = (A \cup C) \cap B$

$\Rightarrow$ $B = (A \cap B) \cup (C \cap B)$ $[\because (A \cup B) \cap B = B]$

$\Rightarrow$ $B = (A \cap B) \cup (B \cap C)$ ...(ii)

From (i) and (ii), we get $B = C$.

EXAMPLE 7 *Let A and B be sets, if $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$ for some set X, prove that $A = B$.*

SOLUTION We have,

$A \cup X = B \cup X$ for some set X

$\Rightarrow$ $A \cap (A \cup X) = A \cap (B \cup X)$

$\Rightarrow$ $A = (A \cap B) \cup (A \cap X)$ $[\because A \cap (A \cup X) = A]$

$\Rightarrow$ $A = (A \cap B) \cup \phi$ $[\therefore A \cap X = \phi \text{(given)}]$

$\Rightarrow$ $A = A \cap B$

$\Rightarrow$ $A \subset B$ ...(i)

Again, $A \cup X = B \cup X$

$\Rightarrow \quad B \cap (A \cup X) = B \cap (B \cup X)$

$\Rightarrow \quad (B \cap A) \cup (B \cap X) = B \qquad [\therefore B \cap (B \cup X) = B]$

$\Rightarrow \quad (B \cap A) \cup \phi = B \qquad [\because B \cap X = \phi \text{(given)}]$

$\Rightarrow \quad B \cap A = B$

$\Rightarrow \quad A \cap B = B$

$\Rightarrow \quad B \subset A \qquad$ ...(ii)

From (i) and (ii), we get $A = B$.

EXAMPLE 8 *For any two sets A and B, prove that:* $P(A) = P(B) \Rightarrow A = B$.

SOLUTION Let x be an arbitrary element of A. Then, there exists a subset, say X, of set A such that $x \in X$.

Now, $\quad X \subset A$

$\Rightarrow \quad X \in P(A)$

$\Rightarrow \quad X \in P(B) \qquad [\because P(A) = P(B)]$

$\Rightarrow \quad X \subset B$

$\Rightarrow \quad x \in B \qquad [\because x \in X \text{ and } X \subset B \therefore x \in B]$

Thus, $\quad x \in A \Rightarrow x \in B$ for all $x \in B$.

$\therefore \quad A \subset B \qquad$ ...(i)

Now, let y be an arbitrary element of B. Then, there exists a subset, say Y, of set B such that $y \in Y$.

Now, $\quad Y \subset B$

$\Rightarrow \quad Y \in P(B)$

$\Rightarrow \quad Y \in P(A) \qquad [\because P(A) = P(B)]$

$\Rightarrow \quad Y \subset A$

$\Rightarrow \quad y \in A$

Thus, $\quad y \in B \Rightarrow y \in A$ for all $y \in B$.

$\therefore \quad B \subset A \qquad$ ...(ii)

From (i) and (ii), we obtain $A = B$.

EXAMPLE 9 *For any two sets A and B prove that:* $P(A \cap B) = P(A) \cap P(B)$.

SOLUTION In order to prove that $P(A \cap B) = P(A) \cap P(B)$, it is sufficient to prove that

$$P(A \cap B) \subset P(A) \cap P(B) \text{ and } P(A) \cap P(B) \subset P(A \cap B).$$

First let

$\quad X \in P(A \cap B)$

$\Rightarrow \quad X \subset A \cap B$

$\Rightarrow \quad X \subset A$ and $X \subset B$

$\Rightarrow \quad X \in P(A)$ and $X \in P(B)$

$\Rightarrow \quad X \in P(A) \cap P(B)$

$\therefore \quad P(A \cap B) \subset P(A) \cap P(B) \qquad$ ...(i)

Now, let

$\quad Y \in P(A) \cap P(B)$. Then,

$\quad Y \in P(A) \cap P(B)$

$\Rightarrow \quad Y \in P(A)$ and $Y \in P(B)$

$\Rightarrow \quad Y \subset A$ and $Y \subset B$

$\Rightarrow \quad Y \subset A \cap B$

$\Rightarrow \quad Y \in P(A \cap B)$

$\therefore \quad P(A) \cap P(B) \subset P(A \cap B) \qquad$ ...(ii)

From (i) and (ii), we get: $P(A \cap B) = P(A) \cap P(B)$.

EXAMPLE 10 *For any two sets A and B prove that* $P(A) \cup P(B) \subset P(A \cup B)$. *But,* $P(A \cup B)$ *is not necessarily a subset of* $P(A) \cup P(B)$.

SOLUTION Let $X \in P(A) \cup P(B)$. Then,

$$\begin{aligned} & X \in P(A) \cup P(B) \\ \Rightarrow\quad & X \in P(A) \text{ or } X \in P(B) \\ \Rightarrow\quad & X \subset A \text{ or } X \subset B \\ \Rightarrow\quad & X \subset A \cup B \\ \Rightarrow\quad & X \in P(A \cup B) \\ \therefore\quad & P(A) \cup P(B) \subset P(A \cup B) \end{aligned}$$

Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$. Then, we find that $X = \{1, 2, 3, 4\} \subset (A \cup B)$. Therefore, $X \in P(A \cup B)$. But, $X \notin P(A)$, $X \notin P(B)$. So, $X \notin P(A) \cup P(B)$. Thus, $P(A \cup B)$ is not necessarily a subset of $P(A) \cup P(B)$.

EXERCISE 1.6

LEVEL-1

1. Find the smallest set A such that $A \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$.
2. Let $A = \{1, 2, 4, 5\}$ $B = \{2, 3, 5, 6\}$ $C = \{4, 5, 6, 7\}$. Verify the following identities:
   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$ (iv) $A - (B \cup C) = (A - B) \cap (A - C)$
   (v) $A - (B \cap C) = (A - B) \cup (A - C)$ (vi) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.
3. If $U = \{2, 3, 5, 7, 9\}$ is the universal set and $A = \{3, 7\}$, $B = \{2, 5, 7, 9\}$, then prove that:
   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

LEVEL-2

4. For any two sets A and B, prove that
   (i) $B \subset A \cup B$ (ii) $A \cap B \subset A$ (iii) $A \subset B \Rightarrow A \cap B = A$
5. For any two sets A and B, show that the following statements are equivalent:
   (i) $A \subset B$ (ii) $A - B = \phi$ (iii) $A \cup B = B$ (iv) $A \cap B = A$.
6. For three sets A, B and C, show that
   (i) $A \cap B = A \cap C$ need not imply $B = C$. (ii) $A \subset B \Rightarrow C - B \subset C - A$
7. For any two sets, prove that:
   (i) $A \cup (A \cap B) = A$ (ii) $A \cap (A \cup B) = A$
8. Find sets A, B and C such that $A \cap B, A \cap C$ and $B \cap C$ are non-empty sets and $A \cap B \cap C = \phi$.
9. For any two sets A and B, prove that: $A \cap B = \phi \Rightarrow A \subseteq B'$.
10. If A and B are sets, then prove that $A - B, A \cap B$ and $B - A$ are pair wise disjoint.
11. Using properties of sets, show that for any two sets A and B, $(A \cup B) \cap (A \cup B') = A$.
12. For any two sets of A and B, prove that:
    (i) $A' \cup B = U \Rightarrow A \subset B$ (ii) $B' \subset A' \Rightarrow A \subset B$
13. Is it true that for any sets A and B, $P(A) \cup P(B) = P(A \cup B)$? Justify your answer.
14. Show that for any sets A and B,
    (i) $A = (A \cap B) \cap (A - B)$ (ii) $A \cup (B - A) = A \cup B$
15. Each set X_r contains 5 elements and each set Y_r contains 2 elements and $\bigcup_{r=1}^{20} X_r = S = \bigcup_{r=1}^{n} Y_r$. If each element of S belongs to exactly 10 of the $X_r's$ and to exactly 4 of $Y_r's$, then find the value of n.

ANSWERS

1. $A = \{3,5,9\}$ 7. $A = \{1, 2\}, B = \{1, 3\}, C = \{2, 3\}$ 12. False 15. 20

HINTS TO SELECTED PROBLEMS

5. (i) $\Rightarrow$ (ii)
We know that, $A - B = \{x \in A : x \notin B\}$
Since $A \subset B$. Therefore, there is no element in A which does not belong to B.
$\therefore \quad A - B = \phi$
Hence, (i) $\Rightarrow$ (ii).
(ii) $\Rightarrow$ (ii)
We have, $A - B = \phi \Rightarrow A \subset B \Rightarrow A \cup B = B$
Hence, (ii) $\Rightarrow$ (iii).
(iii) $\Rightarrow$ (iv)
We have, $A \cup B = B \Rightarrow A \subset B \Rightarrow A \cap B = A$
Hence, (iii) $\Rightarrow$ (iv).
(iv) $\Rightarrow$ (i)
We have, $A \cap B = A \Rightarrow A \subset B$
Hence, (iv) $\Rightarrow$ (i).
Consequently, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

6. (i) Let $A = \{1, 2, 3, 4\}, B = \{3, 4, 5, 6\}$ and $C = \{1, 3, 4, 7, 8\}$. Then, $A \cap B = A \cap C$, but $B \neq C$.
(ii) Let $x \in C - B$. Then,
$x \in C - B$
$\Rightarrow x \in C$ and $x \notin B$
$\Rightarrow x \in C$ and $x \notin A$ $\quad [\because A \subset B]$
$\Rightarrow x \in C - A$
$\therefore \quad C - B \subset C - A$

7. (i) We have,
$A \cup (A \cap B) = (A \cup A) \cap (A \cup B)$ $\quad [\because \cup$ is distributive over $\cap]$
$= A \cap (A \cup B) = A$ $\quad [\because A \subset A \cup B]$
(ii) We have,
$A \cap (A \cup B) = (A \cap A) \cup (A \cap B)$ $\quad [\because \cap$ is distributive over $\cup]$
$= A \cup (A \cap B) = A$ $\quad [\because A \cap B \subset A]$

8. $A = \{1, 2\}, B = \{1, 3\}, C = \{2, 3\}$

9. $x \in A \Rightarrow x \notin B$ $\quad [\because A \cap B = \phi]$
$\Rightarrow x \in B'$.
So, $A \subset B'$

10. We have,
$A - B = \{x : x \in A \text{ and } x \notin B\}, B - A = \{x \in B \text{ and } x \notin A\}$
$\therefore$ $A - B$ and $B - A$ are disjoint sets
Now,
$x \in A - B \Rightarrow x \in A \text{ and } x \notin B \Rightarrow x \notin A \cap B$
$\therefore$ $(A - B)$ and $A \cap B$ are disjoint sets.
Similarly, $B - A$ and $A \cap B$ are disjoint sets.

11. $(A \cup B) \cap (A \cup B') = ((A \cup B) \cap A) \cup ((A \cup B) \cup B')$
$= A \cup ((A \cup B) \cap B') = A \cup (A \cap B') \cup (B \cap B')$
$= A \cup (A \cap B') = A$

12. (i) Let $x \in A$. Then,

$$x \in A \Rightarrow x \in U \Rightarrow x \in A' \cup B \Rightarrow x \in B \qquad [\because\ x \notin A']$$

$\therefore$ $A \subset B$

(ii) Let $x \in A$. Then,

$$x \in A \Rightarrow x \notin A' \Rightarrow x \notin B' \Rightarrow x \in B \qquad [\because\ B' \subset A']$$

$\therefore$ $A \subset B$

1.10 MORE RESULTS ON OPERATIONS ON SETS

THEOREM 1 *If A and B are any two sets, then*

(i) $A - B = A \cap B'$ (ii) $B - A = B \cap A'$

(iii) $A - B = A \Leftrightarrow A \cap B = \phi$ (iv) $(A - B) \cup B = A \cup B$

(v) $(A - B) \cap B = \phi$ (vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$

(vii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

PROOF (i) Let x be an arbitrary element of $A - B$. Then,

$x \in (A - B)$

$\Rightarrow$ $x \in A$ and $x \notin B$

$\Rightarrow$ $x \in A$ and $x \in B'$

$\Rightarrow$ $x \in A \cap B'$

$\therefore$ $A - B \subseteq A \cap B'$ ...(i)

Again, let y be an arbitrary element of $A \cap B'$. Then,

$y \in A \cap B'$

$\Rightarrow$ $y \in A$ and $y \in B'$

$\Rightarrow$ $y \in A$ and $y \notin B$

$\Rightarrow$ $y \in A - B$

$\therefore$ $A \cap B' \subseteq (A - B)$ ...(ii)

Hence, from (i) and (ii), we obtain $A - B = A \cap B'$.

(ii) Proceed as in (i).

(iii) In order to prove that $A - B = A \Leftrightarrow A \cap B = \phi$, we shall prove that :

(a) $A - B = A \Rightarrow A \cap B = \phi$ and, (b) $A \cap B = \phi \Rightarrow A - B = A$.

First, let $A - B = A$. Then we have to prove that $A \cap B = \phi$. If possible, let $A \cap B \neq \phi$. Then,

$A \cap B \neq \phi$

$\Rightarrow$ There exists $x \in A \cap B$

$\Rightarrow$ $x \in A$ and $x \in B$

$\Rightarrow$ $x \in A - B$ and $x \in B$ $[\because A - B = A]$

$\Rightarrow$ $(x \in A$ and $x \notin B)$ and $x \in B$ [By def. of $A - B$]

$\Rightarrow$ $x \in A$ and $(x \notin B$ and $x \in B)$

But $x \notin B$ and $x \in B$ both can never be possible simultaneously. Thus, we arrive at a contradiction. So, our supposition is wrong. Therefore, $A \cap B = \phi$.

Hence, $A - B = A \Rightarrow A \cap B = \phi$ ...(i)

Conversely, let $A \cap B = \phi$. Then we have to prove that $A - B = A$. For this we shall show that $A - B \subseteq A$ and $A \subseteq A - B$.

Let x be an arbitrary element of $A - B$. Then,

$x \in A - B \Rightarrow x \in A$ and $x \notin B \Rightarrow x \in A$

$\therefore$ $A - B \subseteq A$.

Again let y be an arbitrary element of A. Then,

$y \in A$

$\Rightarrow$ $y \in A$ and $y \notin B$ $[\because\ A \cap B = \phi]$

$\Rightarrow \quad y \in A - B$ [By def. of $A - B$]

$\therefore \quad A \subseteq A - B.$

So, we have $A - B \subseteq A$ and $A \subseteq A - B$. Therefore, $A - B = A$.

Thus, $A \cap B = \phi \Rightarrow A - B = A$ ...(ii)

Hence, from (i) and (ii), we have

$$A - B = A \Leftrightarrow A \cap B = \phi.$$

(iv) Let x be an arbitrary element of $(A - B) \cup B$. Then,

$x \in (A - B) \cup B$

$\Rightarrow \quad x \in A - B$ or $x \in B$

$\Rightarrow \quad (x \in A$ and $x \notin B)$ or $x \in B$

$\Rightarrow \quad (x \in A$ or $x \in B)$ and $(x \notin B$ or $x \in B)$

$\Rightarrow \quad x \in A \cup B$

$\therefore \quad (A - B) \cup B \subseteq A \cup B$

Let y be an arbitrary element of $A \cup B$. Then,

$y \in A \cup B$

$\Rightarrow \quad y \in A$ or $y \in B$

$\Rightarrow \quad (y \in A$ or $y \in B)$ and $(y \notin B$ or $y \in B)$

$\Rightarrow \quad (y \in A$ and $y \notin B)$ or $y \in B$

$\Rightarrow \quad y \in (A - B) \cup B$

$\therefore \quad A \cup B \subseteq (A - B) \cup B$

Hence, $(A - B) \cup B = A \cup B$

(v) If possible let $(A - B) \cap B \neq \phi$ Then, there exists at least one element x, (say), in $(A - B) \cap B$.

Now, $x \in (A - B) \cap B$

$\Rightarrow \quad x \in (A - B)$ and $x \in B$

$\Rightarrow \quad (x \in A$ and $x \notin B)$ and $x \in B$

$\Rightarrow \quad x \in A$ and $(x \notin B$ and $x \in B)$

But, $x \notin B$ and $x \in B$ both can never be possible simultaneously. Thus, we arrive at a contradiction. So, our supposition is wrong. Hence, $(A - B) \cap B = \phi$.

(vi) First, let $A \subseteq B$. Then we have to prove that $B' \subseteq A'$. Let x be an arbitrary element of B'. Then,

$x \in B'$

$\Rightarrow \quad x \notin B$

$\Rightarrow \quad x \notin A$ [$\because A \subseteq B$]

$\Rightarrow \quad x \in A'$

$\therefore \quad B' \subseteq A'.$

Thus, $A \subseteq B \Rightarrow B' \subseteq A'$. ...(i)

Conversely, let $B' \subseteq A'$. Then, we have to prove that $A \subseteq B$. Let y be an arbitrary element of A. Then,

$y \in A$

$\Rightarrow \quad y \notin A'$

$\Rightarrow \quad y \notin B'$ [$\because B' \subseteq A'$].

$\Rightarrow \quad y \in B$

$\therefore \quad A \subseteq B.$

Thus, $B' \subseteq A' \Rightarrow A \subseteq B$ ...(ii)

From (i) and (ii), we obtain that $A \subseteq B \Leftrightarrow B' \subseteq A'$.

(vii) Let x be an arbitrary element of $(A - B) \cup (B - A)$. Then,

$$x \in (A - B) \cup (B - A)$$
$\Rightarrow$ $x \in A - B$ or $x \in B - A$
$\Rightarrow$ $(x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$
$\Rightarrow$ $(x \in A$ or $x \notin B)$ and $(x \notin B$ or $x \notin A)$
$\Rightarrow$ $x \in (A \cup B)$ and $x \notin (A \cap B)$
$\Rightarrow$ $x \in (A \cup B) - (A \cap B)$
$\therefore$ $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$. ...(i)

Again, let y be an arbitrary element of $(A \cup B) - (A \cap B)$. Then,

$$y \in (A \cup B) - (A \cap B)$$
$\Rightarrow$ $y \in A \cup B$ and $y \notin A \cap B$
$\Rightarrow$ $(y \in A$ or $y \in B)$ and $(y \notin A$ or $y \notin B)$
$\Rightarrow$ $(y \in A$ and $y \notin B)$ or $(y \in B$ and $y \notin A)$
$\Rightarrow$ $y \in (A - B)$ or $y \in (B - A)$
$\Rightarrow$ $y \in (A - B) \cup (B - A)$
$\therefore$ $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$. ...(ii)

Hence, from (i) and (ii), we have

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

THEOREM 2 *If A, B and C are any three sets, then prove that:*

(i) $A - (B \cap C) = (A - B) \cup (A - C)$ (ii) $A - (B \cup C) = (A - B) \cap (A - C)$
(iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$ (iv) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

PROOF (i) Let x be any element of $A - (B \cap C)$. Then,

$$x \in A - (B \cap C)$$
$\Rightarrow$ $x \in A$ and $x \notin (B \cap C)$
$\Rightarrow$ $x \in A$ and $(x \notin B$ or $x \notin C)$
$\Rightarrow$ $(x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$
$\Rightarrow$ $x \in (A - B)$ or $x \in (A - C)$
$\Rightarrow$ $x \in (A - B) \cup (A - C)$
$\therefore$ $A - (B \cap C) \subseteq (A - B) \cup (A - C)$

Similarly, $(A - B) \cup (A - C) \subseteq A - (B \cap C)$

Hence, $A - (B \cap C) = (A - B) \cup (A - C)$

(ii) Let x be an arbitrary element of $A - (B \cup C)$. Then

$$x \in A - (B \cup C)$$
$\Rightarrow$ $x \in A$ and $x \notin (B \cup C)$
$\Rightarrow$ $x \in A$ and $(x \notin B$ and $x \notin C)$
$\Rightarrow$ $(x \in A$ and $x \notin B)$ and $(x \in A$ and $x \notin C)$
$\Rightarrow$ $x \in (A - B)$ and $x \in (A - C)$
$\Rightarrow$ $x \in (A - B) \cap (A - C)$
$\therefore$ $A - (B \cup C) \subseteq (A - B) \cap (A - C)$

Similarly, $(A - B) \cap (A - C) \subseteq A - (B \cup C)$

Hence, $A - (B \cup C) = (A - B) \cap (A - C)$

(iii) Let x be any arbitrary element of $A \cap (B - C)$. Then

$$x \in A \cap (B - C)$$
$\Rightarrow$ $x \in A$ and $x \in (B - C)$
$\Rightarrow$ $x \in A$ and $(x \in B$ and $x \notin C)$

$\Rightarrow \quad (x \in A \text{ and } x \in B) \text{ and } (x \in A \text{ and } x \notin C)$

$\Rightarrow \quad x \in (A \cap B) \text{ and } x \notin (A \cap C)$

$\Rightarrow \quad x \in (A \cap B) - (A \cap C)$

$\therefore \quad A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$

Similarly, $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$

Hence, $A \cap (B - C) = (A \cap B) - (A \cap C)$.

(iv) $A \cap (B \Delta C) = A \cap [(B - C) \cup (C - B)]$

$= [A \cap (B - C)] \cup [A \cap (C - B)]$ [By distributive law]

$= [(A \cap B) - (A \cap C)] \cup [(A \cap C) - (A \cap B)]$ [Using (iii)]

$= (A \cap B) \Delta (A \cap C)$

ILLUSTRATIVE EXAMPLES

LEVEL-2

EXAMPLE 1 *Let A and B be two sets. Using properties of sets prove that:*

(i) $A \cap B' = \phi \Rightarrow A \subset B$ (ii) $A' \cup B = U \Rightarrow A \subset B$

SOLUTION (i) We have,

$A = (A \cap U)$

$\Rightarrow \quad A = A \cap (B \cup B')$ [$\because B \cup B' = U$]

$\Rightarrow \quad A = (A \cap B) \cup (A \cap B')$ [$\because \cap$ is distributive over union]

$\Rightarrow \quad A = (A \cap B) \cup \phi$ [$\because A \cap B' = \phi$]

$\Rightarrow \quad A = A \cap B$

$\therefore \quad A \subset B$

(ii) From (i), we have

$A \cap B' = \phi$

$\Leftrightarrow \quad (A \cap B')' = \phi'$

$\Leftrightarrow \quad A' \cup (B')' = U$ [$\because \phi' = U$]

$\Leftrightarrow \quad A' \cup B = U$ [$\because (B')' = B$]

Thus, $A \cap B' = \phi \Leftrightarrow A' \cup B = U$ and, $A \cap B' = \phi \Rightarrow A \subset B$

$\therefore \quad A' \cup B = U \Rightarrow A \subset B$

ALITER We have,

$A' \cup B = U$

$\Rightarrow \quad A \cap (A' \cup B) = A \cap U$ [Taking intersection with A]

$\Rightarrow \quad (A \cap A') \cup (A \cap B) = A$ [$\because A \cap U = A$]

$\Rightarrow \quad \phi \cup (A \cap B) = A$

$\Rightarrow \quad A \cap B = A$

$\Rightarrow \quad A \subset B$

EXAMPLE 2 *Let A and B be two sets. Prove that:* $(A - B) \cup B = A$ *if and only if* $B \subset A$.

SOLUTION First let, $(A - B) \cup B = A$. Then, we have to prove that $B \subset A$.

Now, $(A - B) \cup B = A$

$\Rightarrow \quad (A \cap B') \cup B = A$ [$\because A - B = A \cap B'$]

$\Rightarrow \quad (A \cup B) \cap (B' \cup B) = A$

$\Rightarrow \quad (A \cup B) \cap U = A$

$\Rightarrow \quad A \cup B = A$

$\Rightarrow \quad B \subseteq A$

Conversely, let $B \subseteq A$. Then, we have to prove that $(A - B) \cup B = A$.

Now, $(A - B) \cup B = (A \cap B') \cup B$

$= (A \cup B) \cap (B' \cup B)$

$$= (A \cup B) \cap U$$
$$= A \cup B$$
$$= A \qquad [\because B \subset A \therefore A \cup B = A]$$

EXAMPLE 3 *Let A, B and C be three sets such that $A \cup B = C$ and $A \cap B = \phi$. Then, prove that $A = C - B$.*

SOLUTION We have, $A \cup B = C$.

$$\therefore \quad C - B = (A \cup B) - B$$
$$= (A \cup B) \cap B' \qquad [\because X - Y = X \cap Y']$$
$$= (A \cap B') \cup (B \cap B')$$
$$= (A \cap B') \cup \phi$$
$$= A \cap B'$$
$$= A - B$$
$$= A \qquad [\because A \cap B = \phi]$$

EXAMPLE 4 *Let A and B be any two sets. Using properties of sets prove that:*

(i) $(A - B) \cup B = A \cup B$ (ii) $(A - B) \cup A = A$
(iii) $(A - B) \cap B = \phi$ (iv) $(A - B) \cap A = A \cap B'$

SOLUTION (i) We have,

$$(A - B) \cup B = (A \cap B') \cup B \qquad [\because A - B = A \cap B']$$
$$= (A \cup B) \cap (B' \cup B) \qquad [\because \cup \text{ is distributive over } \cap]$$
$$= (A \cup B) \cap U \qquad [\because B' \cup B = U]$$
$$= A \cup B$$

(ii) $(A - B) \cup A = A \qquad [\because A - B \subset A]$

(iii) $(A - B) \cap B = (A \cap B') \cap B = A \cap (B' \cap B) = A \cap \phi = \phi$

(iv) $(A - B) \cap A = A - B \qquad [\because A - B \subset A]$

$$= A \cap B'$$

EXAMPLE 5 *For any two sets A and B prove by using properties of sets that:*

(i) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ (ii) $(A \cap B) \cup (A - B) = A$
(iii) $(A \cup B) - A = B - A$

SOLUTION (i) We have,

$$(A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)' \qquad [\because X - Y = X \cap Y']$$
$$= (A \cup B) \cap (A' \cup B') \qquad [\because (A \cap B)' = A' \cup B']$$
$$= X \cap (A' \cup B'), \text{ where } X = A \cup B$$
$$= (X \cap A') \cup (X \cap B')$$
$$= (B \cap A') \cup (A \cap B') \qquad \left[\begin{array}{l} \because X \cap A' = (A \cup B) \cap A' \\ = (A \cap A') \cup (B \cap A') = \phi \cup (B \cap A') \\ = B \cap A' \text{ Similarly, } X \cap B' = A \cap B' \end{array}\right]$$
$$= (A \cap B') \cup (B \cap A')$$
$$= (A - B) \cup (B - A) \qquad [\because A - B = A \cap B' \text{ and } B - A = B \cap A']$$

(ii) $(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap B')$

$$= X \cup (A \cap B'), \text{ where } X = A \cap B$$
$$= (X \cup A) \cap (X \cup B')$$
$$= A \cap (A \cup B') \qquad \left[\begin{array}{l} \because X \cup A = (A \cap B) \cup A = A \ [\because A \cap B \subset A] \\ X \cup B' = (A \cap B) \cup B' = (A \cup B') \cap (B \cup B') \\ = (A \cup B') \cap U = A \cup B' \end{array}\right]$$
$$= A \qquad [\because A \subset A \cup B']$$

(iii) $(A \cup B) - A = (A \cup B) \cap A' \qquad [\because X - Y = X \cap Y']$

$$= (A \cap A') \cup (B \cap A')$$
$$= \phi \cup (B \cap A') \qquad [\because A \cap A' = \phi]$$

$= B \cap A'$

$= B - A$ $[\because B - A = B \cap A']$

EXAMPLE 6 *For sets A, B and C using properties of sets, prove that:*

(i) $A - (B \cup C) = (A - B) \cap (A - C)$ [NCERT EXEMPLAR]

(ii) $A - (B \cap C) = (A - B) \cup (A - C)$

(iii) $(A \cup B) - C = (A - C) \cup (B - C)$

(iv) $(A \cap B) - C = (A - C) \cap (B - C)$ [NCERT EXEMPLAR]

SOLUTION (i) We have,

$A - (B \cup C) = A \cap (B \cup C)'$ $[\because X - Y = X \cap Y']$

$= A \cap (B' \cap C')$ $[\because (B \cup C)' = B' \cap C']$

$= (A \cap B') \cap (A \cap C')$

$= (A - B) \cap (A - C)$

(ii) $A - (B \cap C) = A \cap (B \cap C)'$ $[\because X - Y = X \cap Y']$

$= A \cap (B' \cup C')$ $[\because (B \cap C)' = B' \cup C']$

$= (A \cap B') \cup (A \cap C')$ [$\because$ $\cap$ is distributive over $\cup$]

$= (A - B) \cup (A - C)$

(iii) $(A \cup B) - C = (A \cup B) \cap C'$ $[\because X - Y = X \cap Y']$

$= (A \cap C') \cup (B \cap C')$

$= (A - C) \cup (B - C)$

(iv) $(A \cap B) - C = (A \cap B) \cap C'$

$= (A \cap C') \cap (B \cap C')$

$= (A - C) \cap (B - C)$

EXAMPLE 7 *For sets A, B and C using properties of sets, prove that:*

(i) $A - (B - C) = (A - B) \cup (A \cap C)$ (ii) $A \cap (B - C) = (A \cap B) - (A \cap C)$

SOLUTION (i) We have,

$A - (B - C) = A - (B \cap C')$ $[\because B - C = B \cap C']$

$= A \cap (B \cap C')'$ $[\because X - Y = X \cap Y']$

$= A \cap (B' \cup C)$ $[\because (B \cap C')' = B' \cup (C')' = B' \cup C]$

$= (A \cap B') \cup (A \cap C)$

$= (A - B) \cup (A \cap C)$

(ii) $A \cap (B - C) = A \cap (B \cap C')$ $[\because B - C = B \cap C']$

$= (A \cap B) \cap C'$

$= \phi \cup ((A \cap B) \cap C')$

$= ((A \cap B) \cap A') \cup ((A \cap B) \cap C')$ $[\because (A \cap B) \cap A' = \phi]$

$= (A \cap B) \cap (A' \cup C')$

$= (A \cap B) \cap (A \cap C)'$

$= (A \cap B) - (A \cap C)$

EXERCISE 1.7

LEVEL-1

1. For any two sets A and B, prove that : $A' - B' = B - A$
2. For any two sets A and B, prove the following :
   (i) $A \cap (A' \cup B) = A \cap B$ (ii) $A - (A - B) = A \cap B$
   (iii) $A \cap (A \cup B)' = \phi$ (iv) $A - B = A \,\Delta\, (A \cap B)$.
3. If A, B, C are three sets such that $A \subset B$, then prove that $C - B \subset C - A$.

4. For any two sets A and B, prove that
   (i) $(A \cup B) - B = A - B$ (ii) $A - (A \cap B) = A - B$
   (iii) $A - (A - B) = A \cap B$ (iv) $A \cup (B - A) = A \cup B$ [NCERT EXEMPLAR]
   (v) $(A - B) \cup (A \cap B) = A$ [NCERT EXEMPLAR]

HINTS TO SELECTED PROBLEMS

1. We know that $X - Y = X \cap Y'$. So $A' - B' = A' \cap (B')' = A' \cap B = B \cap A' = B - A$
2. (ii) $A - (A - B) = A - (A \cap B') = A \cap (A \cap B')' = A \cap (A' \cup (B')') = A \cap (A' \cup B) = A \cap B$
3. Let $x \in C - B$. Then,
   $x \in C - B \Rightarrow x \in C$ and $x \notin B \Rightarrow x \in C$ and $x \notin A \Rightarrow x \in C - A$ $[\because A \subset B]$
   $\therefore\ C - B \subset C - A$.

1.11 SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets.

(iii) $n(A - B) = n(A) - n(A \cap B)$ i.e. $n(A - B) + n(A \cap B) = n(A)$

(iv) $n(A \Delta B) =$ No. of elements which belong to exactly one of A or B

$$= n((A - B) \cup (B - A))$$
$$= n(A - B) + n(B - A) \qquad [\because (A - B) \text{ and } (B - A) \text{ are disjoint}]$$
$$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$$
$$= n(A) + n(B) - 2\,n(A \cap B)$$

(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(vi) Number of elements in exactly two of the sets A, B, C

$$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3\,n(A \cap B \cap C).$$

(vii) Number of elements in exactly one of the sets A, B, C

$$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$$

(viii) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

(ix) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If X and Y are two sets such that $n(X) = 17, n(Y) = 23$ and $n(X \cup Y) = 38$, find $n(X \cap Y)$.*

SOLUTION We have,

$$n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$$
$$\Rightarrow \quad 38 = 17 + 23 - n(X \cap Y) \Rightarrow n(X \cap Y) = 40 - 38 = 2.$$

EXAMPLE 2 *In a group of 800 people, 550 can speak Hindi and 450 can speak English. How many can speak both Hindi and English?*

SOLUTION Let H denote the set of people speaking Hindi and E denote the set of people speaking English. We are given that: $n(H) = 550, n(E) = 450$ and $n(H \cup E) = 800$.
We have to find $n(H \cap E)$.
We know that

$$n(H \cup E) = n(H) + n(E) - n(H \cap E)$$
$$\Rightarrow \quad n(H \cap E) = n(H) + n(E) - n(H \cup E) = 550 + 450 - 800 = 200.$$

Hence, 200 persons can speak both Hindi and English.

EXAMPLE 3 *In a class of 35 students, 24 like to play cricket and 16 like to play football. Also, each student likes to play at least one of the two games. How many students like to play both cricket and football?*

SOLUTION Let C be the set of students who like to play cricket and F be the set of students who like to play football. Then, $C \cup F$ is the set of students who like to play at least one game and, $C \cap F$ is the set of all students who like to play both games. It is given that $n(C) = 24, n(F) = 16$, $n(C \cup F) = 35$ and we have to find $n(C \cap F)$.

Now, $n(C \cup F) = n(C) + n(F) - n(C \cap F)$

$\Rightarrow \quad n(C \cap F) = n(C) + n(F) - n(C \cup F) = 24 + 16 - 35 = 5.$

EXAMPLE 4 *In a group of 50 people, 35 speak Hindi, 25 speak both English and Hindi and all the people speak at least one of the two languages. How many people speak only English and not Hindi ? How many people speak English?*

SOLUTION Let H denote the set of people speaking Hindi and E the set of people speaking English. Then, it is given that: $n(H \cup E) = 50, n(H) = 35, n(H \cap E) = 25$.

Now, $n(E - H) = n(H \cup E) - n(H) = 50 - 35 = 15$

Thus, the number of people speaking English but not Hindi is 15.

Now, $n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$\Rightarrow \quad 50 = 35 + n(E) - 25 \Rightarrow n(E) = 40$

Hence, the number of people who speak English is 40.

EXAMPLE 5 *There are 40 students in a Chemistry class and 60 students in a Physics class. Find the number of students which are either in Physics class or Chemistry class in the following cases:*

(i) *the two classes meet at the same hour.*

(ii) *the two classes meet at different hours and 20 students are enrolled in both the subjects.*

SOLUTION Let A be the set of students in Chemistry class and B be the set of students in Physics class. It is given that $n(A) = 40$ and $n(B) = 60$. We have to find $n(A \cup B)$ in both the cases.

(i) If two classes meet at the same hour, then there will not be a common student sitting in both the classes. Therefore, $n(A \cap B) = 0$.

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 40 + 60 - 0 = 100$

(ii) If two classes meet at different timings then there can be some students attending both the classes. It is given that the number of such students is 20 i.e. $n(A \cap B) = 20$.

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 40 + 60 - 20 = 80.$

EXAMPLE 6 *If A, B and C are three sets and U is the universal set such that $n(U) = 700$, $n(A) = 200, n(B) = 300$ and $n(A \cap B) = 100$. Find $n(A' \cap B')$.*

SOLUTION We have, $A' \cap B' = (A \cup B)'$

$\therefore \quad n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

$= n(U) - [n(A) + n(B) - n(A \cap B)] = 700 - (200 + 300 - 100) = 300.$

EXAMPLE 7 *In a survey of 700 students in a college, 180 were listed as drinking Limca, 275 as drinking Miranda and 95 were listed as both drinking Limca as well as Miranda. Find how many students were drinking neither Limca nor Miranda.*

SOLUTION Let U be the set of all surveyed students, A denote the set of students drinking Limca and B be the set students drinking Miranda. It is given that $n(U) = 700, n(A) = 180, n(B) = 275$ and $n(A \cap B) = 95$. We have to find $n(A' \cap B')$.

Now, $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B) = n(U) - \{n(A) + n(B) - n(A \cap B)\}$

$\Rightarrow \quad n(A' \cap B') = 700 - (180 + 275 - 95) = 700 - 360 = 340.$

LEVEL-2

EXAMPLE 8 *There are 200 individuals with a skin disorder, 120 has been exposed to chemical C_1, 50 to chemical C_2 and 30 to both the chemicals C_1 and C_2. Find the number of individuals exposed to (i) chemical C_1 or chemical C_2 (ii) chemical C_1 but not chemical C_2 (iii) chemical C_2 but not chemical C_1.*

SOLUTION Let U denote the universal set consisting of individuals suffereing from the skin disorder, A denote the set of individuals exposed to chemical C_1 and B denote the set of individuals exposed to chemical C_2. It is given that: $n(U) = 200, n(A) = 120, n(B) = 50$ and $n(A \cap B) = 30$.

(i) The number of individuals exposed to chemical C_1 or chemical C_2 is given by $n(A \cup B)$.
Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 120 + 50 - 30 = 140$
Hence, required number of individuals is 140.

(ii) The number of individuals exposed to chemical C_1 but not chemical C_2 is given by $n(A \cap \overline{B})$.
Now, $n(A \cap \overline{B}) = n(A) - n(A \cap B) = 120 - 30 = 90$
Hence, required number of individuals is 90.

(iii) The number of individuals exposed to chemical C_2 but not chemical C_1 is given by $n(\overline{A} \cap B)$.
Now, $n(\overline{A} \cap B) = n(B) - n(A \cap B) = 50 - 30 = 20$
Hence, required number is 20.

EXAMPLE 9 *Out of 500 car owners investigated, 400 owned Maruti car and 200 owned Hyundai car; 50 owned both cars. Is this data correct?*

SOLUTION Let U be the set of all car owners investigated, M be the set of persons who owned Maruti cars and H be the set of persons who owned Hyundai cars. It is given that $n(U) = 500$, $n(M) = 400, n(H) = 200$ and $n(M \cap H) = 50$.
Now, $n(M \cup H) = n(M) + n(H) - n(M \cap H) = 400 + 200 - 50 = 550$
But, $M \cup H \subseteq U$. Therefore,

$$n(M \cup H) \leq n(U) \Rightarrow n(M \cup H) \leq 500$$

This is a contradication. So, the given data is incorrect.

EXAMPLE 10 *If A and B be two sets containing 3 and 6 elements respectively, what can be the minimum number of elements in $A \cup B$? Find also, the maximum number of elements in $A \cup B$.*

SOLUTION We have, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

This shows that $n(A \cup B)$ is minimum or maximum according as $n(A \cap B)$ is maximum or minimum respectively.

<u>CASE I</u> *When $n(A \cap B)$ is minimum, i.e. $n(A \cap B) = 0$.*

This is possible only when $A \cap B = \phi$. In this case,

$$n(A \cup B) = n(A) + n(B) - 0 = n(A) + n(B) = 3 + 6 = 9$$

So, maximum number of elements in $A \cup B$ is 9.

<u>CASE II</u> *When $n(A \cap B)$ is maximum.*

This is possible only when $A \subseteq B$. In this case, $n(A \cap B) = 3$.

$$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 - 3 = 6$$

So, minimum number of elements in $A \cup B$ is 6.

EXAMPLE 11 *A market research group conducted a survey of 2000 consumers and reported that 1720 consumers liked product P_1 and 1450 consumers liked product P_2. What is the least number that must have liked both the products?*

SOLUTION Let U be the set of all consumers who were questioned, A be the set of consumers who liked product P_1 and B be the set of consumers who liked the product P_2. It is given that $n(U) = 2000, n(A) = 1720, n(B) = 1450$.

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = 1720 + 1450 - n(A \cap B) = 3170 - n(A \cap B)$

Now, $\quad A \cup B \subset U$

$\Rightarrow \quad n(A \cup B) \leq n(U)$

$\Rightarrow \quad 3170 - n(A \cap B) \leq 2000 \Rightarrow 3170 - 2000 \leq n(A \cap B) \Rightarrow n(A \cap B) \geq 1170$

Thus, the least value of $n(A \cap B)$ is 1170.

Hence, the least number of consumer who liked both the products is 1170.

EXAMPLE 12 *A survey shows that 63% of the Americans like cheese whereas 76% like apples. If x% of the Americans like both cheese and apples, find the value of x.*

SOLUTION Let A denote the set of Americans who like cheese and let B denote those who like apples. Let the population of America be 100. Then, $n(A) = 63, n(B) = 76$.

Now, $\quad n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$\Rightarrow \quad n(A \cap B) = 63 + 76 - n(A \cap B) = 139 - n(A \cup B)$

But, $\quad n(A \cup B) \leq 100.$

$\Rightarrow \quad -n(A \cup B) \geq -100$

$\Rightarrow \quad 139 - n(A \cup B) \geq 139 - 100$

$\Rightarrow \quad 139 - n(A \cup B) \geq 39$

$\Rightarrow \quad n(A \cap B) \geq 39 \quad$ ...(i)

Now, $\quad A \cap B \subseteq A$ and $A \cap B \subseteq B$

$\Rightarrow \quad n(A \cap B) \leq n(A)$ and $n(A \cap B) \leq n(B)$

$\Rightarrow \quad n(A \cap B) \leq 63$ and $n(A \cap B) \leq 76$

$\Rightarrow \quad n(A \cap B) \leq 63 \quad$ ...(ii)

From (i) and (ii), we obtain

$39 \leq n(A \cap B) \leq 63 \Rightarrow 39 \leq x \leq 63.$

EXAMPLE 13 *In a class of 35 students, 17 have taken Mathematics, 10 have taken Mathematics but not Economics. Find the number of students who have taken both Mathematics and Economics and the number of students who have taken Economics but not Mathematics, if it is given that each student has taken either Mathematics or Economics or both.*

SOLUTION Let A denote the set of students who have taken Mathematics and B be the set of students who have taken Economics. It is given that $n(A \cup B) = 35, n(A) = 17$ and $n(A - B) = 10$.

Now, $\quad n(A) = n(A - B) + n(A \cap B)$

$\Rightarrow \quad 17 = 10 + n(A \cap B)$

$\Rightarrow \quad n(A \cap B) = 7$

Thus, 7 students have taken both Mathematics and Economics.

Now, $\quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow \quad 35 = 17 + n(B) - 7$

$\Rightarrow \quad n(B) = 25$

Also, $\quad n(B) = n(B - A) + n(A \cap B)$

$\Rightarrow \quad 25 = n(B - A) + 7$

$\Rightarrow \quad n(B - A) = 18.$

Thus, 18 students have taken Economics but not Mathematics.

EXAMPLE 14 *In a town of 10,000 families it was found that 40% families buy newspaper A, 20% families buy newspaper B and 10% families buy newspaper C. 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three news papers, find the number of families which buy (i) A only (ii) B only (iii) none of A, B and C.*

SOLUTION Let P, Q and R denote the sets of families buying newspaper A, B and C respectively. Let U be the universal set. Then,

$n(P) = 40\%$ of $10{,}000 = 4000$, $n(Q) = 20\%$ of $10{,}000 = 2000$, $n(R) = 10\%$ of $10{,}000 = 1000$,
$n(P \cap Q) = 5\%$ of $10{,}000 = 500$, $n(Q \cap R) = 3\%$ of $10{,}000 = 300$, $n(R \cap P) = 4\%$ of $10{,}000 = 400$
$n(P \cap Q \cap R) = 2\%$ of $10{,}000 = 200$ and $n(U) = 10{,}000$.

(i) Required number $= n(P \cap Q' \cap R') = n(P \cap (Q \cup R)')$

$$= n(P) - n[P \cap (Q \cup R)] \qquad [\because n(A \cap B') = n(A) - n(A \cap B)]$$
$$= n(P) - n[(P \cap Q) \cup (P \cap R)]$$
$$= n(P) - [n(P \cap Q) + n(P \cap R) - n\{(P \cap Q) \cap (P \cap R)\}]$$
$$= n(P) - [n(P \cap Q) + n(P \cap R) - n(P \cap Q \cap R)]$$
$$= 4000 - (500 + 400 - 200) = 3300$$

(ii) Required number $= n(P' \cap Q \cap R') = n(Q \cap P' \cap R') = n(Q \cap (P \cup R)')$

$$= n(Q) - n(Q \cap (P \cup R)) \qquad [\because n(A \cap B') = n(A) - n(A \cap B)]$$
$$= n(Q) - n[(Q \cap P) \cup (Q \cap R)]$$
$$= n(Q) - [n(Q \cap P) + n(Q \cap R) - n\{(Q \cap P) \cap (Q \cap R)\}]$$
$$= n(Q) - [n(P \cap Q) + n(Q \cap R) - n(P \cap Q \cap R)]$$
$$= 2000 - (500 + 300 - 200) = 1400$$

(iii) Required number $= n(P' \cap Q' \cap R') = n[(P \cup Q \cup R)']$

$$= n(U) - n(P \cup Q \cup R)$$
$$= n(U) - [n(P) + n(Q) + n(R) - n(P \cap Q) - n(Q \cap R) - n(R \cap P) + n(P \cap Q \cap R)]$$
$$= 10000 - [4000 + 2000 + 1000 - 500 - 300 - 400 + 200] = 4000.$$

EXAMPLE 15 *A college awarded 38 medals in Football, 15 in Basketball and 20 to Cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of the three sports ?*

SOLUTION Let F denote the set of men who received medals in Football, B the set of men who received medals in Basketball and C the set of men who received medals in Cricket. It is given that $n(F) = 38$, $n(B) = 15$, $n(C) = 20$, $n(F \cup B \cup C) = 58$ and $n(F \cap B \cap C) = 3$.

Now,

$$n(F \cup B \cup C) = n(F) + n(B) + n(C) - n(F \cap B) - n(B \cap C) - n(F \cap C) + n(F \cap B \cap C)$$
$$\Rightarrow \quad 58 = 38 + 15 + 20 - n(F \cap B) - n(B \cap C) - n(F \cap C) + 3$$
$$\Rightarrow \quad n(F \cap B) + n(B \cap C) + n(F \cap C) = 76 - 58 = 18$$

Number of men who received medals in exactly two of the three sports

$$= n(F \cap B) + n(B \cap C) + n(F \cap C) - 3\,n(F \cap B \cap C) = 18 - 3 \times 3 = 9$$

Thus, 9 men received medals in exactly two of the three sports.

EXAMPLE 16 *In a survey of 25 students, it was found that 15 had taken Mathematics, 12 had taken Physics and 11 had taken Chemistry, 5 had taken Mathematics and Chemistry, 9 had taken Mathematics and Physics, 4 had taken Physics and Chemistry and 3 had taken all the three subjects. Find the number of students that had*

(i) *only Chemistry.* (ii) *only Mathematics.*
(iii) *only Physics.* (iv) *Physics and Chemistry but not Mathematics.*
(v) *Mathematics and Physics but not Chemistry.*
(vi) *only one of the subjects.* (vii) *at least one of the three subjects.*
(viii) *none of the subjects.*

SOLUTION Let M denote the set of students who had taken Mathematics, P the set of students who had taken Physics and C the set of students who had taken Chemistry. It is given that

$n(U) = 25$, $n(M) = 15$, $n(P) = 12$, $n(C) = 11$, $n(M \cap C) = 5$, $n(M \cap P) = 9$, $n(P \cap C) = 4$
and, $n(M \cap P \cap C) = 3$

(i) Number of students who had opted Chemistry only

$$= n(M' \cap P' \cap C) = n((M \cup P)' \cap C)$$

$= n(C) - n((M \cup P) \cap C)$ $\quad [\because n(A \cap B') = n(A) - n(A \cap B)]$

$= n(C) - n((M \cap C) \cup (P \cap C))$

$= n(C) - \{n(M \cap C) + n(P \cap C) - n(M \cap P \cap C)\} = 11 - (5 + 4 - 3) = 5$

(ii) The number of students who had opted Mathematics only.

$= n(M \cap P' \cap C')$

$= n(M \cap (P \cap C)')$

$= n(M) - n(M \cap (P \cup C))$

$= n(M) - n((M \cap P) \cup (M \cap C))$

$= n(M) - \{n(M \cap P) + n(M \cap C) - n(M \cap P \cap C)\} = 15 - (9 + 5 - 3) = 4$

(iii) The number of students who had opted Physics only

$= n(P \cap M' \cap C')$

$= n(P \cap (M \cup C)')$

$= n(P) - n(P \cap (M \cup C))$

$= n(P) - n((P \cap M) \cup (P \cap C))$

$= n(P) - \{n(P \cap M) + n(P \cap C) - n(P \cap M \cap C)\} = 12 - (9 + 4 - 3) = 2.$

(iv) Required number of students $= n(P \cap C \cap M')$

$= n(P \cap C) - n(P \cap C \cap M)$ $\quad [\because n(A \cap B') = n(A) - n(A \cap B)]$

$= 4 - 3 = 1$

(v) Required number of students $= n(M \cap P \cap C')$

$= n(M \cap P) - n(M \cap P \cap C) = 9 - 3 = 6$

(vi) Required number of students

$= n(M) + n(P) + n(C) - 2\{n(M \cap P) + n(P \cap C) + n(M \cap C) + 3n(M \cap P \cap C)\}$

$= 15 + 12 + 11 - 2(9 + 4 + 5) + 3 \times 3 = 38 - 36 + 9 = 11$

(vii) Required number of students

$= n(M \cup P \cup C)$

$= n(M) + n(P) + n(C) - n(M \cap P) - n(P \cap C) - n(M \cap C) + n(M \cap P \cap C)$

$= 15 + 12 + 11 - 9 - 4 - 5 + 3 = 23$

(viii) Required number of students $= n(M' \cap P' \cap C')$

$= n(M \cup P \cup C)'$

$= n(U) - n(M \cup P \cup C) = 25 - 23 = 2.$

<u>ALITER</u> Consider the Venn diagram shown in Fig. 1.13. Let a, b, c, d, e, f, g denote the number of students in the respective regions.

From the Venn-daigram, we have

$n(M) = a + b + d + e,$

$n(P) = b + c + e + f,$

$n(C) = d + e + f + g,$

$n(M \cap P) = b + e$

$n(P \cap C) = e + f$

$n(M \cap C) = d + e$ and, $n(M \cap P \cap C) = e$

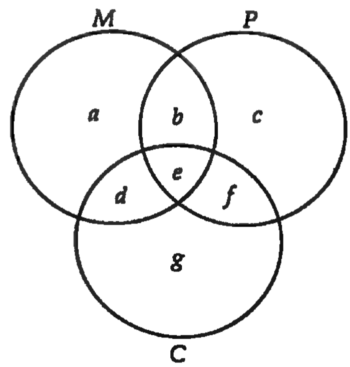

Fig. 1.13

It is given that

$n(M \cap P \cap C) = 3 \Rightarrow e = 3$

$n(M \cap P) = 9 \Rightarrow b + e = 9 \Rightarrow b + 3 = 9 \Rightarrow b = 6$

$n(P \cap C) = 4 \Rightarrow e + f = 4 \Rightarrow 3 + f = 4 \Rightarrow f = 1$

$n(M \cap C) = 5 \Rightarrow d + e = 5 \Rightarrow d + 3 = 5 \Rightarrow d = 2$

$n(M) = 15 \Rightarrow a + b + d + e = 15 \Rightarrow a + 6 + 2 + 3 = 15 \Rightarrow a = 4$

$n(P) = 12 \Rightarrow b + c + e + f = 12 \Rightarrow 6 + c + 3 + 1 = 12 \Rightarrow c = 2$

$n(C) = 11 \Rightarrow d + e + f + g = 11 \Rightarrow 2 + 3 + 1 + g = 11 \Rightarrow g = 5$

Now,

(i) Required number of students = $g = 5$
(ii) Required number of students = $a = 4$
(iii) Required number of students = $c = 2$
(iv) Required number of students = $f = 1$
(v) Required number of students = $b = 6$
(vi) Required number of students = $a + c + g = 4 + 2 + 5 = 11$
(vii) Required number of students = $a + b + c + d + e + f + g = 23$
(viii) Required number of students = $25 - (a + b + c + d + e + f + g) = 25 - 23 = 2.$

EXERCISE 1.8

LEVEL-1

1. If A and B are two sets such that $n(A \cup B) = 50$, $n(A) = 28$ and $n(B) = 32$, find $n(A \cap B)$.
2. If P and Q are two sets such that P has 40 elements, $P \cup Q$ has 60 elements and $P \cap Q$ has 10 elements, how many elements does Q have?
3. In a school there are 20 teachers who teach mathematics or physics. Of these, 12 teach mathematics and 4 teach physics and mathematics. How many teach physics?
4. In a group of 70 people, 37 like coffee, 52 like tea and each person likes at least one of the two drinks. How many like both coffee and tea?
5. Let A and B be two sets such that : $n(A) = 20$, $n(A \cup B) = 42$ and $n(A \cap B) = 4$. Find
   (i) $n(B)$ (ii) $n(A - B)$ (iii) $n(B - A)$
6. A survey shows that 76% of the Indians like oranges, whereas 62% like bananas. What percentage of the Indians like both oranges and bananas?
7. In a group of 950 persons, 750 can speak Hindi and 460 can speak English. Find:
   (i) how many can speak both Hindi and English
   (ii) how many can speak Hindi only (iii) how many can speak English only.
8. In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find:
   (i) how may drink tea and coffee both (ii) how many drink coffee but not tea.
9. In a survey of 60 people, it was found that 25 people read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three newspapers. Find:
   (i) the numbers of people who read at least one of the newspapers.
   (ii) the number of people who read exactly one newspaper.
10. Of the members of three athletic teams in a certain school, 21 are in the basketball team, 26 in hockey team and 29 in the football team. 14 play hockey and basket ball, 15 play hockey and football, 12 play football and basketball and 8 play all the three games. How many members are there in all?
11. In a group of 1000 people, there are 750 who can speak Hindi and 400 who can speak Bengali. How many can speak Hindi only? How many can speak Bengali ? How many can speak both Hindi and Bengali?
12. A survey of 500 television viewers produced the following information; 285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball, 50 do not watch any of the three games. How many watch all the three games? How many watch exactly one of the three games?

LEVEL-2

13. In a survey of 100 persons it was found that 28 read magazine A, 30 read magazine B, 42 read magazine C, 8 read magazines A and B, 10 read magazines A and C, 5 read magazines B and C and 3 read all the three magazines. Find:
    (i) How many read none of three magazines?
    (ii) How many read magazine C only?
14. In a survey of 100 students, the number of students studying the various languages were found to be : English only 18, English but not Hindi 23, English and Sanskrit 8, English 26, Sanskrit 48, Sanskrit and Hindi 8, no language 24. Find:
    (i) How many students were studying Hindi?
    (ii) How many students were studying English and Hindi?
15. In a survey it was found that 21 persons liked product P_1, 26 liked product P_2 and 29 liked product P_3. If 14 persons liked products P_1 and P_2 ; 12 persons liked product P_3 and P_1 ; 14 persons liked products P_2 and P_3 and 8 liked all the three products. Find how many liked product P_3 only.

ANSWERS

1. 10 2. 30 3. 12 4. 19 5. (i) 26 (ii) 16 (iii) 22
6. 38% 7. (i) 260 (ii) 490 (iii) 200 8. (i) 16 (ii) 20
9. (i) 52 (ii) 30 10. 43 11. (i) 600 (ii) 250 (iii) 150
12. 20, 325 13. (i) 20 (ii) 30 14. (i) 18, (ii) 3 15. 11

HINTS TO SELECTED PROBLEMS

5. (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B) \Rightarrow n(B) = 26$
   (ii) $n(A - B) = n(A) - n(A \cap B) \Rightarrow n(A - B) = 16$
   (iii) $n(B - A) = n(B) - n(A \cap B) \Rightarrow 22$
7. (iii) Let A and B denote the sets of persons who can speak Hindi and English respectively. Then, $n(A \cup B) = 950$, $n(A) = 750$ and $n(B) = 460$.
   (i) $n(A \cap B) = n(A) + n(B) - n(A \cup B) = 750 + 460 - 950 = 260$
   (ii) Required number $= n(A - B) = n(A) - n(A \cap B)$
   (iii) Required number $= n(B - A) = n(B) - n(A \cap B)$.
8. (ii) Let A and B be sets of persons who drink tea and coffee respectively. Then $n(A \cup B) = 50, n(A - B) = 14, n(A) = 30$.
   (i) $n(A - B) = 14 \Rightarrow n(A) - n(A \cap B) = 14 \Rightarrow n(A \cap B) = n(A) - 14 = 30 - 14 = 16$
   (ii) Required number $= n(B - A) = n(B) - n(A \cap B)$.
   Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B) \Rightarrow 50 = 30 + n(B) - 16 \Rightarrow n(B) = 36$.
   $\therefore\ n(B - A) = n(B) - n(A \cap B) \Rightarrow n(B - A) = 36 - 16 = 20$
10. Let A, B and C be the sets of members of basketball, hockey and football teams respectively. Then, $n(A) = 21$, $n(B) = 26$, $n(C) = 29$, $n(A \cap B) = 14$, $n(B \cap C) = 15$, $n(A \cap C) = 12$ and $n(A \cap B \cap C) = 8$.
    Required number $= n(A \cup B \cup C)$
    $$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C).$$
11. Let A and B be the sets of persons who can speak Hindi and Bengali respectively. Then, $n(A \cup B) = 1000$, $n(A) = 750$ and $n(B) = 400$.
    No. of persons who can speak Hindi only $= n(A - B) = n(A) - n(A \cap B)$
    No. of persons who can speak Bengali only $= n(B - A) = n(B) - n(B \cap A)$

No. of persons who can speak both Hindi and Bengali $= n(A \cap B) = n(A) + n(B) - n(A \cup B)$.

12. N = Total number of television viewers = 500, $n(F) = 285$, $n(H) = 195$, $n(F \cap B) = 45$, $n(F \cap H) = 70$, $n(H \cap B) = 50$, $n(F' \cap H' \cap B') = 50$.

Now, $n(F' \cap H' \cap B') = 50$

$\Rightarrow \quad n[(F \cup H \cup B)'] = 50$

$\Rightarrow \quad N - n(F \cup H \cup B) = 50$

$\Rightarrow \quad 500 - [n(F) + n(H) + n(B) - n(F \cap H) - n(F \cap B) - n(H \cap B) + n(F \cap H \cap B)] = 50$

$\Rightarrow \quad n(F \cap H \cap B) = 500 - 285 - 195 - 115 + 70 + 50 + 45 - 50 = 20.$

$\therefore$ Required number $= n(F \cap H \cap B) = 20$

Required number $= n(F \cap H' \cap B') + n(F' \cap H' \cap B) + n(F' \cap H \cap B')]$

$= n(F) + n(H) + n(B) - 2[n(F \cap H) + n(H \cap B) + n(B \cap F)\} + 3\,n(F \cap H \cap B)$

14. We have, $a = 18$, $a + b = 23$, $d + e = 8$, $a + b + d + e = 26$, $d + e + f + g = 48$,

and, $a + b + c + d + e + f + g = 100 - 24 = 76$

$\therefore \quad a = 18, b = 0, c = 10, d = 5, e = 3, f = 5$ and, $g = 35$

(i) $n(H) = b + c + e + f = 18$

(ii) $n(H \cap E) = b + e = 3$

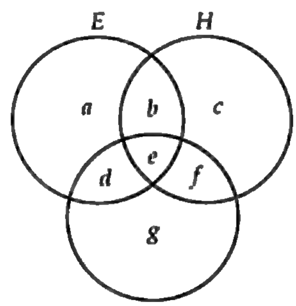

Fig. 1.14

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If a set contains n elements, then write the number of elements in its power set.
2. Write the number of elements in the power set of null set.
3. Let $A = \{x : x \in N, x$ is a multiple of 3\} and $B = \{x : x \in N$ and x is a multiple of 5 \}. Write $A \cap B$.
4. Let A and B be two sets having 3 and 6 elements respectively. Write the minimum number of elements that $A \cup B$ can have.
5. If $A = \{x \in C : x^2 = 1\}$ and $B = \{x \in C : x^4 = 1\}$, then write $A - B$ and $B - A$.
6. If A and B are two sets such that $A \subset B$, then write $B' - A'$ in terms of A and B.
7. Let A and B be two sets having 4 and 7 elements respectively. Then write the maximum number of elements that $A \cup B$ can have.
8. If $A = \{(x, y) : y = \frac{1}{x}, 0 \neq x \in R\}$ and $B = \{(x, y) : y = -x, x \in R\}$, then write $A \cap B$.
9. If $A = \{(x, y) : y = e^x, x \in R\}$ and $B = \{(x, y) : y = e^{-x}, x \in R\}$, then write $A \cap B$.

10. If A and B are two sets such that $n(A)=20, n(B)=25$ and $n(A \cup B)=40$, then write $n(A \cap B)$.

11. If A and B are two sets such that $n(A)=115, n(B)=326, n(A-B)=47$, then write $n(A \cup B)$.

ANSWERS

1. 2^n 2. 1 3. $\{x : x \in N, x$ is a multiple of $15\}$
4. 6 5. $A-B=\phi, B-A=\{i,-i\}$ 6. ϕ 7. 11 8. ϕ
9. $\{(0,1)\}$ 10. 5 11. 373

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. For any set A, $(A')'$ is equal to
(a) A' (b) A (c) ϕ (d) none of these

2. Let A and B be two sets in the same universal set. Then, $A-B=$
(a) $A \cap B$ (b) $A' \cap B$ (c) $A \cap B'$ (d) none of these

3. The number of subsets of a set containing n elements is
(a) n (b) 2^n-1 (c) n^2 (d) 2^n

4. For any two sets A and B, $A \cap (A \cup B)=$
(a) A (b) B (c) ϕ (d) none of these

5. If $A=\{1,3,5,B\}$ and $B=\{2,4\}$, then
(a) $4 \in A$ (b) $\{4\} \subset A$ (c) $B \subset A$ (d) none of these

6. The symmetric difference of A and B is not equal to
(a) $(A-B) \cap (B-A)$ (b) $(A-B) \cup (B-A)$
(c) $(A \cup B)-(A \cap B)$ (d) $\{(A \cup B)-A\} \cup \{(A \cup B)-B\}$

7. The symmetric difference of $A=\{1,2,3\}$ and $B=\{3,4,5\}$ is
(a) $\{1,2\}$ (b) $\{1,2,4,5\}$ (c) $\{4,3\}$ (d) $\{2,5,1,4,3\}$

8. For any two sets A and B, $(A-B) \cup (B-A)=$
(a) $(A-B) \cup A$ (b) $(B-A) \cup B$
(c) $(A \cup B)-(A \cap B)$ (d) $(A \cup B) \cap (A \cap B)$

9. Which of the following statement is false :
(a) $A-B=A \cap B'$ (b) $A-B=A-(A \cap B)$
(c) $A-B=A-B'$ (d) $A-B=(A \cup B)-B$

10. For any three sets A, B and C
(a) $A \cap (B-C)=(A \cap B)-(A \cap C)$ (b) $A \cap (B-C)=(A \cap B)-C$
(c) $A \cup (B-C)=(A \cup B) \cap (A \cup C')$ (d) $A \cup (B-C)=(A \cup B)-(A \cup C)$.

11. Let $A=\{x : x \in R, x>4\}$ and $B=\{x \in R : x<5\}$. Then, $A \cap B=$
(a) $(4,5]$ (b) $(4,5)$ (c) $[4,5)$ (d) $[4,5]$

12. Let U be the universal set containing 700 elements. If A, B are sub-sets of U such that $n(A)=200, n(B)=300$ and $n(A \cap B)=100$. Then, $n(A' \cap B')=$
(a) 400 (b) 600 (c) 300 (d) none of these.

13. Let A and B be two sets such that $n(A) = 16, n(B) = 14, n(A \cup B) = 25$. Then, $n(A \cap B)$ is equal to
(a) 30 (b) 50 (c) 5 (d) none of these

14. If $A = \{1, 2, 3, 4, 5\}$, then the number of proper subsets of A is
(a) 120 (b) 30 (c) 31 (d) 32

15. In set-builder method the null set is represented by
(a) $\{\ \}$ (b) Φ (c) $\{x : x \neq x\}$ (d) $\{x : x = x\}$

16. If A and B are two disjoint sets, then $n(A \cup B)$ is equal to
(a) $n(A) + n(B)$ (b) $n(A) + n(B) - n(A \cap B)$
(c) $n(A) + n(B) + n(A \cap B)$ (d) $n(A)\,n(B)$

17. For two sets $A \cup B = A$ iff
(a) $B \subseteq A$ (b) $A \subseteq B$ (c) $A \neq B$ (d) $A = B$

18. If A and B are two sets such that $n(A) = 70, n(B) = 60, n(A \cup B) = 110$, then $n(A \cap B)$ is equal to
(a) 240 (b) 50 (c) 40 (d) 20

19. If A and B are two given sets, then $A \cap (A \cap B)^c$ is equal to
(a) A (b) B (c) Φ (d) $A \cap B^c$

20. If $A = \{x : x \text{ is a multiple of } 3\}$ and, $B = \{x : x \text{ is a multiple of } 5\}$, then $A - B$ is
(a) $A \cap B$ (b) $A \cap \overline{B}$ (c) $\overline{A} \cap \overline{B}$ (d) $\overline{A \cap B}$

21. In a city 20% of the population travels by car, 50% travels by bus and 10% travels by both car and bus. Then, persons travelling by car or bus is
(a) 80% (b) 40% (c) 60% (d) 70%

22. If $A \cap B = B$, then
(a) $A \subseteq B$ (b) $B \subseteq A$ (c) $A = \Phi$ (d) $B = \Phi$

23. An investigator interviewed 100 students to determine the performance of three drinks: milk, coffee and tea. The investigator reported that 10 students take all three drinks milk, coffee and tea; 20 students take milk and coffee; 25 students take milk and tea; 20 students take coffee and tea; 12 students take milk only; 5 students take coffee only and 8 students take tea only. Then the number of students who did not take any of three drinks is
(a) 10 (b) 20 (c) 25 (d) 30

24. Two finite sets have m and n elements. The number of elements in the power set of first set is 48 more than the total number of elements in power set of the second set. Then, the values of m and n are:
(a) 7, 6 (b) 6, 3 (c) 6, 4 (d) 7, 4

25. In a class of 175 students the following data shows the number of students opting one or more subjects. Mathematics 100; Physics 70; Chemistry 40; Mathematics and Physics 30; Mathematics and Chemistry 28; Physics and Chemistry 23; Mathematics, Physics and Chemistry 18. How many students have offered Mathematics alone?
(a) 35 (b) 48 (c) 60 (d) 22

26. Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each having 5 elements and $B_1, B_2, \ldots, B_n$ are n sets each with 3 elements, let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each element of S belongs to exactly 10 of the $A_i^{'s}$ and exactly 9 of the $B_j^{'s}$, then n is equal to

(a) 15 (b) 3 (c) 45 (d) 35

27. Two finite sets have m and n elements. The number of subsets of the first set is 112 more than that of the second. The values of m and n are respectively

(a) 4, 7 (b) 7, 4 (c) 4, 4 (d) 7, 7

28. For any two sets A and B, $A \cap (A \cup B)'$ is equal to

(a) A (b) B (c) ϕ (d) $A \cap B$

29. The set $(A \cup B')' \cup (B \cap C)$ is equal to

(a) $A' \cup B \cup C$ (b) $A' \cup B$ (c) $A' \cup C'$ (d) $A' \cap B$

30. Let F_1 be the set of all parallelograms, F_2 the set of all rectangles, F_3 the set of all rhombuses, F_4 the set of all squares and F_5 the set of trapeziums in a plane. Then F_1 may be equal to

(a) $F_2 \cap F_3$ (b) $F_3 \cap F_4$ (c) $F_2 \cup F_3$ (d) $F_2 \cup F_3 \cup F_4 \cup F_1$

ANSWERS

1. (b) 2. (c) 3. (d) 4. (a) 5. (d) 6. (a) 7. (b) 8. (c) 9. (c)
10. (a), (b), (c) 11. (b) 12. (c) 13. (c) 14. (c) 15. (c) 16. (a) 17. (a)
18. (d) 19. (d) 20. (b) 21. (c) 22. (b) 23. (b) 24. (c) 25. (c) 26. (c)
27. (b) 28. (c) 29. (b) 30. (d)

SUMMARY

1. A set is a well defined collection of objects.
2. A set is described either in set builder form or tabular form.
3. A set consisting of no element is called the null set and is denoted by ϕ.
4. A set consisting of a single element is called a singleton set.
5. A set consisting of a definite number of elements is called a finite set, otherwise the set is called an infinite set.
6. The number of elements in a finite set A is called its cardinal number or order and is denoted by $n(A)$.
7. Two sets A and B are equal if they have exactly the same elements.
8. A set A is said to be a subset of a set B, if every element of A is also an element of B.
9. If a, b are real numbers such that $a < b$, then the set
   (i) $\{x : x \in R \text{ and } a \le x \le b\}$ is called the closed interval $[a, b]$
   (ii) $\{x : x \in R \text{ and } a < x < b\}$ is called the open interval (a, b)
   (iii) $\{x : x \in R \text{ and } a \le x < b\}$ is called the semi-open or semi-closed interval $[a, b)$.
   (iv) $\{x : x \in R \text{ and } a < x \le b\}$ is called the semi-open or semi-closed interval $(a, b]$.
10. The total number of subsets of a finite set consisting of n elements is 2^n.
11. The collection of all subsets of a set A is called the power set of A and is denoted by $P(A)$.
12. The union of two sets A and B is the set of all those elements which are either in A or in B or in both and is denoted by $A \cup B$. Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$
13. The intersection of two sets A and B is the set of all those elements which are common to both A and B and is denoted by $A \cap B$. Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
14. The difference $A - B$ of two sets A and B is the set of all those elements of A which do not belong to B i.e. $A - B = \{x : x \in A \text{ and } x \notin B\}$. Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$.

15. The symmetric difference of two sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.

16. The complement of a subset A of universal set U is the set of all those elements of U which are not the elements of A. The complement of A is denoted by A' or A^c.

17. For any three sets A, B and C, we have

(i) $A \cup A = A$ and $A \cap A = A$ (Idempotent laws)

(ii) $A \cup \phi = A$ and $A \cap U = A$ (Identity laws)

(iii) $A \cup B = B \cup A$ and $A \cap B = B \cap A$ (Commutative laws)

(iv) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative laws)

(v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive laws)

(vi) $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$ (De' Morgan's laws)

18. If A, B and C are finite sets and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets

(iii) $n(A - B) = n(A) - n(A \cap B)$ i.e., $n(A - B) + n(A \cap B) = n(A)$

(iv) $n(A \Delta B) = n(A - B) + n(B - A) = n(A) + n(B) - 2n(A \cap B)$

(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

(vi) Number of elements in exactly two of sets A, B and C
$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(vii) Number of elements in exactly one of sets A, B and C
$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$.

CHAPTER 2

RELATIONS

2.1 INTRODUCTION

In previous chapter, we have discussed various operations on sets to create more sets out of given sets. In this chapter, we shall study one more operation which is known as the cartesian product of sets. This will finally enable us to introduce the concept of relation.

2.2 ORDERED PAIRS

ORDERED PAIR *An ordered pair consists of two objects or elements in a given fixed order.*

For example, if A and B are any two sets, then by an ordered pair of elements we mean a pair (a, b) in that order, where $a \in A, b \in B$.

NOTE *An ordered pair is not a set consisting of two elements. The ordering of the two elements in an ordered pair is important and the two elements need not be distinct.*

ILLUSTRATION 1 *The position of a point in a two dimensional plane in cartesian coordinates is represented by an ordered pair. Accordingly, the ordered pairs* $(1, 3)$, $(2, 4)$, $(2, 3)$ *and* $(3, 2)$ *represents different points in a plane.*

EQUALITY OF ORDERED PAIRS *Two ordered pairs* (a_1, b_1) *and* (a_2, b_2) *are equal iff* $a_1 = a_2$ and $b_1 = b_2$.

i.e. $\quad (a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2 \text{ and } b_1 = b_2$

It is evident from this definition that $(1, 2) \neq (2, 1)$ and $(1, 1) \neq (2, 2)$.

ILLUSTRATION 2 *Find the values of a and b, if* $(3a - 2, b + 3) = (2a - 1, 3)$.

SOLUTION By the definition of equality of ordered pairs, we obtain

$$(3a - 2, b + 3) = (2a - 1, 3) \Leftrightarrow 3a - 2 = 2a - 1 \text{ and } b + 3 = 3 \Leftrightarrow a = 1 \text{ and } b = 0$$

2.3 CARTESIAN PRODUCT OF SETS

CARTESIAN PRODUCT OF SETS *Let A and B be any two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.*

Thus, $\quad A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

If $\quad A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.

ILLUSTRATION 1 *If $A = \{2, 4, 6\}$ and $B = \{1, 2\}$, then*

$$A \times B = \{2, 4, 6\} \times \{1, 2\} = \{(2, 1), (2, 2), (4, 1), (4, 2), (6, 1), (6, 2)\}$$

and, $\quad B \times A = \{1, 2\} \times \{2, 4, 6\} = \{(1, 2), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6)\}$

It is evident from the above illustration that to write $A \times B$, we take an element from set A and form all ordered pairs with this element as first element and elements of B as second elements. Next we choose another element from A and corresponding to each element in B we form ordered pairs with this element as first element and elements of B as second elements. This process is continued till all elements of A are exhausted.

ILLUSTRATION 2 *If $A = \{a, b\}$ and $B = \{1, 2, 3\}$, find $A \times B$, $B \times A$, $A \times A$, $B \times B$, and $(A \times B) \cap (B \times A)$.*

SOLUTION We have, $A = \{a, b\}$ and $B = \{1, 2, 3\}$

$$\therefore \quad A \times B = \{(a,1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$
$$B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$
$$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$$
$$B \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

Clearly, $(A \times B) \cap (B \times A) = \phi$.

CARTESIAN PRODUCT OF THREE SETS *Let A, B and C be three sets. Then, $A \times B \times C$ is the set of all ordered triplets having first element from A, second element from B and third element from C.*

i.e. $\quad A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$

ILLUSTRATION 3 *If $A = \{1, 2\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$. Then,*

$$A \times B = \{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

and,
$$\begin{aligned} A \times B \times C &= (A \times B) \times C \\ &= \{(1, 3), (1, 4), (2, 3), (2, 4)\} \times \{4, 5, 6\} \\ &= \{(1, 3, 4), (1, 3, 5), (1, 3, 6), (1, 4, 4), (1, 4, 5), (1, 4, 6), \\ &\qquad (2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 4, 4), (2, 4, 5), (2, 4, 6)\} \end{aligned}$$

NOTE *It should be noted that $A \times B \times C = (A \times B) \times C = A \times (B \times C)$.*

If $A_1, A_2, A_3, \ldots, A_n$ are n sets, then the cartesian product $A_1 \times A_2 \times \ldots \times A_n$ of these n sets is the set of all n-tuples of the form $(a_1, a_2, a_3, \ldots, a_n)$, where $a_1 \in A, a_2 \in A_2, \ldots, a_n \in A_n$.

i.e. $\quad A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, a_3, \ldots, a_n) : a_1 \in A_1, a_2 \in A_2, a_3 \in A_3, \ldots, a_n \in A_n\}$

2.3.1 NUMBER OF ELEMENTS IN THE CARTESIAN PRODUCT OF TWO SETS

THEOREM *If A and B are two finite sets, then $n(A \times B) = n(A) \times n(B)$.*

PROOF Let $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b_2, b_3, \ldots, b_n\}$ be two sets having m and n elements respectively. Then,

$$\begin{aligned} A \times B = \{&(a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots, (a_1, b_n) \\ &(a_2, b_1), (a_2, b_2), (a_2, b_3) \ldots, (a_2, b_n) \\ &\quad \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ &(a_m, b_1), (a_m, b_2), (a_m, b_3) \ldots, (a_m, b_n)\} \end{aligned}$$

Clearly, in the tabular representation of $A \times B$ there are m rows of ordered pairs and each row has n distinct ordered pairs. So, $A \times B$ has mn elements.

Hence, $n(A \times B) = mn = n(A) \times n(B)$ Q.E.D.

REMARK (i) *If either A or B is an infinite set, then $A \times B$ is an infinite set.*

(ii) *If A, B, C are finite sets, then $n(A \times B \times C) = n(A) \times n(B) \times n(C)$*

2.3.2 GRAPHICAL REPRESENTATION OF CARTESIAN PRODUCT OF SETS

Let A and B be any two non-empty sets. To represent $A \times B$ graphically, we draw two mutually perpendicular lines, one horizontal and other vertical. On the horizontal line, we represent the elements of set A and on the vertical line, the elements of B. If $a \in A$, $b \in B$, we draw a vertical line through a and a horizontal line through b. These two lines will meet in a point which will denote the ordered pair (a, b). In this manner we mark points corresponding to each ordered pair in $A \times B$. The set of points so obtained represents $A \times B$ graphically as illustrated below.

ILLUSTRATION *If $A = \{1, 2, 3\}$ and $B = \{2, 4\}$, find $A \times B$ and show it graphically.*

SOLUTION Clearly, $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$.

In order to represent $A \times B$ graphically, we draw two perpendicular lines OX and OY as shown in Fig. 2.1. Now, we represent the set A by three points on OX and the set B by two points on OY. The set $A \times B$ is represented by the six points as shown in Fig. 2.1.

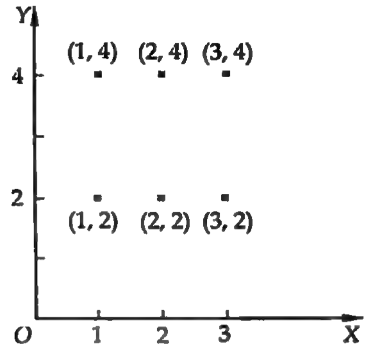

Fig. 2.1

2.3.3 DIAGRAMATIC REPRESENTATION OF CARTESIAN PRODUCT OF TWO SETS

In order to represent $A \times B$ by an arrow diagram, we first draw Venn diagrams representing sets A and B one opposite to the other as shown in Fig. 2.2. Now, we draw line segments starting from each element of A and terminating to each element of set B.

If $A = \{1, 3, 5\}$ and $B = \{a, b\}$, then following figure gives the arrow diagram of $A \times B$.

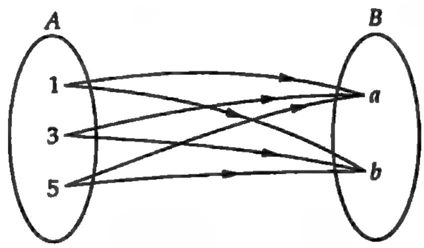

Fig. 2.2

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON EQUALITY OF ORDERED PAIRS**

EXAMPLE 1 *Find x and y, if $(x+3, 5) = (6, 2x+y)$.*

SOLUTION By the definition of equality of ordered pairs

$$
\begin{aligned}
& (x+3, 5) = (6, 2x+y) \\
\Rightarrow \quad & x+3 = 6 \text{ and } 5 = 2x+y \\
\Rightarrow \quad & x = 3 \text{ and } 5 = 2x+y \\
\Rightarrow \quad & x = 3, 5 = 6+y \\
\Rightarrow \quad & x = 3 \text{ and } y = -1
\end{aligned}
$$

Type II **ON FINDING THE CARTESIAN PRODUCT OF TWO SETS**

EXAMPLE 2 *If $A = \{1, 3, 5, 6\}$ and $B = \{2, 4\}$, find $A \times B$ and $B \times A$.*

SOLUTION We have, $A = \{1, 3, 5, 6\}$ and $B = \{2, 4\}$. Therefore,

$$A \times B = \{(1, 2), (1, 4), (3, 2), (3, 4), (5, 2), (5, 4), (6, 2), (6, 4)\}$$

and, $$B \times A = \{(2, 1), (2, 3), (2, 5), (2, 6), (4, 1), (4, 3), (4, 5), (4, 6)\}$$

EXAMPLE 3 *If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{1, 3, 5\}$, find*

(i) $A \times (B \cup C)$ (ii) $A \times (B \cap C)$ (iii) $(A \times B) \cap (A \times C)$

SOLUTION (i) Clearly, $B \cup C = \{1, 3, 4, 5\}$

$\therefore$ $A \times (B \cup C) = \{1, 2, 3\} \times \{1, 3, 4, 5\}$

$= \{(1, 1), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), (2, 4), (2, 5), (3, 1), (3, 3), (3, 4), (3, 5)\}$

(ii) Clearly, $B \cap C = \{3\}$.

$\therefore$ $A \times (B \cap C) = \{1, 2, 3\} \times \{3\} = \{(1, 3), (2, 3), (3, 3)\}$

(iii) $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$,

and, $A \times C = \{(1, 1), (1, 3), (1, 5), (2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (35)\}$

$\therefore$ $(A \times B) \cap (A \times C) = \{(1, 3), (2, 3), (3, 3)\}$.

Type III **ON FINDING SETS A AND B WHEN A × B OR SOME ELEMENTS OF A × B ARE GIVEN**

EXAMPLE 4 *Let $A = \{1, 2, 3\}$ and $B = \{x : x \in N, x$ is prime less than $5\}$. Find $A \times B$ and $B \times A$.*

SOLUTION We have, $A = \{1, 2, 3\}$ and, $B = \{x : x \in N, x \text{ is prime less than } 5\} = \{2, 3\}$

$\therefore$ $A \times B = \{1, 2, 3\} \times \{2, 3\} = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$

and, $B \times A = \{2, 3\} \times \{1, 2, 3\} = \{(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

EXAMPLE 5 *If $A \times B = \{(a, 1), (a, 5), (a, 2), (b, 2), (b, 5), (b, 1)\}$, find $B \times A$.*

SOLUTION Clearly, $B \times A$ can be obtained from $A \times B$ by interchanging the entries (or components) or ordered pair in $A \times B$.

$\therefore$ $B \times A = \{(1, a), (5, a), (2, a), (2, b), (5, b), (1, b)\}$

EXAMPLE 6 *If $A = \{1, 2\}$, form the set $A \times A \times A$.*

SOLUTION We have, $A = \{1, 2\}$.

$\therefore$ $A \times A = \{(1,1), (1, 2), (2, 1), (2, 2)\}$

and, $A \times A \times A = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

EXAMPLE 7 *If R is the set of all real numbers, what do the cartesian products $R \times R$ and $R \times R \times R$ represent?*

SOLUTION The cartesian product of the set R of all real numbers with itself i.e. $R \times R$ is the set of all ordered pairs (x, y) where $x, y \in R$. In other words, $R \times R = \{(x, y) : x, y \in R\}$.

Clearly, $R \times R$ is the set of all points in XY-plane. The set $R \times R$ is also denoted by R^2.

Similarly, we have

$$R \times R \times R = \{(x, y, z) : x, y, z \in R\}$$

Clearly, it represents the set of all points in space. The set $R \times R \times R$ is also denoted by R^3.

EXAMPLE 8 *Express $A = \{(a, b) : 2a + b = 5, a, b \in W\}$ as the set of ordered pairs.*

SOLUTION Here, W denotes the set of whole numbers (non-negative integers).

We have,

$2a + b = 5$, where $a, b \in W$.

$\therefore$ $a = 0 \Rightarrow b = 5, a = 1 \Rightarrow b = 3$ and, $a = 2 \Rightarrow b = 1$

For $a > 3$, the values of b given by the above relation are not whole numbers.

$\therefore$ $A = \{(0, 5), (1, 3), (2, 1)\}$

EXAMPLE 9 *If $A \times B = \{(a, 1), (b, 3), (a, 3), (b, 1), (a, 2), (b, 2)\}$, find A and B.*

SOLUTION Clearly, A is the set of all first entries in ordered pairs in $A \times B$ and B is the set of all second entries in ordered pairs in $A \times B$.

$\therefore$ $A = \{a, b\}$ and $B = \{1, 2, 3\}$

EXAMPLE 10 *Let A and B be two sets such that $A \times B$ consists of 6 elements. If three elements of $A \times B$ are : (1, 4), (2, 6), (3, 6). Find $A \times B$ and $B \times A$.*

SOLUTION Since (1, 4), (2, 6) and (3, 6) are elements of $A \times B$. It follows that 1, 2, 3 are elements of A and 4, 6 are elements of B. It is given that $A \times B$ has 6 elements. So, $A = \{1, 2, 3\}$ and $B = \{4, 6\}$.

Hence, $A \times B = \{1, 2, 3\} \times \{4, 6\} = \{(1, 4), (1, 6), (2, 4), (2, 6), (3, 4), (3, 6)\}$

and, $B \times A = \{4, 6\} \times \{1, 2, 3\} = \{(4, 1), (4, 2), (4, 3), (6, 1), (6, 2), (6, 3)\}$

EXAMPLE 11 *The cartesian product $A \times A$ has 9 elements among which are found $(-1, 0)$ and $(0, 1)$. Find the set A and the remaining elements of $A \times A$.*

SOLUTION Since $(-1, 0) \in A \times A$ and $(0, 1) \in A \times A$. Therefore,

$$(-1, 0) \in A \times A \Rightarrow -1, 0 \in A \text{ and, } (0, 1) \in A \times A \Rightarrow 0, 1 \in A$$

$$\therefore \quad -1, 0, 1 \in A$$

It is given that $A \times A$ has 9 elements. Therefore, A has exactly three elements.

Hence, $A = \{-1, 0, 1\}$.

EXAMPLE 12 *Let A and B be two sets such that $n(A) = 5$ and $n(B) = 2$. If a, b, c, d, e are distinct and $(a, 2), (b, 3), (c, 2), (d, 3), (e, 2)$ are in $A \times B$, find A and B.*

SOLUTION Since $(a, 2), (b, 3), (c, 2), (d, 3), (e, 2)$ are elements of $A \times B$. Therefore, $a, b, c, d, e \in A$ and $2, 3 \in B$.

It is given that $n(A) = 5$ and $n(B) = 2$

$$\therefore \quad a, b, c, d, e \in A \text{ and } n(A) = 5 \Rightarrow A = \{a, b, c, d, e\}$$

$$2, 3 \in B \text{ and } n(B) = 2 \Rightarrow B = \{2, 3\}$$

Type IV ON GRAPHICAL AND DIAGRAMATIC REPRESENTATION OF A × B

EXAMPLE 13 *Let $A = \{-1, 3, 4\}$ and $B = \{2, 3\}$. Represent the following products graphically i.e. by lattices:* (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$

SOLUTION (i) We have, $A = \{-1, 3, 4\}$ and $B = \{2, 3\}$.

$$\therefore \quad A \times B = \{(-1, 2), (-1, 3), (3, 2), (3, 3), (4, 2), (4, 3)\}$$

In order to represent $A \times B$ graphically, we follow the following steps:

(a) Draw two mutually perpendicular line one horizontal and other vertical.

(b) On the horizontal line represent the elements of set A and on the vertical line represent the elements of set B.

(c) Draw vertical dotted lines through points representing elements of A on horizontal line and horizontal lines through points representing elements of B on the vertical line. Points of intersection of these lines will represent $A \times B$ graphically as shown in Fig. 2.3.

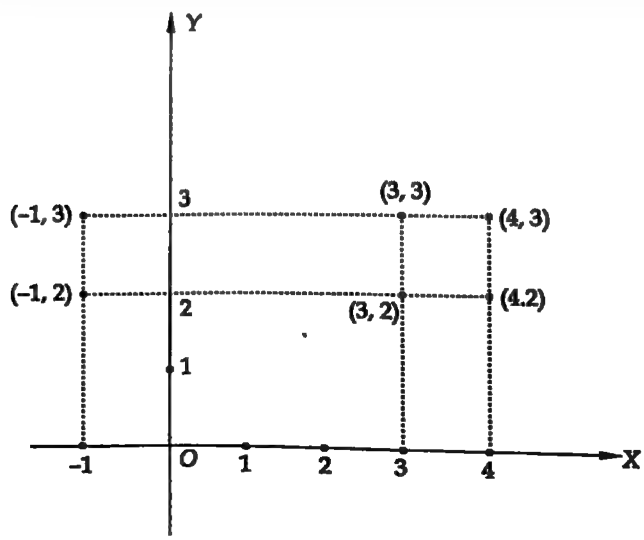

Fig. 2.3

(ii) Clearly, $B \times A = \{2, 3\} \times \{-1, 3, 4\} = \{(2, -1), (2, 3), (2, 4), (3, -1), (3, 3), (3, 4)\}$

Here, we represent B on the horizontal line and A on vertical line. Graphical representation of $B \times A$ is as shown in Fig. 2.4.

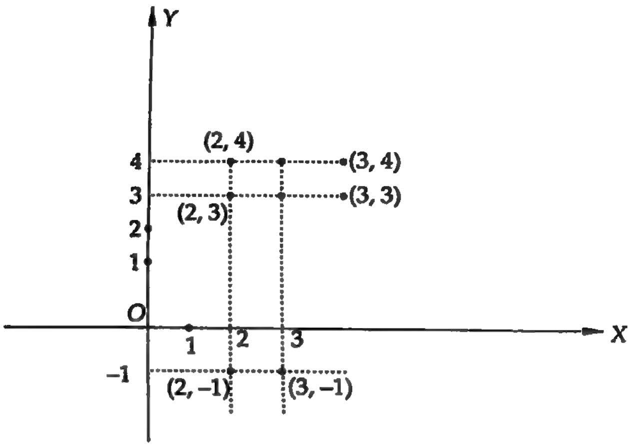

Fig. 2.4

(iii) We have, $A = \{1, 3, 4\}$

$\therefore \quad A \times A = \{-1, 3, 4\} \times \{-1, 3, 4\}$

$= \{(-1, -1), (-1, 3), (-1, 4), (3, -1), (3, 3), (3, 4), (4, -1), (4, 3), (4, 4)\}$

Graphical representation of $A \times A$ is shown in Fig. 2.5.

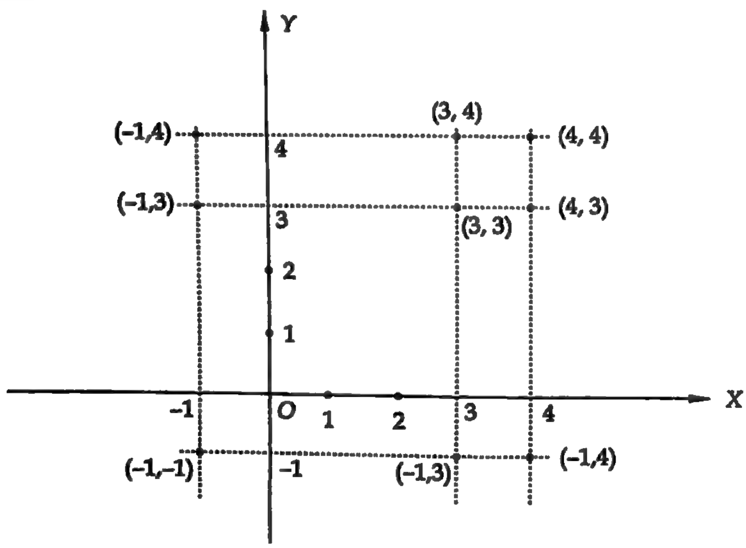

Fig. 2.5

EXAMPLE 14 If $A = \{1, 3, 5\}$, $B = \{x, y\}$ represent the following products by arrow diagrams:

(i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

SOLUTION (i) We have, $A = \{1, 3, 5\}$ and $B = \{x, y\}$

$\therefore \quad A \times B = \{1, 3, 5\} \times \{x, y\} = \{(1, x), (1, y), (3, x), (3, y), (5, x), (5, y)\}$

Following arrow diagram represents $A \times B$.

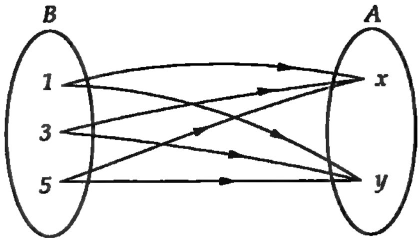

Fig. 2.6

(ii) We have, $B = \{x, y\}$ and $A = \{1, 3, 5\}$.

$\therefore \quad B \times A = \{x, y\} \times \{1, 3, 5\} = \{(x, 1), (x, 3), (x, 5), (y, 1), (y, 3), (y, 5)\}$

It has been represented by the following arrow diagram.

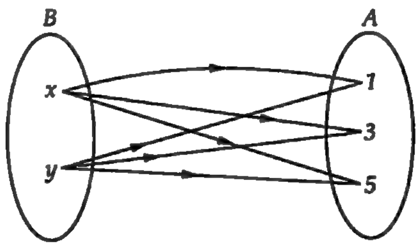

Fig. 2.7

(iii) We have, $A = \{1, 3, 5\}$

$\therefore \; A \times A = \{1, 3, 5\} \times \{1, 3, 5\} = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$

It has been represented by the following arrow diagram.

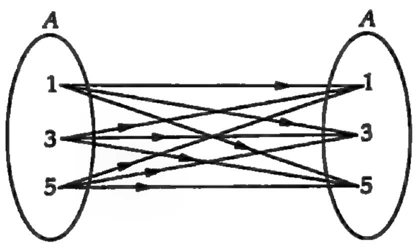

Fig. 2.8

(iv) We have, $B = \{x, y\}$

$\therefore \quad B \times B = \{x, y\} \times \{x, y\} = \{(x, x), (x, y), (y, x), (y, y)\}$

Following is the arrow diagram representing $B \times B$.

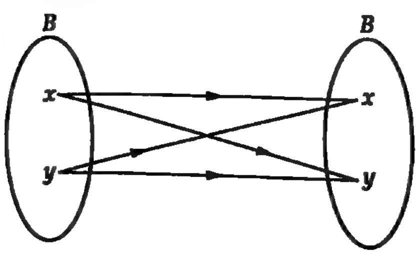

Fig. 2.9

EXERCISE 2.1

LEVEL-1

1. (i) If $\left(\frac{a}{3}+1, b-\frac{2}{3}\right)=\left(\frac{5}{3}, \frac{1}{3}\right)$, find the values of a and b.
   (ii) If $(x+1, 1)=(3, y-2)$, find the values of x and y.
2. If the ordered pairs $(x, -1)$ and $(5, y)$ belong to the set $\{(a, b): b=2a-3\}$, find the values of x and y.
3. If $a \in \{-1, 2, 3, 4, 5\}$ and $b \in \{0, 3, 6\}$, write the set of all ordered pairs (a, b) such that $a+b=5$.
4. If $a \in \{2, 4, 6, 9\}$ and $b \in \{4, 6, 18, 27\}$, then form the set of all ordered pairs (a, b) such that a divides b and $a<b$.
5. If $A=\{1, 2\}$ and $B=\{1, 3\}$, find $A \times B$ and $B \times A$.
6. Let $A=\{1, 2, 3\}$ and $B=\{3, 4\}$. Find $A \times B$ and show it graphically.
7. If $A=\{1, 2, 3\}$ and $B=\{2, 4\}$, what are $A \times B, B \times A, A \times A, B \times B$, and $(A \times B) \cap (B \times A)$?
8. If A and B are two sets having 3 elements in common. *If* $n(A)=5, n(B)=4$, find $n(A \times B)$ and $n[(A \times B) \cap (B \times A)]$.
9. Let A and B be two sets. Show that the sets $A \times B$ and $B \times A$ have an element in common *iff* the sets A and B have an element in common.
10. Let A and B be two sets such that $n(A)=3$ and $n(B)=2$.
    If $(x, 1), (y, 2), (z, 1)$ are in $A \times B$, find A and B, where x, y, z are distinct elements.
11. Let $A=\{1, 2, 3, 4\}$ and $R=\{(a, b): a \in A, b \in A, a \text{ divides } b\}$. Write R explicitly.
12. If $A=\{-1, 1\}$, find $A \times A \times A$.
13. State whether each of the following statements are true or false. If the statement is false, re-write the given statement correctly:
    (i) If $P=\{m, n\}$ and $Q=\{n, m\}$, then $P \times Q=\{(m, n), (n, m)\}$
    (ii) If A and B are non-empty sets, then $A \times B$ is a non-empty set of ordered pairs (x, y) such that $x \in B$ and $y \in A$.
    (iii) If $A=\{1, 2\}, B=\{3, 4\}$, then $A \times (B \cap \phi)=\phi$.
14. If $A=\{1, 2\}$, form the set $A \times A \times A$.
15. If $A=\{1, 2, 4\}$ and $B=\{1, 2, 3\}$, represent following sets graphically:
    (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

ANSWERS

1. (i) $a=2, b=1$ (ii) $x=2, y=3$ 2. $x=1, y=7$ 3. $\{(-1, 6), (2, 3), (5, 0)\}$
4. $\{(2, 4), (2, 6), (2, 18), (6, 18), (9, 18), (9, 27)\}$
5. $A \times B=\{(1, 1), (1, 3), (2, 1), (2, 3)\}$ and $B \times A=\{(1, 1), (1, 2), (3, 1), (3, 2)\}$
6. $A \times B=\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$.
7. $A \times B=\{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$
   $B \times A=\{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2), (4, 3)\}$
   $A \times A=\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
   $B \times B=\{(2, 2), (2, 4), (4, 2), (4, 4)\}$.
   $(A \times B) \cap (B \times A)=\{(2, 2)\}$
8. $n(A \times B)=20, n[(A \times B) \cap (B \times A)]=9$. 10. $A=\{x, y, z\}, B=\{1, 2\}$
11. $R=\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
12. $A \times A \times A=\{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}$
13. (i) F (ii) F (iii) T
14. $A \times A \times A=\{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

8. $n(A \times B) = n(A) \times n(B) = 5 \times 4 = 20$. From theorem 9, *if* A and B have n elements in common, then $(A \times B)$ and $B \times A$ have n^2 elements in common. Therefore,

   $n[(A \times B) \cap (B \times A)] = 3^2 = 9.$

2.4 SOME USEFUL RESULTS

In this section, we intend to study some results on cartesian product of sets which are given as theorems.

THEOREM 1 *For any three sets A, B, C, prove that:*

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

PROOF (i) Let (a, b) be an arbitrary element of $A \times (B \cup C)$. Then,

$(a, b) \in A \times (B \cup C)$

$\Rightarrow$ $a \in A$ and $b \in B \cup C$ [By definition]

$\Rightarrow$ $a \in A$ and ($b \in B$ or $b \in C$) [By definition of union]

$\Rightarrow$ ($a \in A$ and $b \in B$) or ($a \in A$ and $b \in C$)

$\Rightarrow$ $(a, b) \in A \times B$ or $(a, b) \in A \times C$

$\Rightarrow$ $(a, b) \in (A \times B) \cup (A \times C)$

$\therefore$ $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ ...(i)

Now, let (x, y) be an arbitrary element of $(A \times B) \cup (A \times C)$. Then,

$(x, y) \in (A \times B) \cup (A \times C)$

$\Rightarrow$ $(x, y) \in A \times B$ or, $(x, y) \in A \times C$

$\Rightarrow$ ($x \in A$ and $y \in B$) or, ($x \in A$ and $y \in C$)

$\Rightarrow$ $x \in A$ and ($y \in B$ or $y \in C$)

$\Rightarrow$ $x \in A$ and $y \in (B \cup C)$

$\Rightarrow$ $(x, y) \in A \times (B \cap C)$

$\therefore$ $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ ...(ii)

Hence, from (i) and (ii), we obtain

$A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(ii) Let (a, b) be an arbitrary element of $A \times (B \cap C)$. Then,

$(a, b) \in A \times (B \cap C)$

$\Rightarrow$ $a \in A$ and $b \in (B \cap C)$ [By definition]

$\Rightarrow$ $a \in A$ and ($b \in B$ and $b \in C$)

$\Rightarrow$ ($a \in A$ and $b \in B$) and ($a \in A$ and $b \in C$)

$\Rightarrow$ $(a, b) \in A \times B$ and $(a, b) \in A \times C$ [By definition]

$\Rightarrow$ $(a, b) \in (A \times B) \cap (A \times C)$

$\therefore$ $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ ...(i)

Now, let (x, y) be an arbitrary element of $(A \times B) \cap (A \times C)$. Then,

$(x, y) \in (A \times B) \cap (A \times C)$

$\Rightarrow$ $(x, y) \in (A \times B)$ and $(x, y) \in A \times C$

$\Rightarrow$ ($x \in A$ and $y \in B$) and ($x \in A$ and $y \in C$)

$\Rightarrow$ $x \in A$ and ($y \in B$ and $y \in C$)

$\Rightarrow$ $x \in A$ and $y \in (B \cap C)$

$\Rightarrow$ $(x, y) \in A \times (B \cap C)$

$\therefore$ $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ ...(ii)

Hence, from (i) and (ii), we obtain

$A \times (B \cap C) = (A \times B) \cap (A \times C)$ Q.E.D.

THEOREM 2 *For any three sets A, B, C, prove that:* $A \times (B - C) = (A \times B) - (A \times C)$.

PROOF Let (a, b) be an arbitrary element of $A \times (B - C)$. Then,

$(a, b) \in A \times (B - C)$

$\Rightarrow$ $a \in A$ and $b \in (B - C)$

$\Rightarrow$ $a \in A$ and $(b \in B$ and $b \notin C)$

$\Rightarrow$ $(a \in A$ and $b \in B)$ and $(a \in A$ and $b \notin C)$

$\Rightarrow$ $(a, b) \in (A \times B)$ and $(a, b) \notin (A \times C)$

$\Rightarrow$ $(a, b) \in (A \times B) - (A \times C)$

$\therefore$ $A \times (B - C) \subseteq (A \times B) - (A \times C)$ ...(i)

Now, let (x, y) be an arbitrary element of $(A \times B) - (A \times C)$. Then,

$(x, y) \in (A \times B) - (A \times C)$

$\Rightarrow$ $(x, y) \in A \times B$ and $(x, y) \notin A \times C$

$\Rightarrow$ $(x \in A$ and $y \in B)$ and $(x \in A$ and $y \notin C)$

$\Rightarrow$ $x \in A$ and $(y \in B$ and $y \notin C)$

$\Rightarrow$ $x \in A$ and $y \in (B - C)$

$\Rightarrow$ $(x, y) \in A \times (B - C)$

$\therefore$ $(A \times B) - (A \times C) \subseteq A \times (B - C)$ ...(ii)

Hence, from (i) and (ii), we get

$A \times (B - C) = (A \times B) - (A \times C)$ Q.E.D.

THEOREM 3 *If A and B are any two non-empty sets, then prove that:* $A \times B = B \times A \Leftrightarrow A = B$.

PROOF *First, let* $A = B$. *Then we have to prove that* $A \times B = B \times A$.

Now, $A = B$

$\Rightarrow$ $A \times B = A \times A$ and $B \times A = A \times A$ $[\because B = A]$

$\Rightarrow$ $A \times B = B \times A$

Conversely, let $A \times B = B \times A$. Then we have to prove that $A = B$.

Let x be an arbitrary element of A. Then,

$x \in A$

$\Rightarrow$ $(x, b) \in A \times B$ for all $b \in B$

$\Rightarrow$ $(x, b) \in B \times A$ $[\because A \times B = B \times A]$

$\Rightarrow$ $x \in B$ [By definition]

$\therefore$ $A \subseteq B$

Now, let y be an arbitrary element of B. Then,

$y \in B$

$\Rightarrow$ $(a, y) \in A \times B$ for all $a \in A$

$\Rightarrow$ $(a, y) \in B \times A$ $[\because A \times B = B \times A]$

$\Rightarrow$ $y \in A$ [By definition]

$\therefore$ $B \subseteq A$

Hence, $A = B$. Q.E.D.

THEOREM 4 *If* $A \subseteq B$, *show that* $A \times A \subseteq (A \times B) \cap (B \times A)$.

PROOF *Let (a, b) be an arbitrary element of* $A \times A$. *Then,*

$(a, b) \in A \times A$

$\Rightarrow$ $a \in A$ and $b \in A$

$\Rightarrow$ $(a \in A, b \in A)$ and $(a \in A, b \in A)$

$\Rightarrow$ $(a \in A, b \in B)$ and $(a \in B, b \in A)$ $[\because A \subseteq B \therefore a, b \in A \Rightarrow a, b \in B]$

$\Rightarrow$ $(a, b) \in (A \times B)$ and $(a, b) \in (B \times A)$

$\Rightarrow$ $(a, b) \in (A \times B) \cap (B \times A)$

$\therefore$ $A \times A \subseteq (A \times B) \cap (B \times A)$

Hence, $A \subseteq B \Rightarrow A \times A \subseteq (A \times B) \cap (B \times A)$. Q.E.D.

THEOREM 5 *If $A \subseteq B$, prove that $A \times C \subseteq B \times C$ for any set C.*

PROOF Let (a, b) be an arbitrary element of $A \times C$. Then,

$(a, b) \in A \times C$

$\Rightarrow \quad a \in A$ and $b \in C$

$\Rightarrow \quad a \in B$ and $b \in C$ $\quad [\because A \subseteq B \therefore a \in A \Rightarrow a \in B]$

$\Rightarrow \quad (a, b) \in B \times C$

Thus, $(a, b) \in A \times C \Rightarrow (a, b) \in B \times C$ for all $(a, b) \in (A \times C)$.

$\therefore \quad A \times C \subseteq B \times C.$ **Q.E.D.**

THEOREM 6 *If $A \subseteq B$ and $C \subseteq D$, prove that $A \times C \subseteq B \times D$.*

PROOF Let (a, b) be an arbitrary element of $A \times C$. Then,

$(a, b) \in A \times C$

$\Rightarrow \quad a \in A$ and $b \in C$

$\Rightarrow \quad a \in B$ and $b \in D$ $\quad [\because A \subseteq B$ and $C \subseteq D]$

$\Rightarrow \quad (a, b) \in B \times D$

Thus, $(a, b) \in A \times C \Rightarrow (a, b) \in B \times D$ for all $(a, b) \in (A \times C)$.

$\therefore \quad A \times C \subseteq B \times D$ **Q.E.D.**

THEOREM 7 *For any sets A, B, C, D prove that:* $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

PROOF Let (a, b) be an arbitrary element of $(A \times B) \cap (C \times D)$. Then,

$(a, b) \in (A \times B) \cap (C \times D)$

$\Rightarrow \quad (a, b) \in A \times B$ and $(a, b) \in C \times D$

$\Rightarrow \quad (a \in A$ and $b \in B)$ and $(a \in C$ and $b \in D)$

$\Rightarrow \quad (a \in A$ and $a \in C)$ and $(b \in B$ and $b \in D)$

$\Rightarrow \quad a \in (A \cap C)$ and $b \in B \cap D$

$\Rightarrow \quad (a, b) \in (A \cap C) \times (B \cap D)$

$\therefore \quad (A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$

Similarly, $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$

Hence, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ **Q.E.D.**

COROLLARY *For any sets A and B, prove that $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$.*

THEOREM 8 *For any three sets A, B, C prove that:*

(i) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$ (ii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$.

PROOF (i) We have,

$A \times (B' \cup C')' = A \times ((B')' \cap (C')')$ [By De-Morgan's law]

$= A \times (B \cap C) = (A \times B) \cap (A \times C)$ [See Theorem 1]

(ii) $A \times (B' \cap C')' = A \times ((B')' \cup (C')')$ [By De-Morgan's Law]

$= A \times (B \cup C) = (A \times B) \cup (A \times C)$ [See Theorem 1]

Q.E.D.

THEOREM 9 *Let A and B be two non-empty sets having n elements in common, then prove that $A \times B$ and $B \times A$ have n^2 elements in common.*

PROOF We have,

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ [See Theorem 7]

$\Rightarrow \quad (A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$ [On replacing C by B and D by A]

$\Rightarrow \quad (A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$

It is given that $A \cap B$ has n elements, so $(A \cap B) \times (B \cap A)$ has n^2 elements.

But, $(A\times B)\cap(B\times A) = (A\cap B)\times(A\cap B)$ [Proved above]

$\therefore$ $(A\times B)\cap(B\times A)$ has n^2 elements.

Hence, $A\times B$ and $B\times A$ have n^2 elements in common. Q.E.D.

THEOREM 10 *Let A be a non-empty set such that $A\times B = A\times C$. Show that $B = C$.*

PROOF Let b be an arbitrary element of B. Then,

$(a, b)\in A\times B$ for all $a\in A$

$\Rightarrow$ $(a, b)\in A\times C$ for all $a\in A$ [$\because A\times B = A\times C$]

$\Rightarrow$ $b\in C$

Thus, $b\in B \Rightarrow b\in C$

$\therefore$ $B\subset C$ ...(i)

Now, let c be an arbitrary element of C. Then,

$(a, c)\in A\times C$ for all $a\in A$

$\Rightarrow$ $(a, c)\in A\times B$ for all $a\in A$ [$\because A\times B = A\times C$]

$\Rightarrow$ $c\in B$

Thus, $c\in C \Rightarrow c\in B$

$\therefore$ $C\subset B$ ...(ii)

From (i) and (ii), we get $B = C$.

EXERCISE 2.2

LEVEL-1

1. Given $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, find $(A\times B)\cap(B\times C)$.
2. If $A = \{2, 3\}$, $B = \{4, 5\}$, $C = \{5, 6\}$, find $A\times(B\cup C)$, $A\times(B\cap C)$, $(A\times B)\cup(A\times C)$.
3. If $A = \{1, 2, 3\}$, $B = \{4\}$, $C = \{5\}$, then verify that:
   (i) $A\times(B\cup C) = (A\times B)\cup(A\times C)$ (ii) $A\times(B\cap C) = (A\times B)\cap(A\times C)$
   (iii) $A\times(B - C) = (A\times B) - (A\times C)$.
4. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$. Verify that:
   (i) $A\times C\subset B\times D$ (ii) $A\times(B\cap C) = (A\times B)\cap(A\times C)$
5. If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$, find
   (i) $A\times(B\cap C)$ (ii) $(A\times B)\cap(A\times C)$ (iii) $A\times(B\cup C)$ (iv) $(A\times B)\cup(A\times C)$

LEVEL-2

6. Prove that: (i) $(A\cup B)\times C = (A\times C)\cup(B\times C)$ (ii) $(A\cap B)\times C = (A\times C)\cap(B\times C)$
7. If $A\times B\subseteq C\times D$ and $A\times B\neq\phi$, prove that $A\subseteq C$ and $B\subseteq D$.

ANSWERS

1. $\{3, 4\}$.
2. $A\times(B\cup C) = \{(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$
   $A\times(B\cap C) = \{(2, 5), (3, 5)\}$,
   $(A\times B)\cup(A\times C) = \{(2, 4), (2, 5), (3, 4), (3, 5), (2, 6), (3, 6)\}$.
4. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
5. (i) $\{(1, 4), (2, 4), (3, 4)\}$ (ii) $\{(1, 4), (2, 4), (3, 4)\}$
   (iii) $\{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$
   (iv) $\{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$

2.5 RELATIONS

Let A and B denote the sets of all male and female members in the royal family of Dasrath's kingdom. Clearly, A = { Dasrath, Ram, Laxman, Shatrughan, Bharat } and B = { Kaushalya, Kaikai, Sumitra, Sita, Urmila, Shrutkirti, Mandvi }.

If we write R for the relation "was husband of" then the fact that Dasrath was husband of Kaushalya, Kaikai and Sumitra, Ram was husband of Sita, Laxman was husband of Urmila, Bharat was husband of Mandvi and Shatrughan was husband of Shrutkirti can be represented as:

Dasrath R Kaushlya, Dasrath R Kaikai, Dasrath R Sumitra, Ram R Sita, Laxman R Urmila, Bharat R Mandvi and Shatrughan R Shrutkirti.

Now, if we omit the letter R between the pairs of names and write them as ordered pairs, then the above fact can also be written as a set R of ordered pairs as given below:

R = { (Dasrath, Kaushalya), (Dasrath, Kaikai), (Dasrath, Sumitra), (Ram, Sita), (Laxman, Urmila), (Bharat, Mandvi), (Shatrughan, Shrutkirti) }.

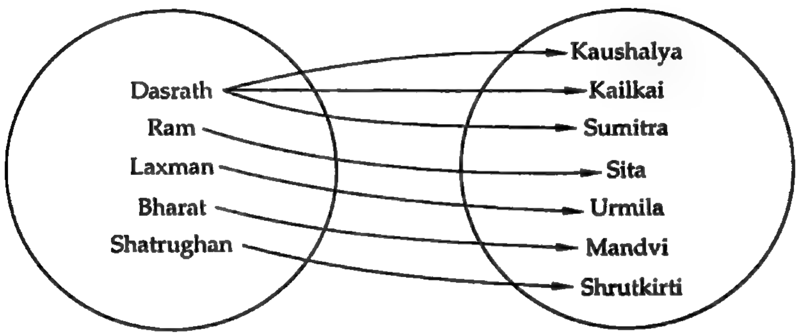

Fig. 2.10

Clearly, $R \subseteq A \times B$.

A visual representation of this relation R in the form of an arrow diagram is as follows:

Thus, we see that the relation "was husband of" from set A to set B gives rise to a subset R of $A \times B$ such that $(x, y) \in R$ *iff* xRy.

Keeping this example in mind, we may define a relation as follows.

RELATION *Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$.*

Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

If R is a relation from a non-void set A to a non-void set B and *if* $(a, b) \in R$, then we write aRb which is read as 'a is related to b by the relation R. *If* $(a, b) \notin R$, then we write $a \not R b$ and we say that a is not related to b by the relation R.

ILLUSTRATION 1 *If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then $R = \{(1, b), (2, c), (1, a), (3, a)\}$, being a subset of $A \times B$, is a relation from A to B. Here, $(1, b)$, $(2, c)$, $(1, a)$ and $(3, a) \in R$, so we write $1Rb$, $2Rc$, $1Ra$ and $3Ra$. But, $(2, b) \notin R$, so we write $2 \not R b$*

ILLUSTRATION 2 *If $A = \{a, b, c, d\}$, $B = \{p, q, r, s\}$, then which of the following are relations from A to B? Give reasons for your answer.*

(i) $R_1 = \{(a, p), (b, r), (c, s)\}$ *(ii)* $R_2 = \{(q, b), (c, s), (d, r)\}$

(iii) $R_3 = \{(a, p), (a, q), (d, p), (c, r), (b, r)\}$ *(iv)* $R_4 = \{(a, p), (q, a), (b, s), (s, b)\}$.

SOLUTION (i) Clearly, $R_1 \subseteq A \times B$. So, R_1 is a relation from A to B.

(ii) Since $(q, b) \in R_2$ but $(q, b) \notin A \times B$. So, $R_2 \not\subseteq A \times B$. Thus, R_2 is not a relation from A to B.

(iii) Clearly, $R_3 \subseteq A \times B$. So it is a relation from A to B.

(iv) R_4 is not a relation from A to B, because (q, a) and (s, b) are elements of R_4 but (q, a) and (s, b) are not in $A \times B$. As such $R_4 \not\subseteq A \times B$.

TOTAL NUMBER OF RELATIONS Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then, $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}. Since each subset of $A \times B$ defines a relation from A to B, so total number of relations from A to B is 2^{mn}. Among these 2^{mn} relations the void relation ϕ and the universal relation $A \times B$ are trivial relations from A to B.

2.5.1 REPRESENTATION OF A RELATION

A relation from a set A to a set B can be represented in any one of the following forms:

(I) ROSTER FORM In this form a relation is represented by the set of all ordered pairs belonging to R.

For example, if R is a relation from set $A = \{-2, -1, 0, 1, 2\}$ to set $B = \{0, 1, 4, 9, 10\}$ by the rule $a R b \Leftrightarrow a^2 = b$. Then, $0 R 0, -2 R 4, -1 R 1, 1 R 1$ and $2 R 4$.

So, R can be described in Roster form as $R = \{(0, 0), (-1, 1), (-2, 4), (1, 1), (2, 4)\}$

(II) SET-BUILDER FORM In this form the relation R from set A to set B is represented as $R = \{(a, b) : a \in A, b \in B$ and a, b satisfy the rule which associates a and $b\}$.

For example, if $A = \{1, 2, 3, 4, 5\}$, $B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\right\}$ and R is a relation from A to B given by $R = \left\{(1, 1), \left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), \left(4, \frac{1}{4}\right), \left(5, \frac{1}{5}\right)\right\}$.

Then, R in set-builder form can be described as: $R = \left\{(a, b) : a \in A, b \in B \text{ and } b = \frac{1}{a}\right\}$.

It should be noted that it is not possible to express every relation from set A to set B in set-builder form. For example, the relation $R = \{(1, a), (1, c), (3, b)\}$ from set $A = \{1, 2, 3, 4\}$ to set $B = \{a, b, c\}$ cannot be described in set-builder form.

(III) BY ARROW DIAGRAM In order to represent a relation from set A to a set B by an arrow diagram, we draw arrows from first components to the second components of all ordered pairs belonging to R.

For example, relation $R = \{(1, 2), (2, 4), (3, 2), (1, 3), (3, 4)\}$ from set $A = \{1, 2, 3, 4, 5\}$ to set $B = \{2, 3, 4, 5, 6, 7\}$ can be represented by the following arrow diagram:

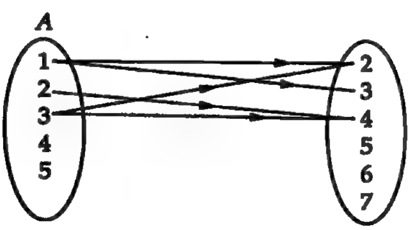

Fig. 2.11

(IV) BY LATTICE In this form, the relation R from set A to set B is represented by darking the dots in the lattice for $A \times B$ which represent the ordered pairs in R.

For example, if $R = \{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$ is a relation from set $A = \{-3, -2, -1, 0, 1, 2, 3\}$ to set $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then R can be represented by the following lattice.

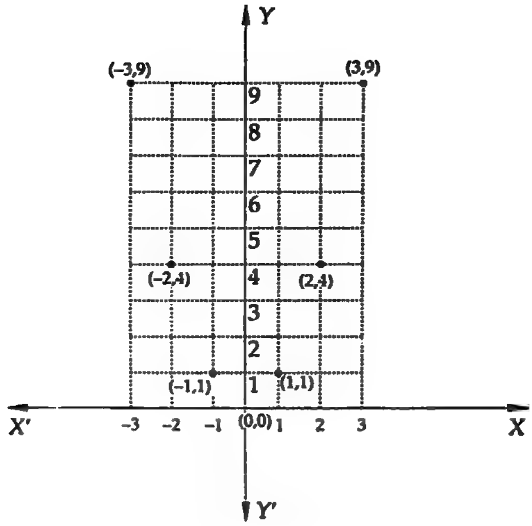

Fig. 2.12

2.5.2 DOMAIN AND RANGE OF A RELATION

Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom $(R) = \{a : (a, b) \therefore \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$.

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B. The set B is called the co-domain of relation R.

ILLUSTRATION 1 *If* $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8, 10\}$ *and let* $R = \{(1, 8), (3, 6), (5, 2), (1, 4)\}$ *be a relation from A to B. Then,*

$$\text{Domain }(R) = \{1, 3, 5\} \text{ and Range}(R) = \{8, 6, 2, 4\}$$

ILLUSTRATION 2 *Let* $A = \{1, 3, 5, 7\}$ *and* $B = \{2, 4, 6, 8\}$ *be two sets and let R be a relation from A to B defined by the phrase "*$(x, y) \in R \Leftrightarrow x > y$*". Under this relation R, we obtain* $3R2, 5R2, 5R4, 7R2, 7R4$ *and* $7R6$.

i.e. $R = \{(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)\}$.

$\therefore$ Domain $(R) = \{3, 5, 7\}$ and Range $(R) = \{2, 4, 6\}$

ILLUSTRATION 3 *If R is a relation from set* $A = \{2, 4, 5\}$ *to set* $B = \{1, 2, 3, 4, 6, 8\}$ *defined by* $xRy \Leftrightarrow x$ *divides* y.

(i) Write R as a set of ordered pairs, *(ii) Find the domain and the range of R.*

SOLUTION (i) Clearly, $2R2, 2R4, 2R6, 2R8, 4R4$, and $4R8$.

$\therefore$ $R = \{(2, 2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8)\}$

(ii) Clearly, Domain $(R) = \{2, 4\}$ and Range $(R) = \{2, 4, 6, 8\}$

RELATION ON A SET *Let A be a non-void set. Then, a relation from A to itself i.e. a subset of* $A \times A$*, is called a relation on set A.*

2.5.3 INVERSE OF A RELATION

INVERSE RELATION *Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by* R^{-1}*, is a relation from B to A and is defined by* $R^{-1} = \{(b, a) : (a, b) \in R\}$.

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, Dom $(R) =$ Range (R^{-1}) and, Range $(R) =$ Dom (R^{-1}).

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ be two sets and let $R = \{(1, a), (1, c), (2, d), (2, c)\}$ be a relation from A to B. Then, $R^{-1} = \{(a, 1), (c, 1), (d, 2), (c, 2)\}$ is a relation from B to A.*

Also, $\quad$ *Dom $(R) = \{1, 2\} = $ Range (R^{-1}), and Range $(R) = \{a, c, d\} = $ Dom (R^{-1}).*

ILLUSTRATION 2 *Let A be the set of first ten natural numbers and let R be a relation on A defined by $(x, y) \in R \Leftrightarrow x + 2y = 10$ i.e. $R = \{(x, y) : x \in A, y \in A$ and $x + 2y = 10\}$. Express R and R^{-1} as sets of ordered pairs. Also, determine (i) domains of R and R^{-1} (ii) ranges of R and R^{-1}.*

SOLUTION We have,

$$(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \frac{10 - x}{2}, x, y \in A \text{ where } A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Now, $\quad x = 1 \Rightarrow y = \dfrac{10-1}{2} = \dfrac{9}{2} \notin A.$

This shows that 1 is not related to any element in A. Similarly we can observe that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation.

Further we find that:

For $x = 2$, $\quad y = \dfrac{10-2}{2} = 4 \in A$. Therefore, $(2, 4) \in R$

For $x = 4$, $\quad y = \dfrac{10-4}{2} = 3 \in A$. Therefore, $(4, 3) \in R$

For $x = 6$, $\quad y = \dfrac{10-6}{2} = 2 \in A$. Therefore, $(6, 2) \in R$

For $x = 8$, $\quad y = \dfrac{10-8}{2} = 1 \in A$. Therefore, $(8, 1) \in R$

Thus, $\quad R = \{(2, 4), (4, 3), (6, 2), (8, 1)\} \Rightarrow R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, Dom$(R) = \{2, 4, 6, 8\} = $ Range (R^{-1}) and, Range $(R) = \{4, 3, 2, 1\} = $ Dom (R^{-1}).

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON EXAMINING WHETHER A SET OF ORDERED PAIRS REPRESENTS A RELATION OR NOT**

EXAMPLE 1 *If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, which of the following are relations from A to B? Give reasons in support of your answer:*

(i) $R_1 = \{ (1, 4), (1, 5), (1, 6) \}$ $\quad$ (ii) $R_2 = \{ (1, 5), (2, 4), (3, 6) \}$

(iii) $R_3 = \{ (1, 4), (1, 5), (3, 6), (2, 6), (3, 4) \}$ $\quad$ (iv) $R_4 = \{ (4, 2), (2, 6), (5, 1), (2, 4) \}$.

SOLUTION (i) Clearly, $R_1 \subseteq A \times B$. So, it is a relation from A to B.

(ii) Clearly, $R_2 \subseteq A \times B$. So, it is a relation from A to B.

(iii) Clearly, $R_3 \subseteq A \times B$. So, it is a relation from A to B.

(iv) Since $(4, 2) \in R_4$ but $(4, 2) \notin A \times B$. So, R_4 is not a relation from A to B.

Type II **ON DESCRIBING A RELATION AND ITS INVERSE AS A SET OF ORDERED PAIRS AND FINDING THEIR DOMAINS AND RANGES**

EXAMPLE 2 *A relation R is defined from a set $A = \{ 2, 3, 4, 5 \}$ to a set $B = \{ 3, 6, 7, 10 \}$ as follows: $(x, y) \in R \Leftrightarrow x$ divides y. Express R as a set of ordered pairs and determine the domain and range of R. Also, find R^{-1}.*

SOLUTION Recall that $a \mid b$ stands for 'a divides b'. For the elements of the given sets A and B, we find that $2 \mid 6, 2 \mid 10, 3 \mid 3, 3 \mid 6$, and $5 \mid 10$.

$\therefore \quad (2, 6) \in R, (2, 10) \in R, (3, 3) \in R, (3, 6) \in R$ and $(5, 10) \in R$.

Thus, $R = \{ (2, 6), (2, 10), (3, 3), (3, 6), (5, 10) \}$.
Clearly, Dom $(R) = \{ 2, 3, 5 \}$ and, Range $(R) = \{ 3, 6, 10 \}$.
Also, $R^{-1} = \{ (6, 2), (10, 2), (3, 3), (6, 3), (10, 5) \}$.

EXAMPLE 3 *If R is the relation "less than" from $A = \{ 1, 2, 3, 4, 5 \}$ to $B = \{ 1, 4, 5 \}$, write down the set of ordered pairs corresponding to R. Find the inverse of R.*

SOLUTION It is given that $(x, y) \in R \Leftrightarrow x < y$, where $x \in A$ and $y \in B$.
For the elements of the given sets A and B, we find that

$$1 < 4,\ 1 < 5,\ 2 < 4,\ 2 < 5,\ 3 < 4,\ 3 < 5 \text{ and } 4 < 5$$

$\therefore$ $(1, 4) \in R, (1, 5) \in R, (2, 4) \in R, (2, 5) \in R, (3, 4) \in R, (3, 5) \in R$ and $(4, 5) \in R$.
Thus, $R = \{ (1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5) \}$.
$\therefore$ $R^{-1} = \{ (4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4) \} = \{(x, y) : x \in B, y \in A \text{ and } x > y\}$.

EXAMPLE 4 *A relation R is defined on the set Z of integers as: $(x, y) \in R \Leftrightarrow x^2 + y^2 = 25$.*

Express R and R^{-1} as the sets of ordered pairs and hence find their respective domains.

SOLUTION We have,

$$(x, y) \in R \Leftrightarrow x^2 + y^2 = 25 \Leftrightarrow y = \pm \sqrt{25 - x^2}$$

We observe that from the above relation $x = 0$ gives $y = \pm 5$.

$\therefore$ $(0, 5) \in R$ and $(0, -5) \in R$

Similarly, $x = \pm 3 \Rightarrow y = \sqrt{25 - 9} = \pm 4$

$\therefore$ $(3, 4) \in R, (-3, 4) \in R, (3, -4) \in R$ and $(-3, -4) \in R$

$x = \pm 4 \Rightarrow y = \sqrt{25 - 16} = \pm 3$

$\therefore$ $(4, 3) \in R, (-4, 3) \in R, (4, -3) \in R$ and $(-4, -3) \in R$

$x = \pm 5 \Rightarrow y = \sqrt{25 - 25} = 0$

$\therefore$ $(5, 0) \in R$ and $(-5, 0) \in R$

We also notice that for any other integral value of x, the value of y given by $y = \pm \sqrt{25 - x^2}$ is not an integer.

$\therefore$ $R = \{ (0, 5), (0, -5), (3, 4), (-3, 4), (3, -4), (-3, -4), (4, 3), (-4, 3), (4, -3), (-4, -3), (5, 0), (-5, 0) \}$

$\Rightarrow$ $R^{-1} = \{(5, 0), (-5, 0), (4, 3), (4, -3), (-4, 3), (-4, -3), (3, 4), (3, -4), (-3, 4), (-3, -4), (0, 5), (0, -5) \}$

Clearly, Domain $(R) = \{ 0, 3, -3, 4, -4, 5, -5 \}$ = Domain (R^{-1}).

EXAMPLE 5 *Let R be the relation on the set N of natural numbers defined by*

$$R = \{(a, b): a + 3b = 12, a \in N, b \in N \}.$$

Find: (i) R (ii) Domain of R (iii) Range of R

SOLUTION (i) We have,

$$a + 3b = 12 \Rightarrow a = 12 - 3b$$

Putting $b = 1, 2, 3$ respectively in the above relation, we get $a = 9, 6, 3$ respectively.
For $b = 4$, $a = 12 - 3b$ gives $a = 0$ which does not belong to N. Also, values of a given by $a = 12 - 3b$ do not belong to N for all $b > 4$.

$\therefore$ $R = \{(9, 1), (6, 2), (3, 3)\}$

(ii) Clearly, Domain of $R = \{9, 6, 3\}$

(iii) Clearly, Range of $R = \{1, 2, 3\}$

Type III ON REPRESENTING A RELATION BY USING AN ARROW DIAGRAM

EXAMPLE 6 *Let $A = \{1, 2, 3, 4, 5, 6\}$. Define a relation R on set A by $R = \{(x, y) : y = x + 1\}$.*
(i) Depict this relation using an arrow diagram (ii) Write down the domain, co-domain and range of R.

SOLUTION (i) Putting $x = 1, 2, 3, 4, 5, 6$ respectively in $y = x + 1$, we get $y = 2, 3, 4, 5, 6, 7$ respectively.

$\therefore \quad (1, 2) \in R, (2, 3) \in R, (3, 4) \in R, (4, 5) \in R, (5, 6) \in R$ and $(6, 7) \notin R$.

For $x = 6$, we get $y = 7$ which does not belong to set A.

Hence, $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

The arrow diagram representing R is as follows.

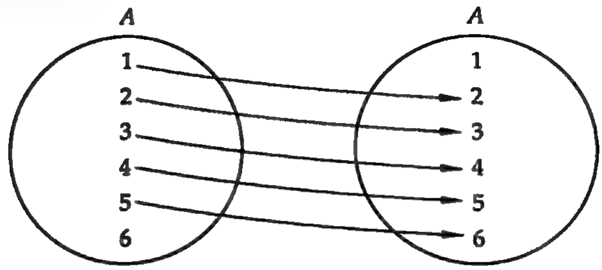

Fig. 2.13

(ii) Clearly, Domain $(R) = \{1, 2, 3, 4, 5\}$, Range $(R) = \{2, 3, 4, 5, 6\}$.

EXAMPLE 7 *Figure 2.14 shows a relation R between the sets P and Q. Write this relation R in (i) Roster form (ii) Set builder form. What is its domain and range?*

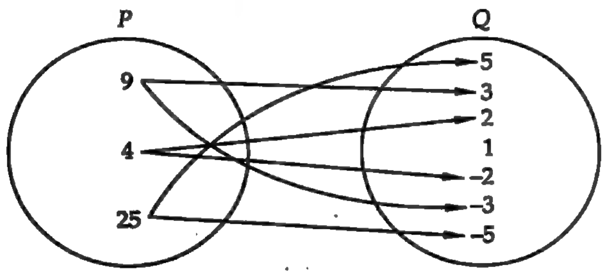

Fig. 2.14

SOLUTION (i) It is evident from the figure that

$$R = \{(9, 3), (9, -3), (4, 2), (4 -2), (25, 5), (25 -5)\}$$

(ii) It is evident from R, that it consists of elements (x, y), where x is the square of y i.e. $x = y^2$.

Therefore, relation R in set builder form is $R = \{(x, y) : x = y^2, x \in P, y \in Q\}$.

The domain and range of R are $\{9, 4, 25\}$ and $\{-5, -3, -2, 2, 3, 5\}$ respectively.

REMARK *In the above example, the range of relation R is not same as the set Q. The set Q is known as the co-domain.*

Type IV **ON PROVING RESULTS BASED ON THE DEFINITION OF A RELATION**

EXAMPLE 8 *Let R be a relation on Q defined by* $R = \{(a, b) : a, b \in Q \text{ and } a - b \in Z\}$.

Show that:

(i) $(a, a) \in R$ *for all* $a \in Q$ (ii) $(a, b) \in R \Rightarrow (b, a) \in R$

(iii) $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$. [NCERT]

SOLUTION (i) For any $a \in Q$, we have

$a - a = 0 \in Z$

$\Rightarrow \quad (a, a) \in R$

Hence, $(a, a) \in R$ for all $a \in Q$.

(ii) Let $(a, b) \in R$. Then,

$(a, b) \in R$

$a - b \in Z$, where $a, b \in Q$

$\Rightarrow$ $b - a \in Z$ $[\because\ b - a = -(a - b)]$

$\Rightarrow$ $(b, a) \in R$

(iii) Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a - b \in Z$ and $b - c \in Z$

$\Rightarrow$ $(a - b) + (b - c) \in Z$

$\Rightarrow$ $a - c \in Z$

$\Rightarrow$ $(a, c) \in R$

EXAMPLE 9 *Let R be a relation on N defined by* $R = \{(a, b) : a, b \in N \text{ and } a = b^2\}$.
Are the following true:

(i) $(a, a) \in R$ *for all* $a \in N$ (ii) $(a, b) \in R \Rightarrow (b, a) \in R$

(iii) $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$ **[NCERT]**

Justify your answer in each case.

SOLUTION (i) We observe that $a = a^2$ is true for $a = 1 \in N$ only. Therefore, $(1, 1) \in R$. But, $(2, 2), (3, 3), (4, 4)$ etc do not belong to R. So, $(a, a) \in R$ for all $a \in N$ is not true.

(ii) We observe that $(4, 2) \in R$, because $4 = 2^2$. But, $(2, 4) \notin R$ as $2 \neq 4^2$.

So, $(a, b) \in R \Rightarrow (b, a) \in R$ is not true for all $a, b \in N$.

(iii) We observe that $(16, 4) \in R$ and $(4, 2) \in R$. However, $(16, 2) \notin R$.

So, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ is not true for all $a, b, c \in N$.

EXAMPLE 10 *Let a relation* R_1 *on the set R of all real numbers be defined as* $(a,b) \in R_1 \Leftrightarrow 1 + ab > 0$ *for all* $a, b \in R$.

Show that: (i) $(a, a) \in R_1$ *for all* $a \in R$ (ii) $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ *for all* $a, b \in R$

SOLUTION (i) For any $a \in R$, we have

$1 + a^2 > 0 \Rightarrow (a, a) \in R_1$

Thus, $(a, a) \in R_1$ for all $a \in R$.

(ii) Let $(a, b) \in R_1$. Then,

$(a, b) \in R_1 \Rightarrow 1 + ab > 0 \Rightarrow 1 + ba > 0 \Rightarrow (b, a) \in R_1$

Thus, $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ for all $a, b \in R$.

EXAMPLE 11 *Let R be the relation on the set Z of all integers defined by* $(x, y) \in R \Rightarrow x - y$ *is divisible by n. Prove that:*

(i) $(x, x) \in R$ *for all* $x \in Z$

(ii) $(x, y) \in R \Rightarrow (y, x) \in R$ *for all* $x, y \in Z$

(iii) $(x, y) \in R$ *and* $(y, z) \in R \Rightarrow (x, z) \in R$ *for all* $x, y, z \in R$.

SOLUTION (i) For any $x \in Z$, we have

$x - x = 0 = 0 \times n$

$\Rightarrow$ $x - x$ is divisible by n

$\Rightarrow$ $(x, x) \in R$

Thus, $(x, x) \in R$ for all $x \in Z$.

(ii) Let $(x, y) \in R$. Then,

$$\begin{aligned}
& (x, y) \in R \\
\Rightarrow\quad & x - y \text{ is divisible by } n \\
\Rightarrow\quad & x - y = \lambda n \text{ for same } \lambda \in Z \\
\Rightarrow\quad & y - x = (-\lambda)\, n \\
\Rightarrow\quad & y - x \text{ is divisible by } n && [\because\ \lambda \in Z \Rightarrow -\lambda \in Z] \\
\Rightarrow\quad & (y, x) \in R
\end{aligned}$$

Thus, $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in Z$.

(iii) Let $(x, y) \in R$ and $(y, z) \in R$. Then,

$$\begin{aligned}
& (x, y) \in R \Rightarrow x - y \text{ is divisible by } n \Rightarrow x - y = \lambda n \text{ for some } \lambda \in Z \\
& (y, z) \in R \Rightarrow y - z \text{ is divisible by } n \Rightarrow y - z = \mu n \text{ for some } \mu \in Z \\
\therefore\quad & (x, y) \in R \text{ and } (y, z) \in R \\
\Rightarrow\quad & x - y = \lambda n \text{ and } y - z = \mu n \\
\Rightarrow\quad & (x - y) + (y - z) = \lambda n + \mu n \\
\Rightarrow\quad & x - z = (\lambda + \mu)\, n \\
\Rightarrow\quad & x - z \text{ is divisible by } n && [\because\ \lambda + \mu \in Z] \\
\Rightarrow\quad & (x, z) \in R
\end{aligned}$$

Thus, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.

EXERCISE 2.3

LEVEL-1

1. If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, which of the following are relations from A to B? Give reasons in support of your answer.
   (i) $\{(1, 6), (3, 4), (5, 2)\}$
   (ii) $\{(1, 5), (2, 6), (3, 4), (3, 6)\}$
   (iii) $\{4, 2), (4, 3), (5, 1)\}$
   (iv) $A \times B$.
2. A relation R is defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows:
   $(x, y) \in R \Leftrightarrow x$ is relatively prime to y
   Express R as a set of ordered pairs and determine its domain and range.
3. Let A be the set of first five natural numbers and let R be a relation on A defined as follows:
   $(x, y) \in R \Leftrightarrow x \leq y$
   Express R and R^{-1} as sets of ordered pairs. Determine also
   (i) the domain of R^{-1}
   (ii) the range of R.
4. Find the inverse relation R^{-1} in each of the following cases:
   (i) $R = \{(1, 2), (1, 3), (2, 3), (3, 2), (5, 6)\}$
   (ii) $R = \{(x, y) : x, y \in N, x + 2y = 8\}$
   (iii) R is a relation from $\{11, 12, 13\}$ to $(8, 10, 12\}$ defined by $y = x - 3$.
5. Write the following relations as the sets of ordered pairs:
   (i) A relation R from the set $\{2, 3, 4, 5, 6\}$ to the set $\{1, 2, 3\}$ defined by $x = 2y$.
   (ii) A relation R on the set $\{1, 2, 3, 4, 5, 6, 7\}$ defined by $(x, y) \in R \Leftrightarrow x$ is relatively prime to y.
   (iii) A relation R on the set $\{0, 1, 2, ..., 10\}$ defined by $2x + 3y = 12$.
   (iv) A relation R from a set $A = \{5, 6, 7, 8\}$ to the set $B = \{10, 12, 15, 16, 18\}$ defined by $(x, y) \in R \Leftrightarrow x$ divides y.
6. Let R be a relation in N defined by $(x, y) \in R \Leftrightarrow x + 2y = 8$. Express R and R^{-1} as sets of ordered pairs.

7. Let $A=\{3,5\}$ and $B=\{7,11\}$. Let $R=\{(a,b): a\in A, b\in B, a-b \text{ is odd }\}$. Show that R is an empty relation from A into B.
8. Let $A=\{1,2\}$ and $B=\{3,4\}$. Find the total number of relations from A into B.
9. Determine the domain and range of the relation R defined by
   (i) $R=\{(x, x+5): x\in\{0,1,2,3,4,5\}\}$ [NCERT]
   (ii) $R=\{(x, x^3): x \text{ is a prime number less than } 10\}$ [NCERT]
10. Determine the domain and range of the following relations:
    (i) $R = \{(a,b): a\in N, a<5, b = 4\}$
    (ii) $S = \{(a,b): b = |a-1|, a\in Z \text{ and } |a|\le 3\}$
11. Let $A=\{a,b\}$. List all relations on A and find their number.
12. Let $A=\{x,y,z\}$ and $B=\{a,b\}$. Find the total number of relations from A into B. [NCERT]
13. Let R be a relation from N to N defined by $R=\{(a,b): a,b\in N \text{ and } a = b^2\}$.
    Are the following statements true?
    (i) $(a,a)\in R$ for all $a\in N$ (ii) $(a,b)\in R \Rightarrow (b,a)\in R$
    (iii) $(a,b)\in R$ and $(b,c)\in R \Rightarrow (a,c)\in R$
14. Let $A = \{1,2,3,\dots,14\}$. Define a relation on a set A by
    $R=\{(x,y): 3x-y=0, \text{ where } x,y\in A\}$.
    Depict this relationship using an arrow diagram. Write down its domain, co-domain and range.
15. Define a relation R on the set N of natural numbers by
    $R=\{(x,y): y=x+5, x \text{ is a natural number less than } 4, x,y\in N\}$.
    Depict this relationship using (i) roster form (ii) an arrow diagram. Write down the domain and range or R. [NCERT]
16. $A=\{1,2,3,5\}$ and $B=\{4,6,9\}$. Define a relation R from A to B by
    $R=\{(x,y):$ the difference between x and y is odd, $x\in A, y\in B\}$.
    Write R in Roster form. [NCERT]

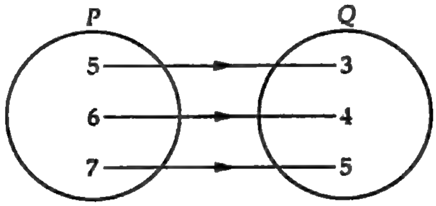

Fig. 2.15

17. Write the relation $R=\{(x,x^3): x \text{ is a prime number less than } 10\}$ in roster form. [NCERT]
18. Let $A=\{1,2,3,4,5,6\}$. Let R be a relation on A defined by
    $R=\{(a,b): a,b\in A, b \text{ is exactly divisible by a}\}$
    (i) Write R in roster form (ii) Find the domain of R (iii) Find the range of R. [NCERT]
19. Figure 2.15 shows a relationship between the sets P and Q. Write this relation in
    (i) set builder form (ii) roster form. What is its domain and range? [NCERT]
20. Let R be the relation on Z defined by $R = \{(a,b): a,b\in Z, a-b \text{ is an integer}\}$. Find the domain and range of R. [NCERT]
21. For the relation R_1 defined on R by the rule $(a,b)\in R_1 \Leftrightarrow 1+ab>0$.
    Prove that: $(a,b)\in R_1$ and $(b,c)\in R_1 \Rightarrow (a,c)\in R_1$ is not true for all $a,b,c\in R$.
22. Let R be a relation on $N\times N$ defined by
    $(a,b)\,R\,(c,d) \Leftrightarrow a+d = b+c$ for all $(a,b),(c,d)\in N\times N$
    Show that:
    (i) $(a,b)\,R\,(a,b)$ for all $(a,b)\in N\times N$
    (ii) $(a,b)\,R\,(c,d) \Rightarrow (c,d)\,R\,(a,b)$ for all $(a,b),(c,d)\in N\times N$
    (iii) $(a,b)\,R\,(c,d)$ and $(c,d)\,R\,(e,f) \Rightarrow (a,b)\,R\,(e,f)$ for all $(a,b),(c,d),(e,f)\in N\times N$

ANSWERS

1. (i) It is not a relation from A to B.
   (ii) It is a subset of $A \times B$, so it is a relation from A to B.
   (iii) It is not a relation from A to B as it is not a subset of $A \times B$.
   (iv) It is a relation from A to B.
2. $R = \{ (2,3), (2,7), (3,7), (3,10), (4,3), (4,7), (5,3), (5,7) \}$
3. $R = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (4,5), (5,5) \}$
   $R^{-1} = \{ (1,1), (2,1), (3,1), (4,1), (5,1), (2,2), (3,2), (4,2), (5,2), (3,3), (4,3), (5,3), (4,4), (5,4), (5,5) \}$
   Domain of $R^{-1} = \{ 1, 2, 3, 4, 5 \}$ = Range of R.
4. (i) $R^{-1} = \{ (2,1), (3,1), (3,2), (2,3), (6,5) \}$ (ii) $R^{-1} = \{ (3,2), (2,4), (1,6) \}$
   (iii) $R^{-1} = \{ (8,11), (10,13) \}$
5. (i) $\{ (2,1), (4,2), (6,3) \}$
   (ii) $\{ (2,3), (2,5), (2,7), (3,2), (3,4), (3,5), (3,7), (4,3), (4,5), (4,7), (5,2), (5,3), (5,4), (5,6), (5,7), (6,5), (6,7), (7,2), (7,3), (7,4), (7,5), (7,6), (7,7) \}$
   (iii) $\{(0,4), (3,2), (6,0) \}$ (iv) $\{ (5,10), (5,15), (6,12), (6,18), (8,16) \}$
6. $R = \{ (2,3), (4,2), (6,1) \}$ $R^{-1} = \{ (3,2), (2,4), (1,6) \}$
8. 16
9. (i) Domain $R = \{0, 1, 2, 3, 4, 5\}$, Range $R = \{5, 6, 7, 8, 9, 10\}$
   (ii) Domain $R = \{2, 3, 5, 7\}$, Range $R = \{8, 27, 125, 343\}$
10. (i) Domain $R = \{1, 2, 3, 4\}$, Range $R = \{4\}$
    Domain $S = \{0, -1, -2, -3, 1, 2, 3\}$, Range $S = \{0, 1, 2, 3, 4\}$
    (ii) $S = \{(0,1)), (-1,2), (-2,3), (-3,4), (1,0), (2,1), (3,2)\}$
11. 16
12. 64
13. (i) No (ii) No (iii) No
14. Domain $(R) = \{1, 2, 3, 4\}$, Co-domain $(R) = A$, Range$(R) = \{3, 6, 9, 12\}$

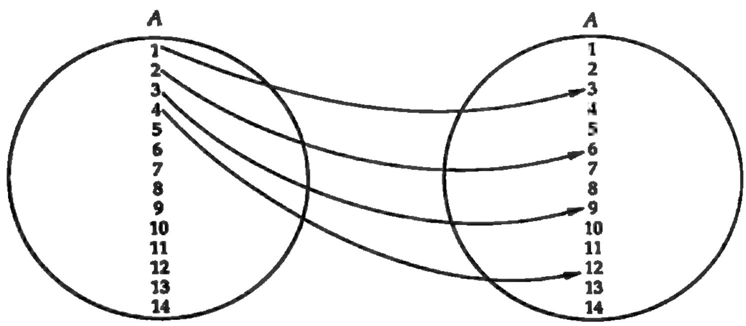

Fig. 2.16

15. (i) $R = \{(1,6), (2,7), (3,8)\}$ (ii) Domain (R) = [1, 2, 3], Range (R) = { 6, 7, 8 }

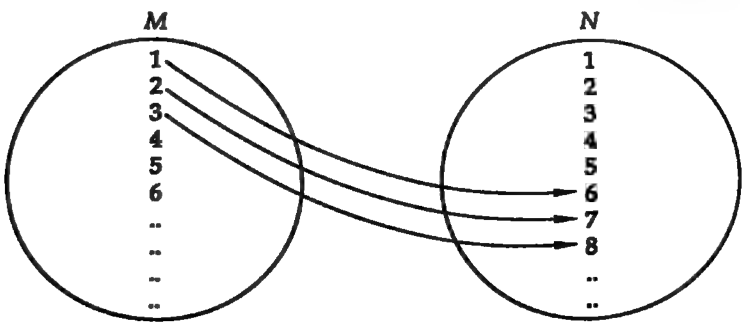

Fig. 2.17

16. $R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

17. $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

18. (i) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5.5), (6, 6)\}$

(ii) Domain $(R) = \{1, 2, 3, 4, 5, 6\}$ (iii) Range$(R) = \{1, 2, 3, 4, 5, 6\}$

19. (i) $R = \{(x, y) : y = x - 2, x \in P, y \in Q\}$ (ii) $R = \{(5, 3), (6, 4), (7, 5)\}$

Domain $(R) = \{5, 6, 7\}$, Range$(R) = 3, 4, 5$

20. (i) Domain$(R) = Z$, Range$(R) = Z$

HINTS TO NCERT & SELECTED PROBLEMS

8. We have, $n(A) = 2, n(B) = 2$

$\therefore \quad n(A \times B) = 2 \times 2 = 4$

So, there are $2^4 = 16$ relations from A to B.

9. (i) We have,

$R = \{(x, x+5) : x \in \{0, 1, 2, 3, 4, 5\}\} = \{(0, 5), (1, 6), (2, 7), (3, 8), (4, 9), (5, 10)\}$

$\therefore$ Domain $(R) = \{0, 1, 2, 3, 4, 5\}$ and, Range$(R) = \{5, 6, 7, 8, 9, 10\}$

(ii) We have,

$R = \{(x, x^3) : x$ is a prime number less than $10\} = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

$\therefore$ Domain $(R) = \{2, 3, 5, 7\}$, and Range$(R) = \{8, 27, 125, 343\}$

10. (i) We have, $R = \{(1, 4), (2, 4), (3, 4), (4, 4)\}$

$\therefore$ Domain $(R) = \{1, 2, 3, 4\}$, Range $(R) = \{4\}$

(ii) We have,

$S = \{(-3, 4), (-2, 3), (-1, 2), (0, 1), (1, 0), (2, 1), (3, 2)\}$

$\therefore$ Domain $(S) = \{-3, -2, -1, 0, 1, 2, 3\}$, and Range $(S) = \{0, 1, 2, 3, 4\}$

12. Here A has 3 elements and B has 2 elements. Therefore, total number of relations from A to B is $2^{3 \times 2} = 64$.

13. (i) No, because $(2, 2) \notin R$.

(ii) No, because $(4, 2) \in R$ but $(2, 4) \notin R$.

(iii) No, because $(16, 4) \in R$ and $(4, 2) \in R$ but $(16, 2) \notin R$.

14. $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$

Domain $(R) = \{1, 2, 3, 4\}$, and Range$(R) = \{3, 6, 9, 12\}$.

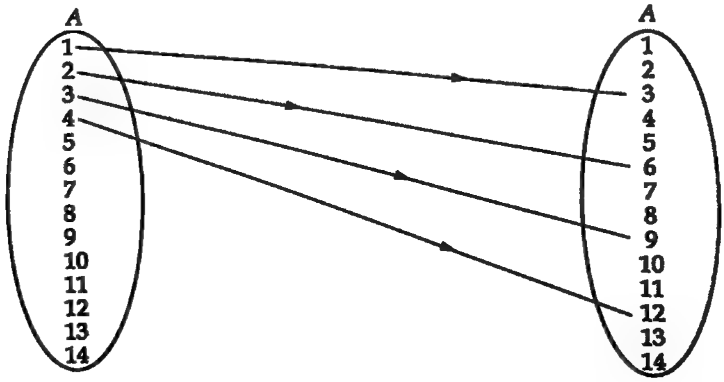

Fig. 2.18

15. (i) We have,

$R = \{(x, y) : y = x + 5, x \text{ is a natural number less than } 4, x, y \in N\} = \{(1, 6), (2, 7), (3, 8)\}$

(ii) Domain $(R) = \{1, 2, 3\}$, and Range $(R) = \{6, 7, 8\}$

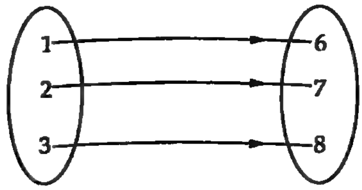

Fig. 2.19

16. We have, $A = \{1, 2, 3, 5\}, B = \{4, 6, 9\}$

$R = \{(x, y) : \text{the difference between } x \text{ and } y \text{ is odd}, x \in A, y \in B\}$

$\therefore \quad R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

17. We have, $R = \{(x, x^3) : x \text{ is a prime number less than } 10\}$

$\therefore \quad R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

18. (i) We have,

$R = \{(a, b) : a, b \in A, b \text{ is exactly divisible by } a\}$, where $A = \{1, 2, 3, 4, 5, 6\}$.

$\therefore R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

(i) Domain $R = \{1, 2, 3, 4, 5, 6\}$ (ii) Range $R = \{1, 2, 3, 4, 5, 6\}$

19. (i) $\{(x, y) : y = x - 2, x \in \{5, 6, 7\}, y \in \{3, 4, 5\}\}$

(ii) $\{(5, 3), (6, 4), (7, 5)\}$

Domain $R = \{5, 6, 7\}$, and Range $R = \{3, 4, 5\}$

20. The relation R on Z is defined by $R = \{(a, b) : a, b \in Z, a - b \text{ is an integer}\}$

Since $a - b$ is an integer for all $a, b \in Z$. So, domain $(R) = Z =$ Range (R).

21. We have,

$$\left(1, -\frac{1}{2}\right) \in R_1 \text{ and } \left(-\frac{1}{2}, -4\right) \in R_1 \text{ as } 1 + \left(\frac{-1}{2}\right) > 0 \text{ and } 1 + \left(\frac{-1}{2}\right)(-4) > 0.$$

But, $1 + 1 \times -4 \not> 0$. So, $(1, -4) \notin R_1$.

22. (i) We know that

$a + b = b + a$ for all $a, b \in N$

$\therefore \quad (a, b)\ R\ (a, b)$ for all $a, b \in N$

(ii) $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \Rightarrow c + b = d + a \Rightarrow (c, d)\ R\ (a, b)$

(iii) $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow a + d = b + c$ and $c + f = d + e$

$\Rightarrow a + d + c + f = b + c + d + e \Rightarrow a + f = b + e \Rightarrow (a, b)\ R\ (e, f)$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $A = \{1, 2, 4\}, B = \{2, 4, 5\}$ and $C = \{2, 5\}$, write $(A - C) \times (B - C)$.
2. If $n(A) = 3, n(B) = 4$, then write $n(A \times A \times B)$.
3. If R is a relation defined on the set Z of integers by the rule $(x, y) \in R \Leftrightarrow x^2 + y^2 = 9$, then write domain of R.

4. If $R = \{(x, y) : x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation defined on the set Z of integers, then write domain of R.
5. If R is a relation from set $A = \{11, 12, 13\}$ to set $B = \{8, 10, 12\}$ defined by $y = x - 3$, then write R^{-1}.
6. Let $A = \{1, 2, 3\}$ and $R = \{(a, b) : |a^2 - b^2| \leq 5, a, b \in A\}$. Then write R as set of ordered pairs.
7. Let $R = \{(x, y) : x, y \in Z, y = 2x - 4\}$. If $(a, -2)$ and $(4, b^2) \in R$, then write the values of a and b.
8. If $R = \{(2, 1), (4, 7), (1, -2), \ldots\}$, then write the linear relation between the components of the ordered pairs of the relation R.
9. If $A = \{1, 3, 5\}$ and $B = \{2, 4\}$, list the elements of R, if $R = \{(x, y) : x, y \in A \times B$ and $x > y\}$.
10. If $R = \{(x, y) : x, y \in W, 2x + y = 8\}$, then write the domain and range of R.
11. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1), (y, 2), (z, 1)$ are in $A \times B$, write A and B.
12. Let $A = \{1, 2, 3, 5\}$, $B = \{4, 6, 9\}$ and R be a relation from A to B defined by $R = \{(x, y) : x - y$ is odd $\}$. Write R in roster form.

ANSWERS

1. $\{(1, 4), (4, 4)\}$ 2. 36 3. Domain $(R) = \{-3, 0, 3\}$
4. Domain $(R) = \{-2, -1, 0, 1, 2\}$ 5. $\{(8, 11), (10, 13)\}$
6. $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$ 7. $a = 1, b = \pm 2$
8. $y = 3x - 5$ 9. $\{(3, 2), (5, 2), (5, 4)\}$
10. Domain $(R) = \{0, 1, 2, 3, 4\}$, Range$(R) = \{0, 2, 4, 6, 8\}$
11. $A = \{x, y, z\}$, $B = \{1, 2\}$ 12. $R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $A = \{1, 2, 4\}$, $B = \{2, 4, 5\}$, $C = \{2, 5\}$, then $(A - B) \times (B - C)$ is
   (a) $\{(1, 2), (1, 5), (2, 5)\}$ (b) $\{(1, 4)\}$
   (c) $(1, 4)$ (d) none of these.
2. If R is a relation on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ given by $x\ R\ y \Leftrightarrow y = 3x$, then $R =$
   (a) $\{(3, 1), (6, 2), (8, 2), (9, 3)\}$ (b) $\{(3, 1), (6, 2), (9, 3)\}$
   (c) $\{(3, 1), (2, 6), (3, 9)\}$ (d) none of these.
3. Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$. If relation R from A to B is given by $R = \{(1, 3), (2, 5), (3, 3)\}$. Then, R^{-1} is
   (a) $\{(3, 3), (3, 1), (5, 2)\}$ (b) $\{(1, 3), (2, 5), (3, 3)\}$
   (c) $\{(1, 3), (5, 2)\}$ (d) none of these.
4. If $A = \{1, 2, 3\}$, $B = \{1, 4, 6, 9\}$ and R is a relation from A to B defined by 'x is greater than y. The range of R is
   (a) $\{1, 4, 6, 9\}$ (b) $\{4, 6, 9\}$ (c) $\{1\}$ (d) none of these.
5. If $R = \{(x, y) : x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation on Z, then domain of R is
   (a) $\{0, 1, 2\}$ (b) $\{0, -1, -2\}$ (c) $\{-2, -1, 0, 1, 2\}$ (d) none of these.
6. A relation R is defined from $\{2, 3, 4, 5\}$ to $\{3, 6, 7, 10\}$ by : $x\ R\ y \Leftrightarrow x$ is relatively prime to y. Then, domain of R is
   (a) $\{2, 3, 5\}$ (b) $\{3, 5\}$ (c) $\{2, 3, 4\}$ (d) $\{2, 3, 4, 5\}$.

7. A relation ϕ from C to R is defined by $x \phi y \Leftrightarrow |x| = y$. Which one is correct?
   (a) $(2+3i)\phi 13$ (b) $3\phi(-3)$ (c) $(1+i)\phi 2$ (d) $i\phi 1$.
8. Let R be a relation on N defined by $x + 2y = 8$. The domain of R is
   (a) $\{2, 4, 8\}$ (b) $\{2, 4, 6, 8\}$ (c) $\{2, 4, 6\}$ (d) $\{1, 2, 3, 4\}$.
9. R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$. Then, R^{-1} is
   (a) $\{(8, 11), (10, 13)\}$ (b) $\{(11, 8), (13, 10)\}$
   (c) $\{(10, 13), (8, 11), (12, 10)\}$ (d) none of these.
10. If the set A has p elements, B has q elements, then the number of elements in $A \times B$ is
   (a) $p+q$ (b) $p+q+1$ (c) pq (d) p^2
11. Let R be a relation from a set A to a set B, then
   (a) $R = A \cup B$ (b) $R = A \cap B$ (c) $R \subseteq A \times B$ (d) $R \subseteq B \times A$.
12. If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is
   (a) 2^{mn} (b) $2^{mn} - 1$ (c) $2mn$ (d) m^n
13. If R is a relation on a finite set having n elements, then the number of relations on A is
   (a) 2^n (b) 2^{n^2} (c) n^2 (d) n^n.

ANSWERS

1. (b) 2. (d) 3. (a) 4. (c) 5. (c) 6. (d) 7. (d) 8. (c) 9. (a)
10. (c) 11. (c) 12. (a) 13. (b)

SUMMARY

1. An ordered pair consists of two objects or elements in a given fixed order.
2. $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$
3. If A and B are two non-empty sets, then $A \times B = \{(a, b) : a \in A, b \in B\}$ is called the cartesian product of A and B. If A and B are finite sets having m and n elements respectively, then $A \times B$ has mn elements.
4. $R \times R = \{(x, y) : x, y \in R\}$ is the set of all points in xy-plane.
5. $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$ set of all points in three dimensional space.
6. For any three sets A, B, C, we have
   (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$. (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   (iii) $A \times (B - C) = A \times B - A \times C$ (iv) $A \times B = B \times A \Leftrightarrow A = B$
   (v) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$ (vi) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
   (vii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$ (viii) $A \times B = A \times C \Rightarrow B = C$
7. Let A and B be two sets. A relation from A to B is a subset of $A \times B$.
8. If A and B are finite sets having m and n elements respectively. Then, 2^{mn} relations can be defined from A to B.
9. If R is a relation from set A to set B, then
   Domain $(R) = \{x : (x, y) \in R\}$, Range $(R) = \{y : (x, y) \in R\}$
10. A relation from a set A to itself is called a relation on A.
11. Let A, B be two sets and let R be a relation from set A to set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.
   Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$
   Domain (R) = Range (R^{-1}), and Range (R) = Domain (R^{-1}).

CHAPTER 3

FUNCTIONS

3.1 INTRODUCTION

In this chapter, we shall study about one of the most important concepts in mathematics known as a function. Functions form one of the most important building blocks of Mathematics. The word "Function" is derived from a Latin word meaning operation and the words mapping and map are synonimus to it. Functions play a very important role in differential and integral calculus which will be studied in XII class. In this chapter, we shall introduce the concept of a function as a correspondence between two sets. We shall also study function as a relation from one set to the other set.

3.2 FUNCTION AS A SPECIAL KIND OF RELATION

DEFINITION *Let A and B be two non-empty sets. A relation f from A to B, i.e., a sub-set of $A \times B$, is called a function (or a mapping or a map) from A to B, if*

(i) *for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$*

(ii) *$(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$.*

Thus, a non-void subset f of $A \times B$ is a function from A to B if each element of A appears in some ordered pair in f and no two ordered pairs in f have the same first element.

If $(a, b) \in f$, then b is called the image of a under f.

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and f_1, f_2 and f_3 be three subsets of $A \times B$ as given below:*

$$f_1 = \{(1, 2), (2, 3), (3, 4)\},\ f_2 = \{(1, 2), (1, 3), (2, 3), (3, 4)\},\ f_3 = \{(1, 3), (2, 4)\}.$$

Then, f_1 is a function from A to B but f_2 and f_3 are not functions from A to B. f_2 is not a function from A to B, because $1 \in A$ has two images 2 and 3 in B and f_3 is not a function from A to B because $3 \in A$ has no image in B.

If a function f is expressed as the set of ordered pairs, the domain of f is the set of all first components of members of f and the range of f is the set of second components of members of f i.e. Domain of $f = \{a : (a, b) \in f\}$, and Range of $f = \{b : (a, b) \in f\}$

ILLUSTRATION 2 If $x, y \in \{1, 2, 3, 4\}$, then which of the following are functions in the given set?

(a) $f_1 = \{(x, y) : y = x + 1\}$ (b) $f_2 = \{(x, y) : x + y > 4\}$

(c) $f_3 = \{(x, y) : y < x\}$ (d) $f_4 = \{(x, y) : x + y = 5\}$

Also, in case of a function give its range.

SOLUTION If we express f_1, f_2, f_3 and f_4 as sets of ordered pairs, then we have

$$f_1 = \{(1, 2), (2, 3), (3, 4)\},$$
$$f_2 = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\},$$
$$f_3 = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\} \text{ and } f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}.$$

(a) We have, $f_1 = \{(1, 2), (2, 3), (3, 4)\}$.

We observe that an element 4 of the given set has not appeared in first place of any ordered pair of f_1. So, f_1 is not a function from the given set to itself.

(b) We have, $f_2 = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\}$.

We observe that 2, 3, 4 have appeared more than once as first components of the ordered pairs in f_2. So, f_2 is not a function.

(c) We have, $f_3 = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$.

We observe that 3 and 4 have appeared more than once as first components of the ordered pairs in f_3. So, f_3 is not a function.

(d) We have, $f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$.

We observe that each element of the given set has appeared as first components in one and only one ordered pair of f_4. So, f_4 is a function in the given set. In this case, Range of $f = \{1, 2, 3, 4\}$.

ILLUSTRATION 3 *Let f be a relation on the set N of natural numbers defined by $f = \{(n, 3n) : n \in N\}$. Is f a function from N to N. If so, find the range of f.*

SOLUTION Since for each $n \in N$, there exists a unique $3n \in N$ such that $(n, 3n) \in f$. Therefore, f is a function from N to N.

Clearly, Range of $f = \{f(n) : n \in N\} = \{3n : n \in N\}$.

ILLUSTRATION 4 *Let f be a subset of $Z \times Z$ defined by $f = \{(ab, (a + b)) : a, b \in Z\}$. Is f a function from Z into Z. Justify your answers.* [NCERT]

SOLUTION We observe that:

$$1 \times 6 = 6 \text{ and } 2 \times 3 = 6$$

$$\Rightarrow \quad (1 \times 6, 1 + 6) \in f \text{ and } (2 \times 3, 2 + 3) \in f$$

$$\Rightarrow \quad (6, 7) \in f \text{ and } (6, 5) \in f$$

So, f is not a function from Z to Z.

3.3 FUNCTION AS A CORRESPONDENCE

DEFINITION *Let A and B be two non-empty sets. Then a function 'f' from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that:*

(i) *all elements of set A are associated to elements in set B.*

(ii) *an element of set A is associated to a unique element in set B.*

In other words, a function 'f' from a set A to a set B associates each element of set A to a unique element of set B.

Terms such as "map" (or "mapping"), "correspondence" are used as synonyms for "function". If f is a function from a set A to a set B, then we write $f : A \to B$ or $A \to B$, which is read as *f is a function from A to B* or *f maps A to B.*

If an element $a \in A$ is associated to an element $b \in B$, then b is called *'the f-image of a'* or *'image of a under f'* or *'the value of the function f at a'*. Also, a is called the pre-image of b under the function f. We write it as: $b = f(a)$

ILLUSTRATION *Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$ be two sets and let f_1, f_2, f_3 and f_4 be rules associating elements (A to elements of) B as shown in the following figures:*

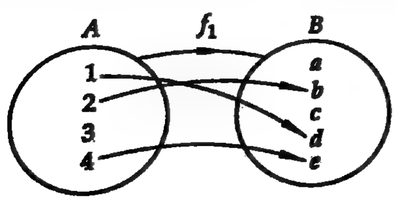

Fig. 3.1

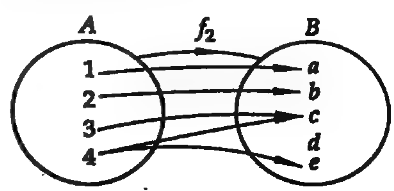

Fig. 3.2

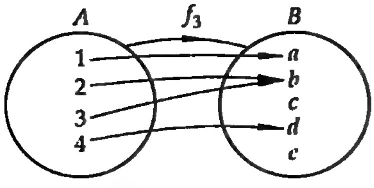

Fig. 3.3

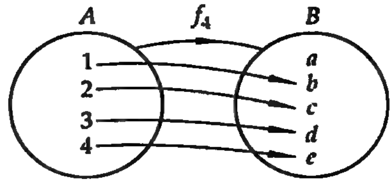

Fig. 3.4

We observe that f_1 is not a function from set A to set B, because there is an element $3 \in A$ which is not associated to any element of B.

Also, f_2 is not a function from A to B because an element $4 \in A$ is associated to two elements c and e in B. But, f_3 and f_4 are functions from A to B, because under f_3 and f_4 each element in A is associated to a unique element in B.

3.3.1 DESCRIPTION OF A FUNCTION

Let $f : A \to B$ be a function such that the set A consists of a finite number of elements. Then, $f(x)$ be described by listing the values which it attains at different points of its domain. For example, if $A = \{-1, 1, 2, 3\}$ and B is the set of real numbers, then a function $f : A \to B$ can be described as $f(-1) = 3, f(1) = 0, f(2) = 3/2$ and $f(3) = 0$. In case, A is an infinite set, then f cannot be described by listing the images at points in its domain. In such cases functions are generally described by a formula. For example, $f : Z \to Z$ given by $f(x) = x^2 + 1$ or $f : R \to R$ given by $f(x) = e^x$ etc.

3.3.2 DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

Let $f : A \to B$. Then, the set A is known as the domain of f and the set B is known as the co-domain of f. The set of all f-images of elements of A is known as the range of f or image set of A under f and is denoted by $f(A)$.

Thus, $f(A) = \{f(x) : x \in A\} =$ Range of f

Clearly, $f(A) \subseteq B$.

ILLUSTRATION 1 *Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4, 5, 6\}$. Consider a rule $f(x) = x^2$. Under this rule, we obtain $f(-2) = (-2)^2 = 4$, $f(-1) = (-1)^2 = 1$, $f(0) = 0^2 = 0$, $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$. We observe that each element of A is associated to a unique element of B. So, $f : A \to B$ given by $f(x) = x^2$ is a function. Clearly, domain $(f) = A = \{-2, -1, 0, 1, 2\}$ and range $(f) = \{0, 1, 4\}$.*

ILLUSTRATION 2 *Consider a rule $f(x) = 2x - 3$ associating elements of N (set of natural numbers) to elements of N. This rule does not define a function from N to itself, because $f(1) = 2 \times 1 - 3 = -1 \notin N$ i.e. $1 \in N$ (domain) is not associated to any element of N (co-domain).*

ILLUSTRATION 3 *Let $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to Z$ be given by $f(x) = x^2 - 2x - 3$. Find:*
(i) the range of f (ii) pre-images of 6, −3 and 5.

SOLUTION (i) We have, $f(x) = x^2 - 2x - 3$.

$\therefore$ $f(-2) = (-2)^2 - 2(-2) - 3 = 5$, $f(-1) = (-1)^2 - 2(-1) - 3 = 0$, $f(0) = -3$,

$f(1) = 1^2 - 2 \times 1 - 3 = -4$ and $f(2) = 2^2 - 2 \times 2 - 3 = -3$.

So, range $(f) = \{f(-2), f(-1), f(0), f(1), f(2)\} = \{0, 5, -3, -4\}$

(ii) Let x be a pre-image of 6. Then,

$$f(x) = 6 \Rightarrow x^2 - 2x - 3 = 6 \Rightarrow x^2 - 2x - 9 = 0 \Rightarrow x = 1 \pm \sqrt{10}$$

Since $x = 1 \pm \sqrt{10} \notin A$. So, there is no pre-image of 6.

Let x be a pre-image of -3. Then,

$$f(x) = -3 \Rightarrow x^2 - 2x - 3 = -3 \Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, 2.$$

Clearly, $0, 2 \in A$. So, 0 and 2 are pre-images of -3.

Let x be a pre-image of 5. Then,

$$f(x)=5 \Rightarrow x^2-2x-3=5 \Rightarrow x^2-2x-8=0 \Rightarrow (x-4)(x+2)=0 \Rightarrow x=4,-2.$$

Since, $-2 \in A$ bet $4 \notin A$. So, -2 is a pre-image of 5.

3.4 EQUAL FUNCTIONS

DEFINITION *Two functions f and g are said to be equal iff*

(i) *domain of f = domain of g* (ii) *co-domain of f = co-domain of g,*

and (iii) *$f(x) = g(x)$ for every x belonging to their common domain.*

If two functions f and g are equal, then we write $f=g$.

ILLUSTRATION 1 *Let $A=\{1, 2\}$, $B=\{3, 6\}$ and $f: A \to B$ given by $f(x)=x^2+2$ and $g: A \to B$ given by $g(x)=3x$. Then, we observe that f and g have the same domain and co-domain. Also we have,* $f(1)=3=g(1)$ *and* $f(2)=6=g(2)$. Hence, $f=g$.

ILLUSTRATION 2 *Let $f: R-\{2\} \to R$ be defined by $f(x)=\dfrac{x^2-4}{x-2}$ and $g: R \to R$ be defined by $g(x)=x+2$. Find whether $f=g$ or not.*

SOLUTION We have, $f(x)=\dfrac{x^2-4}{x-2}$, $x \neq 2$.

$\Rightarrow \quad f(x)=\dfrac{(x-2)(x+2)}{x-2}=x+2$ for all $x \neq 2$.

$\Rightarrow \quad f(x)=g(x)$ for all $x \neq 2$.

Thus, $f(x)=g(x)$ for all $x \in R-\{2\}$. But, $f(x)$ and $g(x)$ have different domains.

Infact, domain of $f=R-\{2\}$ and domain of $g=R$. Therefore, $f \neq g$.

ILLUSTRATION 3 *Let $f: Z \to Z$ and $g: Z \to Z$ be functions defined by $f=\{(n, n^2): n \in Z\}$ and, $g=\{(n, |n|^2): n \in Z\}$. Show that: $f=g$.*

SOLUTION Clearly,

Domain of f = Domain of $g=Z$ and, Co-domain of f = Co-domain of $g=Z$.

We have, $f(n)=n^2$ and $g(n)=|n|^2=n^2$ $\quad [\because |n|^2=n^2]$

$\therefore \quad f(n)=g(n)$ for all $n \in Z$.

Hence, $f=g$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Express the following functions as sets of ordered pairs and determine their ranges*

(i) $f: A \to R, f(x)=x^2+1$, *where* $A=\{-1, 0, 2, 4\}$.

(ii) $g: A \to N, g(x)=2x$, *where* $A=\{x: x \in N, x \leq 10\}$.

SOLUTION (i) We have, $f(x)=x^2+1$.

$\therefore \quad f(-1)=(-1)^2+1=2, f(0)=0^2+1=1, f(2)=2^2+1=5$ and $f(4)=4^2+1=17$

So, $\quad f=\{(x, f(x)): x \in A\}=\{(-1, 2), (0, 1), (2, 5), (4, 17)\}$.

Hence, Range of $(f)=\{2, 1, 5, 17\}$

(ii) We have, $g(x)=2x$ and $A=\{1, 2, 3, \ldots\ldots 10\}$. Therefore,

$g(1)=2\times1=2, \quad g(2)=2\times2=4, \quad g(3)=2\times3=6, \quad g(4)=2\times4=8, g(5)=2\times5=10,$

$g(6)=2\times 6=12,\quad g(7)=2\times 7=14,\quad g(8)=2\times 8=16,\ g(9)=2\times 9=18$
and, $g(10)=2\times 10=20.$

$\therefore\quad g=\{(x, g(x)) : x\in A\} = \{(1, 2), (2, 4), (3, 6), \dots, (10, 20)\}.$

Hence, Range of $g=g(A)=\{g(x) : x\in A\}=\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}.$

EXAMPLE 2 *Find the domain for which the functions $f(x)=2x^2-1$ and $g(x)=1-3x$ are equal.*
[NCERT EXEMPLAR]

SOLUTION The values of x for which $f(x)$ and $g(x)$ are equal are given by

$f(x)=g(x)$

$\Rightarrow\quad 2x^2-1=1-3x \Rightarrow 2x^2+3x-2=0 \Rightarrow (x+2)(2x-1)=0 \Rightarrow x=-2, 1/2.$

Thus, $f(x)$ and $g(x)$ are equal on the set $\{-2, 1/2\}$.

EXAMPLE 3 *Is $g=\{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If this is described by the formula, $g(x)=\alpha x+\beta$, then what values should be assigned to α and β?* **[NCERT EXEMPLAR]**

SOLUTION Since no two ordered pairs in g have the same first component. So, g is a function such that $g(1)=1, g(2)=3, g(3)=5$ and $g(4)=7$.

It is given that $g(x)=\alpha x+\beta$.

$\therefore\quad g(1)=1$ and $g(2)=3 \Rightarrow \alpha+\beta=1$ and $2\alpha+\beta=3 \Rightarrow \alpha=2, \beta=-1.$

EXAMPLE 4 *Given $A=\{-1, 0, 2, 5, 6, 11\}$, $B=\{-2, -1, 0, 18, 28, 108\}$ and $f(x)=x^2-x-2$. Is $f(A)=B$? Find $f(A)$.*

SOLUTION We have, $f(x)=x^2=x-2.$

$\therefore\quad f(-1)=(-1)^2-(-1)-2=0,\quad f(0)=0^2-0-2=-2,\quad f(2)=2^2-2-2=0,$

$f(5)=5^2-5-2=18,\quad f(6)=6^2-6-2=28\quad$ and $\quad f(11)=11^2-11-2=108.$

Hence, $f(A)=\{f(x) : x\in A\} = \{f(-1), f(0), f(2), f(5), f(6), f(11)\}=\{0, -2, 18, 28, 108\}$

We observe that $-1\in B$, but $-1\notin f(A)$. So, $f(A)\neq B$.

EXAMPLE 5 *Let $f: R\to R$ be given by $f(x)=x^2+3$. Find (i) $\{x : f(x)=28\}$ (ii) the pre-images of 39 and 2 under f.*

SOLUTION (i) We have, $f(x)=x^2+3$

$\therefore\quad f(x)=28 \Rightarrow x^2+3=28 \Rightarrow x^2=25 \Rightarrow x=\pm 5$

Hence, $\{x : f(x)=28\}=\{-5, 5\}$.

(ii) Let x be a pre-image of 39. Then,

$f(x)=39 \Rightarrow x^2+3=39 \Rightarrow x^2=36 \Rightarrow x=\pm 6$

So, pre-images of 39 are -6 and 6.

Let x be a pre-image of 2. Then,

$f(x)=2 \Rightarrow x^2+3=2 \Rightarrow x^2=-1$

We find that no real value of x satisfies the equation $x^2=-1$. Therefore, 2 does not have any pre-image under f.

EXAMPLE 6 *Let $f: R\to R$ be a function given by $f(x)=x^2+1$. Find:*

(i) $f^{-1}\{-5\}$ (ii) $f^{-1}\{26\}$ (iii) $f^{-1}\{10, 37\}$

SOLUTION Recall that if $f: A\to B$ is a function and $y\in B$. Then, $f^{-1}\{y\}=\{x\in A : f(x)=y\}$. In other words, $f^{-1}\{y\}$ is the set of pre-images of y.

(i) Let $f^{-1}(-5)=x$. Then,

$$f(x) = -5 \Rightarrow x^2 + 1 = -5 \Rightarrow x^2 = -6$$

Clearly, this equation is not solvable in R. Therefore, there is no pre-image of –5. So, $f^{-1}\{-5\} = \phi$.

(ii) Let $f^{-1}(26) = x$. Then,

$$f(x) = 26 \Rightarrow x^2 + 1 = 26 \Rightarrow x = \pm 5$$

So, pre-image of 26 are –5 and 5.

$\therefore \quad f^{-1}\{26\} = \{-5, 5\}$.

(iii) Let $f^{-1}(10) = x$. Then,

$$f(x) = 10 \Rightarrow x^2 + 1 = 10 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

So, pre-image of 10 are -3 and 3.

Let $f^{-1}(37) = x$. Then,

$$f(x) = 37 \Rightarrow x^2 + 1 = 37 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$$

So, pre-images of 37 are -6 and 6.

Hence, $f^{-1}\{10, 37\} = \{3, -3, 6, -6\}$.

EXAMPLE 7 *Let* $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ *be a function described by the formula* $f(x) = ax + b$ *for some integers* a, b. *Determine* a, b. [NCERT]

SOLUTION Clearly, $f(1) = 1, f(2) = 3, f(0) = -1$ and $f(-1) = -3$.

It is given that $f(x) = ax + b$. Therefore,

$f(1) = 1$ and $f(2) = 3 \Rightarrow a + b = 1$ and $2a + b = 3 \Rightarrow a = 2, b = -1$.

Substituting the values of a and b in $f(x) = ax + b$, we get $f(x) = 2x - 1$.

Clearly, $f(0) = -1$ and $f(-1) = -3$ are true.

Hence, $a = 2$ and $b = -1$.

EXAMPLE 8 *If* $f : R \to R$ *be defined as follows:*

$$f(x) = \begin{cases} 1, & \text{if } x \in Q \\ -1, & \text{if } x \notin Q. \end{cases}$$

Find (i) $f(1/2), f(\pi), f(\sqrt{2})$ (ii) *Range of* f (iii) *pre-images of* 1 *and* -1.

SOLUTION (i) It is evident from the definition of f that at every rational point the function attains value 1 and at every irrational point attains value –1.

$$\therefore \quad \frac{1}{2} \in Q \Rightarrow f\left(\frac{1}{2}\right) = 1, \pi \notin Q \Rightarrow f(\pi) = -1 \text{ and } \sqrt{2} \notin Q \Rightarrow f(\sqrt{2}) = -1.$$

(ii) Range of $f = \{f(x) : x \in R\}$.

Also, by definition $f(x)$ attains values 1 or –1 according as x is rational or irrational and a real number is either rational or irrational. Thus, all rational numbers have image 1 and all irrational numbers have image -1. Hence, Range of $f = \{1, -1\}$.

(iii) Since $f(x) = 1$ for all $x \in Q$. Therefore, pre-images of 1 are rational numbers i.e $f^{-1}(1) = Q$.

Also, –1 is the image of every real number which is not rational.

$\therefore \quad f^{-1}(-1) = R - Q =$ Set of irrational numbers.

EXAMPLE 9 *Let* $f : R \to R$ *be such that* $f(x) = 2^x$. *Determine:*

(i) *Range of* f (ii) $\{x : f(x) = 1\}$ (iii) *whether* $f(x + y) = f(x)\, f(y)$ *holds.*

SOLUTION (i) Since 2^x is positive for every $x \in R$. So, $f(x) = 2^x$ is a positive real number for every $x \in R$. Moreover, for every positive real number x, there exist $\log_2 x \in R$ such that

$$f(\log_2 x) = 2^{\log_2 x} = x \qquad [\because a^{\log_a x} = x]$$

Hence, we conclude that the range of f is the set of all positive real numbers.

(ii) $f(x) = 1 \Rightarrow 2^x = 1 \Rightarrow 2^x = 2^0 \Rightarrow x = 0.$

$\therefore$ $\{x : f(x) = 1\} = \{0\}.$

(iii) We have, $f(x) = 2^x$.

$\therefore$ $f(x + y) = 2^{x+y} = 2^x \times 2^y = f(x) f(y)$

Hence, $f(x + y) = f(x) f(y)$ holds for all $x, y \in R$.

LEVEL-2

EXAMPLE 10 *Let A be the set of two positive integers. Let $f : A \to Z^+$ (set of positive integers) be defined by*

$f(n) = p$, where p is the highest prime factor of n.

If range of $f = \{3\}$. Find set A. Is A uniquely determined?

SOLUTION It is given that the set A consists of two positive integers. So, let $A = \{n, m\}$. Since range of $f = \{3\}$.

$\therefore$ $f(n) = 3$ and $f(m) = 3$

$\Rightarrow$ Highest prime factors of n and m both are equal to 3.

$\Rightarrow$ $(n = 3$ and $m = 6)$ or $(n = 3$ and $m = 9)$ or $(n = 3$ and $m = 12)$ or $(n = 6$ and $m = 12)$ etc.

$\Rightarrow$ $A = \{3, 6\}$, or $A = \{3, 9\}$, or $A = \{3, 12\}$, or $A = \{6, 12\}$ etc.

Clearly, A is not uniquely determined.

EXAMPLE 11 *Let $A \subseteq N$ and $f : A \to A$ be defined by : $f(n) =$ the highest prime factor of n. If range of f is A. Determine A. Is A uniquely determined?*

SOLUTION For any $n \in A$, we have

$f(n) =$ Highest prime factor of n

$\Rightarrow$ $f(n)$ takes prime values only

$\Rightarrow$ Range of f consists of prime numbers only

But, it is given that range of f is A. Therefore, set A consists of prime numbers only.

Hence, A = Set of some prime numbers. Clearly, A is not uniquely determined.

EXERCISE 3.1

LEVEL-1

1. Define a function as a set of ordered pairs.
2. Define a function as a correspondence between two sets.
3. What is the fundamental difference between a relation and a function ? Is every relation a function ?
4. Let $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to Z$ be a function defined by $f(x) = x^2 - 2x - 3$. Find:
   (i) range of f i.e. $f(A)$ (ii) pre-images of 6, –3 and 5.
5. If a function $f : R \to R$ be defined by
$$f(x) = \begin{cases} 3x - 2, & x < 0 \\ 1, & x = 0 \\ 4x + 1, & x > 0 \end{cases}$$
   Find: $f(1), f(-1), f(0), f(2)$.
6. A function $f : R \to R$ is defined by $f(x) = x^2$. Determine
   (i) range of f (ii) $\{x : f(x) = 4\}$ (iii) $\{y : f(y) = -1\}$.
7. Let $f : R^+ \to R$, where R^+ is the set of all positive real numbers, be such that $f(x) = \log_e x$. Determine
   (i) the image set of the domain of f (ii) $\{x : f(x) = -2\}$
   (iii) whether $f(xy) = f(x) + f(y)$ holds.

8. Write the following relations as sets of ordered pairs and find which of them are functions:
   (i) $\{(x, y) : y = 3x, x \in \{1, 2, 3\}, y \in \{3, 6, 9, 12\}\}$.
   (ii) $\{(x, y) : y > x + 1, x = 1, 2$ and $y = 2, 4, 6\}$ (iii) $\{(x, y) : x + y = 3, x, y \in \{0, 1, 2, 3\}\}$
9. Let $f : R \to R$ and $g : C \to C$ be two functions defined as $f(x) = x^2$ and $g(x) = x^2$. Are they equal functions?
10. If f, g, h are three functions defined from R to R as follows:
   (i) $f(x) = x^2$ (ii) $g(x) = \sin x$ (iii) $h(x) = x^2 + 1$
   Find the range of each function.
11. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 5, 9, 11, 15, 16\}$
   Determine which of the following sets are functions from X to Y
   (i) $f_1 = \{(1, 1), (2, 11), (3, 1), (4, 15)\}$ (ii) $f_2 = \{(1, 1), (2, 7), (3, 5)\}$
   (iii) $f_3 = \{(1, 5), (2, 9) (3, 1), (4, 5), (2, 11)\}$. [NCERT]
12. Let $A = \{12, 13, 14, 15, 16, 17\}$ and $f : A \to Z$ be a function given by
   $$f(x) = \text{highest prime factor of } x.$$
   Find range of f.
13. If $f : R \to R$ be defined by $f(x) = x^2 + 1$, then find $f^{-1}\{17\}$ and $f^{-1}\{-3\}$.
14. Let $A = \{p, q, r, s\}$ and $B = \{1, 2, 3\}$. Which of the following relations from A to B is not a function?
   (i) $R_1 = \{(p, 1), (q, 2), (r, 1), (s, 2)\}$ (ii) $R_2 = \{(p, 1), (q, 1), (r, 1), (s, 1)\}$
   (iii) $R_3 = \{(p, 1), (q, 2), (p, 2), (s, 3)\}$ (iv) $R_4 = \{(p, 2), (q, 3), (r, 2), (s, 2)\}$.
15. Let $A = \{9, 10, 11, 12, 13\}$ and let $f : A \to N$ be defined by $f(n)$ = the highest prime factor of n. Find the range of f. [NCERT]
16. The function f is defined by $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$

   The relation g is defined by $g(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 3x, & 2 \le x \le 10 \end{cases}$

   Show that f is a function and g is not a function. [NCERT]
17. If $f(x) = x^2$, find $\dfrac{f(1.1) - f(1)}{(1.1) - 1}$. [NCERT]
18. Express the function $f : X \to R$ given by $f(x) = x^3 + 1$ as set of ordered pairs, where $X = \{-1, 0, 3, 9, 7\}$. [NCERT EXEMPLAR]

ANSWERS

4. (i) $f(A) = \{-4, -3, 0, 5\}$ (ii) ϕ, $\{0, 2\}$, -2. 5. $f(1) = 5$, $f(-1) = -5$, $f(0) = 1$, $f(2) = 9$
6. (i) R^+ (set of all real numbers greater than or equal to zero) (ii) $\{-2, 2\}$ (iii) ϕ
7. (i) R (ii) $\{e^{-2}\}$ (iii) Yes
8. (i) $\{(1, 3), (2, 6), (3, 9)\}$; Function (ii) $\{(1, 4), (1, 6), (2, 4), (2, 6)\}$; Not a function
   (iii) $\{(0, 3), (1, 2), (2, 1), (3, 0)\}$; Function.
9. No, Since domain of $f \neq$ domain of g.
10. (i) $R^+ = \{x \in R \mid x > 0\}$ (ii) $\{x \in R : -1 \le x \le 1\}$ (iii) $\{x \in R : x > 1\}$.
11. (i) 12. $\{3, 13, 7, 5, 2, 17\}$ 13. $f^{-1}(17) = \{-4, 4\}$, $f^{-1}(-3) = \phi$
14. (iii) 15. Range $(f) = \{3, 5, 11, 13\}$. 17. 2.1
18. $f = \{(-1, 0), (0, 1), (3, 28), (9, 730), (7, 344)\}$

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11. (iii) $f_3 = \{(1,5), (2,9), (3,1), (4,5), (2,11)\}$ is not a function because (2, 9) and $(2,11) \in f_3$, which means that 2 is related to two elements in y.
12. Clearly, $f(12)$ = highest prime factor of 12 = 3. Similarly, $f(13)=13$, $f(14)=7$, $f(15)=5$, $f(16)=2$ and $f(17)=17$. Hence, range $(f)=\{3, 13, 7, 5, 2, 17\}$.
15. We have,
$A=\{9, 10, 11, 12, 13\}$ and $f: A \to N$ is defined by $f(n)$ = the highest prime factor of n.
$\therefore \quad f(9)=3, f(10)=5, f(11)=11, f(12)=3$ and $f(13)=13$
Hence, range $(f)=\{f(9), f(10), f(11), f(12), f(13)\}=\{3, 5, 11, 13\}$
16. We observe that $f(x)=\begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$ associates all numbers in [0, 10] to numbers in R and no number in [0, 10] is associated to two or more numbers. Hence, f is a function. But, g is not a function because 2 is associated to two distinct elements viz. 4 and 6.
17. We have, $f(x) = x^2$

$$\therefore \quad \frac{f(1.1)-f(1)}{(1.1)-1} = \frac{(1.1)^2-1^2}{(1.1)-1} = \frac{(1.1+1)(1.1-1)}{(1.1-1)} = 2.1$$

3.5 REAL FUNCTIONS

In this section, we will discuss functions having domain and co-domain both as subsets of the set R of all real numbers. Such functions are called real functions or real valued functions of the real variable as defined below.

REAL VALUED FUNCTION *A function $f: A \to B$ is called a real valued function, if B is a subset of R (set of all real numbers).*

If A and B both are subsets of R, then f is called a real function.

In section 3.3.1, we have discussed the description of a function. Generally, domain and co-domain both are infinite subsets of R in case of real functions of real variable. Therefore, a real function is generally described by some general formula. In other words, images of various elements in the domain of a real function are provided by some general formula. For example, $f: R \to R$ given by $f(x)=x^2+x+1$ or, $f: A \to B$ given by $f(x)=\dfrac{x-1}{x^2-4}$ etc. In practice, real functions are described by giving the general expressions or formulae describing them without mentioning their domains and co-domains. Following are some examples of real functions.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If $f(x)=3x^4-5x^2+9$, find $f(x-1)$.*

SOLUTION We have, $f(x) = 3x^4-5x^2+9$. Replacing x by $(x-1)$, we obtain

$$f(x-1) = 3(x-1)^4-5(x-1)^2+9 = 3x^4-12x^3+13x^2-2x+7$$

EXAMPLE 2 *If $f(x)=x+\dfrac{1}{x}$, prove that $[f(x)]^3=f(x^3)+3f\left(\dfrac{1}{x}\right)$*

SOLUTION We have,

$$f(x)=x+\frac{1}{x}$$

$$\therefore \quad f(x^3)=x^3+\frac{1}{x^3} \text{ and } [f(x)]^3 = \left(x+\frac{1}{x}\right)^3$$

Now, $[f(x)]^3=\left(x+\dfrac{1}{x}\right)^3$

$$\Rightarrow \quad [f(x)]^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$$

$$\Rightarrow \quad [f(x)]^3 = f(x^3) + 3\,f(x) \qquad \left[\because f(x^3) = x^3 + \frac{1}{x^3} \text{ and } f(x) = x + \frac{1}{x}\right]$$

$$\Rightarrow \quad [f(x)]^3 = f(x^3) + 3\,f\left(\frac{1}{x}\right) \qquad \left[\because f(x) = x + \frac{1}{x} \Rightarrow f(x) = f\left(\frac{1}{x}\right)\right]$$

EXAMPLE 3 *If* $f(x) = \dfrac{1}{2x+1}$, $x \neq -\dfrac{1}{2}$, *then show that* $f\big(f(x)\big) = \dfrac{2x+1}{2x+3}$, *provided that* $x \neq -\dfrac{3}{2}$.

SOLUTION We have, $f(x) = \dfrac{1}{2x+1}$

$$\Rightarrow \quad f\big(f(x)\big) = f\left(\frac{1}{2x+1}\right) = \frac{1}{2\left(\frac{1}{2x+1}\right)+1} = \frac{1}{\frac{2}{2x+1}+1} = \frac{2x+1}{2+2x+1} = \frac{2x+1}{2x+3}$$

Clearly, $f\big(f(x)\big) = \dfrac{2x+1}{2x+3}$ is real for $2x+3 \neq 0$ i.e. $f\big(f(x)\big)$ is defined for $2x+3 \neq 0$ i.e. $x \neq -\dfrac{3}{2}$.

Hence, $f\big(f(x)\big) = \dfrac{2x+1}{2x+3}$, provided that $x \neq -\dfrac{3}{2}$.

EXAMPLE 4 *If* $f(x) = \dfrac{x-1}{x+1}$, $x \neq -1$, *then show that* $f\big(f(x)\big) = -\dfrac{1}{x}$, *provided that* $x \neq 0$.

SOLUTION We have,

$$f(x) = \frac{x-1}{x+1}, x \neq -1$$

$$\Rightarrow \quad f\big(f(x)\big) = f\left(\frac{x-1}{x+1}\right)$$

$$\Rightarrow \quad f\big(f(x)\big) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} \qquad \left[\text{Replacing } x \text{ by } \frac{x-1}{x+1} \text{ in the formula for } f(x)\right]$$

$$\Rightarrow \quad f\big(f(x)\big) = \frac{x-1-x-1}{x-1+x+1} = \frac{-2}{2x} = -\frac{1}{x}.$$

Since $-\dfrac{1}{x}$ is meaningful for $x \neq 0$. Hence, $f\big(f(x)\big) = -\dfrac{1}{x}$, provided that $x \neq 0$.

EXAMPLE 5 *Let f be defined by* $f(x) = x - 4$ *and g be defined by* $g(x) = \begin{cases} \dfrac{x^2-16}{x+4}, & x \neq -4 \\ \lambda, & x = -4 \end{cases}$

Find λ *such that* $f(x) = g(x)$ *for all x.*

SOLUTION We have,

$$f(x) = g(x) \quad \text{for all } x \in R$$

$$\Rightarrow \quad f(-4) = g(-4)$$

$$\Rightarrow \quad -4-4 = \lambda$$

$$\Rightarrow \quad \lambda = -8. \qquad [\because f(x) = x-4 \ \therefore f(-4) - 4 = -8]$$

EXAMPLE 6 *If f is a real function defined by* $f(x) = \dfrac{x-1}{x+1}$, *then prove that:* $f(2x) = \dfrac{3\,f(x)+1}{f(x)+3}$.

SOLUTION We have, $f(x) = \dfrac{x-1}{x+1}$

$$\Rightarrow \quad \frac{f(x)}{1} = \frac{x-1}{x+1}$$

$$\Rightarrow \quad \frac{f(x)+1}{f(x)-1} = \frac{x-1+x+1}{x-1-x-1} \qquad \text{[Applying componendo and dividendo]}$$

$$\Rightarrow \quad \frac{f(x)+1}{f(x)-1} = -x$$

$$\Rightarrow \quad x = \frac{f(x)+1}{1-f(x)} \qquad \ldots\text{(i)}$$

Again, $f(x) = \dfrac{x-1}{x+1}$

$$\Rightarrow \quad f(2x) = \frac{2x-1}{2x+1}$$

$$\Rightarrow \quad f(2x) = \frac{2\left\{\dfrac{f(x)+1}{1-f(x)}\right\}-1}{2\left\{\dfrac{f(x)+1}{1-f(x)}\right\}+1} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad f(2x) = \frac{2f(x)+2-1+f(x)}{2f(x)+2+1-f(x)}$$

$$\Rightarrow \quad f(2x) = \frac{3f(x)+1}{f(x)+3}$$

EXERCISE 3.2

LEVEL-1

1. If $f(x) = x^2 - 3x + 4$, then find the values of x satisfying the equation $f(x) = f(2x+1)$.
2. If $f(x) = (x-a)^2 (x-b)^2$, find $f(a+b)$.
3. If $y = f(x) = \dfrac{ax-b}{bx-a}$, show that $x = f(y)$. [NCERT EXEMPLAR]
4. If $f(x) = \dfrac{1}{1-x}$, show that $f[f\{f(x)\}] = x$.
5. If $f(x) = \dfrac{x+1}{x-1}$, show that $f[f(x)] = x$.
6. If $f(x) = \begin{cases} x^2, \text{ when } x < 0 \\ x, \text{ when } 0 \le x < 1 \\ \dfrac{1}{x}, \text{ when } x > 1 \end{cases}$ [NCERT]

   Find: (i) $f(1/2)$ (ii) $f(-2)$ (iii) $f(1)$ (iv) $f(\sqrt{3})$ and (v) $f(\sqrt{-3})$.
7. If $f(x) = x^3 - \dfrac{1}{x^3}$, show that $f(x) + f\left(\dfrac{1}{x}\right) = 0$.
8. If $f(x) = \dfrac{2x}{1+x^2}$, show that $f(\tan\theta) = \sin 2\theta$.

9. If $f(x)=\dfrac{x-1}{x+1}$, then show that

(i) $f\left(\dfrac{1}{x}\right)=-f(x)$ (ii) $f\left(-\dfrac{1}{x}\right)=-\dfrac{1}{f(x)}$

LEVEL-2

10. If $f(x)=(a-x^n)^{1/n}$, $a>0$ and $n\in N$, then prove that $f(f(x))=x$ for all x.

11. If for non-zero x, $a\,f(x)+b\,f\left(\dfrac{1}{x}\right)=\dfrac{1}{x}-5$, where $a\neq b$, then find $f(x)$.

ANSWERS

1. $x=-1, 2/3$ 2. a^2b^2

6. (i) $\dfrac{1}{2}$ (ii) 4 (iii) 1 (iv) $\dfrac{1}{\sqrt{3}}$ (v) does not exist 11. $\dfrac{1}{a^2-b^2}\left\{\dfrac{a}{x}-bx\right\}-\dfrac{5}{a+b}$

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6. We have,

$$f(x)=\begin{cases}x^2, & \text{when } x<0\\ x, & \text{when } 0\le x<1\\ \dfrac{1}{x}, & \text{when } x\ge 1\end{cases}$$

(i) $f\left(\dfrac{1}{2}\right)=\dfrac{1}{2}$ [Using $f(x)=x$, $0\le x<1$]

(ii) $f(-2)=(-2)^2=4$ (iii) $f(1)=\dfrac{1}{1}=1$

(iv) $f(\sqrt{3})=\dfrac{1}{\sqrt{3}}$ (iv) $f(-\sqrt{3})=(-\sqrt{3})^2=3$

11. We have,

$$a\,f(x)+b\,f\left(\frac{1}{x}\right)=\frac{1}{x}-5 \quad \text{... (i)}$$

$$\Rightarrow a\,f\left(\frac{1}{x}\right)+b\,f(x)=x-5 \quad \left[\text{Replacing } x \text{ by } \frac{1}{x}\right] \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$\left\{f(x)+f\left(\frac{1}{x}\right)\right\}(a+b)=x+\frac{1}{x}-10 \Rightarrow f(x)+f\left(\frac{1}{x}\right)=\frac{1}{a+b}\left\{x+\frac{1}{x}-10\right\} \quad \text{...(iii)}$$

Subtracting (ii) from (i), we get $f(x)-f\left(\dfrac{1}{x}\right)=\dfrac{1}{a-b}\left\{\dfrac{1}{x}-x\right\}$ ... (iv)

Add (iii) and (iv) to obtain $f(x)$.

3.6 DOMAIN OF REAL FUNCTIONS

Mathematically to define a function one has to provide its domain, co-domain and the images of elements in its domain either by giving a general formula or by listing them one by one. As the domain and co-domain of real functions are subsets of R. Therefore, conventionally, real functions are described by providing the general formula for finding the images of elements in its domain. In such cases, the domain of the real function $f(x)$ is the set of all those real numbers for which the expression for $f(x)$ or the formula for $f(x)$ assumes real values only. In

otherwords, the domain of $f(x)$ is the set of all those real numbers for which $f(x)$ is meaningful. For example, a real function $f(x)$ described by the general formula $f(x) = \frac{3x-2}{x^2-1}$ assumes real values for all $x \in R$ except for $x = \pm 1$, because denominator of $\frac{3x-2}{x^2-1}$ becomes zero for $x = \pm 1$. So, domain of $f(x)$ is the set of all real numbers other than -1 and 1 i.e. domain$(f) = R - \{-1, 1\}$.

Following examples will illustrate the procedure for finding the domain of a real function of a real variable.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the domain of each of the following real valued functions:*

(i) $f(x) = \frac{1}{x+2}$ (ii) $f(x) = \frac{x-1}{x-3}$

(iii) $f(x) = \frac{2x-3}{x^2-3x+2}$ (iv) $f(x) = \frac{x^2+3x+5}{x^2-5x+4}$ [NCERT]

SOLUTION (i) We have, $f(x) = \frac{1}{x+2}$

Clearly, $f(x)$ assumes real values for all real values of x except for the values of x satisfying $x + 2 = 0$ i.e. $x = -2$. Hence, Domain $(f) = R - \{-2\}$.

(ii) We have, $f(x) = \frac{x-1}{x-3}$

We observe that $f(x)$ is a rational function of x as $\frac{x-1}{x-3}$ is a rational expression. Clearly, $f(x)$ assumes real values for all x except for the values of x for which $x - 3 = 0$ i.e. $x = 3$. Hence,Domain $(f) = R - \{3\}$.

(iii) We have, $f(x) = \frac{2x-3}{x^2-3x+2}$

Clearly, $f(x)$ is a rational function of x as $\frac{2x-3}{x^2-3x+2}$ is a rational expression.

We observe that $f(x)$ assumes real values for all x except for all those values of x for which $x^2 - 3x + 2 = 0$ i.e. $x = 1, 2$. Hence, Domain $(f) = R - \{1, 2\}$.

(iv) We have, $f(x) = \frac{x^2+3x+5}{x^2-5x+4}$

Clearly, $f(x)$ is a rational function of x as $\frac{x^2+3x+5}{x^2-5x+4}$ is a rational expression in x. We observe that $f(x)$ assumes real values for all x except for all those values of x for which $x^2 - 5x + 4 = 0$ i.e. $x = 1, 4$. Hence, Domain $(f) = R - \{1, 4\}$.

EXAMPLE 2 *Find the domain of each of the following functions:*

(i) $f(x) = \sqrt{x-2}$ (ii) $f(x) = \frac{1}{\sqrt{1-x}}$ (iii) $f(x) = \sqrt{4-x^2}$

SOLUTION (i) We have, $f(x) = \sqrt{x-2}$

Clearly, $f(x)$ assumes real values for all x satisfying $x - 2 \geq 0 \Rightarrow x \geq 2 \Rightarrow x \in [2, \infty)$.

Hence, Domain $(f) = [2, \infty)$.

(ii) We have, $f(x) = \dfrac{1}{\sqrt{1-x}}$

Clearly, $f(x)$ assumes real values for all x satisfying $1 - x > 0 \Rightarrow 1 > x \Rightarrow x < 1 \Rightarrow x \in (-\infty, 1)$.

Hence, Domain $(f) = (-\infty, 1)$.

(iii) We have, $f(x) = \sqrt{4 - x^2}$

Clearly, $f(x)$ assumes real values for all x satisfying

$$4 - x^2 \geq 0 \Rightarrow -(x^2 - 4) \geq 0 \Rightarrow x^2 - 4 \leq 0 \Rightarrow (x-2)(x+2) \leq 0 \Rightarrow x \in [-2, 2].$$

Hence, Domain $(f) = [-2, 2]$.

EXAMPLE 3 *Find the domain of the function $f(x)$ defined by $f(x) = \sqrt{4-x} + \dfrac{1}{\sqrt{x^2 - 1}}$.*

SOLUTION Clearly, $f(x)$ is defined for all x satisfying

$$4 - x \geq 0 \text{ and } x^2 - 1 > 0$$

$\Rightarrow \quad x - 4 \leq 0$ and $(x-1)(x+1) > 0$

$\Rightarrow \quad x \leq 4$ and $(x < -1$ or $x > 1)$

$\Rightarrow \quad x \in (-\infty, -1) \cup (1, 4]$.

Hence, Domain $(f) = (-\infty, -1) \cup (1, 4]$.

3.7 RANGE OF REAL FUNCTIONS

The range of a real function of a real variable is the set of all real values taken by $f(x)$ at points in its domain. In order to find the range of a real function $f(x)$, we may use the following algorithm.

ALGORITHM

STEP I *Put $y = f(x)$.*

STEP II *Solve the equation $y = f(x)$ for x in terms of y. Let $x = \phi(y)$.*

STEP III *Find the values of y for which the values of x, obtained from $x = \phi(y)$, are real and in the domain of f.*

STEP IV *The set of values of y obtained in step III is the range of f.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the domain and range of the function $f(x)$ given by $f(x) = \dfrac{x-2}{3-x}$.*

SOLUTION We have, $f(x) = \dfrac{x-2}{3-x}$

Domain of f: Clearly, $f(x)$ is defined for all x satisfying $3 - x \neq 0$ i.e. $x \neq 3$.

Hence, Domain $(f) = R - \{3\}$.

Range of f: Let $y = f(x)$. Then,

$$y=\frac{x-2}{3-x} \Rightarrow 3y-xy=x-2 \Rightarrow x(y+1)=3y+2 \Rightarrow x=\frac{3y+2}{y+1}$$

Clearly, x assumes real values for all y except $y+1=0$ i.e. $y=-1$.
Hence, Range $(f) = R-\{-1\}$.

EXAMPLE 2 *Find the range of each of the following functions:*

(i) $f(x) = \frac{1}{\sqrt{x-5}}$ (ii) $f(x) = \sqrt{16-x^2}$ (iii) $f(x) = \frac{x}{1+x^2}$ (iv) $f(x) = \frac{3}{2-x^2}$

SOLUTION (i) We have, $f(x) = \frac{1}{\sqrt{x-5}}$

Clearly, $f(x)$ takes real values for all x satisfying $x-5>0 \Rightarrow x>5 \Rightarrow x \in (5, \infty)$.

$\therefore$ Domain $(f) = (5, \infty)$

For any $x>5$, we have

$$x-5>0 \Rightarrow \sqrt{x-5} > 0 \Rightarrow \frac{1}{\sqrt{x-5}} > 0 \Rightarrow f(x) > 0.$$

Thus, $f(x)$ takes all real values greater than zero. Hence, Range $(f) = (0, \infty)$.

(ii) We have, $f(x) = \sqrt{16-x^2}$

We observe that $f(x)$ is defined for all x satisfying

$$16-x^2 \geq 0 \Rightarrow x^2-16 \leq 0 \Rightarrow (x-4)(x+4) \leq 0 \Rightarrow -4 \leq x \leq 4 \Rightarrow x \in [-4, 4]$$

$\therefore$ Domain $(f) = [-4, 4]$.

Let $y=f(x)$. Then,

$$y=\sqrt{16-x^2} \Rightarrow y^2=16-x^2 \Rightarrow x^2=16-y^2 \Rightarrow x=\sqrt{16-y^2}$$

Clearly, x will take real values, if

$$16-y^2 \geq 0 \Rightarrow y^2-16 \leq 0 \Rightarrow (y-4)(y+4) \leq 0 \Rightarrow -4 \leq y \leq 4 \Rightarrow y \in [-4, 4]$$

Also, $y=\sqrt{16-x^2} \geq 0$ for all $x \in [-4, 4]$. Therefore, $y \in [0, 4]$ for all $x \in [-4, 4]$.

Hence, Range $(f) = [0, 4]$

(iii) We have, $f(x) = \frac{x}{1+x^2}$

We observe that $f(x)$ takes real values for all $x \in R$. Hence, domain $(f) = R$.

Let $y=f(x)$. Then,

$$y=f(x) \Rightarrow y = \frac{x}{1+x^2} \Rightarrow x^2 y - x + y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$$

Clearly, x will assume real values, if

$1-4y^2 \geq 0$ and $y \neq 0$

$\Rightarrow$ $4y^2-1 \leq 0$ and $y \neq 0$

$\Rightarrow$ $y^2-\frac{1}{4} \leq 0$ and $y \neq 0$

$\Rightarrow$ $\left(y-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \leq 0$ and $y \neq 0$

$\Rightarrow$ $-\frac{1}{2} \leq y \leq \frac{1}{2}$ and $y \neq 0$

$\Rightarrow$ $y \in [-1/2, 1/2]-\{0\}$

Also, $y = 0$ for $x = 0$.

Hence, Range$(f) = [-1/2, 1/2]$

(iv) We have, $f(x) = \dfrac{3}{2 - x^2}$

For $f(x)$ to be real, we must have

$$2 - x^2 \neq 0 \Rightarrow x \neq \pm\sqrt{2}$$

$\therefore$ Domain $(f) = R - \{-\sqrt{2}, \sqrt{2}\}$

Let $f(x) = y$. Then,

$$y = f(x) \Rightarrow y = \frac{3}{2 - x^2} \Rightarrow 2y - x^2 y = 3 \Rightarrow x^2 y = 2y - 3 \Rightarrow x = \pm\sqrt{\frac{2y - 3}{y}}$$

We observe that x will take real values other than $-\sqrt{2}$ and $\sqrt{2}$, if

$$\frac{2y - 3}{y} > 0 \Rightarrow y \in (-\infty, 0) \cup [3/2, \infty) \qquad \text{[See Fig. 3.5]}$$

Fig. 3.5 Signs of $\dfrac{2y - 3}{y}$

Hence, range$(f) = (-\infty, 0) \cup [3/2, \infty)$.

EXAMPLE 3 *Find the domain and range of the function* $f(x) = \dfrac{x^2 - 9}{x - 3}$.

SOLUTION We have, $f(x) = \dfrac{x^2 - 9}{x - 3}$

Domain of f: Clearly, $f(x)$ is not defined for $x - 3 = 0$ i.e. $x = 3$. Therefore, Domain $(f) = R - \{3\}$.

Range of f: Let $f(x) = y$. Then,

$$f(x) = y \Rightarrow \frac{x^2 - 9}{x - 3} = y \Rightarrow x + 3 = y \qquad [\because x \neq 3]$$

It follows from the above relation that y takes all real values except 6 when x takes values in the set $R - \{3\}$. Therefore, Range$(f) = R - \{6\}$.

EXAMPLE 4 *Find the domain and range of the real valued function $f(x)$ given by* $f(x) = \dfrac{4 - x}{x - 4}$.

SOLUTION We have, $f(x) = \dfrac{4 - x}{x - 4}$

Domain of f: We observe that $f(x)$ is defined for all x except at $x = 4$. At $x = 4$, $f(x)$ takes the indeterminate form $\dfrac{0}{0}$. Therefore, Domain $(f) = R - \{4\}$.

Range of f: For any $x \in$ Domain (f) i.e. for any $x \neq 4$, we have

$$f(x) = \frac{4 - x}{x - 4} = \frac{-(x - 4)}{x - 4} = -1$$

Range $(f) = \{-1\}$

EXAMPLE ... $x \in R$, be a function from R into R. Determine the range of f.

[NCERT]

SOLUTION We have, $f(x) = \dfrac{x^2}{x^2+1}$

Domain of f: Clearly, $f(x)$ is defined for all $x \in R$ as $x^2 + 1 \neq 0$ for any $x \in R$. So, Domain$(f) = R$.

Range of f: Let $f(x) = y$. Then,

$$f(x) = y$$

$$\Rightarrow \quad \frac{x^2}{x^2+1} = y \Rightarrow x^2 = x^2 y + y \Rightarrow x^2(1-y) = y \Rightarrow x^2 = \frac{y}{1-y} \Rightarrow x = \pm\sqrt{\frac{y}{1-y}}$$

Clearly, x will take real values, if

$$\frac{y}{1-y} \geq 0$$

$$\Rightarrow \quad \frac{y-0}{y-1} \leq 0$$

$$\Rightarrow \quad 0 \leq y < 1$$

$$\Rightarrow \quad y \in [0, 1)$$ [See Fig. 3.6]

Fig. 3.6 Signs of $\dfrac{y}{1-y}$

Hence, range$(f) = [0, 1)$.

EXAMPLE 6 *Find the domain and range of the function* $f = \left\{\left(x : \dfrac{1}{1-x^2}\right) : x \in R, x \neq \pm 1\right\}$.

SOLUTION We have, $f(x) = \dfrac{1}{1-x^2}$

Domain of f: Clearly, $f(x)$ is defined for all $x \in R$ except for which $x^2 - 1 \neq 0$ i.e. $x = \pm 1$.

Hence, Domain of $f = R - \{-1, 1\}$.

Range of f: Let $f(x) = y$. Then,

$$f(x) = y \Rightarrow \frac{1}{1-x^2} = y \Rightarrow 1 - x^2 = \frac{1}{y} \Rightarrow x^2 = 1 - \frac{1}{y} = \frac{y-1}{y} \Rightarrow x = \pm\sqrt{\frac{y-1}{y-0}}$$

Clearly, x will take real values, if

$$\frac{y-1}{y-0} \geq 0$$

$$\Rightarrow \quad y < 0 \text{ or } y \geq 1$$

$$\Rightarrow \quad y \in (-\infty, 0) \cup [1, \infty)$$ [See Fig. 3.7]

Fig. 3.7 Signs of $\dfrac{y-1}{y-0}$

Hence, range$(f) = (-\infty, 0) \cup [1, \infty)$.

EXAMPLE 7 *Find the domain and range of the function* $f(x) = \dfrac{1}{2 - \sin 3x}$.

SOLUTION We have, $f(x) = \dfrac{1}{2 - \sin 3x}$

Domain of f: We know that

$$-1 \leq \sin 3x \leq 1 \text{ for all } x \in R$$

$$\Rightarrow \quad -1 \leq -\sin 3x \leq 1 \text{ for all } x \in R$$

$$\Rightarrow \quad 1 \leq 2 - \sin 3x \leq 3 \text{ for all } x \in R$$ [Adding 2 throughout]

$$\Rightarrow \quad 2 - \sin 3x \neq 0 \text{ for any } x \in R$$

$$\Rightarrow \quad f(x) = \frac{1}{2 - \sin 3x} \text{ is defined for all } x \in R$$

Hence, domain$(f) = R$.

Range of f: As discussed above

$1 \leq 2 - \sin 3x \leq 3$ for all $x \in R$

$\Rightarrow \quad \frac{1}{3} \leq \frac{1}{2-\sin 3x} \leq 1$ for all $x \in R$

$\Rightarrow \quad \frac{1}{3} \leq f(x) \leq 1$ for all $x \in R$

$\Rightarrow \quad f(x) \in [1/3, 1]$

Hence, range $(f) = [1/3, 1]$

EXERCISE 3.3

LEVEL-1

1. Find the domain of each of the following real valued functions of real variable:

(i) $f(x) = \frac{1}{x}$ (ii) $f(x) = \frac{1}{x-7}$

(iii) $f(x) = \frac{3x-2}{x+1}$ (iv) $f(x) = \frac{2x+1}{x^2-9}$

(v) $f(x) = \frac{x^2+2x+1}{x^2-8x+12}$ [NCERT]

2. Find the domain of each of the following real valued functions of real variable:

(i) $f(x) = \sqrt{x-2}$ (ii) $f(x) = \frac{1}{\sqrt{x^2-1}}$

(iii) $f(x) = \sqrt{9-x^2}$ (iv) $f(x) = \sqrt{\frac{x-2}{3-x}}$

3. Find the domain and range of each of the following real valued functions:

(i) $f(x) = \frac{ax+b}{bx-a}$ (ii) $f(x) = \frac{ax-b}{cx-d}$

(iii) $f(x) = \sqrt{x-1}$ [NCERT] (iv) $f(x) = \sqrt{x-3}$

(v) $f(x) = \frac{x-2}{2-x}$ (vi) $f(x) = |x-1|$ [NCERT]

(vii) $f(x) = -|x|$ [NCERT] (viii) $f(x) = \sqrt{9-x^2}$ [NCERT]

ANSWERS

1. Domain

(i) $R - \{0\}$

(ii) $R - \{7\}$

(iii) $R - \{-1\}$

(iv) $R - \{-3, 3\}$

(v) $R - \{2, 6\}$

2.

	Domain	Range
(i)	$[2, \infty)$	$[0, \infty)$
(ii)	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, -1] \cup (0, \infty)$
(iii)	$[-3, 3]$	$[0, 3]$
(iv)	$[2, 3)$	$[0, \infty)$

3.

Domain	Range	Domain	Range
(i) $R - \left\{\frac{a}{b}\right\}$	$R - \left\{\frac{a}{b}\right\}$	(ii) $R - \left\{\frac{d}{c}\right\}$	$R - \left\{\frac{a}{c}\right\}$
(iii) $[1, \infty)$	$[0, \infty)$	(iv) $[3, \infty)$	$[0, \infty)$
(v) $R - \{2\}$	$\{-1\}$	(vi) R	$[0, \infty)$
(vii) R	$(-\infty, 0)$	(viii) $[-3, 3]$	$[0, 3]$

HINTS TO NCERT & SELECTED PROBLEMS

1. (v) $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12} = \dfrac{(x+1)^2}{(x-6)(x-2)}$ is defined for all x satisfying

$(x-6)(x-2) \neq 0$ i.e. $x \neq 2, 6$.

$\therefore$ Domain $(f) = R - \{2, 6\}$

3. (iii) $f(x) = \sqrt{x-1}$ is defined for all x satisfying $x - 1 \geq 0$ i.e. $x \geq 1$. So, domain $(f) = [1, \infty)$.

Let $y = \sqrt{x-1}$. Clearly, $y \geq 0$ for all $x \in [1, \infty)$. So, range $(f) = [0, \infty)$.

(vi) We have, $f(x) = |x-1|$. Clearly, $f(x)$ is defined for all $x \in R$. So, domain $(f) = R$.

Also, $f(x) = |x-1| \geq 0$ for all $x \in R$. So, range $(f) = [0, \infty)$.

(vii) We have $f(x) = -|x|$

We observe that $f(x)$ is defined for all $x \in R$. So, domain $(f) = R$.

Also, $|x| \geq 0$ for all $x \in R \Rightarrow -|x| \leq 0$ for all $x \in R \Rightarrow f(x) \leq 0$ for all $x \in R$.

So, range $(f) = (-\infty, 0]$.

(viii) We have, $f(x) = \sqrt{9 - x^2}$. Clearly, $f(x)$ takes real values if

$9 - x^2 \geq 0 \Rightarrow x^2 - 9 \leq 0 \Rightarrow (x-3)(x+3) \leq 0 \Rightarrow x \in [-3, 3]$

$\therefore$ Domain $(f) = [-3, 3]$

Also, $f(x) = \sqrt{9 - x^2} \geq 0$ for all $x \in [-3, 3]$.

Let $y = \sqrt{9 - x^2}$. Then, $y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9 \Rightarrow x = \sqrt{9 - y^2}$

Clearly, $x \in R$ if $y \in [-3, 3]$. But, $y \geq 0$. Therefore, $y \in [0, 3]$.

Hence, range $(f) = [0, 3]$

3.8 SOME STANDARD REAL FUNCTIONS AND THEIR GRAPHS

In this section, we shall discuss some standard real functions which frequently occur in the study of calculus.

CONSTANT FUNCTION *If k is a fixed real number, then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function.*

Sometimes we also call it the constant function k.

We observe that the domain of the constant function $f(x) = k$ is the set R of all real numbers and range of f is the singleton set $\{k\}$.

The graph of a constant function $f(x) = k$ is a straight line parallel to x-axis (See Fig. 3.8) which is above or below x-axis according as k is positive or negative. If $k = 0$, then the straight line is coincident to x-axis.

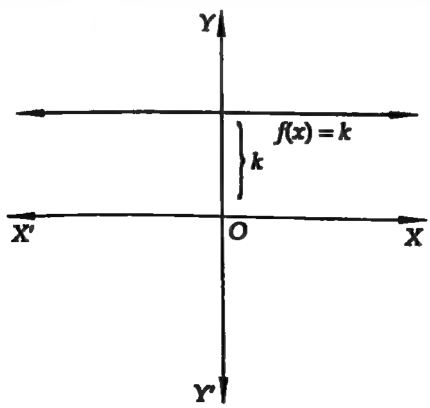


Fig. 3.8 Constant function

IDENTITY FUNCTION *The function that associates each real number to itself is called the identity function and is usually denoted by I.*

Thus, the function $I : R \to R$ defined by $I(x) = x$ for all $x \in R$ is called the identity function.

Clearly, the domain and range of the identity function are both equal to R.

The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with X-axis.

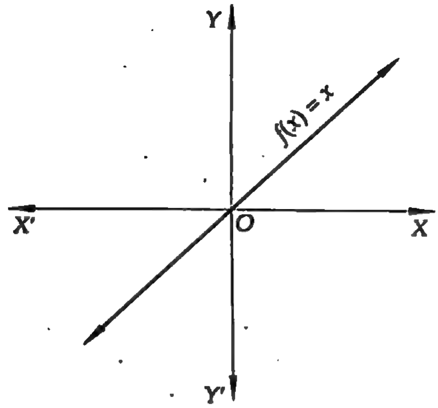


Fig. 3.9 Identity function

MODULUS FUNCTION *The function $f(x)$ defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$ is called the modulus function.*

It is also called the absolute value function.

We observe that the domain of the modulus function is the set R of all real numbers and the range is the set of all non-negative real numbers i.e. $R^+ = \{x \in R : x \geq 0\}$.

The graph of the modulus function is as shown in Fig. 3.10. for $x \geq 0$, the graph coincides with the graph of the identity function i.e. the line $y = x$ and for $x < 0$, it is coincident to the line $y = -x$.

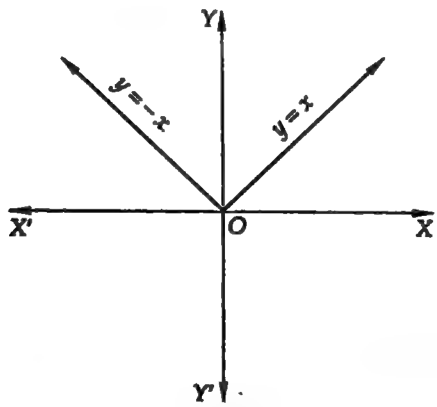


Fig. 3.10 Modulus function

PROPERTIES OF MODULUS FUNCTION The modulus function has the following properties:

(i) For any real number x, $\sqrt{x^2} = |x|$.

For example, $\sqrt{\cos^2 x} = |\cos x| = \begin{cases} \cos x, & 0 \leq x \leq \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} < x \leq \pi \end{cases}$

(ii) If a, b are positive real numbers, then

$$x^2 \le a^2 \Leftrightarrow |x| \le a \Leftrightarrow -a \le x \le a$$
$$x^2 \ge a^2 \Leftrightarrow |x| \ge a \Leftrightarrow x \le -a \text{ or, } x \ge a$$
$$x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$$
$$x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a \text{ or, } x > a$$
$$a^2 \le x^2 \le b^2 \Leftrightarrow a \le |x| \le b \Leftrightarrow x \in [-b, -a] \cup [a, b]$$
$$a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$$

(iii) For real numbers x and y, we have

$$|x+y| = |x| + |y| \Leftrightarrow (x \ge 0 \text{ and } y \ge 0) \text{ or, } (x < 0 \text{ and } y < 0)$$
$$|x-y| = |x| - |y| \Leftrightarrow (x \ge 0, y \ge 0 \text{ and } |x| \ge |y|) \text{ or, } (x \le 0, y \le 0 \text{ and } |x| \ge |y|)$$
$$|x \pm y| \le |x| + |y|$$
$$|x \pm y| \ge \big||x| - |y|\big|$$

GREATEST INTEGER FUNCTION (FLOOR FUNCTION) For any real number x, we use the symbol $[x]$ or, $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x.

For example, $[2.75] = 2, [3] = 3, [0.74] = 0, [-7.45] = -8$ etc.

The function $f: R \to R$ defined by $f(x) = [x]$ for all $x \in R$ is called the greatest integer function or the floor function.

It is also called a step function.

Clearly, domain of the greatest integer function is the set R of all real numbers and the range is the set Z of all integers as it attains only integer values.

The graph of the greatest integer function is shown in Fig 3.11.

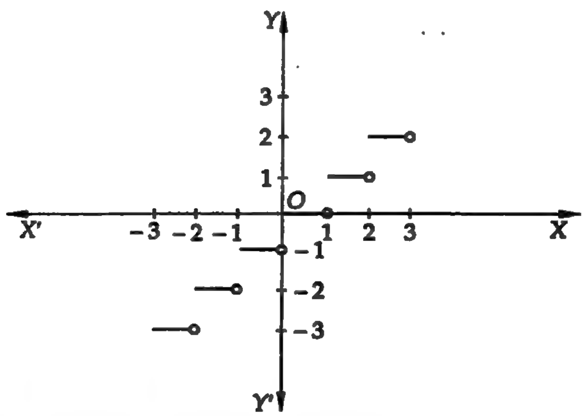


Fig. 3.11 Greatest integer function

PROPERTIES OF GREATEST INTEGER FUNCTION *If n is an integer and x is a real number between n and $n+1$, then*

(i) $[-n] = -[n]$

(ii) $[x+k] = [x] + k$ for any integer k.

(iii) $[-x] = -[x] - 1$

(iv) $[x] + [-x] = \begin{cases} -1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(v) $[x] - [-x] = \begin{cases} 2[x] + 1, & \text{if } x \notin Z \\ 2[x], & \text{if } x \in Z \end{cases}$

(vi) $[x] \ge k \Rightarrow x \ge k$, where $k \in Z$

(vii) $[x] \le k \Rightarrow x < k+1$, where $k \in Z$ (viii) $[x] > k \Rightarrow x > k+1$, were $k \in Z$

(ix) $[x] < k \Rightarrow x < k$, where $k \in Z$ (x) $[x+y] = [x]+[y+x-[x]]$ for all $x, y \in R$

(xi) $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right] = [nx], n \in N.$

SMALLEST INTEGER FUNCTION (CEILING FUNCTION) For any real number x, we use the symbol $\lceil x \rceil$ to denote the smallest integer greater than or equal to x.

For example, $\lceil 4.7 \rceil = 5, \lceil -7.2 \rceil = -7, \lceil 5 \rceil = 5, \lceil 0.75 \rceil = 1$ etc.

The function $f: R \to R$ defined by $f(x) = \lceil x \rceil$ for all $x \in R$ is called the smallest integer function or the ceiling function.

It is also a step function.

We observe that the domain of the smallest integer function is the set R of all real numbers and its range is the set Z of all integers.

The graph of the smallest integer function is as shown in Fig. 3.12.

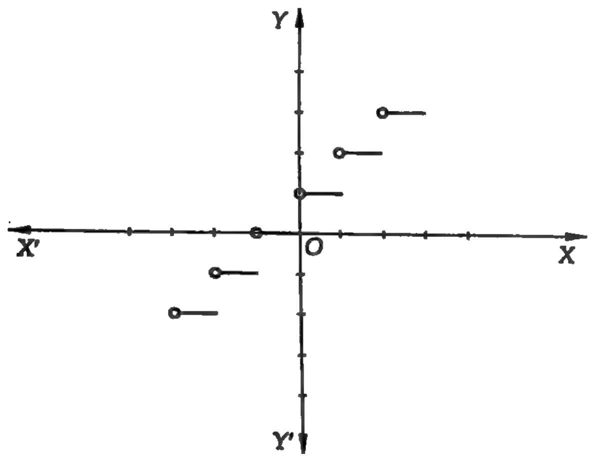


Fig. 3.12 Smallest integer function

PROPERTIES OF SMALLEST INTEGER FUNCTION Following are some properties of smallest integer function:

(i) $\lceil -n \rceil = -\lceil n \rceil$, where $n \in Z$

(ii) $\lceil -x \rceil = -\lceil x \rceil + 1$, where $x \in R - Z$

(iii) $\lceil x+n \rceil = \lceil x \rceil + n$, where $x \in R - Z$ and $n \in Z$

(iv) $\lceil x \rceil + \lceil -x \rceil = \begin{cases} 1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(v) $\lceil x \rceil - \lceil -x \rceil = \begin{cases} 2\lceil x \rceil - 1, & \text{if } x \notin Z \\ 2\lceil x \rceil, & \text{if } x \in Z \end{cases}$

FRACTIONAL PART FUNCTION For any real number x, we use the symbol $\{x\}$ to denote the fractional part or decimal part of x.

For example, $\{3.45\} = 0.45, \{-2.75\} = 0.25, \{-0.55\} = 0.45, \{3\} = 0, \{-7\} = 0$ etc.

The function $f: R \to R$ defined by $f(x) = \{x\}$ for all $x \in R$ is called the fractional part function.

We observe that the domain of the fractional part function is the set R of all real numbers and the range is the set $[0, 1)$.

It is evident from the definition that $f(x) = \{x\} = x - [x]$ for all $x \in R$

The graph of the fractional part function is as shown in Fig. 3.13.

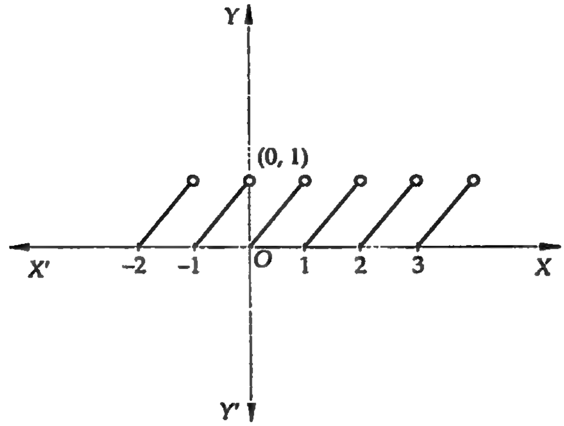


Fig. 3.13 Fractional part function

SIGNUM FUNCTION *The function f defined by*

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{or,} \quad f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

is called the signum function.

The domain of the signum function is the set R of all real numbers and the range is the set $\{-1, 0, 1\}$

The graph of the signum function is as shown in Fig. 3.14.

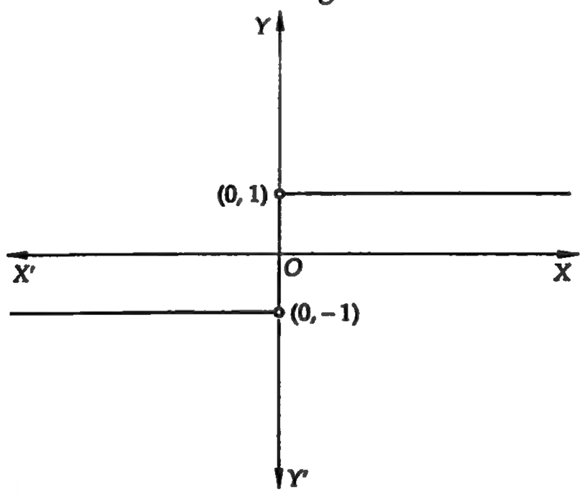


Fig. 3.14 Signum function

EXPONENTIAL FUNCTION *If a is a positive real number other than unity, then a function that associates each $x \in R$ to a^x is called the exponential function.*

In other words, a function $f: R \to R$ defined by $f(x) = a^x$, where $a > 0$ and $a \neq 1$ is called the exponential function.

We observe that the domain of an exponential function is R the set of all real numbers and the range is the set $(0, \infty)$ as it attains only positive values.

As $a > 0$ and $a \neq 1$. So, we have the following cases:

CASE I *When* $a > 1$

We observe that the values of $y = f(x) = a^x$ increase as the values of x increase.

Also, $f(x) = a^x \begin{cases} < 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ > 1 & \text{for } x > 0. \end{cases}$

Thus, the graph of $f(x) = a^x$ for $a > 1$ as shown in Fig. 3.15.

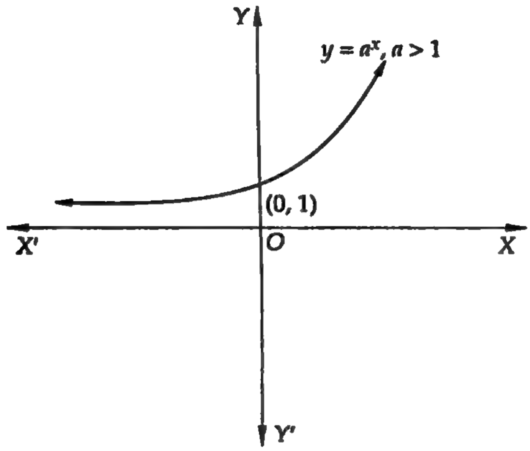


Fig. 3.15 Exponential function $f(x) = a^x$ for $a > 1$

We also observe that:

$2^x < 3^x < 4^x < \ldots$ for all $x > 1$

$2^x = 3^x = 4^x = \ldots = 1$ for $x = 0$

$2^x > 3^x > 4^x > \ldots$ for all $x < 1$

So, the graphs of $f(x) = 2^x$, $f(x) = 3^x$, $f(x) = 4^x$ etc. are as shown in Fig. 3.16.

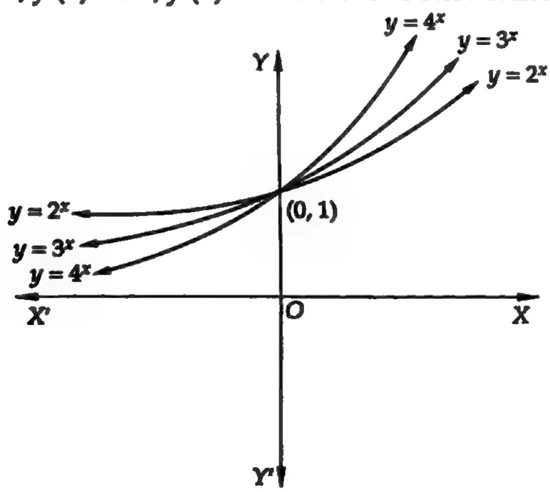


Fig. 3.16 Graphs of exponential functions

CASE II *When* $0 < a < 1$

In this case, the values of $y = f(x) = a^x$ decrease with the increase in x and $y > 0$ for all $x \in R$.

Also,

$$y = f(x) = a^x \begin{cases} > 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ < 1 & \text{for } x > 0 \end{cases}$$

Thus, the graph of $f(x) = a^x$ for $0 < a < 1$ is as shown in Fig. 3.17.

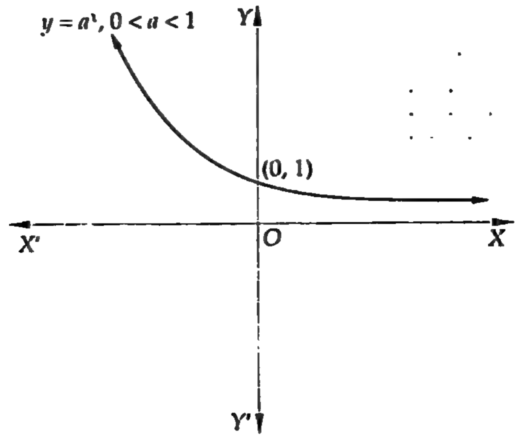


Fig. 3.17 Graph of exponential function $f(x) = a^x$ for $0<a<1$

The graphs of $f(x) = a^x$, $0 < a < 1$ for different values of a are shown in Fig. 3.18.

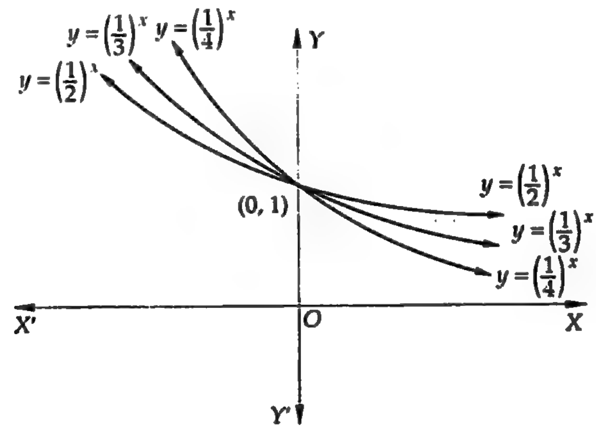


Fig. 3.18

REMARK *We have, $2 < e < 3$. Therefore, graph of $f(x) = e^x$ is identical to that of $f(x) = a^x$ for $a > 1$ and the graph of $f(x) = e^{-x}$ is identical to that of $f(x) = a^x$ for $0 < a < 1$.*

LOGARITHMIC FUNCTION *If $a > 0$ and $a \neq 1$, then the function defined by $f(x) = \log_a x$, $x > 0$ is called the logarithmic function.*

Previously we have learnt that the logarithmic function and the exponential function are inverse functions i.e. $\log_a x = y \Leftrightarrow x = a^y$.

We observe that the domain of the logarithmic function is the set of all non-negative real numbers i.e. $(0, \infty)$ and the range is the set R of all real numbers.

As $a > 0$ and $a \neq 1$. So, we have the following cases.

CASE I *When $a > 1$*

In this case, we have

$$y = \log_a x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1 \end{cases}$$

Also, the values of y increase with the increase in x.

So, the graph of $y = \log_a x$ is as shown in Fig. 3.19.

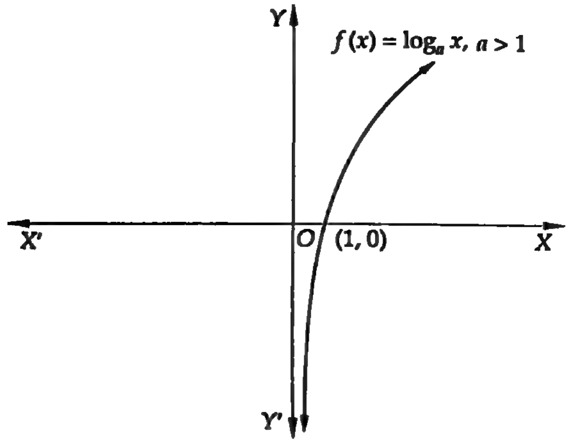


Fig. 3.19 Logarithmic function $f(x) = \log_a x$ for $a > 1$

CASE II *When* $0 < a < 1$

In this case, we have

$$y = \log_a x \begin{cases} > 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ < 0 & \text{for } x > 1 \end{cases}$$

Also, the values of y decrease with the increase in x.

So, the graph of $y = \log_a x$ is as shown in Fig. 3.20.

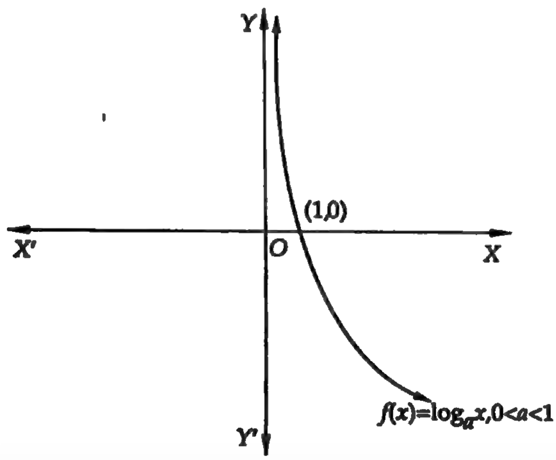


Fig. 3.20 Logarithmic function $f(x) = \log_a x, 0 < a < 1$

PROPERTIES OF LOGARITHMIC FUNCTION Following are some useful properties of logarithmic function:

(i) $\log_a 1 = 0$, where $a > 0, a \neq 1$

(ii) $\log_a a = 1$, where $a > 0, a \neq 1$

(iii) $\log_a (xy) = \log_a |x| + \log_a |y|$, where $a > 0, a \neq 1$ and $xy > 0$

(iv) $\log_a \left(\frac{x}{y}\right) = \log_a |x| - \log_a |y|$, where $a > 0, a \neq 1$ and $\frac{x}{y} > 0$

(v) $\log_a (x^n) = n \log_a |x|$, where $a > 0, a \neq 1$ and $x^n > 0$

(vi) $\log_{a^n} x^m = \frac{m}{n} \log_a x$, where $a > 0, a \neq 1$ and $x > 0$

(vii) $x^{\log_a y} = y^{\log_a x}$, where $x > 0, y > 0, a > 0, a \neq 1$

(viii) If $a > 1$, then the values of $f(x) = \log_a x$ increase with the increase in x.

i.e $x < y \Leftrightarrow \log_a x < \log_a y$

$$\text{Also, } \log_a x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1. \end{cases}$$

(ix) If $0 < a < 1$, then the values of $f(x) = \log_a x$ decrease with the increase in x.

i.e $x < y \Leftrightarrow \log_a x > \log_a y$

$$\text{Also, } \log_a x \begin{cases} > 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ < 0 & \text{for } x > 1 \end{cases}$$

(x) $\log_a x = \dfrac{1}{\log_x a}$ for $a > 0, a \neq 1$ and $x > 0, x \neq 1$.

<u>REMARK</u> *Functions $f(x) = \log_a x$ and $g(x) = a^x$ are inverse of each other. So, their graphs are mirror images of each other in the line mirror $y = x$.*

RECIPROCAL FUNCTION The function that associates a real number x to its reciprocal $1/x$ is called the reciprocal function. Since $1/x$ is not defined for $x = 0$. So, we define the reciprocal function as follows:

DEFINITION *The function $f : R - \{0\} \to R$ defined by $f(x) = \dfrac{1}{x}$ is called the reciprocal function.*

Clearly, domain of the reciprocal function is $R - \{0\}$ and its range is also $R - \{0\}$.

We observe that the sign of $\dfrac{1}{x}$ is same as that of x and $\dfrac{1}{x}$ decreases with the increase in x.

So, the graph of $f(x) = \dfrac{1}{x}$ is as shown in Fig. 3.21.

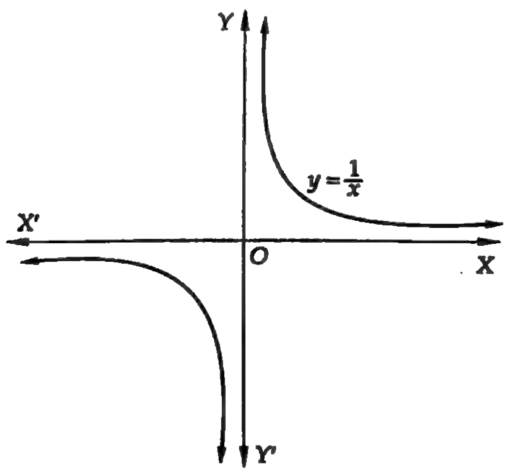


Fig. 3.21 Reciprocal function

SQUARE ROOT FUNCTION The function that associates a real number x *to* $+\sqrt{x}$ is called the square root function. Since $\sqrt{x}$ is real for $x > 0$. So, we defined the square root function as follows:

DEFINITION *The function $f: R^+ \to R$ defined by $f(x) = +\sqrt{x}$ is called the square root function.*

Clearly, domain of the square root function is R^+ *i.e.* $[0, \infty)$ and its range is also $[0, \infty)$.

We observe that the values of $f(x) = +\sqrt{x}$ increase with the increase in x. So, the graph of $f(x) = +\sqrt{x}$ is as shown in Fig. 3.22.

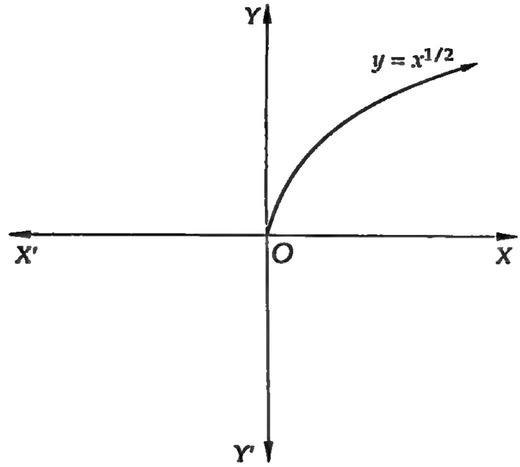


Fig. 3.22 Square root function

SQUARE FUNCTION The function that associates a real number x to its square i.e. x^2 is called the square function. Since x^2 is defined for all $x \in R$. So, we define the square function as follows:

DEFINITION *The function $f: R \to R$ defined by $f(x) = x^2$ is called the square function.*

Clearly, domain of the square function is R and its range is the set of all non-negative real numbers i.e. $[0, \infty)$. The graph of $f(x) = x^2$ is parabola as shown in Fig. 3.23.

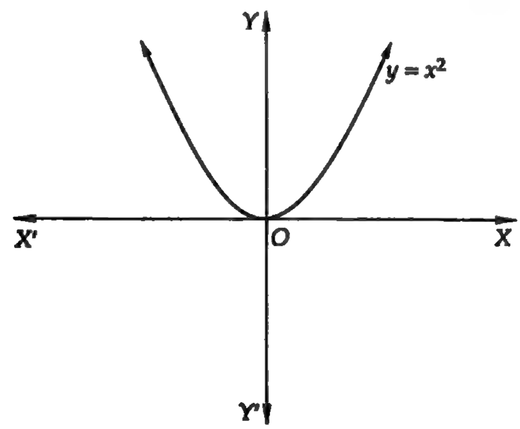


Fig. 3.23 Square function

CUBE FUNCTION The function that associate a real number x to its cube is called the cube function. We observe that x^3 is meaningful for all $x \in R$. So, we define the cube function as follows:

DEFINITION *The function $f: R \to R$ defined by $f(x) = x^3$ is called the cube function.*

We observe that the sign of x^3 is same as that of x and the values of x^3 increase with the increase in x. So, the graph of $f(x) = x^3$ is as shown in Fig. 3.24. Clearly, the graph is symmetrical in opposite quadrants.

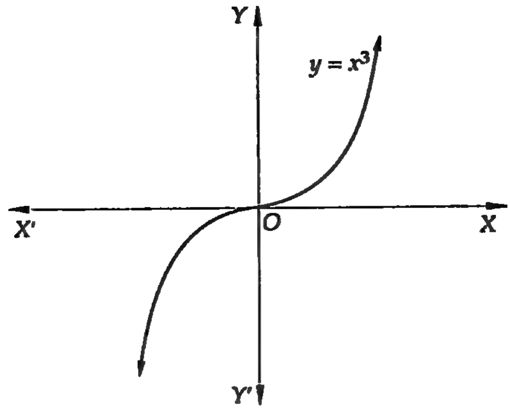


Fig. 3.24 Cube function

CUBE ROOT FUNCTION The function that associates a real number x to its cube root $x^{1/3}$ is called the cube root function. Clearly, $x^{1/3}$ is defined for all $x \in R$. So, we define the cube root function as follows:

DEFINITION *The function $f : R \to R$ defined by $f(x) = x^{1/3}$ is called the cube root function.*

Clearly, domain and range of the cube root function are both equal to R.

Also, the sign of $x^{1/3}$ is same as that of x and $x^{1/3}$ increase with the increase in x. So, the graph of $f(x) = x^{1/3}$ is as shown in Fig. 3.25.

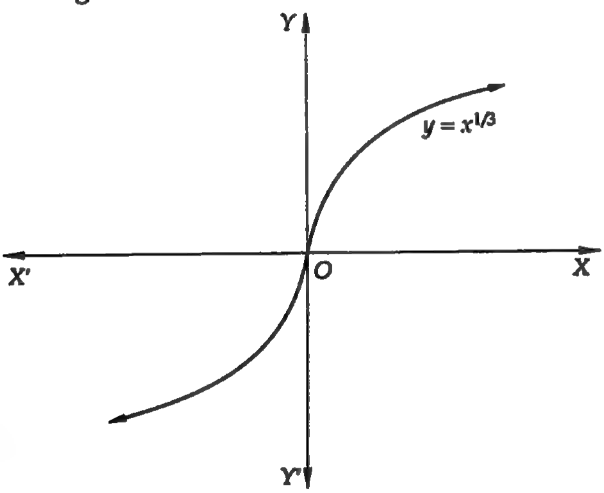


Fig. 3.25 Cube root function

REMARK 1 *A function $f : R \to R$ is said to be a polynomial function if $f(x)$ is a polynomial in x. For example, $f(x) = x^2 - x + 4$, $g(x) = x^3 + 3x^2 + \sqrt{2}\,x - 1$ etc are polynomial functions.*

REMARK 2 *A function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function. The domain of a rational function $f(x) = \frac{p(x)}{q(x)}$ is the set of all real numbers, except points where $q(x) = 0$.*

3.9 OPERATIONS ON REAL FUNCTIONS

In this section, we shall introduce various operations, namely addition, subtraction, multiplication, division etc. on real functions.

ADDITION *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then, their sum $f + g$ is defined as that function from $D_1 \cap D_2$ to R which associates each $x \in D_1 \cap D_2$ to the number $f(x) + g(x)$.*

In other words, if $f : D_1 \to R$ and $g : D_2 \to R$ are two real functions, then their sum $f+g$ is a function from $D_1 \cap D_2$ to R such that

$$(f+g)(x) = f(x) + g(x) \quad \text{for all } x \in D_1 \cap D_2.$$

PRODUCT *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then, their product (or pointwise multiplication) fg is a function from $D_1 \cap D_2$ to R and is defined as*

$$(fg)(x) = f(x)\, g(x) \quad \text{for all } x \in D_1 \cap D_2$$

DIFFERENCE (SUBTRACTION) *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then the difference of g from f is denoted by $f-g$ and is defined as*

$$(f-g)(x) = f(x) - g(x) \quad \text{for all } x \in D_1 \cap D_2$$

QUOTIENT *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then the quotient of f by g is denoted by $\frac{f}{g}$ and it is a function from $D_1 \cap D_2 - \{x : g(x) = 0\}$ to R defined by*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{for all } x \in D_1 \cap D_2 - \{x : g(x) = 0\}$$

MULTIPLICATION OF A FUNCTION BY A SCALAR *Let $f : D \to R$ be a real function and α be a scalar (real number). Then the product αf is a function from D to R and is defined as*

$$(\alpha f)(x) = \alpha f(x) \quad \text{for all } x \in D.$$

RECIPROCAL OF A FUNCTION *If $f : D \to R$ is a real function, then its reciprocal function $\frac{1}{f}$ is a function from $D - \{x : f(x) = 0\}$ to R and is defined as $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$.*

<u>REMARK 1</u> *The sum, difference product and quotient are defined for real functions only on their common domain. These operations do not make any sense for general functions even if their domains are same, because the sum, difference, product and quotient may or may not be meaningful for the elements in their common domain.*

<u>REMARK 2</u> *For any real function $f : D \cap R$ and $n \in N$, we define*

$$(\underbrace{fff \ldots f}_{n\text{-times}})(x) = \underbrace{f(x)\, f(x) \ldots f(x)}_{n\text{-times}} = \{f(x)\}^n \quad \text{for all } x \in D$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the sum and difference of the identity function and the modulus function.* [NCERT]

SOLUTION We know that $f : R \to R$ defined by $f(x) = x$ is the identity function and $g : R \to R$ defined by $g(x) = |x|$ is the modulus function. Clearly, f and g have the same domain. Therefore, $f+g : R \to R$ and $f-g : R \to R$.

Now,

$$(f+g)(x) = f(x) + g(x)$$

$$\Rightarrow \quad (f+g)(x) = x + |x|$$

$$\Rightarrow \quad (f+g)(x) = \begin{cases} x+x, & \text{if } x \geq 0 \\ x-x, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad (f+g)(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

and,

$$(f-g)(x) = f(x) - g(x)$$

$$\Rightarrow \quad (f-g)(x) = x - |x|$$

$$\Rightarrow \quad (f-g)(x) = \begin{cases} x-x, & \text{if } x \geq 0 \\ x-(-x), & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad (f-g)(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ 2x, & \text{if } x < 0 \end{cases}$$

Thus, $f+g: R \to R$ and $f-g: R \to R$ are defined as

$$(f+g)(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \text{ and, } (f-g)(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ 2x, & \text{if } x < 0 \end{cases}$$

EXAMPLE 2 *What are the sum and difference of the identity function and the reciprocal function?* **[NCERT]**

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then, $f: R \to R$ and $g: R - \{0\} \to R$ such that $f(x) = x$ for all $x \in R$ and, $g(x) = \frac{1}{x}$ for all $x \in R - \{0\}$.

The domains of f and g are R and $R - \{0\}$ respectively. Also, we have $R \cap R - \{0\} = R - \{0\}$.

Therefore, $f + g: R - \{0\} \to R$ and $f - g: R - \{0\} \to R$ are given by

$$(f+g)(x) = f(x) + g(x) = x + \frac{1}{x} \text{ and, } (f-g)(x) = f(x) - g(x) = x - \frac{1}{x}$$

EXAMPLE 3 *Let $f: [2, \infty) \to R$ and $g: [-2, \infty) \to R$ be two real functions defined by $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt{x+2}$. Find $f+g$ and $f-g$.*

SOLUTION Let $D_1 = [2, \infty)$ and $D_2 = [-2, \infty)$. Then, $D_1 \cap D_2 = [2, \infty)$.

Thus, $f+g: [2, \infty) \to R$ and $f-g: [2, \infty) \to R$ are given by

$$(f+g)(x) = f(x) + g(x) = \sqrt{x-2} + \sqrt{x+2} \quad \text{for all } x \in [2, \infty)$$

and, $(f-g)(x) = f(x) - g(x) = \sqrt{x-2} - \sqrt{x+2}$ for all $x \in [2, \infty)$.

EXAMPLE 4 *Find the product of the identity function and the modulus function.*

SOLUTION Let f and g denote respectively the identity function and modulus function. Then, $f: R \to R$ such that $f(x) = x$ for all x and, $g: R \to R$ such that $g(x) = |x|$ for all x.

Clearly, f and g have the same domain. Therefore, the product $f\,g$ is a function from R to itself and is given by

$$(f\,g)(x) = f(x)\,g(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

EXAMPLE 5 *Find the quotient of the identity function by the modulus function.*

SOLUTION Let f and g denote respectively the identity function and the modulus function. Then, $f: R \to R$ is defined as $f(x) = x$ and, $g: R \to R$ is defined as $g(x) = |x|$.

Clearly, f and g have the same domain.

And, $g(x) = 0 \Rightarrow |x| = 0 \Rightarrow x = 0$.

Therefore, the quotient of f by g i.e. $\frac{f}{g}$ is a function from $R - \{0\} \to R$ and is defined as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{|x|} = \begin{cases} \frac{x}{x} = 1, & x > 0 \\ \frac{x}{-x} = -1, & x < 0 \end{cases}$$

EXAMPLE 6 *Find the product of the identity function and the reciprocal function.*

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then, $f: R \to R$ is defined as $f(x) = x$ for all $x \in R$ and, $g: R - \{0\} \to R$ is defined as $g(x) = \frac{1}{x}$ for all $x \in R - \{0\}$.

Now, Domain $(f) \cap$ Domain $(g) = R \cap R - \{0\} = R - \{0\}$

Therefore, the product fg is a function from $R - \{0\}$ to R and is defined as

$$(fg)(x) = f(x)\,g(x) = x \times \frac{1}{x} = 1 \text{ for all } x \in R - \{0\}$$

Thus, $fg : R - \{0\} \to R$ is given by $(fg)(x) = 1$ for all $x \in R - \{0\}$.

EXAMPLE 7 *Find the quotient of the identity function by the reciprocal function.*

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then, $f : R \to R$ is defined as $f(x) = x$ for all $x \in R$ and, $g : R - \{0\} - R$ is defined as $g(x) = \frac{1}{x}$ for all $x \in R - \{0\}$

Now, Domain $(f) \cap$ Domain $(g) = R \cap R - \{0\} = R - \{0\}$

And, $g(x) \neq 0$ for any $x \in R - \{0\}$

$\therefore$ $\frac{f}{g}$ is a function from $R - \{0\} \to R$ and is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{1/x} = x^2$$

Hence, $\frac{f}{g} : R - \{0\} \to R$ is given by $\left(\frac{f}{g}\right)(x) = x^2$ for all $x \in R - \{0\}$.

EXAMPLE 8 *Let c be a non-zero real number and $f : R \to R$ be a function defined by $f(x) = \frac{x}{c}$ for all $x \in R$. Find* (i) cf (ii) $c^2 f$ (iii) $\left(\frac{1}{c}\right) f$.

SOLUTION Clearly, cf, $c^2 f$ and $\left(\frac{1}{c}\right) f$ are functions from R to itself.

Now,

(i) $(cf)(x) = cf(x) = c \times \frac{x}{c} = x$ for all $x \in R$

(ii) $(c^2 f)(x) = c^2 f(x) = c^2 \times \frac{x}{c} = cx$ for all $x \in R$

(iii) $\left(\left(\frac{1}{c}\right) f\right)(x) = \left(\frac{1}{c}\right) f(x) = \frac{1}{c} \times \frac{x}{c} = \frac{x}{c^2}$ for all $x \in R$.

EXAMPLE 9 *Let f and g be two real functions defined by $f(x) = \frac{1}{x+4}$ and $g(x) = (x+4)^3$.*

Find the following: (i) $f+g$ (ii) $f-g$ (iii) fg (iv) $\frac{f}{g}$ (v) $2f$ (vi) $\frac{1}{f}$ (vii) $\frac{1}{g}$

SOLUTION We observe that $f(x) = \frac{1}{x+4}$ is defined for all $x \neq -4$. So, domain $(f) = R - \{-4\}$.

Clearly, $g(x) = (x+4)^3$ is defined for all $x \in R$. So, domain $(g) = R$.

$\therefore$ Domain $(f) \cap$ Domain $(g) = R - \{-4\}$.

(i) $f + g : R - \{-4\} \to R$ is given by

$$(f+g)(x) = f(x) + g(x) = \frac{1}{x+4} + (x+4)^3 = \frac{(x+4)^4 + 1}{x+4}.$$

(ii) $f - g : R - \{-4\} \to R$ is defined as

$$(f-g)(x) = f(x) - g(x) = \frac{1}{x+4} - (x+4)^3 = \frac{1-(x+4)^4}{x+4}$$

(iii) $fg : R - \{-4\} \to R$ is given by

$$(fg)(x) = f(x)\,g(x) = \frac{1}{x+4} \times (x+4)^3 = (x+4)^2$$

(iv) $g(x) = 0 \Rightarrow (x+4)^3 = 0 \Rightarrow x = -4.$

$\therefore$ Domain $\left(\frac{f}{g}\right)$ = Domain $(f) \cap$ Domain $(g) - \{x : g(x) = 0\} = R - \{-4\}$

Thus, $\frac{f}{g} : R - \{-4\} \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1}{(x+4)^4}$

(v) $2f : R - \{-4\} \to R$ is given by

$(2f)(x) = 2(f(x)) = \frac{2}{x+4}$ for all $x \in R - \{-4\}$.

(vi) We observe that $f(x) \neq 0$ for any $x \in R - \{-4\}$. Therefore, $\frac{1}{f} : R - \{-4\} \to R$ is given by

$$\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} = \frac{1}{1/(x+4)} = (x+4)$$

(vii) We observe that $g(x) = (x+4)^3 = 0$ for $x = -4$. Therefore, $\frac{1}{g} : R - \{-4\} \to R$ is given by

$$\left(\frac{1}{g}\right)(x) = \frac{1}{g(x)} = \frac{1}{(x+4)^3}$$

EXAMPLE 10 *Let f and g be real functions defined by $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4-x^2}$. Then, find each of the following functions:*

(i) $f+g$ (ii) $f-g$ (iii) fg (iv) $\frac{f}{g}$ (v) ff (vi) gg

SOLUTION We have, $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4-x^2}$.

Clearly, $f(x)$ is defined for all x satisfying

$x + 2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, \infty)$

$\therefore$ Domain $(f) = [-2, \infty)$

We observe that $g(x)$ is defined for all x satisfying

$4 - x^2 \geq 0 \Rightarrow x^2 - 4 \leq 0 \Rightarrow (x-2)(x+2) \leq 0 \Rightarrow x \in [-2, 2]$

$\therefore$ Domain $(g) = [-2, 2]$

Now,

Domain $(f) \cap$ Domain $(g) = [-2, \infty) \cap [-2, 2] = [-2, 2]$

(i) $f + g : [-2, 2] \to R$ is given by

$(f+g)(x) = f(x) + g(x) = \sqrt{x+2} + \sqrt{4-x^2}$

(ii) $f - g : [-2, 2] \to R$ is given by

$(f-g)(x) = f(x) - g(x) = \sqrt{x+2} - \sqrt{4-x^2}$

(iii) $fg : [-2, 2] \to R$ is given by

$(fg)(x) = f(x)\,g(x) = \sqrt{x+2} \times \sqrt{4-x^2} = \sqrt{(x+2)^2(2-x)} = (x+2)\sqrt{2-x}$

(iv) We have, $g(x) = \sqrt{4-x^2}$

$\therefore$ $g(x) = 0 \Rightarrow 4 - x^2 = 0 \Rightarrow x = \pm 2.$

So, Domain $\left(\frac{f}{g}\right) = [-2, 2] - \{-2, 2\} = (-2, 2)$

$\therefore \quad \dfrac{f}{g} : (-2, 2) \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{(x+2)}}{\sqrt{4-x^2}} = \frac{1}{\sqrt{2-x}}$$

(v) Since domain $(f) = [-2, \infty)$. Therefore,

$$(ff)(x) = f(x)\, f(x) = [f(x)]^2 = (\sqrt{x+2})^2 = x+2 \text{ for all } x \in [-2, \infty)$$

(vi) Since domain $(g) = [-2, 2)$. Therefore,

$$(gg)(x) = g(x)\, g(x) = [g(x)]^2 = \left(\sqrt{4-x^2}\right)^2 = 4 - x^2 \text{ for all } x \in [-2, 2]$$

EXAMPLE 11 *Let f be the exponential function and g be the logarithmic function. Find:*

(i) $(f+g)(1)$ (ii) $(fg)(1)$ (iii) $(3f)(1)$ (iv) $(5g)(1)$

SOLUTION We have,

$f: R \to R$ given by $f(x) = e^x$ and, $g: R^+ \to R$ given by $g(x) = \log_e x$.

(i) Since Domain $(f) \cap$ Domain $(g) = R \cap R^+ = R^+$. Therefore,

$f + g: R^+ \to R$ is given by

$(f+g)(x) = f(x) + g(x) = e^x + \log_e x$ for all $x \in R^+$

$\therefore \quad (f+g)(1) = e^1 + \log_e 1 = e + 0 = e.$

(ii) Domain $(f) \cap$ Domain $(g) = R \cap R^+ = R^+$. Therefore, $fg: R^+ \to R$ is given by

$(fg)(x) = f(x)\, g(x) = e^x . \log_e x.$

$\therefore \quad (fg)(1) = e^1 \times \log_e 1 = e \times 0 = 0$

(iii) Clearly, $(3f)(x) = 3(f(x)) = 3e^x$

$\therefore \quad (3f)(1) = 3e^1 = 3e$

(iv) Clearly, $(5g)(x) = 5(g(x)) = 5 \log_e x$

$\therefore \quad (5g)(1) = 5 \log_e 1 = 5 \times 0 = 0.$

LEVEL-2

EXAMPLE 12 *Find the domain of each of the following functions given by*

(i) $f(x) = \dfrac{1}{\sqrt{x - |x|}}$ [NCERT EXEMPLAR] (ii) $f(x) = \dfrac{1}{\sqrt{x + |x|}}$ [NCERT EXEMPLAR]

(iii) $f(x) = \dfrac{1}{\sqrt{x - [x]}}$ (iv) $f(x) = \dfrac{1}{\sqrt{x + [x]}}$

SOLUTION (i) We have, $f(x) = \dfrac{1}{\sqrt{x - |x|}}$

We know that $|x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}$

$\therefore \quad x - |x| = \begin{cases} x - x = 0, & \text{if } x \ge 0 \\ x + x = 2x, & \text{if } x < 0 \end{cases}$

$\Rightarrow \quad x - |x| \le 0$ for all x

$\Rightarrow \quad \dfrac{1}{\sqrt{x - |x|}}$ does not take real values for any $x \in R$

$\Rightarrow \quad f(x)$ is not defined for any $x \in R$.

Hence, Domain $(f) = \phi$

(ii) We have, $f(x) = \dfrac{1}{\sqrt{x + |x|}}$

We know that $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$$\therefore \quad x + |x| = \begin{cases} x + x, & \text{if } x \geq 0 \\ x - x, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad x + |x| = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \qquad ...(i)$$

Thus, $f(x) = \dfrac{1}{\sqrt{x + |x|}}$ assumes real values, if

$x + |x| > 0$

$\Rightarrow \quad x > 0$ [Using (i)]

$\Rightarrow \quad x \in (0, \infty)$.

Hence, Domain $(f) = (0, \infty)$.

(iii) We have, $f(x) = \dfrac{1}{\sqrt{x - [x]}}$

We know that $0 \leq x - [x] < 1$ for all $x \in R$. Also, $x - [x] = 0$ for $x \in Z$.

Thus, $f(x) = \dfrac{1}{\sqrt{x - [x]}}$ is defined, if

$x - [x] > 0 \Rightarrow x \in R - Z$ $\quad [\because x - [x] = 0 \text{ for } x \in Z \text{ and } 0 < x - [x] < 1 \text{ for } x \in R - Z]$

Hence, Domain $(f) = R - Z$.

(iv) We have, $f(x) = \dfrac{1}{\sqrt{x + [x]}}$

We know that

$$\left.\begin{array}{ll} x + [x] > 0 & \text{for all } x > 0 \\ x + [x] = 0 & \text{for } x = 0 \\ x + [x] < 0 & \text{for all } x < 0 \end{array}\right\} \qquad ...(i)$$

Also, $f(x) = \dfrac{1}{\sqrt{x + [x]}}$ is defined for all x satisfying $x + [x] > 0$. Therefore, from (i), we obtain that, Domain $(f) = (0, \infty)$.

EXAMPLE 13 *Find the domain of definition of the function $f(x)$ given by*

$$f(x) = \log_4 \left\{ \log_5 \left(\log_3 (18x - x^2 - 77 \right) \right\}$$

SOLUTION We have,

$$f(x) = \log_4 \left\{ \log_5 \left(\log_3 (18x - x^2 - 77) \right) \right\}$$

Since $\log_a x$ is defined for all $x > 0$. Therefore, $f(x)$ is defined if

$\log_5 \{\log_3 (18x - x^2 - 77)\} > 0$ and $18x - x^2 - 77 > 0$

$\Rightarrow \quad \log_3 (18x - x^2 - 77) > 5^0$ and $x^2 - 18x + 77 < 0$

$\Rightarrow \quad \log_3 (18x - x^2 - 77) > 1$ and $(x - 11)(x - 7) < 0$

$\Rightarrow \quad 18x - x^2 - 77 > 3^1$ and $7 < x < 11$

$\Rightarrow \quad 18x - x^2 - 80 > 0$ and $7 < x < 11$

$\Rightarrow \quad x^2 - 18x + 80 < 0$ and $7 < x < 11$

$\Rightarrow \quad (x - 10)(x - 8) < 0$ and $7 < x < 11$

$\Rightarrow \quad 8 < x < 10$ and $7 < x < 11 \Rightarrow 8 < x < 10 \Rightarrow x \in (8, 10)$.

Hence, the domain of $f(x)$ is $(8, 10)$.

EXAMPLE 14 *Find the domain of definition of the function $f(x)$ given by* $f(x)=\dfrac{1}{\log_{10}(1-x)}+\sqrt{x+2}$.

SOLUTION We have, $f(x) = \dfrac{1}{\log_{10}(1-x)}+\sqrt{x+2}$

Let $g(x) = \dfrac{1}{\log_{10}(1-x)}$ and $h(x) = \sqrt{x+2}$. Then, $f(x) = g(x) + h(x)$

$\therefore$ Domain (f) = Domain $(g) \cap$ Domain (h)

Now, $g(x) = \dfrac{1}{\log_{10}(1-x)}$ is defined for all x for which $\log_{10}(1-x)$ is defined and

$\log_{10}(1-x) \neq 0 \Rightarrow 1-x>0$ and $1-x \neq 1 \Rightarrow x<1$ and $x \neq 0 \Rightarrow x \in (-\infty, 0) \cup (0, 1)$

$\therefore$ Domain $(g) = (-\infty, 0) \cup (0, 1)$.

And, $h(x) = \sqrt{x+2}$ is defined for all x satisfying.

$x+2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, \infty)$.

$\therefore$ Domain $(h) = [-2, \infty)$.

Hence, Domain $(f) = (-\infty, 0) \cup (0, 1) \cap [-2, \infty) = [-2, 0) \cup (0, 1)$

EXAMPLE 15 *Find the range of each of the following functions:*

(i) $f(x) = |x-3|$ [NCERT EXEMPLAR]

(ii) $f(x) = 1-|x-2|$ [NCERT EXEMPAR]

(iii) $f(x) = \dfrac{|x-4|}{x-4}$ [NCERT EXEMPLAR]

SOLUTION (i) We have, $f(x) = |x-3|$

Clearly, $f(x)$ is defined for all $x \in R$. Therefore, Domain $(f) = R$.

$\because$ $|x-3| > 0$ for all $x \in R$

$\therefore$ $0 \leq |x-3| < \infty$ for all $x \in R$

$\Rightarrow$ $0 \leq f(x) < \infty$ for all $x \in R$

$\Rightarrow$ $f(x) \in [0, \infty)$ for all $x \in R$

Hence, Range $(f) = [0, \infty)$.

(ii) We have, $f(x) = 1-|x-2|$.

We observe that $f(x)$ is defined for all $x \in R$. Therefore, Domain $(f) = R$.

$\because$ $0 \leq |x-2| < \infty$ for all $x \in R$

$\therefore$ $-\infty < -|x-2| \leq 0$ for all $x \in R$

$\Rightarrow$ $-\infty < 1-|x-2| \leq 1$ for all $x \in R$

$\Rightarrow$ $-\infty < f(x) \leq 1$ for all $x \in R$

$\Rightarrow$ $f(x) \in (-\infty, 1]$

Hence, Range $(f) = (-\infty, 1]$

(iii) We have, $f(x) = \dfrac{|x-4|}{x-4}$

Clearly, $f(x)$ is defined for all $x \in R$ except at $x = 4$. Therefore, Domain $(f) = R - \{4\}$

Now,

$$f(x) = \frac{|x-4|}{x-4}$$

$$\Rightarrow \quad f(x) = \begin{cases} \dfrac{x-4}{x-4} = 1 \quad , & \text{if } x > 4 \\ \dfrac{-(x-4)}{x-4} = -1 \ , & \text{if } x < 4 \end{cases}$$

Hence, Range $(f) = \{-1, 1\}$

EXAMPLE 16 *Find the domain and range of each of the following functions given by*

(i) $f(x) = \dfrac{1}{\sqrt{x-[x]}}$ (ii) $f(x) = 1 - |x-3|$

SOLUTION (i) We have, $f(x) = \dfrac{1}{\sqrt{x-[x]}}$

Domain of f: We know that

$0 \le x - [x] < 1$ for all $x \in R$

And, $x - [x] = 0$ for $x \in Z$.

$\therefore$ $0 < x - [x] < 1$ for all $x \in R - Z$

$\Rightarrow$ $f(x) = \dfrac{1}{\sqrt{x-[x]}}$ exists for all $x \in R - Z$.

Hence, Domain $(f) = R - Z$.

Range of f: We have,

$0 < x - [x] < 1$ for all $x \in R - Z$

$\Rightarrow$ $0 < \sqrt{x-[x]} < 1$ for all $x \in R - Z$

$\Rightarrow$ $1 < \dfrac{1}{\sqrt{x-[x]}} < \infty$ for all $x \in R - Z$

$\Rightarrow$ $1 < f(x) < \infty$ for all $x \in R - Z$

$\Rightarrow$ Range$(f) = (1, \infty)$.

(ii) We have, $f(x) = 1 - |x-3|$

Clearly, $f(x)$ is defined for all $x \in R$. Therefore, Domain $(f) = R$.

Range of f: For any $x \in R$, we have

$|x-3| \ge 0 \Rightarrow -|x-3| \le 0 \Rightarrow 1 - |x-3| \le 1 \Rightarrow f(x) \le 1 \Rightarrow f(x) \in (-\infty, 1]$

Hence, Range$(f) = (-\infty, 1]$.

EXAMPLE 17 *Find the domain of the real function f (x) defined by* $f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$.

SOLUTION We have, $f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$

We observe that $f(x)$ is defined for all x satisfying $\dfrac{1-|x|}{2-|x|} \ge 0$.

Fig. 3.26 Signs of $\frac{1-|x|}{2-|x|}$

Now, $\dfrac{1-|x|}{2-|x|} \ge 0$

$\Rightarrow$ $\dfrac{|x|-1}{|x|-2} \ge 0$

$\Rightarrow$ $|x| \le 1$ or, $|x| > 2$ [$\because$ See Fig. 3.26]

$\Rightarrow$ $x \in [-1, 1]$ or, $x \in (-\infty, -2) \cup (2, \infty)$

$\Rightarrow$ $x \in (-\infty, -2) \cup (2, \infty) \cup [-1, 1]$

Hence, domain $(f) = (-\infty, -2) \cup (2, \infty) \cup [-1, 1]$.

EXAMPLE 18 *Find the domain of the function f given by* $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$ [NCERT EXEMPALR]

SOLUTION We have,

$$f(x) = \frac{1}{\sqrt{[x]^2 - [x] - 6}}$$

Clearly, $f(x)$ is defined for all x satisfying

$$[x]^2 - [x] - 6 > 0$$

$\Rightarrow \quad ([x] - 3)([x] + 2) > 0$

$\Rightarrow \quad [x] < -2$ or $[x] > 3$

$\Rightarrow \quad x \in (-\infty, -2)$ or $x \in [4, \infty)$

$\Rightarrow \quad x \in (-\infty, -2) \cup [4, \infty)$

Hence, domain $(f) = (-\infty, -2) \cup [4, \infty)$.

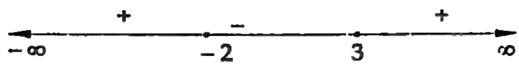


Fig. 3.27 Signs of $[x]^2 - [x] - 6$ for different values of x

EXERCISE 3.4

LEVEL-1

1. Find $f+g$, $f-g$, cf $(c \in R, c \neq 0)$, fg, $\frac{1}{f}$ and $\frac{f}{g}$ in each of the following:

   (i) $f(x) = x^3 + 1$ and $g(x) = x + 1$ (ii) $f(x) = \sqrt{x-1}$ and $g(x) = \sqrt{x+1}$.

2. Let $f(x) = 2x + 5$ and $g(x) = x^2 + x$. Describe (i) $f+g$ (ii) $f-g$ (iii) fg (iv) f/g. Find the domain in each case.

3. If $f(x)$ be defined on $[-2, 2]$ and is given by $f(x) = \begin{cases} -1 & , \ -2 \le x \le 0 \\ x-1 & , \ 0 < x \le 2 \end{cases}$

   and $g(x) = f(|x|) + |f(x)|$. Find $g(x)$.

4. Let f, g be two real functions defined by $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{9 - x^2}$. Then, describe each of the following functions:

   (i) $f+g$ (ii) $g-f$ (iii) $f\,g$ (iv) f/g

   (v) $\frac{g}{f}$ (vi) $2f - \sqrt{5}\,g$ (vii) $f^2 + 7f$ (viii) $\frac{5}{g}$

5. If $f(x) = \log_e(1-x)$ and $g(x) = [x]$, then determine each of the following functions:

   (i) $f+g$ (ii) fg (iii) $\frac{f}{g}$ (iv) $\frac{g}{f}$

   Also, find $(f+g)(-1)$, $(fg)(0)$, $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$, $\left(\frac{g}{f}\right)\left(\frac{1}{2}\right)$.

6. If f, g, h are real functions defined by $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x}$ and $h(x) = 2x^2 - 3$, then find the values of $(2f + g - h)(1)$ and $(2f + g - h)(0)$.

7. The function f is defined by $f(x) = \begin{cases} 1 - x, & x < 0 \\ 1 & , \ x = 0 \\ x + 1, & x > 0 \end{cases}$. Draw the graph of $f(x)$. [NCERT]

8. Let $f, g : R \to R$ be defined, respectively by $f(x) = x + 1$, $g(x) = 2x - 3$. Find $f+g$, $f-g$ and $\frac{f}{g}$. [NCERT]

9. Let $f : [0, \infty) \to R$ and $g : R \to R$ be defined by $f(x) = \sqrt{x}$ and $g(x) = x$. Find $f+g$, $f-g$, fg and $\frac{f}{g}$. [NCERT]

10. Let $f(x) = x^2$ and $g(x) = 2x + 1$ be two real functions. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$. [NCERT]

ANSWERS

1. (i) $f+g: R \to R$ given by $(f+g)(x) = x^3 + x + 2$

$f-g: R \to R$ given by $(f-g)(x) = x^3 - x$

$cf: R \to R$ given by $(cf)(x) = c(x^3+1)$

$fg: R \to R$ given by $(fg)(x) = (x+1)^2(x^2 - x + 1)$

$\frac{1}{f}: R - \{-1\} \to R$ given by $\left(\frac{1}{f}\right)(x) = \frac{1}{x^3+1}$

$\frac{f}{g}: R - \{-1\} \to R$ given by $\left(\frac{f}{g}\right)(x) = x^2 + x + 1$

(ii) $f \pm g: [1, \infty) \to R$ defined by $(f+g)(x) = \sqrt{x-1} \pm \sqrt{x+1}$

$cf: [1, \infty) \to R$ defined by $(cf)(x) = c\sqrt{x-1}$

$fg: [1, \infty) \to R$ defined by $(fg)(x) = \sqrt{x^2-1}$

$\frac{1}{f}: (1, \infty) \to R$ defined by $\left(\frac{1}{f}\right)(x) = \frac{1}{\sqrt{x-1}}$

$\frac{f}{g}: [1, \infty) \to R$ defined by $\left(\frac{f}{g}\right)(x) = \sqrt{\frac{x-1}{x+1}}$

2. (i) $(f+g)(x) = x^2 + 3x + 5$; dom $(f+g) = R$ (ii) $(f-g)(x) = 5 + x - x^2$; dom $(f-g) = R$

(iii) $(fg)(x) = 2x^3 + 7x^2 + 5x$; dom $(fg) = R$ (iv) $\left(\frac{f}{g}\right)(x) = \frac{2x+5}{x^2+x}$, dom $\left(\frac{f}{g}\right) = R - \{0, 1\}$

3. $g(x) = \begin{cases} -x, & -2 \le x \le 0 \\ 0, & 0 < x < 1 \\ 2(x-1), & 1 \le x \le 2 \end{cases}$

4. (i) $f+g: [-1, 3] \to R$ defined by $(f+g)(x) = \sqrt{x+1} + \sqrt{9-x^2}$

(ii) $g-f: [-1, 3] \to R$ defined by $(g-f)(x) = \sqrt{9-x^2} - \sqrt{x+1}$

(iii) $fg: [-1, 3] \to R$ defined by $(fg)(x) = \sqrt{9 + 9x - x^2 - x^3}$

(iv) $\frac{f}{g}: [-1, 3] \to R$ defined by $\left(\frac{f}{g}\right)(x) = \sqrt{\frac{x+1}{9-x^2}}$

(v) $\frac{g}{f}: (-1, 3] \to R$ defined by $\left(\frac{g}{f}\right)(x) = \sqrt{\frac{9-x^2}{x+1}}$

(vi) $2f - \sqrt{5}\,g: [-1, 3] \to R$ defined by $\left(2f - \sqrt{5}\,g\right)(x) = 2\sqrt{x+1} - \sqrt{45 - 5x^2}$

(vii) $f^2 + 7f: [-1, \infty) \to R$ defined by $(f^2 + 7f)(x) = x + 1 + 7\sqrt{x+1}$

(viii) $\frac{5}{g}: (-3, 3) \to R$ defined by $\left(\frac{5}{g}\right)(x) = \frac{5}{\sqrt{9-x^2}}$

5. (i) $f+g: (-\infty, 1) \to R$ defined by $(f+g)(x) = \log_e(1-x) + [x]$

(ii) $fg: (-\infty, 1) \to R$ defined by $(fg)(x) = [x]\log_e(1-x)$

(iii) $\frac{f}{g}: (-\infty, 0) \to R$ defined by $\left(\frac{f}{g}\right)(x) = \frac{\log_e(1-x)}{[x]}$

(iv) $\frac{g}{f}:(-\infty, 0)\cup(0,1)\to R$ defined by $\left(\frac{g}{f}\right)(x)=\frac{[x]}{\log_e(1-x)}$

$(f+g)(-1) = \log_e 2-1$ and, $(fg)(0) = 0$, $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$ does not exist $\left(\frac{g}{f}\right)\left(\frac{1}{2}\right)=0$

6. 0, does not exist.

8. $f+g:R\to R$ defined by $(f+g)(x)=3x-2$; $f-g:R\to R$ defined by $(f-g)(x)=-x+4$

$\frac{f}{g}:R-\left\{\frac{3}{2}\right\}\to R$ defined by $\left(\frac{f}{g}\right)(x)=\frac{x+1}{2x-3}$

9. $f+g:[0,\infty)\to R$ defined by $(f+g)(x)=\sqrt{x}+x$; $f-g:[0,\infty)\to$ defined by $(f-g)(x)=\sqrt{x}-x$

$fg:[0,\infty)\to R$ defined by $(fg)(x)=x^{3/2}$; $\frac{f}{g}:(0,\infty)\to R$ defined by $\left(\frac{f}{g}\right)(x)=\frac{1}{\sqrt{x}}$

10. $(f+g)(x)=(x+1)^2$, $(f-g)(x)=x^2-2x-1$, $(fg)(x)=2x^3+x^2$, $\left(\frac{f}{g}\right)(x)=\frac{x^2}{2x+1}$.

HINTS TO NCERT & SELECTED PROBLEMS

7. We have,

$$f(x) = \begin{cases} 1-x, & x<0 \\ 1, & x=0 \\ x+1, & x>0 \end{cases}$$

Let $f(x)=y$. Then,

$y = 1-x$ or $x+y = 1$ for $x<0$

$y = 1$ for $x = 1$

and $y = x+1$ or $-x+y = 1$ for $x>0$.

To draw the graph of $f(x)$, draw the line $x+y=1$ and take its that part for which $x<0$ i.e. take that portion of the line which lies on the left side of y-axis.

For $x>0$, draw the line $-x+y=1$ and take its that portion which lies on the right side of y-axis. For $x=0$, mark the point (0, 1).

So, the graph of $f(x)$ is as shown in Fig. 3.27.

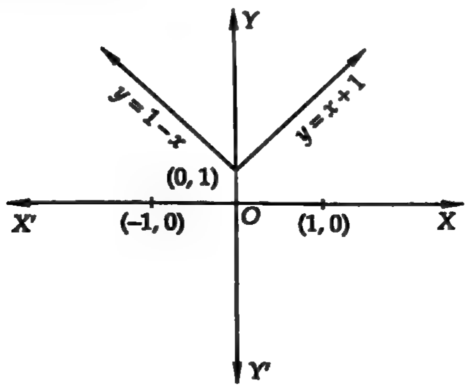


Fig. 3.27

8. $f:R\to R$ and $g:R\to R$ are given by $f(x)=x+1$ and $g(x)=2x-3$.

Clearly, $D(f)=R$ and $D(g)=R$. Therefore,

(i) $D(f+g)=D(f)\cap D(g)=R$ and, $(f+g)(x)=f(x)+g(x)=x+1+2x-3=3x-2$

(ii) $D(f-g)=D(f)\cap D(g)=R$ and, $(f-g)(x)=f(x)-g(x)=x+1-2x+3=-x+4$

(iii) $D(fg) = D(f) \cap D(g) = R$ and, $(fg)(x) = f(x)\,g(x) = (x+1)(2x-3) = 2x^2 - x - 3$

(iv) $D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = R - \left\{\frac{3}{2}\right\}$ and, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x+1}{2x-3}$

9. It is given that $f : [0, \infty) \to R$ and $g : R \to R$ are such that $f(x) = \sqrt{x}$ and $g(x) = x$.

$D(f+g) = [0, \infty) \cap R = [0, \infty)$

So, $f + g : [0, \infty) \to R$ is given by $(f+g)(x) = f(x) + g(x) = \sqrt{x} + x$

$D(f-g) = D(f) \cap D(g) = [0, \infty) \cap R = [0, \infty)$

So, $f - g : [0, \infty) \to R$ is given by $(f-g)(x) = f(x) - g(x) = \sqrt{x} - x$

$D(fg) = D(f) \cap D(g) = [0, \infty) \cap R = [0, \infty)$

So, $fg : [0, \infty) \to R$ is given by $(fg)(x) = f(x)\,g(x) = \sqrt{x}\ x = x^{3/2}$

$$D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = (0, \infty)$$

So, $\frac{f}{g} : (0, \infty) \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

10. We have, $f(x) = x^2$ and $g(x) = 2x + 1$. Clearly, $D(f) = R$ and $D(g) = R$.

$\therefore\quad D(f \pm g) = D(f) \cap D(g) = R \cap R = R$

$D(fg) = D(f) \cap D(g) = R \cap R = R$

$$D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = R \cap R - \left\{-\frac{1}{2}\right\} = R - \left\{-\frac{1}{2}\right\}$$

Thus,

$f + g : R \to R$ is given by $(f+g)(x) = f(x) + g(x) = x^2 + 2x + 1$

$f - g : R \to R$ is given by $(f-g)(x) = f(x) - g(x) = x^2 - 2x - 1$

$(fg) : R \to R$ is given by $(fg)(x) = f(x)\,g(x) = x(2x+1)$

$\left(\frac{f}{g}\right) : R - \left\{\frac{1}{2}\right\} \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{2x+1}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the range of the real function $f(x) = |x|$.
2. If f is a real function satisfying $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ for all $x \in R - \{0\}$, then write the expression for $f(x)$.
3. Write the range of the function $f(x) = \sin[x]$, where $\frac{-\pi}{4} \leq x \leq \frac{\pi}{4}$.
4. If $f(x) = \cos[\pi^2]\,x + \cos[-\pi^2]\,x$, where $[x]$ denotes the greatest integer less than or equal to x, then write the value of $f(\pi)$.
5. Write the range of the function $f(x) = \cos[x]$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
6. Write the range of the function $f(x) = e^{x - [x]}, x \in R$.
7. Let $f(x) = \frac{\alpha x}{x+1}, x \neq -1$. Then write the value of α satisfying $f(f(x)) = x$ for all $x \neq -1$.

8. If $f(x) = 1 - \frac{1}{x}$, then write the value of $f\left(f\left(\frac{1}{x}\right)\right)$.
9. Write the domain and range of the function $f(x) = \frac{x-2}{2-x}$.
10. If $f(x) = 4x - x^2, x \in R$, then write the value of $f(a+1) - f(a-1)$.
11. If f, g, h are real functions given by $f(x) = x^2$, $g(x) = \tan x$ and, $h(x) = \log_e x$, then write the value of (hogof) $\left(\sqrt{\frac{\pi}{4}}\right)$.
12. Write the domain and range of function $f(x)$ given by $f(x) = \frac{1}{\sqrt{x - |x|}}$.
13. Write the domain and range of $f(x) = \sqrt{x - [x]}$.
14. Write the domain and range of function $f(x)$ given by $f(x) = \sqrt{[x] - x}$.
15. Let A and B be two sets such that $n(A) = p$ and $n(B) = q$, write the number of functions from A to B.
16. Let f and g be two functions given by
$f = \{(2, 4), (5, 6), (8, -1), (10, -3)\}$ and $g = \{(2, 5), (7, 1), (8, 4), (10, 13), (11, -5)\}$.
Find the domain of $f + g$.
17. Find the set of values of x for which the functions $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal.
18. Let f and g be two real functions given by
$f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 1)\}$ and $g = \{(1, 0), (2, 2), (3, -1), (4, 4), (5, 3)\}$.
Find the domain of fg.

ANSWERS

1. $[0, \infty)$
2. $f(x) = x^2 - 2$, where $|x| > 2$
3. $\{-\sin 1, 0, \sin 1\}$
4. 0
5. $\{1, \cos 1, \cos 2\}$
6. $[1, e)$
7. $\alpha = -1$
8. $\frac{x}{x-1}$
9. $D(f) = R - \{2\}$, $R(f) = \{-1\}$
10. $4(2-a)$
11. 0
12. $D(f) = \phi = R(f)$
13. $D(f) = R$, $R(f) = [0, 1)$
14. $D(f) = \phi = R(f)$
15. q^p
16. $\{2, 8, 10\}$
17. $\{-1, 4/3\}$
18. $\{2, 3, 4, 5\}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. Let $A = \{1, 2, 3\}, B = \{2, 3, 4\}$, then which of the following is a function from A to B?
(a) $\{(1, 2), (1, 3), (2, 3), (3, 3)\}$
(b) $\{(1, 3), (2, 4)\}$
(c) $\{(1, 3), (2, 2), (3, 3)\}$
(d) $\{(1, 2), (2, 3), (3, 2), (3, 4)\}$.
2. If $f: Q \to Q$ is defined as $f(x) = x^2$, then $f^{-1}(9)$ is equal to
(a) 3
(b) -3
(c) $\{-3, 3\}$
(d) ϕ
3. Which one of the following is not a function?
(a) $\{(x, y): x, y \in R, x^2 = y\}$
(b) $\{(x, y): x, y \in R, y^2 = x\}$
(c) $\{(x, y): x, y \in R, x = y^3\}$
(d) $\{(x, y): x, y \in R, y = x^3\}$
4. If $f(x) = \cos(\log x)$, then $f(x^2) f(y^2) - \frac{1}{2}\left\{f\left(\frac{x^2}{y^2}\right) + f(x^2 y^2)\right\}$ has the value
(a) -2
(b) -1
(c) $1/2$
(d) none of these

5. If $f(x) = \cos(\log x)$, then $f(x)\,f(y) - \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(xy)\right\}$ has the value

(a) -1 (b) $1/2$ (c) -2 (d) none of these

6. Let $f(x) = |x-1|$. Then,

(a) $f(x^2) = [f(x)]^2$ (b) $f(x+y) = f(x)\,f(y)$

(c) $f(|x|) = |f(x)|$ (d) none of these

7. The range of $f(x) = \cos[x]$, for $-\pi/2 < x < \pi/2$ is

(a) $\{-1, 1, 0\}$ (b) $\{\cos 1, \cos 2, 1\}$ (c) $\{\cos 1, -\cos 1, 1\}$ (d) $[-1, 1]$

8. Which of the following are functions?

(a) $\{(x, y) : y^2 = x, x, y \in R\}$ (b) $\{(x, y) : y = |x|, x, y \in R\}$

(c) $\{(x, y) : x^2 + y^2 = 1, x, y \in R\}$ (d) $\{(x, y) : x^2 - y^2 = 1, x, y \in R\}$

9. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $g(x) = \frac{3x + x^3}{1 + 3x^2}$, then $f(g(x))$ is equal to

(a) $f(3x)$ (b) $\{f(x)\}^3$ (c) $3f(x)$ (d) $-f(x)$

10. If $A = \{1, 2, 3\}$, $B = \{x, y\}$, then the number of functions that can be defined from A into B is

(a) 12 (b) 8 (c) 6 (d) 3

11. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to

(a) $\{f(x)\}^2$ (b) $\{f(x)\}^3$ (c) $2f(x)$ (d) $3f(x)$

12. If $f(x) = \cos(\log x)$, then value of $f(x)\,f(4) - \frac{1}{2}\left\{f\left(\frac{x}{4}\right) + f(4x)\right\}$ is

(a) 1 (b) -1 (c) 0 (d) ± 1

13. If $f(x) = \frac{2^x + 2^{-x}}{2}$, then $f(x+y)\,f(x-y)$ is equals to

(a) $\frac{1}{2}\{f(2x) + f(2y)\}$ (b) $\frac{1}{2}\{f(2x) - f(2y)\}$

(c) $\frac{1}{4}\{f(2x) + f(2y)\}$ (d) $\frac{1}{4}\{f(2x) - f(2y)\}$

14. If $2f(x) - 3f\left(\frac{1}{x}\right) = x^2$ $(x \neq 0)$, then $f(2)$ is equal to

(a) $-\frac{7}{4}$ (b) $\frac{5}{2}$ (c) -1 (d) none of these

15. Let $f : R \to R$ be defined by $f(x) = 2x + |x|$. Then $f(2x) + f(-x) - f(x) =$

(a) $2x$ (b) $2|x|$ (c) $-2x$ (d) $-2|x|$

16. The range of the function $f(x) = \frac{x^2 - x}{x^2 + 2x}$ is

(a) R (b) $R - \{1\}$ (c) $R - \{-1/2, 1\}$ (d) none of these

17. If $x \neq 1$ and $f(x) = \frac{x+1}{x-1}$ is a real function, then $f(f(f(2)))$ is

(a) 1 (b) 2 (c) 3 (d) 4

18. If $f(x) = \cos(\log_e x)$, then $f\left(\frac{1}{x}\right) f\left(\frac{1}{y}\right) - \frac{1}{2}\left\{f(xy) + f\left(\frac{x}{y}\right)\right\}$ is equal to

(a) $\cos(x-y)$ (b) $\log(\cos(x-y))$ (c) 1 (d) $\cos(x+y)$

19. Let $f(x)=x,\ g(x)=\dfrac{1}{x}$ and $h(x)=f(x)\,g(x)$. Then, $h(x)=1$ for

(a) $x\in R$ (b) $x\in Q$ (c) $x\in R-Q$ (d) $x\in R, x\neq 0$

20. If $f(x)=\dfrac{\sin^4 x+\cos^2 x}{\sin^2 x+\cos^4 x}$ for $x\in R$, then $f(2002)=$

(a) 1 (b) 2 (c) 3 (d) 4

21. The function $f:R\to R$ is defined by $f(x)=\cos^2 x+\sin^4 x$. Then, $f(R)=$

(a) $[3/4,1)$ (b) $(3/4,1]$ (c) $[3/4,1]$ (d) $(3/4,1)$

22. Let $A=\{x\in R: x\neq 0, -4\leq x\leq 4\}$ and $f:A\to R$ be defined by $f(x)=\dfrac{|x|}{x}$ for $x\in A$. Then A is

(a) $\{1,-1\}$ (b) $\{x:0\leq x\leq 4\}$ (c) $\{1\}$ (d) $\{x:-4\leq x\leq 0\}$

23. If $f:R\to R$ and $g:R\to R$ are defined by $f(x)=2x+3$ and $g(x)=x^2+7$, then the values of x such that $g(f(x))=8$ are

(a) $1, 2$ (b) $-1, 2$ (c) $-1,-2$ (d) $1,-2$

24. If $f:[-2,2]\to R$ is defined by $f(x)=\begin{cases}-1, & \text{for } -2\leq x\leq 0\\ x-1, & \text{for } 0\leq x\leq 2\end{cases}$, then

$\{x\in[-2,2]: x\leq 0 \text{ and } f(|x|)=x\}=$

(a) $\{-1\}$ (b) $\{0\}$ (c) $\{-1/2\}$ (d) ϕ

25. If $e^{f(x)}=\dfrac{10+x}{10-x}$, $x\in(-10,10)$ and $f(x)=k\,f\left(\dfrac{200x}{100+x^2}\right)$, then $k=$

(a) 0.5 (b) 0.6 (c) 0.7 (d) 0.8

26. If f is a real valued function given by $f(x)=27x^3+\dfrac{1}{x^3}$ and α,β are roots of $3x+\dfrac{1}{x}=12$. Then,

(a) $f(\alpha)\neq f(\beta)$ (b) $f(\alpha)=10$ (c) $f(\beta)=-10$ (d) none of these

27. If $f(x)=64x^3+\dfrac{1}{x^3}$ and α,β are the roots of $4x+\dfrac{1}{x}=3$. Then,

(a) $f(\alpha)=f(\beta)=-9$ (b) $f(\alpha)=f(\beta)=63$
(c) $f(\alpha)\neq f(\beta)$ (d) none of these

28. If $3f(x)+5f\left(\dfrac{1}{x}\right)=\dfrac{1}{x}-3$ for all non-zero x, then $f(x)=$

(a) $\dfrac{1}{14}\left(\dfrac{3}{x}+5x-6\right)$ (b) $\dfrac{1}{14}\left(-\dfrac{3}{x}+5x-6\right)$

(c) $\dfrac{1}{14}\left(-\dfrac{3}{x}+5x+6\right)$ (d) none of these

29. If $f:R\to R$ be given by $f(x)=\dfrac{4^x}{4^x+2}$ for all $x\in R$. Then,

(a) $f(x)=f(1-x)$ (b) $f(x)+f(1-x)=0$
(c) $f(x)+f(1-x)=1$ (d) $f(x)+f(x-1)=1$

30. If $f(x)=\sin[\pi^2]x+\sin[-\pi^2]x$, where $[x]$ denotes the greatest integer less than or equal to x, then

(a) $f(\pi/2) = 1$ (b) $f(\pi) = 2$ (c) $f(\pi/4) = -1$ (d) none of these

31. The domain of the function $f(x) = \sqrt{2 - 2x - x^2}$ is
(a) $[-\sqrt{3}, \sqrt{3}]$ (b) $[-1-\sqrt{3}, -1+\sqrt{3}]$
(c) $[-2, 2]$ (d) $[-2-\sqrt{3}, -2+\sqrt{3}]$

32. The domain of definition of $f(x) = \sqrt{\dfrac{x+3}{(2-x)(x-5)}}$ is
(a) $(-\infty, -3] \cup (2, 5)$ (b) $(-\infty, -3) \cup (2, 5)$
(c) $(-\infty, -3] \cup [2, 5]$ (d) none of these

33. The domain of the function $f(x) = \sqrt{\dfrac{(x+1)(x-3)}{x-2}}$ is
(a) $[-1, 2) \cup [3, \infty)$ (b) $(-1, 2) \cup [3, \infty)$
(c) $[-1, 2] \cup [3, \infty)$ (d) none of these

34. The domain of definition of the function $f(x) = \sqrt{x-1} + \sqrt{3-x}$ is
(a) $[1, \infty)$ (b) $(-\infty, 3)$ (c) $(1, 3)$ (d) $[1, 3]$

35. The domain of definition of the function $f(x) = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{1+x}}$ is
(a) $(-\infty, -2] \cup [2, \infty)$ (b) $[-1, 1]$
(c) ϕ (d) none of these

36. The domain of definition of the function $f(x) = \log|x|$ is
(a) R (b) $(-\infty, 0)$ (c) $(0, \infty)$ (d) $R - \{0\}$

37. The domain of definition of $f(x) = \sqrt{4x - x^2}$ is
(a) $R - [0, 4]$ (b) $R - (0, 4)$ (c) $(0, 4)$ (d) $[0, 4]$

38. The domain of definition of $f(x) = \sqrt{x - 3 - 2\sqrt{x-4}} - \sqrt{x - 3 + 2\sqrt{x-4}}$ is
(a) $[4, \infty)$ (b) $(-\infty, 4]$ (c) $(4, \infty)$ (d) $(-\infty, 4)$

39. The domain of the function $f(x) = \sqrt{5|x| - x^2 - 6}$ is
(a) $(-3, -2) \cup (2, 3)$ (b) $[-3, -2) \cup [2, 3)$
(c) $[-3, -2] \cup [2, 3]$ (d) none of these

40. The range of the function $f(x) = \dfrac{x}{|x|}$ is
(a) $R - \{0\}$ (b) $R - \{-1, 1\}$ (c) $\{-1, 1\}$ (d) none of these

41. The range of the function $f(x) = \dfrac{x+2}{|x+2|}, x \neq -2$ is
(a) $\{-1, 1\}$ (b) $\{-1, 0, 1\}$ (c) $\{1\}$ (d) $(0, \infty)$

42. The range of the function $f(x) = |x-1|$ is
(a) $(-\infty, 0)$ (b) $[0, \infty)$ (c) $(0, \infty)$ (d) R

43. Let $f(x) = \sqrt{x^2 + 1}$. Then, which of the following is correct?
(a) $f(xy) = f(x)f(y)$ (b) $f(xy) \geq f(x)f(y)$ (c) $f(xy) \leq f(x)f(y)$ (d) none of these

44. If $[x]^2 - 5[x] + 6 = 0$, where $[\cdot]$ denotes the greatest integer function, then
(a) $x \in [3, 4]$ (b) $x \in (2, 3]$ (c) $x \in [2, 3]$ (d) $x \in [2, 4)$

45. The range of $f(x) = \dfrac{1}{1 - 2\cos x}$ is
(a) $[1/3, 1]$ (b) $[-1, 1/3]$
(c) $(-\infty, -1) \cup [1/3, \infty)$ (d) $[-1/3, 1]$

ANSWERS

1. (c)	2. (c)	3. (b)	4. (d)	5. (d)	6. (d)	7. (b)	8. (b)
9. (c)	10. (b)	11. (c)	12. (c)	13. (a)	14. (a)	15. (b)	16. (c)
17. (c)	18. (d)	19. (d)	20. (a)	21. (c)	22. (a)	23. (c)	24. (c)
25. (a)	26. (c)	27. (a)	28. (b)	29. (c)	30. (a)	31. (b)	32. (a)
33. (a)	34. (d)	35. (c)	36. (d)	37. (d)	38. (a)	39. (c)	40. (c)
41. (a)	42. (b)	43. (c)	44. (b)	45. (b)			

SUMMARY

1. Let A and B be two non-empty sets. Then a relation f from A to B is a function, if
   (i) for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$
   (ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$.
   In other words, f is a function from A to B if each element of A appears in some ordered pair in f and no two ordered pairs in f have the same first element.
   If $(a, b) \in f$, then b is called the image of a under f.
2. A function f from a set A to a set B is a rule associating elements of set A to elements of set B such that every element in set A is associated to a unique elments in set B.
   The set A is called the domain of f and the set B is called its co-domain.
3. The range of a function f is the set of images of elements in the domain.
4. A real function has the domain and co-domain both as subsets of set R.
5. If $f : D_1 \to R$ and $g : D_2 \to R$ are two real functions and $c \in R$, then
   (i) $f \pm g : D_1 \cap D_2 \to R$ is defined as $(f \pm g)(x) = f(x) \pm g(x)$
   (ii) $fg : D_1 \cap D_2 \to R$ is defined as $(fg)(x) = f(x)\, g(x)$
   (iii) $\frac{f}{g} : D_1 \cap D_2 - \{x : g(x) = 0\} \to R$ is defined as $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
   (iv) $cf : D_1 \cap D_2 \to R$ is defined as $(cf)(x) = c\, f(x)$.

CHAPTER 4

MEASUREMENT OF ANGLES

4.1 INTRODUCTION

The word *'Trigonometry'* is derived from two Greek words : (i) trigonon and, (ii) metron. The word trigonon means a triangle and the word metron means a measure. Hence, trigonometry means the science of measuring triangles. In broader sense it is that branch of Mathematics which deals with the measurement of the sides and the angles of a triangle and the problems *allied* with angles.

4.2 ANGLES

ANGLE *Consider a ray $\overrightarrow{OA}$. If this ray rotates about its end point O and takes the position OB, then we say that the angle $\angle AOB$ has been generated.*

Thus, an angle is considered as the figure obtained by rotating a given ray about its end-point.

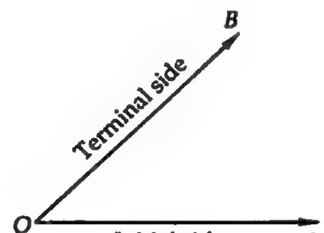


Fig. 4.1

The revolving ray is called the generating line of the angle. The initial position OA is called the *initial side* and the final position OB is called *terminal side* of the angle. The end point O about which the ray rotates is called the vertex of the angle.

MEASURE OF AN ANGLE *The measure of an angle is the amount of rotation from the initial side to the terminal side.*

SENSE OF AN ANGLE *The sense of an angle is determined by the direction of rotation of the initial side into the terminal side. The sense of an angle is said to be positive or negative according as the initial side rotates in anticlockwise or clockwise direction to get to the terminal side.*

RIGHT ANGLE *If the revolving ray starting from its initial position to final position describes one quarter of a circle, then we say that the measure of the angle formed is a right angle.*

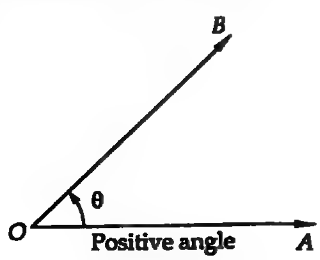


Fig. 4.2

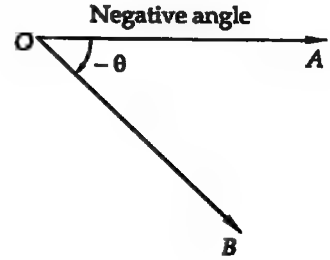


Fig. 4.3

4.3 SOME USEFUL TERMS

QUADRANTS *Let $X'OX$ and YOY' be two lines at right angles in the plane of the paper. These lines divide the plane of the paper into four equal parts which are known as quadrants. The lines $X'OX$ and YOY' are known as x-axis and y-axis respectively. These two lines taken together are known as the coordinate axes. The regions XOY, YOX', $X'OY'$ and $Y'OX$ are known as the first, the second, the third and the fourth quadrant respectively.*

ANGLE IN STANDARD POSITION *An angle is said to be in standard position if its vertex coincides with the origin O and the initial side coincides with OX i.e. the positive direction of x-axis.*

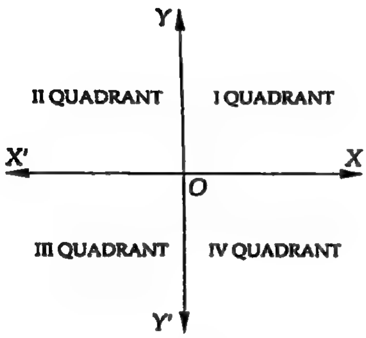


Fig. 4.4

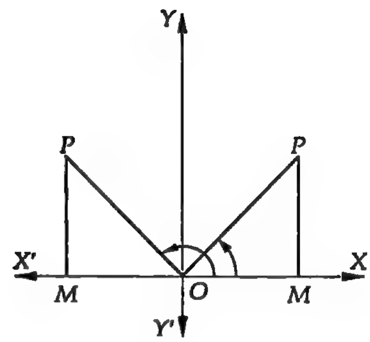


Fig. 4.5

ANGLE IN A QUADRANT *An angle in standard position is said to be in a particular quadrant, if the terminal side of the angle in standard position lies in that quadrant.*

QUADRANT ANGLE *An angle in standard position is said to be a quadrant angle, if the terminal side coincides with one of the axes.*

TRIANGLE OF REFERENCE *If from any point P on the terminal side of an angle in standard position a perpendicular PM is drawn on x-axis, then the right angled triangle OMP, thus formed, is called the triangle of reference of the $\angle XOP$. (See Fig. 4.5)*

CO-TERMINAL ANGLES *Two angles with different measures but having the same initial sides and the same terminal sides are known as co-terminal angles.*

4.4 SYSTEMS OF MEASUREMENT OF ANGLES

There are three systems for measuring angles, viz. (i) Sexagesimal or English system, (ii) Centesimal or French system, (iii) Circular system.

SEXAGESIMAL SYSTEM In this system a right angle is divided into 90 equal parts, called degrees. The symbol 1° is used to denote one degree. Thus, one degree is one-ninetieth part of a right angle. Each degree is divided into 60 equal parts, called minutes. The symbol $1'$ is used to denote one minute. And each minute is divided into 60 equal parts, called seconds. The symbol $1''$ is used to denote one second.

Thus, 1 right angle = 90 degrees (90°)

$$1^\circ = 60 \text{ minutes } (= 60')$$

$$1' = 60 \text{ seconds } (= 60'')$$

CENTESIMAL SYSTEM In this system a right angle is divided into 100 equal parts, called grades; each grade is subdivided into 100 minutes, and each minute into 100 seconds.

The symbols 1^g, $1'$ and $1''$ are used to denote a grade, a minute, and a second respectively.

Thus, 1 right angle = 100 grades ($= 100^g$)

$$1 \text{ grade} = 100 \text{ minutes } (=100')$$
$$1 \text{ minute} = 100 \text{ seconds } (=100'')$$

CIRCULAR SYSTEM In this system the unit of measurement is radian as defined below.

RADIAN *One radian, written as 1^c, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.*

Consider a circle of radius r having centre at O. Let A be a point on the circle. Now, cut off an arc AP whose length is equal to the radius r of the circle. Then by the definition the measure of $\angle AOP$ is 1 radian $(=1^c)$.

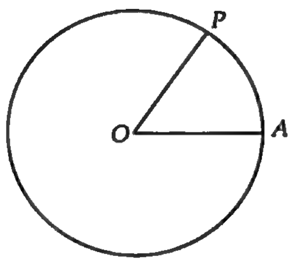


Fig. 4.6

Since a radian is chosen as the unit of measurement of an angle, therefore it should be a constant quantity. This we shall show in the following two theorems.

THEOREM 1 *Radian is a constant angle.*

PROOF Consider a circle with centre O and radius r. Take a point A on the circle and cut off an arc AP whose length is equal to the radius r. Join OA and OP. Then, by definition $\angle AOP = 1^c$.

Produce AO to meet the circle at B so that

$\angle AOB$ = a straight angle = 2 right angles.

Since the angles at the centre of a circle are proportional to the arcs subtending them.

$$\therefore \quad \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } APB}$$

$$\Rightarrow \quad \frac{\angle AOP}{\angle AOB} = \frac{r}{\pi r} \qquad \left[\because \text{ arc } APB = \frac{1}{2}(2\pi r) = \pi r\right]$$

$$\Rightarrow \quad \frac{\angle AOP}{\angle AOB} = \frac{1}{\pi}$$

$$\Rightarrow \quad \angle AOP = \frac{1}{\pi}\angle AOB = \frac{\text{a straight angle}}{\pi}$$

$$\Rightarrow \quad 1^c = \frac{\text{a straight angle}}{\pi} \qquad [\because \angle AOP = 1^c]$$

$$\Rightarrow \quad 1^c = \text{constant} \qquad [\because \text{ A straight angle and } \pi \text{ both are constants}]$$

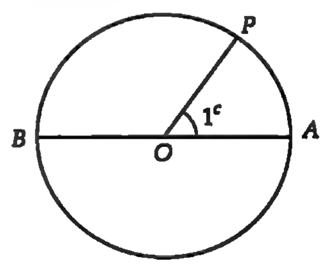


Fig. 4.7

Hence, radian is a constant angle **Q.E.D.**

THEOREM 2 *The number of radians in an angle subtended by an arc of a circle at the centre is equal to* $\dfrac{arc}{radius}$.

PROOF Consider a circle with centre O and radius r. Let $\angle AOQ = \theta^c$ and let arc $AQ = s$. Let P be a point on the arc AQ such that arc $AP = r$. Then, $\angle AOP = 1^c$.

Since angles at the centre of a circle are proportional to the arcs subtending them.

$$\therefore \quad \frac{\angle AOQ}{\angle AOP} = \frac{\text{arc } AQ}{\text{arc } AP}$$

$$\Rightarrow \quad AOQ = \left(\frac{\text{arc } AQ}{\text{arc } AP} \times 1\right)^c \qquad [\because \angle AOP = 1^c]$$

$$\Rightarrow \quad \theta = \frac{s}{r} \text{ radians}.$$

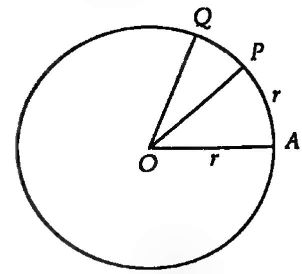


Fig. 4.8

Q.E.D.

4.5 RELATION BETWEEN DEGREES AND RADIANS

Consider a circle with centre O and radius r. Let A be a point on the circle. Join OA and cut off an arc OP of length equal to the radius of the circle. Then, $\angle AOP = 1$ radian. Produce AO to meet the circle at B.

$\therefore \quad \angle AOB$ = a straight angle = 2 right angles

We know that the angles at the centre of a circle are proportional to the arcs subtending them.

$$\therefore \quad \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } APB}$$

$$\Rightarrow \quad \frac{\angle AOP}{2 \text{ right angles}} = \frac{r}{\pi r} \qquad \left[\because \text{ arc } APB = \frac{1}{2}(\text{Circumference})\right]$$

$$\Rightarrow \quad 2 = \frac{\text{rightangles}}{\pi}$$

$$\Rightarrow \quad 1^c = \frac{180^\circ}{\pi}$$

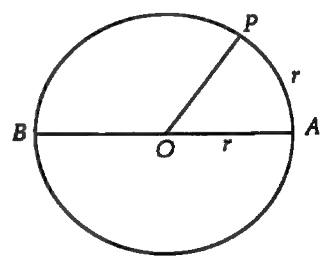


Fig. 4.9

Hence, One radian $= \dfrac{180^\circ}{\pi} \Rightarrow \pi$ radians $= 180^\circ$.

<u>REMARK 1</u> *When an angle is expressed in radians, the word radian is generally omitted.*

<u>REMARK 2</u> *Since* $180^\circ = \pi$ *radians. Therefore,* $1^\circ = \dfrac{\pi}{180}$ *radian.*

Hence, $30^\circ = \dfrac{\pi}{180} \times 30 = \dfrac{\pi}{6}$ *radians,* $45^\circ = \dfrac{\pi}{180} \times 45 = \dfrac{\pi}{4}$ *radians,*

$60^\circ = \dfrac{\pi}{180} \times 60 = \dfrac{\pi}{3}$ *radians,* $90^\circ = \dfrac{\pi}{180} \times 90 = \dfrac{\pi}{2}$ *radians etc.*

<u>REMARK 3</u> *We have,* π *radians* $= 180^\circ$

$$\therefore \quad 1 \text{ radian} = \frac{180^\circ}{\pi} = \left(\frac{180}{22} \times 7\right)^\circ = 57^\circ 16' 22'' \textit{ (approx).}$$

<u>REMARK 4</u> *We have,* $180^\circ = \pi$ radians

$$\therefore \quad 1^\circ = \frac{\pi}{180} \text{ radian} = \left(\frac{22}{7 \times 180}\right) \text{radian} = 0.01746 \text{ radian}.$$

4.6 RELATION BETWEEN THREE SYSTEMS OF MEASUREMENT OF AN ANGLE

Let D be the number of degrees, R be the number of radians and G be the number of grades in an angle θ.

$$\because \quad 90^\circ = 1 \text{ right angle}$$

$$\Rightarrow \quad 1^\circ = \frac{1}{90} \text{ right angle}$$

$\Rightarrow \quad D^\circ = \dfrac{D}{90}$ right angles

$\Rightarrow \quad \theta = \dfrac{D}{90}$ right angles ... (ii)

Also, π radians = 2 right angles

$\Rightarrow \quad 1$ radian $= \dfrac{2}{\pi}$ right angles

$\Rightarrow \quad R$ radians $= \dfrac{2R}{\pi}$ right angles

$\Rightarrow \quad \theta = \dfrac{2R}{\pi}$ right angles ... (ii)

And, 100 grades = 1 right angle

$\Rightarrow \quad 1$ grade $= \dfrac{1}{100}$ right angle

$\Rightarrow \quad$ G grades $= \dfrac{G}{100}$ right angles

$\Rightarrow \quad \theta = \dfrac{G}{100}$ right angles ... (iii)

From (i), (ii) and (iii), we get

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

This is the required relation between the three systems of measurement of an angle.

SOME USEFUL POINTS

(i) The angle between two consecutive digits in a clock is $30^\circ (= \pi/6$ radians).
(ii) The hour hand rotates through an angle of 30° in one hour i.e. $(1/2)^\circ$ in one minute.
(iii) The minute hand rotates through an angle of 6° in one minute.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the degree measure corresponding to the following radian measures:*

(i) $\left(\dfrac{2\pi}{15}\right)^c$ (ii) $\left(\dfrac{\pi}{8}\right)^c$ (iii) $\left(\dfrac{1}{4}\right)^c$ (iv) -2^c (v) 6^c (vi) $\left(\dfrac{11}{16}\right)^c$

SOLUTION We have, π radians $= 180^\circ$

$\therefore \quad 1^c = \left(\dfrac{180}{\pi}\right)^\circ$

(i) $\left(\dfrac{2\pi}{15}\right)^c = \left(\dfrac{2\pi}{15} \times \dfrac{180}{\pi}\right)^\circ = 24^\circ$

(ii) $\left(\dfrac{\pi}{8}\right)^c = \left(\dfrac{\pi}{8} \times \dfrac{180}{\pi}\right)^\circ = \left(\dfrac{45}{2}\right)^\circ = \left(22\dfrac{1}{2}\right)^\circ = 22^\circ\left(\dfrac{1}{2} \times 60\right)' = 22^\circ\, 30'$

(iii) $\left(\dfrac{1}{4}\right)^c = \left(\dfrac{1}{4} \times \dfrac{180}{\pi}\right)^\circ = \left(\dfrac{1}{4} \times \dfrac{180}{22} \times 7\right)^\circ = \left(\dfrac{315}{22}\right)^\circ = \left(14\dfrac{7}{22}\right)^\circ$

$= 14^\circ\left(\dfrac{7}{22} \times 60\right)' = 14^\circ\left(19\dfrac{1}{11}\right)' = 14^\circ\, 19'\left(\dfrac{1}{11} \times 60\right)'' = 14^\circ 19'\, 5''$

(iv) $(-2)^c = \left(\frac{180}{\pi} \times -2\right)^\circ = \left(\frac{180}{22} \times 7 \times (-2)\right)^\circ = \left(-114\frac{6}{11}\right)^\circ = \left\{-114^\circ\left(\frac{6}{11} \times 60\right)'\right\}$

$= -\left[114^\circ\left(32\frac{8}{11}\right)'\right] = -\left[114^\circ\ 32'\left(\frac{8}{11} \times 60\right)''\right] = -(114^\circ\ 32'\ 44'')$

(v) $6^c = \left(\frac{180}{\pi} \times 6\right)^\circ = \left(\frac{180}{22} \times 7 \times 6\right)^\circ = \left(\frac{90 \times 7 \times 6}{11}\right)^\circ = \left(\frac{3780}{11}\right)^\circ = \left(343\frac{7}{11}\right)^\circ$

$= 343^\circ\left(\frac{7}{11} \times 60\right)' = 343^\circ\left(\frac{420}{11}\right)' = 343'\ 38'\left(\frac{2}{11} \times 60\right)'' = 343^\circ\ 38'\ 11''$

(vi) $\left(\frac{11}{16}\right)^c = \left(\frac{180}{\pi} \times \frac{11}{16}\right)^\circ = \left(\frac{180}{22} \times 7 \times \frac{11}{16}\right)^\circ = \left(\frac{315}{8}\right)^\circ = \left(39\frac{3}{8}\right)^\circ$

$= 39^\circ\left(\frac{3}{8} \times 60\right)' = 39^\circ\ 22'\left(\frac{1}{2} \times 60\right)'' = 39^\circ\ 22'\ 30''$

EXAMPLE 2 *Find the radian measures corresponding to the following degree measures :*
(i) 340° (ii) 75° (iii) $-37^\circ\ 30'$ (iv) $5^\circ\ 37'\ 30''$ (v) $40^\circ\ 20'$ (vi) 520°

SOLUTION We have,

$180^\circ = \pi^c$. Therefore, $1^\circ = \left(\frac{\pi}{180}\right)^c$

(i) $340^\circ = \left(340 \times \frac{\pi}{180}\right)^c = \left(\frac{17\pi}{9}\right)^c$

(ii) $75^\circ = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$

(iii) Clearly, $30' = \left(\frac{30}{60}\right)^\circ = \frac{1}{2}^\circ$

$\therefore$ $-37^\circ\ 30' = -\left(37\frac{1}{2}\right)^\circ = -\left(\frac{75}{2}\right)^\circ = -\left(\frac{75}{2} \times \frac{\pi}{180}\right)^c = -\left(\frac{5\pi}{24}\right)^c$

(iv) Clearly, $30'' = \left(\frac{30}{60}\right)' = \left(\frac{1}{2}\right)'$

$\therefore$ $37'\ 30'' = \left(37\frac{1}{2}\right)' = \left(\frac{75}{2}\right)' = \left(\frac{75}{2} \times \frac{1}{60}\right)^\circ = \left(\frac{5}{8}\right)^\circ$

So, $5^\circ\ 37'\ 30'' = \left(5\frac{5}{8}\right)^\circ = \left(\frac{45}{8}\right)^\circ = \left(\frac{45}{8} \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{32}\right)^c$

(v) Clearly, $20' = \left(\frac{20}{60}\right)^\circ = \frac{1}{3}^\circ$

$\therefore$ $40^\circ\ 20' = \left(40\frac{1}{3}\right)^\circ = \left(\frac{121}{3}\right)^\circ = \left(\frac{121}{3} \times \frac{\pi}{180}\right)^c = \left(\frac{121\pi}{540}\right)^c$

(vi) Clearly, $520^\circ = \left(520 \times \frac{\pi}{180}\right)^c = \left(\frac{26\pi}{9}\right)^c$

EXAMPLE 3 *Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring* 15°.

SOLUTION Let s be the length of the arc subtending an angle θ^c at the centre of a circle of radius r.

Then, $\theta = \frac{s}{r}$. Here, $r = 5$ cm and $\theta = 15° = \left(15 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{12}\right)^c$

$$\therefore \quad \theta = \frac{s}{r} \Rightarrow \frac{\pi}{12} = \frac{s}{5} \Rightarrow s = \frac{5\pi}{12} \text{ cm.}$$

EXAMPLE 4 *Find in degrees the angle subtended at the centre of a circle of diameter 50 cm by an arc of length 11 cm.*

SOLUTION Here, $r = 25$ cm and $s = 11$ cm

$$\therefore \quad \theta = \left(\frac{s}{r}\right)^c$$

$$\Rightarrow \quad \theta = \left(\frac{11}{25}\right)^c = \left(\frac{11}{25} \times \frac{180}{\pi}\right)^\circ = \left(\frac{11}{25} \times \frac{180}{22} \times 7\right)^\circ = \left(\frac{126}{5}\right)^\circ = \left(25\frac{1}{5}\right)^\circ = 25°\left(\frac{1}{5} \times 60\right)' = 25°\,12'$$

EXAMPLE 5 *In a circle of diameter 40 cm the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.*

SOLUTION Let arc $AB = s$. It is given that $OA = 20$ cm and chord $AB = 20$ cm. Therefore, $\Delta\, OAB$ is an equilateral triangle. Hence,

$$\angle AOB = 60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$$

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$$

$$\Rightarrow \quad \frac{\pi}{3} = \frac{s}{20} \Rightarrow s = \frac{20\pi}{3} \text{ cm.}$$

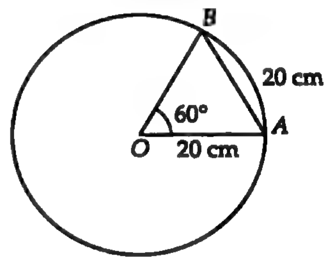


Fig. 4.10

EXAMPLE 6 *The angles of a triangle are in A.P. The number of degrees in the least is to the number of radians in the greatest as* $60 : \pi$. *Find the angles in degrees.*

SOLUTION Let the angles of the triangle be $(a-d)°$, $a°$ and $(a+d)°$ Then,

$$(a-d) + a + (a+d) = 180° \Rightarrow 3a = 180° \Rightarrow a = 60°$$

So, the angles are $(60-d)°$, $60°$, $(60+d)°$.

Clearly, $(60-d)°$ is the least angle and $(60+d)°$ is the greatest angle.

Now, Greatest angle $= (60+d)° = \left\{(60+d)\frac{\pi}{180}\right\}^c$

It is given that:

$$\frac{\text{Number of degrees in the least angle}}{\text{Number of radians in the greatest angle}} = \frac{60}{\pi}$$

$$\Rightarrow \quad \frac{(60-d)}{(60+d)\frac{\pi}{180}} = \frac{60}{\pi} \Rightarrow 3(60-d) = (60+d) \Rightarrow 120 = 4d \Rightarrow d = 30.$$

Hence, measures of the angles are $(60-30)°$, $60°$, $(60+30)°$ i.e. $30°$, $60°$, $90°$.

EXAMPLE 7 *The angles of a triangle are in A.P. The number of grades in the least, is to the number of radians in the greatest as* $40 : \pi$. *Find the angles in degrees.*

SOLUTION Let measures of the angles of the triangle in degrees be $(a-d)°$, $a°$ and $(a+d)°$. Then,

$$(a-d) + a + (a+d) = 180 \Rightarrow 3a = 180 \Rightarrow a = 60$$

So, measures of the angles are $(60-d)°$, $60°$ and $(60+d)°$.

Clearly, measure of the least angle is $(60° - d)°$ and that of the greatest angle is $(60 + d)°$.
Now,

$$\text{Measure of the least angle} = (60-d)° = \left\{(60-d)\times\frac{100}{90}\right\}^g = \left\{(60-d)\times\frac{10}{9}\right\}^g \qquad [\because 90° = 100^g]$$

$$\text{Measure of the greatest angle} = (60+d)° = \left\{(60+d)\times\frac{\pi}{180}\right\}^c$$

It is given that:

$$\frac{\text{Number of grades in the least angle}}{\text{Number of radians in the greatest angle}} = \frac{40}{\pi}$$

$$\Rightarrow \quad \frac{(60-d)\times\frac{10}{9}}{(60+d)\frac{\pi}{180}} = \frac{40}{\pi}$$

$$\Rightarrow \quad \frac{600-10d}{9}\times\frac{180}{(60+d)\pi} = \frac{40}{\pi}$$

$$\Rightarrow \quad 600-10d = 120+2d \Rightarrow 12d = 480 \Rightarrow d = 40.$$

Hence, measures the angles of the triangle are 20°, 60° and 100°.

EXAMPLE 8 *Express the angular measurement of the angle of a regular decagon in degrees, grades and radians.*

SOLUTION We know that the angle of an n sided regular polygon is equal to $\left(\frac{2n-4}{n}\right)$ right angles.

Let θ be the angle of a regular decagon. Then,

$$\theta = \left(\frac{2\times 10-4}{10}\right) = \frac{8}{5} \text{ right angles} = \left(\frac{8}{5}\times 90\right)° = 144° \qquad [\because 1 \text{ right angle} = 90°]$$

$$\text{Again, } \theta = \frac{8}{5} \text{ right angles} = \left(\frac{8}{5}\times 100\right)^g = 160^g \qquad [\because 1 \text{ right angle} = 100^g]$$

$$\text{And, } \theta = \frac{8}{5} \text{ right angles} = \left(\frac{8}{5}\times\frac{\pi}{2}\right)^c = \left(\frac{4\pi}{5}\right)^c \qquad \left[\because 1 \text{ right angle} = \frac{\pi^c}{2}\right]$$

EXAMPLE 9 *If the arcs of same length in two circles subtend angles of 60° and 75° at their centres. Find the ratio of their radii.*

SOLUTION Let r_1 and r_2 be the radii of the given circles and let their arcs of same length s subtend angles of 60° and 75° at their centres.

$$\text{Now, } 60° = \left(60\times\frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c, \text{ and } 75° = \left(75\times\frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$$

$$\therefore \quad \frac{\pi}{3} = \frac{s}{r_1}, \text{ and } \frac{5\pi}{12} = \frac{s}{r_2} \qquad \left[\because \theta = \left(\frac{s}{r}\right)^c\right]$$

$$\Rightarrow \quad \frac{\pi}{3}r_1 = s, \text{ and } \frac{5\pi}{12}r_2 = s$$

$$\Rightarrow \quad \frac{\pi}{3}r_1 = \frac{5\pi}{12}r_2 \Rightarrow 4r_1 = 5r_2 \Rightarrow r_1 : r_2 = 5:4$$

Hence, $r_1 : r_2 = 5:4$

EXAMPLE 10 *Find in degrees the angle through which a pendulum swings if its length is 50 cm and the tip describes an arc of length 10 cm.*

SOLUTION Here, $r = 50$ cm and $s = 10$ cm.

$\therefore \quad \theta = \left(\frac{s}{r}\right)^c$

$\Rightarrow \quad \theta = \left(\frac{10}{50}\right)^c = \left(\frac{1}{5}\right)^c = \left(\frac{1}{5} \times \frac{180}{\pi}\right)^\circ = \left(\frac{36}{22} \times 7\right)^\circ = \left(\frac{126}{11}\right)^\circ = \left(11\frac{5}{11}\right)^\circ = 11^\circ\left(\frac{5}{11} \times 60\right)' = 11^\circ 27' 16''$

EXAMPLE 11 *A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 metres when it has traced out 72° at the centre, find the length of the rope.*

SOLUTION Let the post be at point P and let PA be the length of the rope in tight position. Suppose the horse moves along the arc AB so that $\angle APB = 72^\circ$ and arc $AB = 88$ m. Let r be the length of the rope i.e. $PA = r$ metres.

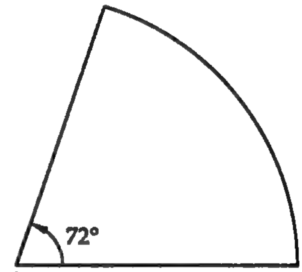


Fig. 4.11

Here, $\theta = 72^\circ = \left(72 \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c$ and $s = 88$ m

$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$

$\Rightarrow \quad \frac{2\pi}{5} = \frac{88}{r} \Rightarrow r = 88 \times \frac{5}{2\pi} = 70$ metres.

EXAMPLE 12 *A circular wire of radius 7.5 cm is cut and bent so as to lie along the circumference of a hoop whose radius is 120 cm. Find in degrees the angle which is subtended at the centre of the hoop.*

SOLUTION It is given that the radius of the circular wire is 7.5 cm.

$\therefore$ Length of the circular wire $= 2\pi \times 7.5 = 15\pi$ cm [$\because$ Circumference $= 2\pi r$]

Radius of the hoop = 120 cm.

Let θ be the angle subtended by the wire at the centre of the hoop. Then,

$$\theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \theta = \left(\frac{15\pi}{120}\right)^c = \left(\frac{\pi}{8}\right)^c = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^\circ = 22^\circ 30'$$

EXAMPLE 13 *The moon's distance from the earth is 360,000 kms and its diameter subtends an angle of 31' at the eye of the observer. Find the diameter of the moon.*

SOLUTION Let AB be the diameter of the moon and let E be the eye of the observer. Since the distance between the earth and the moon is quite large, so we take diameter AB as arc AB. Let d be the diameter of the moon. Then, d = arc AB.

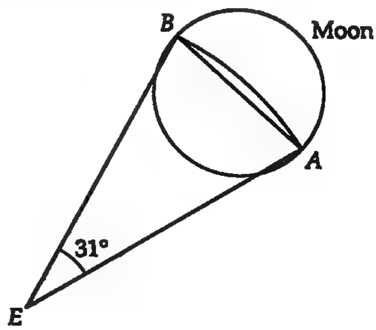


Fig. 4.12

We have,

$$\theta = 31' = \left(\frac{31}{60}\right)^\circ = \left(\frac{31}{60} \times \frac{\pi}{180}\right)^c, \text{ and } r = 360000 \text{ kms}$$

$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$

$\Rightarrow \quad \frac{31}{60} \times \frac{\pi}{180} = \frac{d}{360000}$

$\Rightarrow \quad d = \left(\frac{31}{60} \times \frac{\pi}{180} \times 360000\right)$ km

$\Rightarrow \quad d = 3247.62$ kms

Hence, the diameter of the moon is 3247.62 km.

EXAMPLE 14 *If the angular diameter of the moon be 30′, how far from the eye a coin of diameter 2.2 cm be kept to hide the moon ?*

SOLUTION Suppose the coin is kept at a distance r from the eye to hide the moon completely. Let E be the eye of the observer and let AB be the diameter of the coin.

Then, arc AB = diameter AB = 2.2 cm.

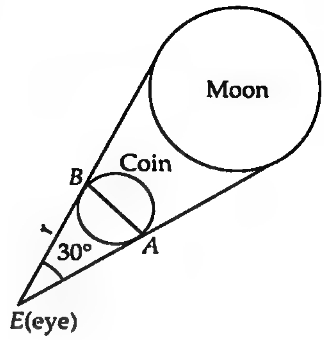


Fig. 4.13

We have, $\theta = 30' = \left(\frac{30}{60}\right)^\circ = \left(\frac{1}{2}\times\frac{\pi}{180}\right)^c = \left(\frac{\pi}{360}\right)^c$

$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$

$\Rightarrow \quad \frac{\pi}{360} = \frac{2.2}{r}$

$\Rightarrow \quad r = \frac{2.2\times 360}{\pi}\text{ cm} \Rightarrow r = \frac{2.2\times 360\times 7}{22} = 252\text{ cm.}$

EXAMPLE 15 *Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find what is the height of the letters that he can read at a distance of 12 metres.*

SOLUTION Let h be the required height in metres. Here h can be considered as the arc of a circle of radius 12 m, which subtends an angle of 5′ at its centre.

Here, $\theta = 5' = \left(\frac{5}{60}\right)^\circ = \left(\frac{1}{12}\times\frac{\pi}{180}\right)^c$, and $r = 12$ m.

$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{\pi}{12\times 180} = \frac{h}{12} \Rightarrow h = \left(\frac{\pi}{180}\right)\text{metre} = 1.7\text{ cm.}$

EXAMPLE 16 *The perimeter of a certain sector of a circle is equal to the length of the arc of a semi-circle having the same radius. Express the angle of the sector in degrees, minutes and seconds.*

SOLUTION Let r be the radius of the circle and θ be the sector angle. Then,

Perimeter of the sector $= 2r + r\theta$

Length of the arc of a semi-circle of radius $r = \pi r$

It is given that

$$2r + r\theta = \pi r$$

$$\Rightarrow \quad 2 + \theta = \pi$$

$$\Rightarrow \quad \theta = (\pi - 2)\text{ radians} = \left\{(\pi-2)\times\frac{180}{\pi}\right\}^\circ = 180^\circ - \left(\frac{360}{\pi}\right)^\circ = 180^\circ - 114^\circ\,32'\,44'' = 65^\circ\,27'\,16''$$

EXAMPLE 17 *The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes? (Use $\pi = 3.14$)*

SOLUTION In 60 minutes, the minute hand of a watch completes one rotation i.e., it rotates through 360°.

$\therefore$ Angle traced by the minute hand in 1 minute $= \left(\frac{360}{60}\right)^\circ = 6^\circ$

$\Rightarrow$ Angle traced by the minute hand in 40 minutes $= (40\times 6)^\circ = 240^\circ = \left(240\times\frac{\pi}{180}\right)^c = \left(\frac{4\pi}{3}\right)^c$

Now,

$$\theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{4\pi}{3} = \frac{\text{arc}}{1.5} \Rightarrow \text{arc} = \left(\frac{4\pi}{3}\times 1.5\right)\text{cm} = 2\pi\text{ cm} = 2\times 3.14\text{ cm} = 6.28\text{ cm}$$

Hence, the tip of the minute hand travels 6.28 cm in 40 minutes.

EXAMPLE 18 *Find the angle between the minute hand of a clock and the hour hand when the time is 7 : 20 AM.*

SOLUTION We know that the hour hand completes one rotation in 12 hours while the minute hand completes one rotation in 60 minutes.

$\therefore$ Angle traced by the hour hand in 12 hours = 360°

$\Rightarrow$ Angle traced by the hour hand in 7 hrs 20 min. i.e $\frac{22}{3}$ hrs $=\left(\frac{360}{12}\times\frac{22}{3}\right)^\circ = 220°$.

Also, the angle traced by the minute hand in 60 min = 360°.

$\Rightarrow$ Angle traced by the minute hand in 20 min $=\left(\frac{360}{60}\times 20\right)^\circ = 120°$

Hence, the required angle between two hands = 220° − 120° = 100°.

EXAMPLE 19 *Find in degrees and radians the angle between the hour hand and the minute-hand of a clock at half past three.*

SOLUTION The angle traced by the hour hand in 12 hours = 360°

$\therefore$ The angle traced by the hour hand in 3 hrs 30 min. i.e. $\frac{7}{2}$ hrs $=\left(\frac{360}{12}\times\frac{7}{2}\right)^\circ = 105°$

The angle traced by the minute hand in 60 min = 360°

$\Rightarrow$ The angle traced by the minute hand in 30 min $=\left(\frac{360}{60}\times 30\right) = 180°$

Hence, the required angle between two hands $= 180° - 105° = 75° = \left(75\times\frac{\pi}{180}\right) = \frac{5\pi}{12}$ radians.

LEVEL-2

EXAMPLE 20 *For each natural number k, let C_k denote the circle with radius k centimetres and centre at the origin. On the circle C_k, a particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves on C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n, then find the value of n.*

SOLUTION The path of the particle is shown by bold line segments and arcs. It is given that on the circle C_k of radius k centimetres the particle moves k centimetres. Therefore, angular displacement on kth circle is given by

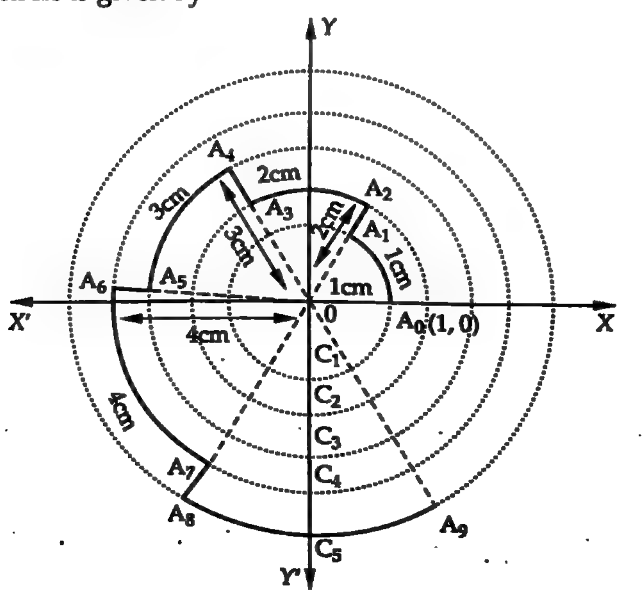


Fig. 4.14

$$\theta = \frac{k}{k} \text{ radian } = 1 \text{ radian.}$$

Thus, angular displacement on each circle is 1 radian.

If the particle crosses the x-axis for the first time on circle C_n then

Total angular displacement $= n$ radians.

As the particle crosses the positive direction of the x-axis for the first time on the n^{th} circle C_n.

Total angular displacement $> 2\pi$ radians

$\Rightarrow \quad n > 2\pi$

$\Rightarrow \quad n = 7 \qquad$ [$\because$ n is the natural number such that $n > 2\pi$]

EXERCISE 4.1

LEVEL-1

1. Find the degree measure corresponding to the following radian measures (Use $\pi = 22/7$:
   (i) $\frac{9\pi}{5}$ (ii) $-\frac{5\pi}{6}$ (iii) $\left(\frac{18\pi}{5}\right)^c$ (iv) $(-3)^c$ (v) 11^c (vi) 1^c
2. Find the radian measure corresponding to the following degree measures:
   (i) $300°$ (ii) $35°$ (iii) $-56°$ (iv) $135°$ (v) $-300°$ (vi) $7° 30'$
   (vii) $125° 30'$ (viii) $-47° 30'$
3. The difference between the two acute angles of a right-angled triangle is $\frac{2\pi}{5}$ radians. Express the angles in degrees.
4. One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{3}{2}x$ degrees while the third is $\frac{\pi x}{75}$ radians. Express all the angles in degrees.
5. Find the magnitude, in radians and degrees, of the interior angle of a regular
   (i) pentagon (ii) octagon (iii) heptagon (iv) duodecagon.
6. The angles of a quadrilateral are in A.P. and the greatest angle is 120°. Express the angles in radians.
7. The angles of a triangle are in *A.P.* and the number of degrees in the least angle is to the number of degrees in the mean angle as 1 : 120. Find the angles in radians.
8. The angle in one regular polygon is to that in another as 3 : 2 and the number of sides in first is twice that in the second. Determine the number of sides of two polygons.
9. The angles of a triangle are in A.P. such that the greatest is 5 times the least. Find the angles in radians.
10. The number of sides of two regular polygons are as 5 : 4 and the difference between their angles is 9°. Find the number of sides of the polygons.
11. A rail road curve is to be laid out on a circle. What radius should be used if the track is to change direction by 25° in a distance of 40 metres?
12. Find the length which at a distance of 5280 m will subtend an angle of 1′ at the eye.
13. A wheel makes 360 revolutions per minute. Through how many radians does it turn in 1 second?
14. Find the angle in radians through which a pendulum swings if its length is 75 cm and the tip describes an arc of length (i) 10 cm (ii) 15 cm (iii) 21 cm.

15. The radius of a circle is 30 cm. Find the length of an arc of this circle, if the length of the chord of the arc is 30 cm.
16. A railway train is travelling on a circular curve of 1500 metres radius at the rate of 66 km/hr. Through what angle has it turned in 10 seconds?
17. Find the distance from the eye at which a coin of 2 cm diameter should be held so as to conceal the full moon whose angular diameter is 31′.
18. Find the diameter of the sun in km supposing that it subtends an angle of 32′ at the eye of an observer. Given that the distance of the sun is 91×10^6 km.
19. If the arcs of the same length in two circles subtend angles 65° and 110° at the centre, find the ratio of their radii.
20. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (Use $\pi = 22/7$).

ANSWERS

1. (i) 324° (ii) −150° (iii) 648° (iv) −171° 49′ 5″ (v) 630° (vi) 57° 16′ 21″
2. (i) $\frac{5\pi}{3}$ (ii) $\frac{7\pi}{36}$ (iii) $-\frac{14\pi}{45}$ (iv) $\frac{3\pi}{4}$ (v) $-\frac{5\pi}{3}$ (vi) $\frac{\pi}{24}$ (vii) $\frac{251\pi}{360}$ (viii) $-\frac{19\pi}{72}$
3. 81°, 9°
4. 24°, 60°, 96°
5. (i) $\left(\frac{3\pi}{5}\right)^c$; 108° (ii) $\left(\frac{3\pi}{4}\right)^c$; 135° (iii) $\left(\frac{5\pi}{7}\right)^c$, 128° 34′ 17″ (iv) $\left(\frac{5\pi}{6}\right)^c$; 150°
6. $\frac{\pi}{3}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{2\pi}{3}$
7. $\frac{\pi}{360}, \frac{\pi}{3}, \frac{239\pi}{360}$
8. 8, 4
9. $\frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}$
10. 10, 8
11. 91.64 m
12. 1.5365
13. 12π
14. (i) $\left(\frac{11}{90}\right)^c$ (ii) $\frac{1}{5}$ (iii) $\frac{7}{25}$
15. 10π cm
16. $\left(\frac{11}{90}\right)^c$
17. 2.217m
18. 847407.4 km
19. 22 : 13
20. 12°, 36′

HINTS TO NCERT & SELECTED PROBLEMS

4. $$\left(\frac{2}{3}x\right)^g = \left(\frac{2}{3}x \times \frac{90}{100}\right)^\circ = \left(\frac{3x}{5}\right)^\circ \text{ and, } \left(\frac{\pi x}{75}\right)^c = \left(\frac{\pi}{75} \times \frac{180}{\pi}\right)^c = \left(\frac{12x}{5}\right)^\circ$$
$$\therefore \quad \left(\frac{3}{5}x\right)^\circ + \left(\frac{3}{2}x\right)^\circ + \left(\frac{12x}{5}\right)^\circ = 180^\circ \Rightarrow x = 40^\circ$$
5. A heptagon has seven sides and the number of sides of a dudecagon is twelve .
6. Let the measures of angles in degrees be $a - 3d, a - d, a + d, a + 3d$. Then,
Sum of the angles = $360°° \Rightarrow 4a = 360° \Rightarrow a = 90°$.
Also, Greatest angle = $120° \Rightarrow a + 3d = 120° \Rightarrow d = 10°$.
11. Here, $\theta = 25° = \left(25 \times \frac{\pi}{180}\right)^c$ and arc = 40 meters.

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If D, G and R denote respectively the number of degrees, grades and radians in an angle, then

(a) $\frac{D}{100} = \frac{G}{90} = \frac{2R}{\pi}$ (b) $\frac{D}{90} = \frac{G}{100} = \frac{R}{\pi}$

(c) $\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$ (d) $\frac{D}{90} = \frac{G}{100} = \frac{R}{2\pi}$

2. If the angles of a triangle are in A.P., then the measures of one of the angles in radians is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

3. The angle between the minute and hour hands of a clock at 8:30 is

(a) 80° (b) 75° (c) 60° (d) 105°

4. At 3:40, the hour and minute hands of a clock are inclined at

(a) $\frac{2\pi^c}{3}$ (b) $\frac{7\pi^c}{12}$ (c) $\frac{13\pi^c}{18}$ (d) $\frac{3\pi^c}{4}$

5. If the arcs of the same length in two circles subtend angles 65° and 110° at the centre, then the ratio of the radii of the circles is

(a) 22 : 13 (b) 11 : 13 (c) 22 : 15 (d) 21 : 13

6. If OP makes 4 revolutions in one second, the angular velocity in radians per second is

(a) π (b) 2π (c) 4π (d) 8π

7. A circular wire of radius 7 cm is cut and bent again into an arc of a circle of radius 12 cm. The angle subtended by the arc at the centre is

(a) 50° (b) 210° (c) 100° (d) 60° (e) 195°

8. The radius of the circle whose arc of length 15π cm makes an angle of $3\pi/4$ radian at the centre is

(a) 10 cm (b) 20 cm (c) $11\frac{1}{4}$ cm (d) $22\frac{1}{2}$ cm

ANSWERS

1. (c) 2. (b) 3. (b) 4. (c) 5. (a) 6. (d) 7. (b) 8. (b)

SUMMARY

1. The measure of an angle is the amount of rotation from the initial side to the terminal side.
2. The sense of an angle is positive or negative according as the initial side rotates in anti-clockwise or clockwise direction to get the terminal side.
3. Three systems of measuring angles are:

   (i) Sexagesimal system (ii) Centesimal system (iii) Circular system

   In sexagesimal system:

   1 right angle = 90 degrees (= 90°)

   1° = 60 minutes (= 60′)

   1′ = 60 seconds (= 60″)

   In centesimal system:

   1 right angle = 100 grades ($=100^g$)

   1^g = 100 minutes (= 100′)

   1′ = 100 seconds (=100″)

   In circular system, the unit of measurement is radian. One radian is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

   π radians = 180°

4. The relation between three systems of measurement of an angle is

$$\frac{D}{90^\circ} = \frac{G}{100} = \frac{2R}{\pi}$$

CHAPTER 5

TRIGONOMETRIC FUNCTIONS

5.1 INTRODUCTION

In the present chapter, we will first introduce trigonometric ratios which are also known as trigonometric functions and then the identities involving them.

5.2 TRIGONOMETRIC RATIOS OR FUNCTIONS

Consider an angle $\theta = \angle XOA$ as shown in Fig. 5.1. Let P be any point other than O on its terminal side OA and let PM be perpendicular from P on x-axis. Let length $OP = r$, $OM = x$ and $MP = y$. We take the length $OP = r$ always positive while x and y can be positive or negative depending upon the position of the terminal side OA of $\angle XOA$.

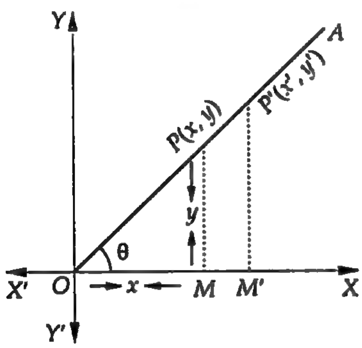


Fig. 5.1

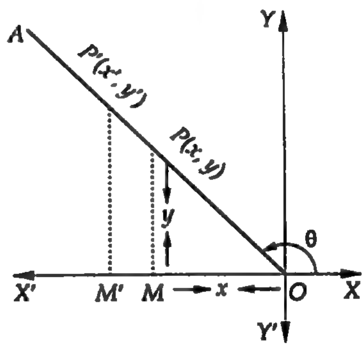


Fig. 5.2

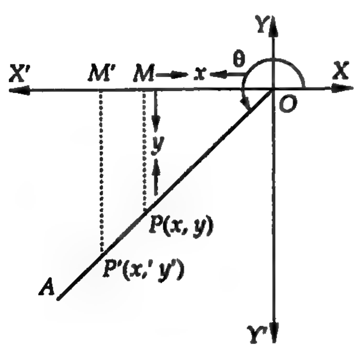


Fig. 5.3

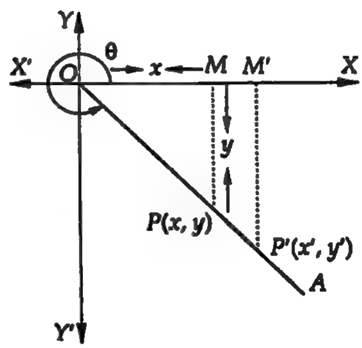


Fig. 5.4

In the right angled triangle OMP, we have

Base $= OM = x$, Perpendicular $= PM = y$, and Hypotenues $= OP = r$

We define the following trigonometric ratios which are also known as trgonometric functions.

$$\text{Sine } \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{y}{r}, \text{ and is written as } \sin\theta$$

$$\text{Cosine } \theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{x}{r}, \text{ and is written as } \cos\theta$$

$$\text{Tangent } \theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{y}{x}, \text{ and is written as } \tan\theta$$

$$\text{Cosecant } \theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{r}{y}, \text{ and is written as } \operatorname{cosec}\theta$$

$$\text{Secant } \theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{r}{x}, \text{ and is written as } \sec\theta$$

$$\text{Cotangent } \theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{x}{y}, \text{ and is written as } \cot\theta$$

NOTE 1 *The triangle OMP is known as the triangle of reference.*

NOTE 2 *It should be noted that* $\sin\theta$ *does not mean the product of sin and θ. The* $\sin\theta$ *is correctly read as sine of angle θ.*

These functions depend only on the value of the angle θ and not on the position of the point P chosen on the terminal side of the angle θ as proved in the following theorem.

THEOREM *The trigonometric ratios are same for the same angle.*

PROOF Let $P'(x', y')$ be any point other than $P(x, y)$ on the terminal side OA with $OP' = r'$. Let $P'M'$ be perpendicular on x-axis. Clearly, triangles OMP and $OM'P'$ are similar. Therefore,

$$\frac{y}{r} = \frac{y'}{r'}, \quad \frac{x}{r} = \frac{x'}{r'} \quad \text{and} \quad \frac{y}{x} = \frac{y'}{x'}$$

$$\Rightarrow \quad \sin\theta = \frac{y}{r} = \frac{y'}{r'}, \quad \cos\theta = \frac{x}{r} = \frac{x'}{r'} \quad \text{and} \quad \tan\theta = \frac{y}{x} = \frac{y'}{x'}$$

Thus, sine, cosine and tangent are the same whatever point be taken on the terminal side OA. Similarly, it can be proved for the other ratios.

Hence, the trigonometric ratios or trigonometric functions are independent of the choice of the size of triangle of reference.

Q.E.D.

REMARK 1 *If the terminal side coincides with x-axis, then* $\operatorname{cosec}\theta$ *and* $\cot\theta$ *are not defined. If it coincides with y-axis, then* $\sec\theta$ *and* $\tan\theta$ *are not defined.*

REMARK 2 *The following relations are obvious from the definitions of trigonometric ratios :*

(i) $\sin\theta \times \operatorname{cosec}\theta = 1 \Rightarrow \sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ and $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

(ii) $\cos\theta \times \sec\theta = 1 \Rightarrow \cos\theta = \dfrac{1}{\sec\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\tan\theta \times \cot\theta = 1 \Rightarrow \cot\theta = \dfrac{1}{\tan\theta}$ and $\tan\theta = \dfrac{1}{\cot\theta}$

(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

REMARK 3 *The trigonometric ratios may be positive or negative depending upon x and/or y as discussed in section 5.4.*

5.3 TRIGONOMETRIC IDENTITIES

IDENTITY *An equation involving trigonometric functions which is true for all those angles for which the functions are defined is called a trigonometric identity.*

For example, $\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ etc. are trigonometric identities as they hold for all those values of θ except those values for which $\sec\theta$ and $\operatorname{cosec}\theta$ are not defined.

But, $\sin\theta = \cos\theta$ is a trigonometric equation not a trigonometric identity because it does not hold for all values of θ.

5.3.1 FUNDAMENTAL TRIGONOMETRIC IDENTITIES

In this subsection, we shall state and prove some fundamental trigonometrical identities.

THEOREM *Prove that:*

(i) $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ *or,* $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ *(ii)* $\cos\theta = \dfrac{1}{\sec\theta}$ *or,* $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\cot\theta = \dfrac{1}{\tan\theta}$ *or,* $\tan\theta = \dfrac{1}{\cot\theta}$ *(iv)* $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(v) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ *(vi)* $\sin^2\theta + \cos^2\theta = 1$

(vii) $1 + \tan^2\theta = \sec^2\theta$ *(viii)* $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

PROOF Let a revolving ray start from OX and revolve into the position OP to trace out any angle θ in any of the four quadrants. From P drawn PM perpendicular to x-axis. In the right angled triangle OMP, we have

$$OP^2 = OM^2 + PM^2,$$

$$\sin\theta = \frac{PM}{OP}, \cos\theta = \frac{OM}{OP}, \tan\theta = \frac{PM}{OM}, \operatorname{cosec} = \frac{OP}{PM}, \sec\theta = \frac{OP}{OM} \text{ and, } \cot\theta = \frac{OM}{PM}.$$

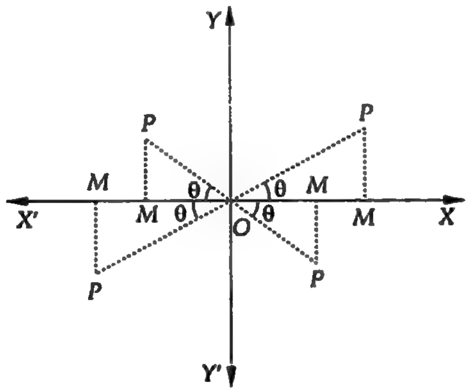


Fig. 5.5

Clearly, identities in (i) to (v) are trivial So, let us prove identity (vi).

(vi) $$\sin^2\theta + \cos^2\theta = \left(\frac{PM}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = \frac{PM^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2} = 1$$

Hence, $\sin^2\theta + \cos^2\theta = 1$

(vii) We have,

$$1 + \tan^2\theta = 1 + \left(\frac{PM}{OM}\right)^2 = 1 + \frac{PM^2}{OM^2} = \frac{OM^2 + PM^2}{OM^2} = \frac{OP^2}{OM^2} = \left(\frac{OP}{OM}\right)^2 = \sec^2\theta$$

Hence, $1 + \tan^2\theta = \sec^2\theta$.

(viii) We have,

$$1 + \cot^2\theta = 1 + \left(\frac{OM}{PM}\right)^2 = 1 + \frac{OM^2}{PM^2} = \frac{PM^2 + OM^2}{PM^2} = \frac{OP^2}{PM^2} = \left(\frac{OP}{PM}\right)^2 = \operatorname{cosec}^2\theta$$

Hence, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ Q.E.D.

NOTE *It should be noted that* $(\sin\theta)^2$ *is written as* $\sin^2\theta$, $(\cos\theta)^2$ *is written as* $\cos^2\theta$ *etc.*

We shall now discuss more identities involving the trigonometrical functions in the following examples.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove the following identities:*

(i) $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

(ii) $\cot^4\theta + \cot^2\theta = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta$

(iii) $2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta = \cot^4\theta - \tan^4\theta$

(iv) $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = \tan^2\theta + \cot^2\theta + 7$

SOLUTION (i) We have,

$$\begin{aligned}
\text{LHS} &= (\sin^8\theta - \cos^8\theta) = (\sin^4\theta)^2 - (\cos^4\theta)^2 \\
&= (\sin^4\theta - \cos^4\theta)(\sin^4\theta + \cos^4\theta) \\
&= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta) \\
&= (\sin^2\theta - \cos^2\theta)(\sin^4\theta + \cos^4\theta) \\
&= (\sin^2\theta - \cos^2\theta)(\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta) \\
&= (\sin^2\theta - \cos^2\theta)\left\{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta\right\} \\
&= (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta) = \text{RHS}
\end{aligned}$$

(ii)
$$\begin{aligned}
\text{LHS} &= \cot^4\theta + \cot^2\theta = (\cot^2\theta)^2 + \cot^2\theta \\
&= (\operatorname{cosec}^2\theta - 1)^2 + (\operatorname{cosec}^2\theta - 1) \qquad [\because 1 + \cot^2\theta = \operatorname{cosec}^2\theta] \\
&= \operatorname{cosec}^4\theta - 2\operatorname{cosec}^2\theta + 1 + \operatorname{cosec}^2\theta - 1 = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta = \text{RHS}
\end{aligned}$$

(iii)
$$\begin{aligned}
\text{LHS} &= 2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta \\
&= 2\sec^2\theta - (\sec^2\theta)^2 - 2\operatorname{cosec}^2\theta + (\operatorname{cosec}^2\theta)^2 \\
&= 2(1 + \tan^2\theta) - (1 + \tan^2\theta)^2 - 2(1 + \cot^2\theta) + (\cot^2\theta + 1)^2 \\
&= 2 + 2\tan^2\theta - (1 + \tan^4\theta + 2\tan^2\theta) - 2 - 2\cot^2\theta + (\cot^4\theta - 2\cot^2\theta + 1) \\
&= \cot^4\theta - \tan^4\theta = \text{RHS}
\end{aligned}$$

(iv)
$$\begin{aligned}
\text{LHS} &= (\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 \\
&= \sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta\operatorname{cosec}\theta + \cos^2\theta + \sec^2\theta + 2\cos\theta\sec\theta \\
&= (\sin^2\theta + \cos^2\theta) + (\operatorname{cosec}^2\theta + \sec^2\theta) + 2 + 2 \\
&= 1 + (1 + \cot^2\theta) + (1 + \tan^2\theta) + 4 = \tan^2\theta + \cot^2\theta + 7 = \text{RHS}
\end{aligned}$$

EXAMPLE 2 *Prove the following identities:*

(i) $(1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta) = 2$

(ii) $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1 + \sin\theta}{\cos\theta}$ [NCERT EXEMPLAR]

SOLUTION (i) We have,

$$\begin{aligned}
\text{LHS} &= (1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta) \\
&= \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right) \\
&= \frac{(\sin\theta + \cos\theta - 1)}{\sin\theta}\,\frac{(\sin\theta + \cos\theta + 1)}{\cos\theta}
\end{aligned}$$

$$= \frac{(\sin\theta+\cos\theta)^2-1}{\sin\theta\cos\theta} = \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{RHS}$$

(ii) $$\text{LHS} = \frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \frac{(\tan\theta+\sec\theta)-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1} \quad [\because \sec^2\theta-\tan^2\theta=1]$$

$$= \frac{(\sec\theta+\tan\theta)\{1-(\sec\theta-\tan\theta)\}}{\tan\theta-\sec\theta+1} = \frac{(\sec\theta+\tan\theta)(\tan\theta-\sec\theta+1)}{\tan\theta-\sec\theta+1}$$

$$= \sec\theta+\tan\theta = \frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta} = \text{RHS}$$

EXAMPLE 3 *If* $\tan\theta+\sin\theta = m$ *and* $\tan\theta-\sin\theta = n$, *show that* $m^2-n^2 = 4\sqrt{mn}$.

SOLUTION We have, **[NCERT EXEMPLAR]**

$$\text{LHS} = m^2-n^2 = (\tan\theta+\sin\theta)^2-(\tan\theta-\sin\theta)^2 = 4\tan\theta\sin\theta \quad ...(i)$$

$$\text{RHS} = 4\sqrt{mn} = 4\sqrt{(\tan\theta+\sin\theta)(\tan\theta-\sin\theta)} = 4\sqrt{\tan^2\theta-\sin^2\theta}$$

$$\Rightarrow \quad \text{RHS} = 4\sqrt{\frac{\sin^2\theta}{\cos^2\theta}-\sin^2\theta} = 4\sqrt{\frac{\sin^2\theta-\sin^2\theta\cos^2\theta}{\cos^2\theta}} = 4\sqrt{\frac{\sin^2\theta(1-\cos^2\theta)}{\cos^2\theta}}$$

$$\Rightarrow \quad \text{RHS} = 4\sqrt{\frac{\sin^4\theta}{\cos^2\theta}} = 4\frac{\sin^2\theta}{\cos\theta} = 4\tan\theta\sin\theta \quad ...(ii)$$

From (i) and (ii), we obtain that LHS = RHS.

EXAMPLE 4 *If* $\cos\theta+\sin\theta = \sqrt{2}\cos\theta$, *prove that* $\cos\theta-\sin\theta = \sqrt{2}\sin\theta$.

SOLUTION We have,

$$\cos\theta+\sin\theta = \sqrt{2}\cos\theta$$

$$\Rightarrow \quad (\cos\theta+\sin\theta)^2 = (\sqrt{2}\cos\theta)^2$$

$$\Rightarrow \quad \cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta = 2\cos^2\theta$$

$$\Rightarrow \quad \cos^2\theta-\sin^2\theta = 2\sin\theta\cos\theta$$

$$\Rightarrow \quad (\cos\theta+\sin\theta)(\cos\theta-\sin\theta) = 2\sin\theta\cos\theta$$

$$\Rightarrow \quad \cos\theta-\sin\theta = \frac{2\sin\theta\cos\theta}{\cos\theta+\sin\theta}$$

$$\Rightarrow \quad \cos\theta-\sin\theta = \frac{2\sin\theta\cos\theta}{\sqrt{2}\cos\theta} \quad [\because \cos\theta+\sin\theta = \sqrt{2}\cos\theta]$$

$$\Rightarrow \quad \cos\theta-\sin\theta = \sqrt{2}\sin\theta$$

ALITER We know that

$$(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2 = 2$$

$$\Rightarrow \quad (\sqrt{2}\cos\theta)^2+(\cos-\sin\theta)^2 = 2 \quad [\because \cos\theta+\sin\theta = \sqrt{2}\cos\theta]$$

$$\Rightarrow \quad (\cos\theta-\sin\theta)^2 = 2-2\cos^2\theta$$

$$\Rightarrow \quad (\cos\theta-\sin\theta)^2 = 2\sin^2\theta \Rightarrow \cos\theta-\sin\theta = \sqrt{2}\sin\theta.$$

EXAMPLE 5 *If* $a\cos\theta+b\sin\theta = x$ *and* $a\sin\theta-b\cos\theta = y$, *prove that* $a^2+b^2 = x^2+y^2$.

[NCERT EXEMPLAR]

SOLUTION We have,

$x = a\cos\theta+b\sin\theta$ and $y = a\sin\theta-b\cos\theta$.

$\therefore \quad x^2+y^2=(a\cos\theta+b\sin\theta)^2+(a\sin\theta-b\cos\theta)^2$

$$=(a^2\cos^2\theta+b^2\sin^2\theta+2ab\sin\theta\cos\theta)+(a^2\sin^2\theta+b^2\cos^2\theta-2ab\sin\theta\cos\theta)$$

$$=a^2(\cos^2\theta+\sin^2\theta)+b^2(\sin^2\theta+\cos^2\theta)=a^2+b^2$$

EXAMPLE 6 *If* $a\cos\theta-b\sin\theta=c$, *show that* $a\sin\theta+b\cos\theta=\pm\sqrt{a^2+b^2-c^2}$.

SOLUTION Clearly,

$$(a\cos\theta-b\sin\theta)^2+(a\sin\theta+b\cos\theta)^2$$

$$=a^2(\cos^2\theta+\sin^2\theta)+b^2(\sin^2\theta+\cos^2\theta)-2ab\sin\theta\cos\theta+2ab\sin\theta\cos\theta$$

$$=a^2+b^2$$

$\therefore \quad (a\sin\theta+b\cos\theta)^2=a^2+b^2-(a\cos\theta-b\sin\theta)^2$

$\Rightarrow \quad (a\sin\theta+b\cos\theta)^2=a^2+b^2-c^2 \qquad [\because a\cos\theta-b\sin\theta=c]$

$\Rightarrow \quad a\sin\theta+b\cos\theta=\pm\sqrt{a^2+b^2-c^2}$

EXAMPLE 7 *If* $\sec\theta+\tan\theta=p$, *obtain the values of* $\sec\theta$, $\tan\theta$ *and* $\sin\theta$ *in terms of* p.

SOLUTION We know that: $\sec^2\theta-\tan^2\theta=1$.

$\therefore \quad (\sec\theta+\tan\theta)(\sec\theta-\tan\theta)=1\Rightarrow p(\sec\theta-\tan\theta)=1\Rightarrow \sec\theta-\tan\theta=\dfrac{1}{p}$

Now, $\sec\theta+\tan\theta=p$ and, $\sec\theta-\tan\theta=\dfrac{1}{p}$

$\Rightarrow \quad (\sec\theta+\tan\theta)+(\sec\theta-\tan\theta)=p+\dfrac{1}{p}$

$\Rightarrow \quad 2\sec\theta=p+\dfrac{1}{p}\Rightarrow \sec\theta=\dfrac{p^2+1}{2p} \qquad \text{...(i)}$

Again, $\sec\theta+\tan\theta=p$ and $\sec\theta-\tan\theta=\dfrac{1}{p}$

$\Rightarrow \quad (\sec\theta+\tan\theta)-(\sec\theta-\tan\theta)=p-\dfrac{1}{p}$

$\Rightarrow \quad 2\tan\theta=p-\dfrac{1}{p}\Rightarrow \tan\theta=\dfrac{p^2-1}{2p} \qquad \text{...(ii)}$

Dividing (ii) by (i), we get: $\sin\theta=\dfrac{p^2-1}{p^2+1}$

EXAMPLE 8 *Prove that:* $2\sec^2\theta-\sec^4\theta-2\,\text{cosec}^2\theta+\text{cosec}^4\theta=\dfrac{1-\tan^8\theta}{\tan^4\theta}$.

SOLUTION We have,

$$2\sec^2\theta-\sec^4\theta-2\,\text{cosec}^2\theta+\text{cosec}^4\theta$$

$$=2(1+\tan^2\theta)-(1+\tan^2\theta)^2-2(1+\cot^2\theta)+(1+\cot^2\theta)^2$$

$$=2(1+\tan^2\theta-1-\cot^2\theta)+(1+2\cot^2\theta+\cot^4\theta)-(1+2\tan^2\theta+\tan^4\theta)$$

$$=2(\tan^2\theta-\cot^2\theta)+(2\cot^2\theta-2\tan^2\theta)+\cot^4\theta-\tan^4\theta$$

$$=\cot^4\theta-\tan^4\theta=\frac{1}{\tan^4\theta}-\tan^4\theta=\frac{1-\tan^8\theta}{\tan^4\theta}$$

EXAMPLE 9 *Prove that:* $3(\sin\theta-\cos\theta)^4+6(\sin\theta+\cos\theta)^2+4(\sin^6\theta+\cos^6\theta)-13=0$.

SOLUTION We have,

$$3(\sin\theta-\cos\theta)^4+6(\sin\theta+\cos\theta)^2+4(\sin^6\theta+\cos^6\theta)-13$$

$$=3\left\{(\sin\theta-\cos\theta)^2\right\}^2+6(\sin\theta+\cos\theta)^2$$

$$+4\left\{(\sin^2\theta+\cos^2\theta)^3-3\sin^2\theta\cos^2\theta(\sin^2\theta+\cos^2\theta)\right\}-13$$

$$=3(1-2\sin\theta\cos\theta)^2+6(1+2\sin\theta\cos\theta)+4(1-3\sin^2\theta\cos^2\theta)-13$$

$$=3(1-4\sin\theta\cos\theta+4\sin^2\theta\cos^2\theta)+6(1+2\sin\theta\cos\theta)+4(1-3\sin^2\theta\cos^2\theta)-13$$

$$=3+6+4-13=0$$

EXAMPLE 10 *Given that:* $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)=(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$. *Show that one of the values of each member of this equality is* $\sin\alpha\sin\beta\sin\gamma$.

SOLUTION We have,

$$(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)=(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$$

Multiplying both sides by $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$, we get

$$(1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2=(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$$

$$\Rightarrow\quad (1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2=(1-\cos^2\alpha)(1-\cos^2\beta)(1-\cos^2\gamma)$$

$$\Rightarrow\quad (1+\cos\alpha)^2(1+\cos\beta)^2(1+\cos\gamma)^2=\sin^2\alpha\sin^2\beta\sin^2\gamma$$

$$\Rightarrow\quad (1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)=\pm\sin\alpha\sin\beta\sin\gamma$$

Hence, one of the values of $(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)$ is $\sin\alpha\sin\beta\sin\gamma$.

Similarly, by multiplying both sides by $(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$, we find that one of the values of $(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$ is also $\sin\alpha\sin\beta\sin\gamma$.

LEVEL-2

EXAMPLE 11 *Prove that:* $\sec^2\theta+\mathrm{cosec}^2\theta\ \geq\ 4$.

SOLUTION We have,

$$\sec^2\theta+\mathrm{cosec}^2\theta=(1+\tan^2\theta)+1+(\cot^2\theta)=2+\tan^2\theta+\cot^2\theta$$

$$=2+\tan^2\theta+\cot^2\theta-2\tan\theta\cot\theta+2\tan\theta\cot\theta$$

$$=2+(\tan\theta-\cot\theta)^2+2$$

$$=4+(\tan\theta-\cot\theta)^2\geq 4 \qquad [\because (\tan\theta-\cot\theta)^2\geq 0]$$

EXAMPLE 12 *If* $10\sin^4\alpha+15\cos^4\alpha=6$, *find the value of* $27\,\mathrm{cosec}^6\alpha+8\sec^6\alpha$.

SOLUTION We have,

$$10\sin^4\alpha+15\cos^4\alpha=6$$

$$\Rightarrow\quad 10\sin^4\alpha+15\cos^4\alpha=6(\sin^2\alpha+\cos^2\alpha)^2$$

$$\Rightarrow\quad 10\tan^4\alpha+15=6(\tan^2\alpha+1)^2 \qquad \text{[Dividing boths sides by } \cos^4\alpha]$$

$$\Rightarrow\quad (2\tan^2\alpha-3)^2=0\Rightarrow \tan^2\alpha=\frac{3}{2}$$

$$\therefore\quad 27\,\mathrm{cosec}^6\alpha+8\sec^6\alpha=27(1+\cot^2\alpha)^3+8(1+\tan^2\alpha)^3$$

$$=27\left(1+\frac{2}{3}\right)^3+8\left(1+\frac{3}{2}\right)^3=27\times\frac{125}{27}+8\times\frac{125}{8}=250.$$

EXAMPLE 13 *If* $\frac{\sin A}{\sin B} = p$ *and* $\frac{\cos A}{\cos B} = q$, *find* $\tan A$ *and* $\tan B$.

SOLUTION We have,

$$\frac{\sin A}{\sin B} = p \text{ and } \frac{\cos A}{\cos B} = q$$

$$\Rightarrow \quad \frac{\sin A}{\sin B} \cdot \frac{\cos B}{\cos A} = \frac{p}{q}$$

$$\Rightarrow \quad \frac{\tan A}{\tan B} = \frac{p}{q} \Rightarrow \frac{\tan A}{p} = \frac{\tan B}{q} = \lambda \text{ (say)} \Rightarrow \tan A = p\lambda \text{ and } \tan B = q\lambda \quad \text{...(i)}$$

Now, $\sin A = p \sin B$

$$\Rightarrow \quad \frac{\tan A}{\sqrt{1+\tan^2 A}} = p \frac{\tan B}{\sqrt{1+\tan^2 B}}$$

$$\Rightarrow \quad \frac{p\lambda}{\sqrt{1+p^2\lambda^2}} = p \frac{q\lambda}{\sqrt{1+q^2\lambda^2}}$$

$$\Rightarrow \quad p^2(1+q^2\lambda^2) = p^2 q^2 (1+p^2\lambda^2)$$

$$\Rightarrow \quad \lambda^2 (q^2 - p^2 q^2) = q^2 - 1$$

$$\Rightarrow \quad \lambda^2 = \frac{q^2-1}{q^2(1-p^2)} \Rightarrow \lambda = \pm \frac{1}{q}\sqrt{\frac{q^2-1}{1-p^2}}$$

$$\therefore \quad \tan A = \pm \frac{p}{q}\sqrt{\frac{q^2-1}{1-p^2}} \text{ and, } \tan B = \pm\sqrt{\frac{q^2-1}{1-p^2}} \quad \text{[Using (i)]}$$

EXAMPLE 14 *If* $\tan^2\theta = 1 - a^2$, *prove that* $\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2-a^2)^{3/2}$. *Also, find the values of* a *for which the above result holds true.*

SOLUTION We have,

$$\sec\theta + \tan^3\theta \operatorname{cosec}\theta = \sec\theta \left\{1 + \tan^3\theta \frac{\operatorname{cosec}\theta}{\sec\theta}\right\}$$

$$= \sqrt{1+\tan^2\theta}\left\{1 + \tan^3\theta \times \cot\theta\right\}$$

$$= (1+\tan^2\theta)^{3/2}$$

$$= (1+1-a^2)^{3/2} = (2-a^2)^{3/2} \quad [\because \tan^2\theta = 1-a^2]$$

Now,

$$\tan^2\theta \geq 0 \text{ for all } \theta \Rightarrow 1 - a^2 \geq 0 \Rightarrow a^2 - 1 \leq 0 \Rightarrow -1 \leq a \leq 1 \quad \text{...(i)}$$

Since LHS of $\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2-a^2)^{3/2}$ is real for all $\theta \in R$. So, RHS must also be real.

$$\therefore \quad 2 - a^2 \geq 0 \Rightarrow a^2 - 2 \leq 0 \Rightarrow -\sqrt{2} \leq a \leq \sqrt{2} \quad \text{...(ii)}$$

From (i) and (ii), we find that the given relation holds true for all $a \in [-1, 1]$.

EXAMPLE 15 *If* $a\cos^3\theta + 3a\cos\theta\sin^2\theta = m$ *and* $a\sin^3\theta + 3a\cos^2\theta\sin\theta = n$, *then prove that:* $(m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$.

SOLUTION We have,

$$a\cos^3\theta + 3a\cos\theta\sin^2\theta = m \quad \text{and} \quad a\sin^3\theta + 3a\cos^2\theta\sin\theta = n$$

$$\Rightarrow \quad a\cos^3\theta + 3a\cos\theta\sin^2\theta + a\sin^3\theta + 3a\cos^2\theta\sin\theta = m+n$$

and, $\quad a\cos^3\theta + 3a\cos\theta\sin^2\theta - a\sin^3\theta - 3a\cos^2\theta\sin\theta = m-n$

$\Rightarrow \quad a(\cos\theta+\sin\theta)^3 = m+n$ and, $a(\cos\theta-\sin\theta)^3 = m-n$

$\Rightarrow \quad \cos\theta+\sin.\theta = \left(\frac{m+n}{a}\right)^{1/3}$ and, $\cos\theta-\sin\theta = \left(\frac{m-n}{a}\right)^{1/3}$

$\Rightarrow \quad (\cos\theta+\sin\theta)^2 + (\cos\theta-\sin\theta)^2 = \left(\frac{m+n}{a}\right)^{2/3} + \left(\frac{m-n}{a}\right)^{2/3}$

$\Rightarrow \quad 2(\cos^2\theta+\sin^2\theta) = \frac{(m+n)^{2/3}}{a^{2/3}} + \frac{(m-n)^{2/3}}{a^{2/3}}$

$\Rightarrow \quad (m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$

EXAMPLE 16 *If* $2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1$, *prove that* $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$.

SOLUTION We have,

$$2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1$$

Dividing throughout by $\tan^2\alpha\tan^2\beta\tan^2\gamma$, we get

$\Rightarrow \quad 2+\cot^2\gamma+\cot^2\alpha+\cot^2\beta = \cot^2\alpha\cot^2\beta\cot^2\gamma$

$\Rightarrow \quad 2+\text{cosec}^2\gamma-1+\text{cosec}^2\alpha-1+\text{cosec}^2\beta-1 = (\text{cosec}^2\alpha-1)(\text{cosec}^2\beta-1)(\text{cosec}^2\gamma-1)$

$\Rightarrow \quad \text{cosec}^2\alpha+\text{cosec}^2\beta+\text{cosec}^2\gamma-1$

$= \text{cosec}^2\alpha\,\text{cosec}^2\beta\,\text{cosec}^2\gamma - \text{cosec}^2\alpha\,\text{cosec}^2\beta - \text{cosec}^2\beta\,\text{cosec}^2\gamma$
$\quad - \text{cosec}^2\gamma\,\text{cosec}^2\alpha + \text{cosec}^2\alpha + \text{cosec}^2\beta + \text{cosec}^2\gamma - 1$

$\Rightarrow \quad \text{cosec}^2\alpha\,\text{cosec}^2\beta\,\text{cosec}^2\gamma = \text{cosec}^2\alpha\,\text{cosec}^2\beta + \text{cosec}^2\beta\,\text{cosec}^2\gamma + \text{cosec}^2\gamma\,\text{cosec}^2\alpha$

$\Rightarrow \quad 1 = \sin^2\gamma+\sin^2\alpha+\sin^2\beta$ [Multiplying throughout by $\sin^2\alpha\sin^2\beta\sin^2\gamma$]

$\Rightarrow \quad \sin^2\alpha+\sin^2\beta+\sin^2\gamma = 1$

EXAMPLE 17 *If* $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2-b^2$, *and* $\frac{ax\sin\theta}{\cos^2\theta}-\frac{by\cos\theta}{\sin^2\theta}=0$, *prove that*

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$$

SOLUTION We have,

$$\frac{ax\sin\theta}{\cos^2\theta}-\frac{by\cos\theta}{\sin^2\theta}=0$$

$\Rightarrow \quad ax\sin^3\theta - by\cos^3\theta = 0$

$\Rightarrow \quad \frac{\sin^3\theta}{by} = \frac{\cos^3\theta}{ax}$

$\Rightarrow \quad \left(\frac{\sin^3\theta}{by}\right)^{2/3} = \left(\frac{\cos^3\theta}{ax}\right)^{2/3}$

$\Rightarrow \quad \frac{\sin^2\theta}{(by)^{2/3}} = \frac{\cos^2\theta}{(ax)^{2/3}}$

$\Rightarrow \quad \frac{\sin^2\theta}{(by)^{2/3}} = \frac{\cos^2\theta}{(ax)^{2/3}} = \frac{\sin^2\theta+\cos^2\theta}{(by)^{2/3}+(ax)^{2/3}}$ [Using ratio and proportions]

$$\Rightarrow \quad \frac{\sin^2\theta}{(by)^{2/3}} = \frac{\cos^2\theta}{(ax)^{2/3}} = \frac{1}{(ax)^{2/3}+(by)^{2/3}}$$

$$\Rightarrow \quad \sin^2\theta = \frac{(by)^{2/3}}{(ax)^{2/3}+(by)^{2/3}} \text{ and, } \cos^2\theta = \frac{(ax)^{2/3}}{(ax)^{2/3}+(by)^{2/3}}$$

$$\Rightarrow \quad \sin\theta = \frac{(by)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}} \text{ and, } \cos\theta = \frac{(ax)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}}$$

Substituting these values in $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2-b^2$, we get

$$(ax)^{2/3}\sqrt{(ax)^{2/3}+(by)^{2/3}}+(by)^{2/3}\sqrt{(ax)^{2/3}+(by)^{2/3}} = a^2-b^2$$

$$\Rightarrow \quad \left\{\sqrt{(ax)^{2/3}+(by)^{2/3}}\right\}\left\{(ax)^{2/3}+(by)^{2/3}\right\} = a^2-b^2$$

$$\Rightarrow \quad \left\{(ax)^{2/3}+(by)^{2/3}\right\}^{3/2} = a^2-b^2$$

$$\Rightarrow \quad (ax)^{2/3}+(by)^{2/3} = (a^2-b^2)^{2/3}$$

EXAMPLE 18 *If* $m^2+m'^2+2mm'\cos\theta=1$, $n^2+n'^2+2nn'\cos\theta=1$, *and* $mn+m'n'+(mn'+m'n)\cos\theta=0$, *prove that* (i) $m^2+n^2=\text{cosec}^2\theta$ (ii) $m'^2+n'^2=\text{cosec}^2\theta$

SOLUTION (i) We have,

$$m^2+m'^2+2mm'\cos\theta = 1 \text{ and } n^2+n'^2+2nn'\cos\theta = 1$$

$$\Rightarrow \quad m'^2+2mm'\cos\theta+m^2\cos^2\theta-m^2\cos^2\theta+m^2 = 1$$

and, $\quad n'^2+2nn'\cos\theta+n^2\cos^2\theta-n^2\cos^2\theta+n^2=1$

$$\Rightarrow \quad (m'+m\cos\theta)^2+m^2(1-\cos^2\theta)=1 \text{ and } (n'+n\cos\theta)^2+n^2(1-\cos^2\theta)=1$$

$$\Rightarrow \quad (m'+m\cos\theta)^2 = 1-m^2\sin^2\theta \quad ...(i) \quad \text{and,} \quad (n'+n\cos\theta)^2 = 1-n^2\sin^2\theta \quad ...(ii)$$

Now, $\quad (m'+m\cos\theta)(n'+n\cos\theta) = m'n'+(mn'+m'n)\cos\theta+mn\cos^2\theta$

$$\Rightarrow \quad (m'+m\cos\theta)(n'+n\cos\theta) = -mn+mn\cos^2\theta \quad [\because mn+m'n'+(mn'+m'n)\cos\theta=0]$$

$$\Rightarrow \quad (m'+m\cos\theta)(n'+n\cos\theta) = -mn(1-\cos^2\theta)$$

$$\Rightarrow \quad (m'+m\cos\theta)(n'+n\cos\theta) = -mn\sin^2\theta$$

$$\Rightarrow \quad (m'+m\cos\theta)^2(n'+n\cos\theta)^2 = m^2n^2\sin^4\theta \quad \text{[On squaring both sides]}$$

$$\Rightarrow \quad (1-m^2\sin^2\theta)(1-n^2\sin^2\theta) = m^2n^2\sin^4\theta \quad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad 1-(m^2+n^2)\sin^2\theta+m^2n^2\sin^4\theta = m^2n^2\sin^4\theta$$

$$\Rightarrow \quad 1 = (m^2+n^2)\sin^2\theta$$

$$\Rightarrow \quad m^2+n^2 = \text{cosec}^2\theta$$

(ii) As the given relations do not alter by replacing m by m' and n by n'. Therefore, on replacing m by m' and n by n' in $m^2+n^2=\text{cosec}^2\theta$, we get $m'+n'^2=\text{cosec}^2\theta$.

EXAMPLE 19 *If* $\frac{\sin^4\theta}{a}+\frac{\cos^4\theta}{b}=\frac{1}{a+b}$, *prove that*

(i) $\frac{\sin^8\theta}{a^3}+\frac{\cos^8\theta}{b^3}=\frac{1}{(a+b)^3}$ (ii) $\frac{\sin^{4n}\theta}{a^{2n-1}}+\frac{\cos^{4n}\theta}{b^{2n-1}}=\frac{1}{(a+b)^{2n-1}}$, $n\in N$

SOLUTION We have,

$$\frac{\sin^4\theta}{a}+\frac{\cos^4\theta}{b}=\frac{1}{a+b}$$

$$\Rightarrow \quad (a+b)\left(\frac{\sin^4\theta}{a}+\frac{\cos^4\theta}{b}\right)=1$$

$$\Rightarrow \quad (a+b)\left(\frac{\sin^4\theta}{a}+\frac{\cos^4\theta}{b}\right)=(\sin^2\theta+\cos^2\theta)^2$$

$$\Rightarrow \quad \sin^4\theta+\cos^4\theta+\frac{b}{a}\sin^4\theta+\frac{a}{b}\cos^4\theta=\sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad \frac{b}{a}\sin^4\theta+\frac{a}{b}\cos^4\theta-2\sin^2\theta\cos^2\theta=0$$

$$\Rightarrow \quad \left(\sqrt{\frac{b}{a}}\sin^2\theta-\sqrt{\frac{a}{b}}\cos^2\theta\right)^2=0$$

$$\Rightarrow \quad \sqrt{\frac{b}{a}}\sin^2\theta=\sqrt{\frac{a}{b}}\cos^2\theta$$

$$\Rightarrow \quad \tan^2\theta=\frac{a}{b}$$

$$\Rightarrow \quad \frac{\sin^2\theta}{a}=\frac{\cos^2\theta}{b}$$

$$\Rightarrow \quad \frac{\sin^2\theta}{a}=\frac{\cos^2\theta}{b}=\frac{\sin^2\theta+\cos^2\theta}{a+b}$$

$$\Rightarrow \quad \frac{\sin^2\theta}{a}=\frac{\cos^2\theta}{b}=\frac{1}{a+b}$$

$$\Rightarrow \quad \sin^2\theta=\frac{a}{a+b},\ \cos^2\theta=\frac{b}{a+b} \qquad \text{...(i)}$$

(i) $$\frac{\sin^8\theta}{a^3}+\frac{\cos^8\theta}{b^3}=\frac{1}{a^3}(\sin^2\theta)^4+\frac{1}{b^3}(\cos^2\theta)^4$$

$$=\frac{1}{a^3}\left(\frac{a}{a+b}\right)^4+\frac{1}{b^3}\left(\frac{b}{a+b}\right)^4 \qquad \text{[Using (i)]}$$

$$=\frac{a}{(a+b)^4}+\frac{b}{(a+b)^4}=\frac{a+b}{(a+b)^4}=\frac{1}{(a+b)^3}$$

(ii) $$\frac{\sin^{4n}\theta}{a^{2n-1}}+\frac{\cos^{4n}\theta}{b^{2n-1}}=\frac{(\sin^2\theta)^{2n}}{a^{2n-1}}+\frac{(\cos^2\theta)^{2n}}{b^{2n-1}}$$

$$=\frac{1}{a^{2n-1}}\left(\frac{a}{a+b}\right)^{2n}+\frac{1}{b^{2n-1}}\left(\frac{b}{a+b}\right)^{2n}$$

$$=\frac{a}{(a+b)^{2n}}+\frac{b}{(a+b)^{2n}}=\frac{a+b}{(a+b)^{2n}}=\frac{1}{(a+b)^{2n-1}}$$

EXAMPLE 20 *If* $\dfrac{\cos^4\alpha}{\cos^2\beta}+\dfrac{\sin^4\alpha}{\sin^2\beta}=1$, *prove that*

(i) $\sin^4\alpha+\sin^4\beta=2\sin^2\alpha\sin^2\beta$ (ii) $\dfrac{\cos^4\beta}{\cos^2\alpha}+\dfrac{\sin^4\beta}{\sin^2\alpha}=1$

SOLUTION We have,

$$\frac{\cos^4\alpha}{\cos^2\beta}+\frac{\sin^4\alpha}{\sin^2\beta}=1$$

$$\Rightarrow \quad \cos^4\alpha\sin^2\beta+\sin^4\alpha\cos^2\beta=\cos^2\beta\sin^2\beta$$

$$\Rightarrow \quad \cos^4\alpha(1-\cos^2\beta)+\cos^2\beta(1-\cos^2\alpha)^2=\cos^2\beta(1-\cos^2\beta)$$

$$\Rightarrow \quad \cos^4\alpha-\cos^4\alpha\cos^2\beta+\cos^2\beta-2\cos^2\alpha\cos^2\beta+\cos^4\alpha\cos^2\beta=\cos^2\beta-\cos^4\beta$$

$$\Rightarrow \quad \cos^4\alpha-2\cos^2\alpha\cos^2\beta+\cos^4\beta=0$$

$$\Rightarrow \quad (\cos^2\alpha-\cos^2\beta)^2=0$$

$$\Rightarrow \quad \cos^2\alpha-\cos^2\beta=0$$

$$\Rightarrow \quad \cos^2\alpha=\cos^2\beta \qquad \text{...(i)}$$

$$\Rightarrow \quad 1-\sin^2\alpha=1-\sin^2\beta$$

$$\Rightarrow \quad \sin^2\alpha=\sin^2\beta \qquad \text{... (ii)}$$

(i) $$\sin^4\alpha+\sin^4\beta=(\sin^2\alpha-\sin^2\beta)^2+2\sin^2\alpha\sin^2\beta$$
$$=2\sin^2\alpha\sin^2\beta \qquad [\because \sin^2\alpha=\sin^2\beta]$$

(ii) $$\frac{\cos^4\beta}{\cos^2\alpha}+\frac{\sin^4\beta}{\sin^2\alpha}=\frac{\cos^2\beta\cos^2\beta}{\cos^2\alpha}+\frac{\sin^2\beta\sin^2\beta}{\sin^2\alpha}$$
$$=\frac{\cos^2\beta\ \cos^2\alpha}{\cos^2\alpha}+\frac{\sin^2\beta\ \sin^2\alpha}{\sin^2\alpha} \qquad \text{[Using (i)and (ii)]}$$
$$=\cos^2\beta+\sin^2\beta$$

EXAMPLE 21 *If x is any non-zero real number, show that* $\cos\theta$ *and* $\sin\theta$ *can never be equal to* $x+\frac{1}{x}$.

SOLUTION We have following cases:

CASE I *When* $x>0$: In this case, we have

$$x+\frac{1}{x}=(\sqrt{x})^2+\left(\frac{1}{\sqrt{x}}\right)^2-2\times\sqrt{x}\times\frac{1}{\sqrt{x}}+2\times\sqrt{x}\times\frac{1}{\sqrt{x}}$$

$$\Rightarrow \quad x+\frac{1}{x}=\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2+2\ge 2 \qquad \left[\because \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2\ge 0\right]$$

CASE II *When* $x<0$: Let $x=-y$. Then, $y>0$

$$\Rightarrow \quad x+\frac{1}{x}=-y-\frac{1}{y}=-\left(y+\frac{1}{y}\right)$$

But, $y+\frac{1}{y}\ge 2$ [From Case I]

$$\Rightarrow \quad -\left(y+\frac{1}{y}\right)\le -2 \Rightarrow x+\frac{1}{x}\le -2$$

$x+\frac{1}{x}\ge 2$ for $x>0$ and, $x+\frac{1}{x}\le -2$ for $x<0$

But, $-1\le\sin\theta\le 1$ and $-1\le\cos\theta\le 1$ for all θ.

Hence, $\sin\theta$ and [illegible] equal to $x+\frac{1}{x}$ for any non-zero [illegible]

EXAMPLE 22 *If $A=\cos^2\theta+\sin^4\theta$, prove that $\frac{3}{4}\le A\le 1$ for all values of θ.*

SOLUTION We have

$$A=\cos^2\theta+\sin^4\theta=\cos^2\theta+(\sin^2\theta)^2$$

Now, $-1\le\sin\theta\le1$ for all θ

$\Rightarrow$ $0\le\sin^2\theta\le1$ for all θ

$\Rightarrow$ $(\sin^2\theta)^2\le\sin^2\theta$ [For $0<x<1$, $x^n<x$ for all $n\in N-\{1\}$]

$\Rightarrow$ $\cos^2\theta+(\sin^2\theta)^2\le\cos^2\theta+\sin^2\theta$ for all θ

$\Rightarrow$ $A\le1$ for all θ ...(i)

Again,

$$A=\cos^2\theta+\sin^4\theta$$

$\Rightarrow$ $A=1-\sin^2\theta+(\sin^2\theta)^2$

$\Rightarrow$ $A=1-\frac{1}{4}+\left\{\frac{1}{4}-\sin^2\theta+(\sin^2\theta)^2\right\}$

$\Rightarrow$ $A=\frac{3}{4}+\left(\frac{1}{2}-\sin^2\theta\right)^2$

Now,

$$\left(\frac{1}{2}-\sin^2\theta\right)^2\ge0 \quad \text{for all } \theta$$

$\Rightarrow$ $\frac{3}{4}+\left(\frac{1}{2}-\sin^2\theta\right)^2\ge\frac{3}{4}$ for all θ

$\Rightarrow$ $A\ge\frac{3}{4}$ for all θ ...(ii)

From (i) and (ii), we obtain $\frac{3}{4}\le A\le1$ for all θ

EXERCISE 5.1

LEVEL-1

Prove the following identities (1-16)

1. $\sec^4\theta-\sec^2\theta=\tan^4\theta+\tan^2\theta$
2. $\sin^6\theta+\cos^6\theta=1-3\sin^2\theta\cos^2\theta$
3. $(\text{cosec}\,\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1$
4. $\text{cosec}\,\theta(\sec\theta-1)-\cot\theta(1-\cos\theta)=\tan\theta-\sin\theta$
5. $\dfrac{1-\sin A\cos A}{\cos A(\sec A-\text{cosec}\,A)}\cdot\dfrac{\sin^2A-\cos^2A}{\sin^3A+\cos^3A}=\sin A$
6. $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}=(\sec A\,\text{cosec}\,A+1)$
7. $\dfrac{\sin^3A+\cos^3A}{\sin A+\cos A}+\dfrac{\sin^3A-\cos^3A}{\sin A-\cos A}=2$
8. $(\sec A\sec B+\tan A\tan B)^2-(\sec A\tan B+\tan A\sec B)^2=1$
9. $\dfrac{\cos\theta}{1-\sin\theta}=\dfrac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta}$

10. $\dfrac{\tan^3\theta}{1+\tan^2\theta}+\dfrac{\cot^3\theta}{1+\cot^2\theta}=\dfrac{1-2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}$

11. $1-\dfrac{\sin^2\theta}{1+\cot\theta}+\dfrac{\cos^2\theta}{1+\tan\theta}=\sin\theta\cos\theta$

12. $\left(\dfrac{1}{\sec^2\theta-\cos^2\theta}+\dfrac{1}{\operatorname{cosec}^2\theta-\sin^2\theta}\right)\sin^2\theta\cos^2\theta=\dfrac{1-\sin^2\theta\cos^2\theta}{2+\sin^2\theta\cos^2\theta}$

13. $(1+\tan\alpha\tan\beta)^2+(\tan\alpha-\tan\beta)^2=\sec^2\alpha\sec^2\beta$

14. $\dfrac{(1+\cot\theta+\tan\theta)(\sin\theta-\cos\theta)}{\sec^3\theta-\operatorname{cosec}^3\theta}=\sin^2\theta\cos^2\theta$

15. $\dfrac{2\sin\theta\cos\theta-\cos\theta}{1-\sin\theta+\sin^2\theta-\cos^2\theta}=\cot\theta$

16. $\cos\theta(\tan\theta+2)(2\tan\theta+1)=2\sec\theta+5\sin\theta$

17. If $x=\dfrac{2\sin\theta}{1+\cos\theta+\sin\theta}$, then prove that $\dfrac{1-\cos\theta+\sin\theta}{1+\sin\theta}$ is also equal to x.

[NCERT EXEMPLAR]

18. If $\sin\theta=\dfrac{a^2-b^2}{a^2+b^2}$, find the values of $\tan\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$

19. If $\tan x=\dfrac{b}{a}$, then find the value of $\sqrt{\dfrac{a+b}{a-b}}+\sqrt{\dfrac{a-b}{a+b}}$. [NCERT EXEMPLAR]

20. If $\tan\theta=\dfrac{a}{b}$, show that $\dfrac{a\sin\theta-b\cos\theta}{a\sin\theta+b\cos\theta}=\dfrac{a^2-b^2}{a^2+b^2}$

LEVEL-2

21. If $\operatorname{cosec}\theta-\sin\theta=a^3$, $\sec\theta-\cos\theta=b^3$, then prove that $a^2b^2(a^2+b^2)=1$.

22. If $\cot\theta(1+\sin\theta)=4m$ and $\cot\theta(1-\sin\theta)=4n$, prove that $(m^2-n^2)^2=mn$.

23. If $\sin\theta+\cos\theta=m$, then prove that

$$\sin^6\theta+\cos^6\theta=\frac{4-3(m^2-1)^2}{4},\text{ where } m^2\le 2$$

24. If $a=\sec\theta-\tan\theta$ and $b=\operatorname{cosec}\theta+\cot\theta$, then show that $ab+a-b+1=0$.

[NCERT EXEMPLAR]

25. Prove that: $\left|\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}+\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}\right|=-\dfrac{2}{\cos\theta}$, where $\dfrac{\pi}{2}<\theta<\pi$

26. If $T_n=\sin^n\theta+\cos^n\theta$, prove that

(i) $\dfrac{T_3-T_5}{T_1}=\dfrac{T_5-T_7}{T_3}$ (ii) $2T_6-3T_4+1=0$ (iii) $6T_{10}-15T_8+10T_6-1=0$

ANSWERS

18. $\tan\theta=\dfrac{a^2-b^2}{2ab}$, $\sec\theta=\dfrac{a^2+b^2}{2ab}$, $\operatorname{cosec}\theta=\dfrac{a^2+b^2}{a^2-b^2}$ 19. $\dfrac{2\cos x}{\sqrt{\cos^2x-\sin^2x}}$.

HINTS TO SELECTED PROBLEMS

21. We have, $\operatorname{cosec}\theta - \sin\theta = a^3$, $\sec\theta - \cos\theta = b^3$

$$\Rightarrow \quad \frac{1-\sin^2\theta}{\sin\theta} = a^3, \frac{1-\cos^2\theta}{\cos\theta} = b^3$$

$$\Rightarrow \quad \frac{\cos^2\theta}{\sin\theta} = a^3, \frac{\sin^2\theta}{\cos\theta} = b^3$$

$$\Rightarrow \quad \frac{\sin^2\theta}{\cos\theta} \div \frac{\cos^2\theta}{\sin\theta} = \frac{b^3}{a^3}$$

$$\Rightarrow \quad \tan^3\theta = \frac{b^3}{a^3} \Rightarrow \tan\theta = \frac{b}{a} \Rightarrow \sin\theta = \frac{b}{\sqrt{a^2+b^2}} \text{ and } \cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

Substituting these values of $\sin\theta$ and $\cos\theta$ in $\frac{\cos^2\theta}{\sin\theta} = a^3$, we obtain

$$\frac{a^2}{b\sqrt{(a^2+b^2)}} = a^3 \Rightarrow ab\sqrt{a^2+b^2} = 1 \Rightarrow a^2b^2(a^2+b^2) = 1$$

22. We have, $\cot\theta(1+\sin\theta) = 4m$ and $\cot\theta(1-\sin\theta) = 4n$

$$\Rightarrow \quad \cot\theta + \cos\theta = 4m \text{ and } \cot\theta - \cos\theta = 4n$$

$$\Rightarrow \quad (\cot\theta+\cos\theta)^2 - (\cot\theta-\cos\theta)^2 = 16m^2 - 16n^2 \text{ and } (\cot\theta+\cos\theta)(\cot\theta-\cos\theta) = 16mn$$

$$\Rightarrow \quad 4\cot\theta\cos\theta = 16(m^2-n^2) \text{ and } \cot^2\theta - \cos^2\theta = 16mn$$

$$\Rightarrow \quad \frac{\cos^2\theta}{\sin\theta} = 4(m^2-n^2) \text{ and } \frac{\cos^4\theta}{\sin^2\theta} = 16mn$$

$$\Rightarrow \quad \frac{\cos^4\theta}{\sin^2\theta} = 16(m^2-n^2)^2 \text{ and } \frac{\cos^4\theta}{\sin^2\theta} = 16mn$$

$$\Rightarrow \quad 16(m^2-n^2)^2 = 16mn$$

$$\Rightarrow \quad (m^2-n^2) = mn$$

5.4 SIGNS OF THE TRIGONOMETRIC RATIOS OR FUNCTIONS

In the previous section, we have introduced six trigonometric ratios. Their signs depend on the quadrant in which the terminal side of the angle lies. We always take the length $OP = r$ (see Fig. 5.1 — 5.4) to be positive.

Thus, $\sin\theta = \frac{y}{r}$ has the sign of y, $\cos\theta = \frac{x}{r}$ has the sign of x. The sign of $\tan\theta$ depends on the signs of x and y and similarly the signs of other trigonometric ratios are decided by the signs of x and/or y. Thus, we have the following :

In first quadrant: We have, $x > 0$ and $y > 0$

$\therefore \quad \sin\theta = \frac{y}{r} > 0$, $\cos\theta = \frac{x}{r} > 0$, $\tan\theta = \frac{y}{x} > 0$, $\operatorname{cosec}\theta = \frac{r}{y} > 0$, $\sec\theta = \frac{r}{x} > 0$ and $\cot\theta = \frac{x}{y} > 0$.

Thus, in the first quadrant all trigonometric functions are positive.

In second quadrant: We have, $x < 0$ and $y > 0$

$\therefore \quad \sin\theta = \frac{y}{r} > 0$, $\cos\theta = \frac{x}{r} < 0$, $\tan\theta = \frac{y}{x} < 0$, $\operatorname{cosec}\theta = \frac{r}{y} > 0$, $\sec\theta = \frac{r}{x} < 0$ and $\cot\theta = \frac{x}{y} < 0$.

Thus, in the second quadrant sine and cosecant functions are positive and all others are negative.

In third quadrant: We have, $x < 0$ and $y < 0$

$\therefore$ $\sin\theta = \frac{y}{r} < 0$, $\cos\theta = \frac{x}{r} < 0$, $\tan\theta = \frac{y}{x} > 0$, $\text{cosec}\,\theta = \frac{r}{y} < 0$, $\sec\theta = \frac{r}{x} < 0$ and $\cot\theta = \frac{x}{y} > 0$.

Thus, in the third quadrant all trigonometric functions are negative except tangent and its reciprocal cotangent.

In fourth quadrant: We have, $x > 0$ and $y < 0$

$\therefore$ $\sin\theta = \frac{y}{r} > 0$, $\cos\theta = \frac{x}{r} > 0$, $\tan\theta = \frac{y}{x} > 0$, $\text{cosec}\,\theta = \frac{r}{y} < 0$, $\sec\theta = \frac{r}{x} > 0$ and $\cot\theta = \frac{x}{y} < 0$.

Thus, in the fourth quadrant all trigonometric functions are negative except cosine and its reciprocal secant.

It follows from the above discussion that the signs of the trigonometric ratios in different quadrants are as under :

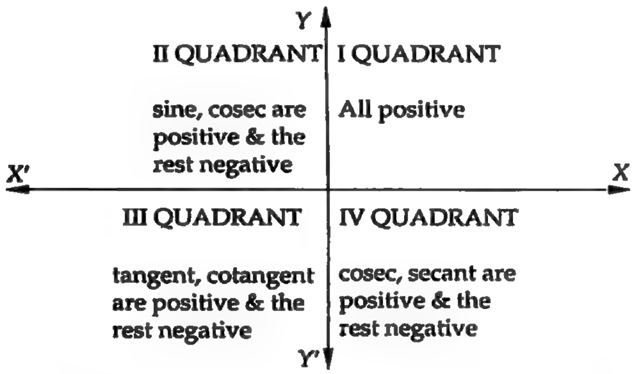


Fig. 5.6

SIMPLE RULE TO REMEMBER *A crude aid to memorise the signs of trigonometrical ratios in different quadrants is the four-word phrase "ALL SCHOOL TO COLLEGE". The first letter of the first word in this phrase is 'A'. This may be taken to indicate that all trigonometric ratios are positive in the first quadrant. The first letter of the second word is 'S'. This indicates that sine and its reciprocal are positive in the second quadrant. The first letter of third word is 'T'. This may be taken as to indicate that tangent and its reciprocal are positive in the third quadrant. The first letter of the fourth word in the phrase is 'C' which may be taken as to indicate that only cosine and its reciprocal are positive in the fourth quadrant.*

5.5 VARIATIONS IN VALUES OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS

DIFFERENT QUADRANTS Let $X'OX$ and YOY' be the coordinate axes. Draw a circle with centre at origin O and radius unity. Suppose the circle cuts the coordinate axes at A, B, A' and B' as shown in Fig. 5.7. Let $P(x, y)$ be a point on the circle such that $\angle AOP = \theta$. Then, $x = \cos\theta$ and $y = \sin\theta$.

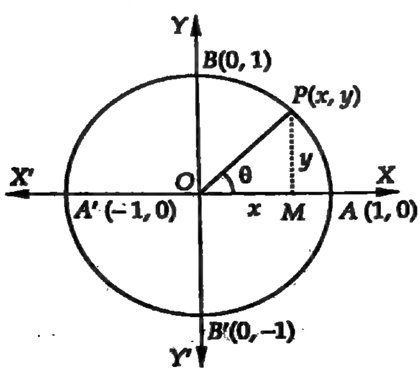


Fig. 5.7

It is evident from the Fig. 5.7 that

$$-1 \le x \le 1 \text{ and } -1 \le y \le 1$$

$$\Rightarrow \quad -1 \le \cos\theta \le 1 \text{ and } -1 \le \sin\theta \le 1 \text{ for all values of } \theta.$$

In the first quadrant as the angle θ increases from 0° to 90°, we observe that the values of y increase from 0 to 1. Consequently $\sin\theta$ increases from 0 to 1. In the second quadrant as θ increases from 90° to 180°, y decreases from 1 to 0. Consequently, $\sin\theta$ decreases from 1 to 0. In the third quadrant as θ increases from 180° to 270°, $\sin\theta$ $(= y)$ decreases from 0 to –1 and finally in the fourth quadrant $\sin\theta$ increases from –1 to 0 as θ increases from 270° to 360°. Similarly, we can observe the variations in the values of other trigonometric functions. The following table exhibits the variations in the values of all trigonometrical ratios.

	I QUADRANT		II QUADRANT
$\sin\theta$	increases from 0 to 1	$\sin\theta$	decreases from 1 to 0
$\cos\theta$	decreases from 1 to 0	$\cos\theta$	decreases from 0 to –1
$\tan\theta$	increases from 0 to ∞	$\tan\theta$	increases from $-\infty$ to 0
$\cot\theta$	decreases from ∞ to 0	$\cot\theta$	decreases from 0 to $-\infty$
$\sec\theta$	increases from 1 to ∞	$\sec\theta$	increases from $-\infty$ to –1
$\operatorname{cosec}\theta$	decreases from ∞ to 1	$\operatorname{cosec}\theta$	decreases from 1 to ∞
	III QUADRANT		IV QUADRANT
$\sin\theta$	decreases from 0 to –1	$\sin\theta$	increases from –1 to 0
$\cos\theta$	decreases from –1 to 0	$\cos\theta$	increases from 0 to 1
$\tan\theta$	increases from 0 to ∞	$\tan\theta$	increases from $-\infty$ to 0
$\cot\theta$	decreases from ∞ to 0	$\cot\theta$	decreases from 0 to $-\infty$
$\sec\theta$	decreases from –1 to $-\infty$	$\sec\theta$	decreases from ∞ to 1
$\operatorname{cosec}\theta$	decreases from $-\infty$ to –1	$\operatorname{cosec}\theta$	decreases from –1 to ∞

<u>REMARK</u> *Note that* $+\infty$ and $-\infty$ *are two symbols. These are not real numbers. When we say that* $\tan\theta$ *increases from 0 to ∞ as θ varies from 0 to $\pi/2$, it means that* $\tan\theta$ *increases in the interval* $(0, \pi/2)$ *and it attains arbitrarily large positive values as θ tends to $\pi/2$. Similarly, we interpret for other trigonometrical functions.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find* $\sin\theta$ and $\tan\theta$ *if* $\cos\theta = -\frac{12}{13}$ *and θ lies in the third quadrant.*

SOLUTION We have,

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \sin\theta = \pm\sqrt{1-\cos^2\theta}$$

In third quadrant $\sin\theta$ is negative.

$$\therefore \quad \sin\theta = -\sqrt{1-\cos^2\theta} \Rightarrow \sin\theta = -\sqrt{1-\left(-\frac{12}{13}\right)^2} = -\frac{5}{13}$$

Now, $$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\frac{5}{13}\times\frac{13}{-12} = \frac{5}{12}$$

EXAMPLE 2 *Find the values of* $\cos\theta$ *and* $\tan\theta$, *if* $\sin\theta = -\frac{3}{5}$ *and* $\pi < \theta < \frac{3\pi}{2}$.

SOLUTION We have,

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

In the third quadrant $\cos\theta$ is negative and $\tan\theta$ is positive.

$$\therefore \quad \cos\theta = -\sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = -\sqrt{1-\frac{9}{25}} = -\frac{4}{5}$$

and, $$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\frac{3}{5} \times -\frac{5}{4} = \frac{3}{4}$$

EXAMPLE 3 *Find all other trigonometrical ratios if* $\sin\theta = -\frac{2\sqrt{6}}{5}$ *and* θ *lies in quadrant III.*

SOLUTION We have,

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

In the third quadrant $\cos\theta$ is negative.

$$\therefore \quad \cos\theta = -\sqrt{1-\sin^2\theta} \Rightarrow \cos\theta = -\sqrt{1-\frac{24}{25}} = -\frac{1}{5}$$

In the third quadrant $\tan\theta$ is positive.

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\frac{2\sqrt{6}}{5} \times -\frac{5}{1} = 2\sqrt{6}$$

Now, $$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \Rightarrow \operatorname{cosec}\theta = -\frac{5}{2\sqrt{6}}$$

$$\sec\theta = \frac{1}{\cos\theta} \Rightarrow \sec\theta = -5$$

and, $$\cot\theta = \frac{1}{\tan\theta} \Rightarrow \cot\theta = \frac{1}{2\sqrt{6}}.$$

EXAMPLE 4 *If* $\cos\theta = -\frac{1}{2}$ *and* $\pi < \theta < \frac{3\pi}{2}$, *find the value of* $4\tan^2\theta - 3\operatorname{cosec}^2\theta$.

SOLUTION Since θ lies in the third quadrant. Therefore, $\sin\theta$ is negative and $\tan\theta$ is positive.

Now, $$\sin\theta = \pm\sqrt{1-\cos^2\theta} \Rightarrow \sin\theta = -\sqrt{1-\frac{1}{4}} = -\frac{\sqrt{3}}{2} \Rightarrow \operatorname{cosec}\theta = \frac{-2}{\sqrt{3}}$$

And, $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \sqrt{3}.$$

Hence, $$4\tan^2\theta - 3\operatorname{cosec}^2\theta = 4\times 3 - 3\times\frac{4}{3} = 8.$$

EXAMPLE 5 *If* $\sec\theta = \sqrt{2}$ *and* $\frac{3\pi}{2} < \theta < 2\pi$, *find the value of* $\frac{1+\tan\theta+\operatorname{cosec}\theta}{1+\cot\theta-\operatorname{cosec}\theta}$.

SOLUTION We have,

$$\sec\theta = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$$

But, θ lies in the fourth quadrant in which $\sin\theta$ is negative.

$$\therefore \quad \sin\theta = -\frac{1}{\sqrt{2}} \Rightarrow \operatorname{cosec}\theta = -\sqrt{2}$$

Now, $\tan\theta = \dfrac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{1} = -1 \Rightarrow \cot\theta = -1$

$$\therefore \quad \frac{1+\tan\theta+\operatorname{cosec}\theta}{1+\cot\theta-\operatorname{cosec}\theta} = \frac{1-1-\sqrt{2}}{1-1+\sqrt{2}} = -1$$

LEVEL-2

EXAMPLE 6 *Prove that :*

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \begin{cases} \sec\theta-\tan\theta & , \text{ if } -\frac{\pi}{2}<\theta<\frac{\pi}{2} \\ -\sec\theta+\tan\theta, & \text{if } \frac{\pi}{2}<\theta<\frac{3\pi}{2} \end{cases}$$

SOLUTION We have,

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}} = \frac{1-\sin\theta}{\sqrt{\cos^2\theta}} = \frac{1-\sin\theta}{|\cos\theta|}$$

$$\Rightarrow \quad \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \begin{cases} \dfrac{1-\sin\theta}{\cos\theta}, & \text{if } -\frac{\pi}{2}<\theta<\frac{\pi}{2} \\ \dfrac{1-\sin\theta}{-\cos\theta}, & \text{if } \frac{\pi}{2}<\theta<\frac{3\pi}{2} \end{cases}$$

$$\Rightarrow \quad \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \begin{cases} \sec\theta-\tan\theta & , \text{ if } -\frac{\pi}{2}<\theta<\frac{\pi}{2} \\ -\sec\theta+\tan\theta, & \text{if } \frac{\pi}{2}<\theta<\frac{3\pi}{2} \end{cases}$$

EXAMPLE 7 Prove that : $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \begin{cases} \operatorname{cosec}\theta+\cot\theta & , \text{ if } 0<\theta<\pi \\ -\operatorname{cosec}\theta-\cot\theta, & \text{if } \pi<\theta<2\pi \end{cases}$

SOLUTION We have,

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$$

$$\Rightarrow \quad \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{1+\cos\theta}{\sqrt{\sin^2\theta}} = \frac{1+\cos\theta}{|\sin\theta|}$$

$$\Rightarrow \quad \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \begin{cases} \dfrac{1+\cos\theta}{\sin\theta}, & \text{if } 0<\theta<\pi \\ \dfrac{1+\cos\theta}{-\sin\theta}, & \text{if } \pi<\theta<\theta<2\pi \end{cases}$$

$$\Rightarrow \quad \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \begin{cases} \operatorname{cosec}\theta+\cot\theta & , \text{ if } 0<\theta<\pi \\ -\operatorname{cosec}\theta-\cot\theta, & \text{if } \pi<\theta<2\pi \end{cases}$$

EXAMPLE 8 Prove that :

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \begin{cases} \dfrac{2}{\cos\theta}, & \text{if } 0\le\theta<\frac{\pi}{2} \\ -\dfrac{2}{\cos\theta}, & \text{if } \frac{\pi}{2}<\theta\le\pi \end{cases}$$

SOLUTION We have,

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \frac{(1-\sin\theta)+(1+\sin\theta)}{\sqrt{1-\sin^2\theta}}$$

$$\Rightarrow \quad \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}+\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\frac{2}{\sqrt{\cos^2\theta}}=\frac{2}{|\cos\theta|} \qquad [\because \sqrt{x^2}=|x|]$$

$$\Rightarrow \quad \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}+\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\begin{cases}\dfrac{2}{\cos\theta}, & \text{if } 0\le\theta<\dfrac{\pi}{2}\\ -\dfrac{2}{\cos\theta}, & \text{if } \dfrac{\pi}{2}<\theta\le\pi\end{cases}$$

EXERCISE 5.2

LEVEL-1

1. Find the values of the other five trigonometric functions in each of the following:
   (i) $\cot\theta=\frac{12}{5}$, θ in quadrant III (ii) $\cos\theta=-\frac{1}{2}$, θ in quadrant II
   (iii) $\tan\theta=\frac{3}{4}$, θ in quadrant III (iv) $\sin\theta=\frac{3}{5}$, θ in quadrant I
2. If $\sin\theta=\frac{12}{13}$ and θ lies in the second quadrant, find the value of $\sec\theta+\tan\theta$.
3. If $\sin\theta=\frac{3}{5}$, $\tan\phi=\frac{1}{2}$ and $\frac{\pi}{2}<\theta<\pi<\phi<\frac{3\pi}{2}$, find the value of $8\tan\theta-\sqrt{5}\sec\phi$.
4. If $\sin\theta+\cos\theta=0$ and θ lies in the fourth quadrant, find $\sin\theta$ and $\cos\theta$.
5. If $\cos\theta=-\frac{3}{5}$ and $\pi<\theta<\frac{3\pi}{2}$, find the values of other five trigonometric functions and hence evaluate $\frac{\operatorname{cosec}\theta+\cot\theta}{\sec\theta-\tan\theta}$.

ANSWERS

1. (i) $\sin\theta=-\frac{5}{13}$, $\cos\theta=-\frac{12}{13}$, $\tan\theta=\frac{5}{12}$, $\operatorname{cosec}\theta=-\frac{13}{5}$, $\sec\theta=-\frac{13}{12}$
   (ii) $\sin\theta=\frac{\sqrt{3}}{2}$, $\tan\theta=-\sqrt{3}$, $\operatorname{cosec}\theta=\frac{2}{\sqrt{3}}$, $\cot\theta=\frac{-1}{\sqrt{3}}$, $\sec\theta=-2$
   (iii) $\sin\theta=-\frac{3}{5}$, $\cos\theta=-\frac{4}{5}$, $\operatorname{cosec}\theta=-\frac{5}{3}$, $\sec\theta=-\frac{5}{4}$, $\cot\theta=\frac{4}{3}$
   (iv) $\cos\theta=\frac{4}{5}$, $\tan\theta=\frac{3}{4}$, $\sec\theta=\frac{5}{4}$, $\cot\theta=\frac{4}{3}$, $\operatorname{cosec}\theta=\frac{5}{3}$
2. -5 3. $-\frac{7}{2}$ 4. $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ 5. $\frac{1}{6}$

HINTS TO SELECTED PROBLEM

4. We have,
$$\sin\theta+\cos\theta=0 \Rightarrow \sin\theta=-\cos\theta \Rightarrow \tan\theta=-1$$
$$\therefore \sec^2\theta=1+\tan^2\theta \Rightarrow \sec^2\theta=1+(-1)^2=2 \Rightarrow \sec\theta=\sqrt{2}\Rightarrow \cos\theta=\frac{1}{\sqrt{2}}$$

5.6 TRIGONOMETRIC RATIOS OF ALLIED ANGLES

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90°.

The angles $-\theta$, $90°\pm\theta$, $180°\pm\theta$, $360°\pm\theta$ etc are angles allied to the angle θ if θ is measured in degrees. However, if θ is measured in radians, then the angles allied to θ are $-\theta$, $\frac{\pi}{2}\pm\theta$, $\pi\pm\theta$, $2\pi\pm\theta$ etc. Using trigonometric ratios of allied angles we can find the trigonometric ratios of angles of any magnitude.

5.6.1 TRIGONOMETRIC RATIOS OF $(-\theta)$ IN TERMS OF THAT OF θ

Let a revolving ray starting from its initial position OX, trace out an angle $\angle XOA = \theta$. Let $P(x, y)$ be a point on OA such that $OP = r$. Draw PM perpendicular from P on x-axis. Let there be another revolving ray OA' which starts from the initial position OX and describes an angle $\angle XOA' = -\theta$ in the clockwise sense. Let P' be a point on OA' such that $OP' = OP$. Draw $P'M'$ perpendicular from P' on x-axis. Clearly, M and M' coincide and ΔOMP is congruent to $\Delta OMP'$. Then the coordinates of P' are $(x, -y)$.

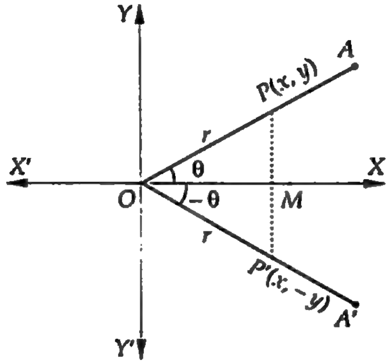


Fig. 5.8

$\therefore \quad \sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta, \ \cos(-\theta) = \frac{x}{r} = \cos\theta, \ \tan(-\theta) = -\frac{y}{x} = -\tan\theta.$

Taking the reciprocals of these trigonometric ratios, we have

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta,\ \sec(-\theta) = \sec\theta \text{ and } \cot(-\theta) = -\cot\theta.$$

5.6.2 TRIGONOMETRIC RATIOS OF $(90°-\theta)$ IN TERMS OF THAT OF θ

Let a revolving ray starting from its initial position OX, trace out an angle $\angle XOA = \theta$. Let $P(x, y)$ be a point on OA such that $OP = r$. Draw PM perpendicular from P on x-axis. Let another revolving ray OA' starting from the initial position OX, trace out angle of $90°$ to coincide with OY and then it rotates in the clockwise direction through an angle θ. Then, OA' in its final position traces out an angle $\angle XOA' = 90° - \theta$. Let P' be a point on OA' such that $OP' = OP = r$. From P' draw perpendicular $P'M'$ on x-axis. Then, ΔOMP and $OM'P'$ are congruent.

Clearly, $M'P' = OM = x$ and $OM' = MP = y$

So, the coordinates of P' are (y, x).

$$\therefore \quad \sin(90-\theta) = \frac{P'M'}{OP'} = \frac{x}{r} = \cos\theta.$$

$$\cos(90-\theta) = \frac{OM'}{OP'} = \frac{y}{r} = \sin\theta$$

$$\tan(90-\theta) = \frac{P'M'}{OM'} = \frac{x}{y} = \cot\theta$$

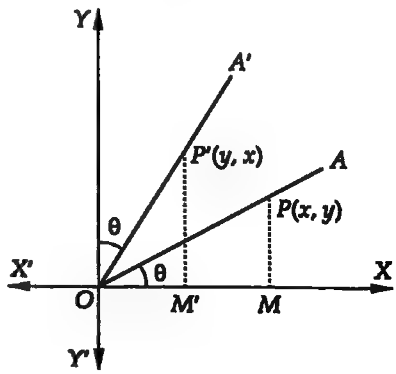


Fig. 5.9

Similarly, we obtain that

$$\operatorname{cosec}(90-\theta) = \sec\theta,\ \sec(90-\theta) = \operatorname{cosec}\theta \text{ and, } \cot(90-\theta) = \tan\theta.$$

5.6.3 TRIGONOMETRIC RATIOS OF (90° + θ) IN TERMS OF THAT OF θ

Let a revolving ray OA starting from its initial position OX, trace out an angle $\angle XOA = \theta$ and let another revolving ray OA' starting from the same initial position OX, first trace out an angle θ so as to coincide with OA and the it revolves through an angle of 90° in anticlockwise direction to form an angle $\angle XOA' = 90° + \theta$. Let P and P' be points on OA and OA' respectively such that $OP = OP' = r$. Draw perpendiculars PM and $P'M'$ from P and P' respectively on OX. Let the coordinates of P be (x, y). Then, $OM = x$ and $P'M' = y$.

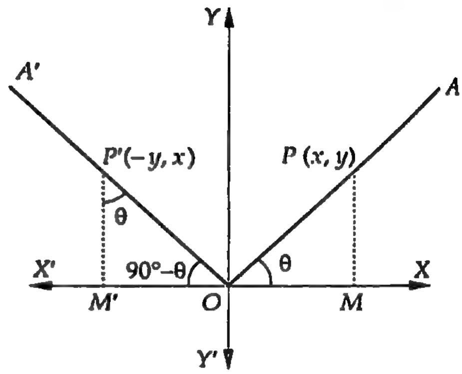


Fig. 5.10

Clearly, $\Delta P'M'O$ is congruent to ΔOMP.

$\therefore \quad OM' = PM = y$ and $P'M' = OM = x$.

So, the coordinates of P' are $(-y, x)$.

Hence,

$$\sin(90+\theta) = \frac{M'P'}{OP'} = \frac{x}{r} = \cos\theta.$$

$$\cos(90+\theta) = \frac{OM'}{OP'} = \frac{-y}{r} = -\sin\theta$$

$$\tan(90+\theta) = \frac{M'P'}{OM'} = \frac{x}{-y} = -\frac{x}{y} = -\cot\theta$$

Similarly, we obtain

$\operatorname{cosec}(90°+\theta) = \sec\theta$, $\sec(90°+\theta) = -\operatorname{cosec}\theta$ and $\cot(90°+\theta) = -\tan\theta$.

5.6.4 TRIGONOMETRIC RATIOS OF (180° − θ) IN TERMS OF THAT OF θ

Let a revolving ray OA starting from its initial position OX, trace out an angle $\angle XOA = \theta$. Let another revolving ray OA' starting from its initial position OX, trace out angle of 180° to coincide with OX' and then rotates in clockwise sense through an angle θ. Let P and P' be points on OA and OA' respectively such that $OP = OP' = r$. Draw perpendiculars PM and $P'M'$ from P and P' on OX and OX' respectively.

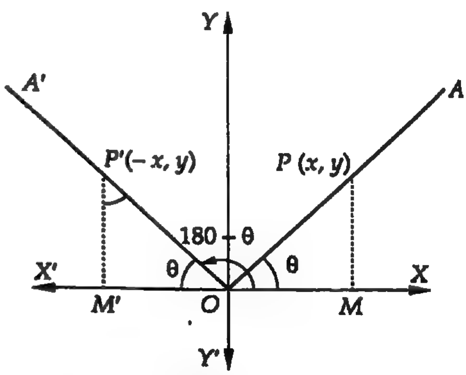


Fig. 5.11

Clearly, triangles OMP and $OM'P'$ are congruent.

$\therefore \quad M'P' = MP = y$ and $OM' = OM$.

Hence, the coordinates of P' are $(-x, y)$.

Now, $\sin(180-\theta) = \dfrac{P'M'}{OP'} = \dfrac{y}{r} = \sin\theta.$

$$\cos(180-\theta) = \frac{OM'}{OP'} = -\frac{x}{r} = -\cos\theta$$

and, $\tan(180-\theta) = \dfrac{P'M'}{OM'} = \dfrac{y}{-x} = -\tan\theta$

Similarly, we obtain

$\operatorname{cosec}(180°-\theta) = \operatorname{cosec}\theta$, $\sec(180°-\theta) = -\sec\theta$ and, $\cot(180°-\theta) = -\cot\theta$

5.6.5 TRIGONOMETRICAL RATIOS OF (180° + θ) IN TERMS OF THAT OF θ

Let a revolving ray OA starting from its initial position OX, trace out an angle $\angle XOA = \theta$. Let another revolving line OA' starting from its position OX, trace out an angle $\angle XOA' = 180° + \theta$. Let P and P' be points on OA and OA' respectively such that $OP = OP' = r$. Draw perpendiculars PM and $P'M'$ from P and P' respectively on OX and OX'. Clearly, triangles OPM and $OP'M'$ are congruent.

$\therefore \quad OM' = OM = x, \ PM = P'M' = y$

Hence, the coordiantes of P' are $(-x, -y)$.

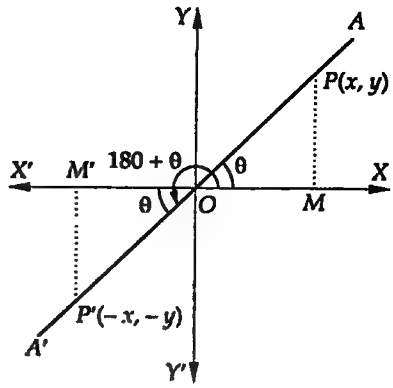


Fig. 5.12

Clearly, $\sin(180+\theta) = \dfrac{P'M'}{OP'} = \dfrac{-y}{r} = -\dfrac{y}{r} = -\sin\theta$

$\cos(180+\theta) = \dfrac{OM'}{OP'} = \dfrac{-x}{r} = -\dfrac{x}{r} = -\cos\theta$

$\tan(180+\theta) = \dfrac{P'M'}{OM'} = \dfrac{-y}{-x} = \dfrac{y}{x} = \tan\theta$

Similarly, we obtain

$\operatorname{cosec}(180°+\theta) = -\operatorname{cosec}\theta$, $\sec(180°+\theta) = -\sec\theta$ and $\cot(180°+\theta) = \cot\theta$.

5.6.6 TRIGONOMETRIC RATIOS OF (360° − θ) IN TERMS OF THAT OF θ

The terminal sides of co-terminal angles coincide, so their trigonometrical ratios are same.

Clearly, $360° - \theta$ and $-\theta$ are co-terminal angles.

$\therefore \quad \sin(360°-\theta) = \sin(-\theta) = -\sin\theta$, $\cos(360°-\theta) = \cos(-\theta) = \cos\theta$,

and, $\quad \tan(360°-\theta) = \tan(-\theta) = -\tan\theta$

Similarly, we have

$\operatorname{cosec}(360°-\theta) = -\operatorname{cosec}\theta$, $\sec(360°-\theta) = \sec\theta$ and $\cot(360°-\theta) = -\cot\theta$.

We know that the terminal sides of co-terminal angles always coincide and θ and $360° + \theta$ are co-terminal angles. Therefore,

$\sin(360°+\theta) = \sin\theta$, $\cos(360°+\theta) = \cos\theta$, $\tan(360°+\theta) = \tan\theta$,

$\sec(360°+\theta) = \sec\theta$, $\operatorname{cosec}(360°+\theta) = \operatorname{cosec}\theta$ and $\cot(360°+\theta) = \cot\theta$

In fact, for any positive integer n, angle $(360° \times n + \theta)$ is coterminal to angle θ. Therefore, for any positive integer n, we have

$\sin(360° \times n+\theta) = \sin\theta$, $\cos(360° \times n+\theta) = \cos\theta$, $\tan(360° \times n+\theta) = \tan\theta$,

$\operatorname{cosec}(360° \times n+\theta) = \operatorname{cosec}\theta$, $\sec(360° \times n+\theta) = \sec\theta$ and, $\cot(360° \times n+\theta) = \cot\theta$.

If θ is in radians, then the above results may be written as

$\sin(2n\pi+\theta) = \sin\theta$, $\cos(2n\pi+\theta) = \cos\theta$, $\tan(2n\pi+\theta) = \tan\theta$,

$\operatorname{cosec}(2n\pi+\theta) = \operatorname{cosec}\theta$, $\sec(2n\pi+\theta) = \sec\theta$ and $\cot(2n\pi+\theta) = \cot\theta$

5.7 PERIODIC FUNCTION

A function $f(x)$ is said to be a periodic function if there exists a real number $T > 0$ such that $f(x+T) = f(x)$ for all x.

If T is the smallest positive real number such that $f(x+T) = f(x)$ for all x, then T is called the fundamental period of $f(x)$.

Since $\sin(2n\pi+\theta) = \sin\theta$, $\cos(2n\pi+\theta) = \cos\theta$ for all values of θ and $n \in N$.

Therefore, sine and cosine functions are periodic functions.

We find that 2π is the smallest positive real number such that $\sin(2\pi+\theta) = \sin\theta$, and $\cos(2\pi+\theta) = \cos\theta$ for all values of θ.

So, sine and cosine functions are periodic with period 2π.

We also know that $\tan(\pi + \theta) = \tan\theta$ and $\cot(\pi + \theta) = \cot\theta$. Therefore, $\tan\theta$ and $\cot\theta$ are periodic with period π.

Similarly, $\text{cosec}\,\theta$ and $\sec\theta$ are periodic functions with period 2π.

Following table provides periods of all trigonometric functions as a ready reference.

Function:	Sine	Cosine	Tangent	Cosecant	Secant	Cotangent
Period:	2π	2π	π	2π	2π	π

5.8 EVEN AND ODD FUNCTIONS

EVEN FUNCTION *A function $f(x)$ is said to be an even function, if $f(-x) = f(x)$ for all x in its domain.*

ODD FUNCTION *A function $f(x)$ is an odd function, if $f(-x) = -f(x)$ for all x in its domain.*

We have seen that $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$ and $\tan(-\theta) = -\tan\theta$.

Therefore, $\sin\theta$ and $\tan\theta$ and their reciprocals $\text{cosec}\,\theta$ and $\cot\theta$ are odd functions whereas $\cos\theta$ and its reciprocal $\sec\theta$ are even functions.

In earlier classes, we have learnt about the values of trigonometrical ratios for 0°, 30°, 45°, 60° and 90°. The values of trigonometric functions for these angles are same as that of trigonometrical ratios studied in earlier classes. In fact, the value of a trigonometrical ratio for any angle is same as the value of trigonometrical function for the same angle. Thus, we have the following table:

Angle / *Trigonometric Function*	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0
cosec	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	not defined	-1	not defined
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined	-1	not defined	1
cot	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	not defined	0	not defined

In order to find the values of trigonometrical functions for any angle in terms of those of positive acute angle, we may follow the following algorithm:

ALGORITHM

<u>STEP I</u> *See whether the given angle α is positive or negative if it is negative, make it positive by using the following :*

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$ *etc.*

<u>STEP II</u> *Express the positive angle α obtained in step I in the form $\alpha = 90° \times n \pm \theta$, where θ is an acute angle.*

STEP III *Determine the quadrant in which the terminal side of the angle α lies.*

STEP IV *Determine the sign of the given trigonometrical function in the quadrant obtained in step III.*

STEP V *If n in step II is an odd integer, then* $\sin\alpha = \pm\cos\theta$, $\cos\alpha = \pm\sin\theta$, $\tan\alpha = \pm\cot\theta$, $\sec\alpha = \pm\operatorname{cosec}\theta$ *and* $\operatorname{cosec}\theta = \pm\sec\theta$. *The sign on RHS will be the sign obtained in step IV.*

If n in step II is an even integer, then $\sin\alpha = \pm\sin\theta$, $\cos\alpha = \pm\cos\theta$ $\tan\alpha = \pm\tan\theta$, $\sec\alpha = \pm\sec\theta$ *and* $\operatorname{cosec}\alpha = \pm\operatorname{cosec}\theta$. *The sign on RHS is the sign obtained in step IV.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the values of the following trigonometric ratios:*

(i) $\sin 315°$ (ii) $\cos 210°$ (iii) $\cos(-480°)$ (iv) $\sin(-1125°)$

SOLUTION (i) Clearly,

$$\sin 315° = \sin(90° \times 3 + 45°)$$

Since 315° lies in the IVth quadrant in which sine function is negative and 3 is an odd integer.

$$\therefore \quad \sin 315° = \sin(90° \times 3 + 45) = -\cos 45° = -\frac{1}{\sqrt{2}}$$

(ii) Clearly,

$$\cos 210° = \cos(90° \times 2 + 30°).$$

Since 210° is in the III quadrant in which cosine function is negative. Also the multiple of 90° is even.

$$\therefore \quad \cos 210° = \cos(90° \times 2 + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$$

(iii) Clearly,

$$\cos(-480°) = \cos 480° = \cos(90° \times 5 + 30°)$$

Since 480° is in the II quadrant in which cosine function is negative. Also the multiple of 90° is odd.

$$\therefore \quad \cos(-480°) = \cos 480° = \cos(90° \times 5 + 30°) = -\sin 30° = -\frac{1}{2}$$

(iv) We have,

$$\sin(-1125°) = -\sin 1125° = -\sin(90° \times 12 + 45°)$$

Clearly, 1125° lies in the first quadrant. The multiple of 90° in this expression is even.

$$\therefore \quad \sin(-1125°) = -\sin(90° \times 12 + 45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$$

EXAMPLE 2 *Find the values of the following trigonometric ratios.*

(i) $\operatorname{cosec} 390°$ (ii) $\cot 570°$ (iii) $\tan 480°$ (iv) $\cos 270°$

(v) $\tan\frac{19\pi}{3}$ (vi) $\sin\left(\frac{-11\pi}{3}\right)$ (vii) $\cot\left(\frac{-15\pi}{4}\right)$

SOLUTION (i) We have,

$$390° = 90° \times 4 + 30°$$

Clearly, 390° is in I quadrant and the multiple of 90° is even.

$$\therefore \quad \operatorname{cosec} 390° = \operatorname{cosec}(90° \times 4 + 30°) = \operatorname{cosec} 30° = 2.$$

(ii) We have,

$$570° = 90° \times 6 + 30°$$

Clearly, 570° is in the IIIrd quadrant and the multiple of 90° is even.

$\therefore \quad \cot 570° = \cot(90° \times 6 + 30°) = \cot 30° = \sqrt{3}$

(iii) We have,

$480° = 90° \times 5 + 30°$

Clearly, 480° is in the second quadrant and the multiple of 90° is odd.

$\therefore \quad \tan 480° = \tan(90° \times 5 + 30°) = -\cot 30° = -\sqrt{3}$

(iv) We have,

$270° = 90° \times 3 + 0°$

Clearly, 270° is in the negative direction of y-axis i.e. on the boundary line of II and III quadrant. Also, the multiple of 90° is an odd integer.

$\therefore \quad \cos 270° = \cos(90° \times 3 + 0°) = \pm \sin 0° = 0$

(v) We have,

$$\frac{19\pi}{3} = \left(\frac{19}{3} \times 180\right)^{\circ} = 1140° = 90° \times 12 + 60°$$

Clearly, this angle lies in first quadrant.

$$\therefore \quad \tan \frac{19\pi}{3} = \tan(90° \times 12 + 60°) = \tan 60° = \sqrt{3}$$

(vi) We have,

$$\frac{11\pi}{3} = \left(\frac{11 \times 180°}{3}\right)^{\circ} = 660° = 90° \times 7 + 30°$$

$$\therefore \quad \sin\left(-\frac{11\pi}{3}\right) = -\sin\frac{11\pi}{3} = -\sin(90° \times 7 + 30°) = -(-\cos 30°) = -\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$$

(vii) We have,

$$\frac{15\pi}{4} = \left(\frac{15 \times 180}{4}\right)^{\circ} = 675° = 90° \times 7 + 45°$$

$$\therefore \quad \cot\left(-\frac{15\pi}{4}\right) = -\cot\left(\frac{15\pi}{4}\right) = -\cot(90° \times 7 + 45°) = -(-\cot 45°) = -(-1) = 1.$$

EXAMPLE 3 *Prove that:* $\cos 510° \cos 330° + \sin 390° \cos 120° = -1$.

SOLUTION We have,

$$\begin{aligned} \text{LHS} &= \cos 510° \cos 330° + \sin 390° \cos 120° \\ &= \cos(90° \times 5 + 60°)\cos(90° \times 3 + 60°) + \sin(90° \times 4 + 30°°)\cos(90° \times 1 + 30°) \\ &= (-\sin 60°)(\sin 60°) + (\sin 30°)(-\sin 30°) \\ &= -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{3}{4} - \frac{1}{4} = -1 = \text{RHS} \end{aligned}$$

EXAMPLE 4 *Prove that :* $\sin(-420°)(\cos 390°) + \cos(-660°)(\sin 330°) = -1$.

SOLUTION We have,

$$\begin{aligned} \text{LHS} &= \sin(-420°)(\cos 390°) + \cos(-660°)(\sin 330°) \\ &= -\sin 420° \cos 390° + \cos 660° \sin 330° \qquad [\because \sin(-\theta) = -\sin\theta, \cos(-\theta) = \cos\theta] \\ &= -\sin(90° \times 4 + 60°)\cos(90° \times 4 + 30°) + \cos(90° \times 7 + 30°)\sin(90° \times 3 + 60°) \\ &= -(\sin 60°)(\cos 30°) + (\sin 30°)(-\cos 60°) = -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = -1 = \text{RHS} \end{aligned}$$

EXAMPLE 5 *Prove that:*

(i) $\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}$ (ii) $2\sin^2\frac{\pi}{6} + \operatorname{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3} = \frac{3}{2}$

(iii) $\cot^2\frac{\pi}{6} + \operatorname{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6} = 6$ (iv) $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} = 10$

SOLUTION (i) LHS $= \sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4}$

$$= \left(\sin \frac{\pi}{6}\right)^2 + \left(\cos \frac{\pi}{3}\right)^2 - \left(\tan \frac{\pi}{4}\right)^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - (1)^2 = \frac{1}{4} + \frac{1}{4} - 1 = -\frac{1}{2}$$

(ii) LHS $= 2 \sin^2 \frac{\pi}{6} + \operatorname{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3}$

$$= 2\left(\sin \frac{\pi}{6}\right)^2 + \left(\operatorname{cosec} \frac{7\pi}{6}\right)^2 \left(\cos \frac{\pi}{3}\right)^2$$

$$= 2\left(\sin \frac{\pi}{6}\right)^2 + \left\{\operatorname{cosec}\left(\pi + \frac{\pi}{6}\right)\right\}^2 \left(\cos \frac{\pi}{3}\right)^2$$

$$= 2\left(\sin \frac{\pi}{6}\right)^2 + \left\{-\operatorname{cosec} \frac{\pi}{6}\right\}^2 \left(\cos \frac{\pi}{3}\right)^2 \quad [\because \operatorname{cosec}(\pi + \theta) = -\operatorname{cosec} \theta]$$

$$= 2\left(\frac{1}{2}\right)^2 + (-2)^2 \times \left(\frac{1}{2}\right)^2 = \frac{1}{2} + 1 = \frac{3}{2} = \text{RHS}$$

(iii) LHS $= \cot^2 \frac{\pi}{6} + \operatorname{cosec} \frac{5\pi}{6} + 3 \tan^2 \frac{\pi}{6}$

$$= \left(\cot \frac{\pi}{6}\right)^2 + \operatorname{cosec}\left(\pi - \frac{\pi}{6}\right) + 3\left(\tan \frac{\pi}{6}\right)^2 = (\sqrt{3})^2 + 2 + 3\left(\frac{1}{\sqrt{3}}\right)^2 = 3 + 2 + 1 = 6 = \text{RHS}$$

(iv) LHS $= 2 \sin^2 \frac{3\pi}{4} + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{\pi}{3}$

$$= 2\left(\sin \frac{3\pi}{4}\right)^2 + 2\left(\cos \frac{\pi}{4}\right)^2 + 2\left(\sec \frac{\pi}{3}\right)^2$$

$$= 2\left(\sin \frac{\pi}{4}\right)^2 + 2\left(\cos \frac{\pi}{4}\right)^2 + 2\left(\sec \frac{\pi}{3}\right)^2 \quad \left[\because \sin \frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin \frac{\pi}{4}\right]$$

$$= 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2(2)^2 = 1 + 1 + 8 = 10 = \text{RHS}$$

EXAMPLE 6 *Prove that:* $\dfrac{\cos(90° + \theta) \sec(-\theta) \tan(180° - \theta)}{\sec(360° - \theta) \sin(180° + \theta) \cot(90° - \theta)} = -1.$

SOLUTION We have,

$$\text{LHS} = \frac{\cos(90° + \theta) \sec(-\theta) \tan(180° - \theta)}{\sec(360° - \theta) \sin(180° + \theta) \cot(90° - \theta)} = \frac{(-\sin\theta)(\sec\theta)(-\tan\theta)}{(\sec\theta)(-\sin\theta)(\tan\theta)} = -1 = \text{RHS}$$

EXAMPLE 7 *Prove that :*

(i) $\dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\frac{\pi}{2} + x\right)} = \cot^2 x$

(ii) $\cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\left\{\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\right\} = 1$

SOLUTION (i) LHS $= \dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\frac{\pi}{2} + x\right)} = \dfrac{(-\cos x) \times (\cos x)}{(\sin x)(-\sin x)} = \dfrac{-\cos^2 x}{-\sin^2 x} = \cot^2 x = \text{RHS}$

(ii) LHS $= \cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\left\{\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\right\}$

$$= (\sin x)(\cos x)(\tan x + \cot x) \quad \left[\begin{array}{l} \because \cos\left(\dfrac{3\pi}{2}+x\right) = \sin x, \cos(2\pi + x) = \cos x \\ \cot\left(\dfrac{3\pi}{2}-x\right) = \tan x \text{ and } \cot(2\pi + x) = \cot x \end{array}\right]$$

$$= \sin x \cos x \left\{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right\}$$

$$= \sin x \cos x \left\{\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right\} = \sin x \cos x \times \frac{1}{\sin x \cos x} = 1$$

EXAMPLE 8 *If A, B, C, D are angles of a cyclic quadrilateral, prove that*

$$\cos A + \cos B + \cos C + \cos D = 0.$$

SOLUTION We know that the opposite angles of a cyclic quadrilateral are supplementary i.e. $A + C = \pi$ and $B + D = \pi$.

$\therefore \quad A = \pi - C$ and $B = \pi - D$

$\Rightarrow \quad \cos A = \cos(\pi - C) = -\cos C$ and, $\cos B = \cos(\pi - D) = -\cos D$

$\therefore \quad \cos A + \cos B + \cos C + \cos D = -\cos C - \cos D + \cos C + \cos D = 0$

EXAMPLE 9 *In any quadrilateral ABCD, prove that*

(i) $\sin(A + B) + \sin(C + D) = 0$ (ii) $\cos(A + B) = \cos(C + D)$

SOLUTION We have,

(i) $A + B + C + D = 2\pi$

$\Rightarrow \quad A + B = 2\pi - (C + D)$

$\Rightarrow \quad \sin(A + B) = \sin\{2\pi - (C + D)\}$

$\Rightarrow \quad \sin(A + B) = -\sin(C + D)$

$\Rightarrow \quad \sin(A + B) + \sin(C + D) = 0 \quad [\because \sin(2\pi - \theta) = -\sin\theta]$

(ii) We have,

$A + B + C + D = 2\pi$

$\Rightarrow \quad A + B = 2\pi - (C + D)$

$\Rightarrow \quad \cos(A + B) = \cos\{2\pi - (C + D)\}$

$\Rightarrow \quad \cos(A + B) = \cos(C + D) \quad [\because \cos(2\pi - \theta) = \cos\theta]$

LEVEL-2

EXAMPLE 10 *Find the value of the expression*

Imp.
$$3\left\{\sin^4\left(\frac{3\pi}{2}-\theta\right)+\sin^4(3\pi+\theta)\right\}-2\left\{\sin^6\left(\frac{\pi}{2}+\theta\right)+\sin^6(5\pi-\theta)\right\}$$

SOLUTION The given expression is [NCERT EXEMPLAR]

$$3\left\{\sin^4\left(\frac{3\pi}{2}-\theta\right)+\sin^4(3\pi+\theta)\right\}-2\left\{\sin^6\left(\frac{\pi}{2}+\theta\right)+\sin^6(5\pi-\theta)\right\}$$

$$= 3\left\{(-\cos\theta)^4 + (-\sin\theta)^4\right\} - 2\left\{(\cos\theta)^6 + (\sin\theta)^6\right\}$$

$$= 3(\cos^4\theta + \sin^4\theta) - 2(\cos^6\theta + \sin^6\theta)$$

$$= 3\left\{(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta\right\}$$

$$-2\left\{(\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)\right\}$$

$$= 3(1 - 2\sin^2\theta\cos^2\theta) - 2(1 - 3\cos^2\theta\sin^2\theta)$$

$$= 3 - 6\sin^2\theta\cos^2\theta - 2 + 6\sin^2\theta\cos^2\theta = 1.$$

EXERCISE 5.3

LEVEL-1

1. Find the values of the following trigonometric ratios:

(i) $\sin \frac{5\pi}{3}$ (ii) $\sin 3060°$ (iii) $\tan \frac{11\pi}{6}$ (iv) $\cos(-1125°)$

(v) $\tan 315°$ (vi) $\sin 510°$ (vii) $\cos 570°$ (viii) $\sin(-330°)$

(ix) $\operatorname{cosec}(-1200°)$ (x) $\tan(-585°)$ (xi) $\cos 855°$ (xii) $\sin 1845°$

(xiii) $\cos 1755°$ (xiv) $\sin 4530°$

2. Prove that:

(i) $\tan 225° \cot 405° + \tan 765° \cot 675° = 0$

(ii) $\sin \frac{8\pi}{3} \cos \frac{23\pi}{6} + \cos \frac{13\pi}{3} \sin \frac{35\pi}{6} = \frac{1}{2}$

(iii) $\cos 24° + \cos 55° + \cos 125° + \cos 204° + \cos 300° = \frac{1}{2}$

(iv) $\tan(-225°) \cot(-405°) - \tan(-765°) \cot(675°) = 0$

(v) $\cos 570° \sin 510° + \sin(-330°) \cos(-390°) = 0$

(vi) $\tan \frac{11\pi}{3} - 2 \sin \frac{4\pi}{6} - \frac{3}{4} \operatorname{cosec}^2 \frac{\pi}{4} + 4 \cos^2 \frac{17\pi}{6} = \frac{3 - 4\sqrt{3}}{2}$

(vii) $3 \sin \frac{\pi}{6} \sec \frac{\pi}{3} - 4 \sin \frac{5\pi}{6} \cot \frac{\pi}{4} = 1$

3. Prove that:

(i) $\dfrac{\cos(2\pi + \theta) \operatorname{cosec}(2\pi + \theta) \tan(\pi/2 + \theta)}{\sec(\pi/2 + \theta) \cos\theta \cot(\pi + \theta)} = 1$

(ii) $\dfrac{\operatorname{cosec}(90° + \theta) + \cot(450° + \theta)}{\operatorname{cosec}(90° - \theta) + \tan(180° - \theta)} + \dfrac{\tan(180° + \theta) + \sec(180° - \theta)}{\tan(360° + \theta) - \sec(-\theta)} = 2$

(iii) $\dfrac{\sin(180° + \theta) \cos(90° + \theta) \tan(270° - \theta) \cot(360° - \theta)}{\sin(360° - \theta) \cos(360° + \theta) \operatorname{cosec}(-\theta) \sin(270° + \theta)} = 1$

(iv) $\left\{1 + \cot\theta - \sec\left(\frac{\pi}{2} + \theta\right)\right\} \left\{1 + \cot\theta + \sec\left(\frac{\pi}{2} + \theta\right)\right\} = 2 \cot\theta$

(v) $\dfrac{\tan(90° - \theta) \sec(180° - \theta) \sin(-\theta)}{\sin(180° + \theta) \cot(360° - \theta) \operatorname{cosec}(90° - \theta)} = 1$

4. Prove that : $\sin^2 \frac{\pi}{18} + \sin^2 \frac{\pi}{9} + \sin^2 \frac{7\pi}{18} + \sin^2 \frac{4\pi}{9} = 2$

5. Prove that : $\sec\left(\frac{3\pi}{2} - \theta\right) \sec\left(\theta - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + \theta\right) \tan\left(\theta - \frac{3\pi}{2}\right) = -1.$

6. In a ΔABC, prove that :

(i) $\cos(A + B) + \cos C = 0$ (ii) $\cos\left(\frac{A + B}{2}\right) = \sin \frac{C}{2}$ (iii) $\tan \frac{A + B}{2} = \cot \frac{C}{2}$

7. If A, B, C, D be the angles of a cyclic quadrilateral, taken in order, prove that :

$\cos(180° - A) + \cos(180° + B) + \cos(180° + C) - \sin(90° + D) = 0$

8. Find x from the following equations :

(i) $\operatorname{cosec}(90° + \theta) + x \cos\theta \cot(90° + \theta) = \sin(90° + \theta)$

(ii) $x \cot(90° + \theta) + \tan(90° + \theta) \sin\theta + \operatorname{cosec}(90° + \theta) = 0$

9. Prove that:

(i) $\tan 720° - \cos 270° - \sin 150° \cos 120° = \frac{1}{4}$

(ii) $\sin 780° \sin 480° + \cos 120° \sin 150° = \frac{1}{2}$

(iii) $\sin 780° \sin 120° + \cos 240° \sin 390° = \frac{1}{2}$

(iv) $\sin 600° \cos 390° + \cos 480° \sin 150° = -1$

(v) $\tan 225° \cot 405° + \tan 765° \cot 675° = 0$

ANSWERS

1. (i) $-\frac{\sqrt{3}}{2}$ (ii) 0 (iii) $-\frac{1}{\sqrt{3}}$ (iv) $\frac{1}{\sqrt{2}}$ (v) -1 (vi) $\frac{1}{2}$
(vii) $-\frac{\sqrt{3}}{2}$ (viii) $\frac{1}{2}$ (ix) $-\frac{2}{\sqrt{3}}$ (x) -1 (xi) $-\frac{1}{\sqrt{2}}$ (xii) $\frac{1}{\sqrt{2}}$
(xiii) $1/\sqrt{2}$ (xiv) $-1/2$ 8. (i) $\tan\theta$ (ii) $\sin\theta$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the maximum and minimum values of $\cos(\cos x)$.
2. Write the maximum and minimum values of $\sin(\sin x)$.
3. Write the maximum value of $\sin(\cos x)$.
4. If $\sin x = \cos^2 x$, then write the value of $\cos^2 x(1 + \cos^2 x)$.
5. If $\sin x + \operatorname{cosec} x = 2$, then write the value of $\sin^n x + \operatorname{cosec}^n x$.
6. If $\sin x + \sin^2 x = 1$, then write the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x$.
7. If $\sin x + \sin^2 x = 1$, then write the value of $\cos^8 x + 2\cos^6 x + \cos^4 x$.
8. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then write the value of $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$.
9. Write the value of $\sin 10° + \sin 20° + \sin 30° + \ldots + \sin 360°$.
10. A circular wire of radius 15 cm is cut and bent so as to lie along the circumference of a loop of radius 120 cm. Write the measure of the angle subtended by it at the centre of the loop.
11. Write the value of $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$.
12. Write the value of $\cos 1° + \cos 2° + \cos 3° + \ldots + \cos 180°$.
13. If $\cot(\alpha + \beta) = 0$, then write the value of $\sin(\alpha + 2\beta)$.
14. If $\tan A + \cot A = 4$, then write the value of $\tan^4 A + \cot^4 A$.
15. Write the least value of $\cos^2\theta + \sec^2\theta$.
16. If $x = \sin^{14}\theta + \cos^{20}\theta$, then write the smallest interval in which the value of x lie.
17. If $3\sin\theta + 5\cos\theta = 5$, then write the value of $5\sin\theta - 3\cos\theta$.

ANSWERS

1. 1, cos 1 2. sin 1, − sin 1 3. sin 1 4. 1 5. 2 6. 1
7. 1 8. 0 9. 0 10. 45° 11. 0 12. 0 13. $\sin\alpha$ or $\cos\beta$
14. 194 15. 2 16. $(0, 1]$ 17. 3 or − 3

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $\tan\theta = x - \frac{1}{4x}$, then $\sec\theta - \tan\theta$ is equal to

(a) $-2x, \frac{1}{2x}$ (b) $-\frac{1}{2x}, 2x$ (c) $2x$ (d) $2x, \frac{1}{2x}$

2. If $\sec\theta = x + \frac{1}{4x}$, then $\sec\theta + \tan\theta =$

(a) $x, \frac{1}{x}$ (b) $2x, \frac{1}{2x}$ (c) $-2x, \frac{1}{2x}$ (d) $-\frac{1}{x}, x$

3. If $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, then $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$ is equal to

(a) $\sec\theta - \tan\theta$ (b) $\sec\theta + \tan\theta$ (c) $\tan\theta - \sec\theta$ (d) none of these

4. If $\pi < \theta < 2\pi$, then $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$ is equal to

(a) $\text{cosec}\,\theta + \cot\theta$ (b) $\text{cosec}\,\theta - \cot\theta$ (c) $-\text{cosec}\,\theta + \cot\theta$ (d) $-\text{cosec}\,\theta - \cot\theta$

5. If $0 < \theta < \frac{\pi}{2}$, and if $\frac{y+1}{1-y} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$, then y is equal to

(a) $\cot\frac{\theta}{2}$ (b) $\tan\frac{\theta}{2}$ (c) $\cot\frac{\theta}{2} + \tan\frac{\theta}{2}$ (d) $\cot\frac{\theta}{2} - \tan\frac{\theta}{2}$

6. If $\frac{\pi}{2} < \theta < \pi$, then $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$ *is equal to*

(a) $2\sec\theta$ (b) $-2\sec\theta$ (c) $\sec\theta$ (d) $-\sec\theta$

7. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, then $x^2 + y^2 + z^2$ is independent of

(a) θ, ϕ (b) r, θ (c) r, ϕ (d) r

8. If $\tan\theta + \sec\theta = \sqrt{3}$, $0 < \theta < \pi$, then θ is equal to

(a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

9. If $\tan\theta = -\frac{1}{\sqrt{5}}$ and θ lies in the IV quadrant, then the value of $\cos\theta$ is

(a) $\frac{\sqrt{5}}{\sqrt{6}}$ (b) $\frac{2}{\sqrt{6}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{6}}$

10. If $\frac{3\pi}{4} < \alpha < \pi$, then $\sqrt{2\cot\alpha + \frac{1}{\sin^2\alpha}}$ is equal to

(a) $1 - \cot\alpha$ (b) $1 + \cot\alpha$ (c) $-1 + \cot\alpha$ (d) $-1 - \cot\alpha$

11. $\sin^6 A + \cos^6 A + 3\sin^2 A\cos^2 A =$

(a) 0 (b) 1 (c) 2 (d) 3

12. If $\text{cosec}\,\theta - \cot\theta = \frac{1}{2}$, $0 < \theta < \frac{\pi}{2}$, then $\cos\theta$ is equal to

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $-\frac{3}{5}$ (d) $-\frac{5}{3}$

13. If $\text{cosec}\,\theta + \cot\theta = \frac{11}{2}$, then $\tan\theta =$

(a) $\frac{21}{22}$ (b) $\frac{15}{16}$ (c) $\frac{44}{117}$ (d) $\frac{117}{44}$

14. $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true if and only if

(a) $x + y \neq 0$ (b) $x = y, x \neq 0$ (c) $x = y$ (d) $x \neq 0, y \neq 0$

15. If θ is an acute angle and $\tan\theta = \frac{1}{\sqrt{7}}$, then the value of $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$ is

(a) 3/4 (b) 1/2 (c) 2 (d) 5/4

16. The value of $\sin^2 5° + \sin^2 10° + \sin^2 15° + \ldots + \sin^2 85° + \sin^2 90°$ is

(a) 7 (b) 8 (c) 9.5 (d) 10

17. $\sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9} =$

(a) 1 (b) 4 (c) 2 (d) 0

18. If $\tan A + \cot A = 4$, then $\tan^4 A + \cot^4 A$ is equal to

(a) 110 (b) 191 (c) 80 (d) 194

19. If $x \sin 45° \cos^2 60° = \frac{\tan^2 60° \operatorname{cosec} 30°}{\sec 45° \cot^{2°} 30°}$, then $x =$

(a) 2 (b) 4 (c) 8 (d) 16

20. If $\operatorname{cosec}\theta - \cot\theta = \frac{1}{2}$, $0 < \theta < \frac{\pi}{2}$, then $\cos\theta$ is equal to

(a) $-3/5$ (b) $-5/3$ (c) 5/3 (d) 3/5

21. If $\operatorname{cosec} A + \cot A = \frac{11}{2}$, then $\tan A =$

(a) 21/22 (b) 15/16 (c) 44/117 (d) 117/43

22. If $\tan\theta + \sec\theta = e^x$, then $\cos\theta$ equals

(a) $\frac{e^x + e^{-x}}{2}$ (b) $\frac{2}{e^x + e^{-x}}$ (c) $\frac{e^x - e^{-x}}{2}$ (d) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$

23. If $\sec\theta + \tan\theta = k$, $\cos\theta =$

(a) $\frac{k^2+1}{2k}$ (b) $\frac{2k}{k^2+1}$ (c) $\frac{k}{k^2+1}$ (d) $\frac{k}{k^2-1}$

24. If $f(x) = \cos^2 x + \sec^2 x$, then

(a) $f(x) < 1$ (b) $f(x) = 1$ (c) $2 < f(x) < 1$ (d) $f(x) \geq 2$

25. Which of the following is incorrect?

(a) $\sin\theta = -1/5$ (b) $\cos\theta = 1$ (c) $\sec\theta = 1/2$ (d) $\tan\theta = 20$

26. The value of $\cos 1° \cos 2° \cos 3° \ldots \cos 179°$ is

(a) $1/\sqrt{2}$ (b) 0 (c) 1 (d) -1

27. The value of $\tan 1° \tan 2° \tan 3° \ldots \tan 89°$ is

(a) 0 (b) 1 (c) 1/2 (d) not defined

28. Which of the following is correct?

(a) $\sin 1° > \sin 1$ (b) $\sin 1° < \sin 1$ (c) $\sin 1° = \sin 1$ (d) $\sin 1° = \frac{\pi}{180}\sin 1$

29. If A lies in second quadrant and $3\tan A + 4 = 0$, then the value of $2\cot A - 5\cos A + \sin A$ is equal to

(a) $-53/10$ (b) 23/10 (c) 37/10 (d) 7/10

ANSWERS

1. (a)	2. (b)	3. (c)	4. (d)	5. (b)	6. (b)	7. (a)	8. (c)
9. (a)	10. (d)	11. (b)	12. (b)	13. (c)	14. (b)	15. (a)	16. (c)
17. (c)	18. (d)	19. (c)	20. (d)	21. (c)	22. (b)	23. (b)	24. (d)
25. (c)	26. (b)	27. (b)	28. (b)	29. (b)			

SUMMARY

1. Following are some of the fundamental trigonometric identities:

(i) $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ or, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

(ii) $\cos\theta = \dfrac{1}{\sec\theta}$ or, $\sec\theta = \dfrac{1}{\cos\theta}$ (iii) $\cot\theta = \dfrac{1}{\tan\theta}$ or, $\tan\theta = \dfrac{1}{\cot\theta}$

(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ or, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ (v) $\sin^2\theta + \cos^2\theta = 1$

(vi) $1 + \tan^2\theta = \sec^2\theta$ or, $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta}$

(vii) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ or, $\operatorname{cosec}\theta - \cot\theta = \dfrac{1}{\operatorname{cosec}\theta + \cot\theta}$

2. (i) $\sin(-\theta) = -\sin\theta$ or, $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

(ii) $\cos(-\theta) = \cos\theta$ or, $\sec(-\theta) = \sec\theta$

(iii) $\tan(-\theta) = -\tan\theta$ or, $\cot(-\theta) = -\cot\theta$

(iv) $\sin(90° - \theta) = \cos\theta$, $\cos(90° - \theta) = \sin\theta$
$\tan(90° - \theta) = \cot\theta$, $\sec(90° - \theta) = \operatorname{cosec}\theta$
$\operatorname{cosec}(90° - \theta) = \sec\theta$, $\cot(90° - \theta) = \tan\theta$

(v) $\sin(90° + \theta) = \cos\theta$, $\cos(90° + \theta) = -\sin\theta$
$\tan(90° + \theta) = -\cot\theta$, $\cot(90° + \theta) = -\tan\theta$
$\sec(90° + \theta) = -\operatorname{cosec}\theta$, $\operatorname{cosec}(90° + \theta) = \sec\theta$

(vi) $\sin(180° - \theta) = \sin\theta$, $\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$, $\cot(180° - \theta) = -\cot\theta$
$\sec(180° - \theta) = -\sec\theta$, $\operatorname{cosec}(180° - \theta) = \operatorname{cosec}\theta$

(vii) $\sin(270° - \theta) = -\cos\theta$, $\cos(270° - \theta) = -\sin\theta$
$\tan(270° - \theta) = \cot\theta$, $\cot(270° - \theta) = \tan\theta$
$\operatorname{cosec}(270° - \theta) = -\sec\theta$, $\sec(270° - \theta) = -\operatorname{cosec}\theta$

(viii) $\sin(270° + \theta) = -\cos\theta$, $\cos(270° + \theta) = \sin\theta$
$\tan(270° + \theta) = -\cot\theta$, $\cot(270° + \theta) = -\tan\theta$
$\operatorname{cosec}(270° + \theta) = -\sec\theta$, $\sec(270° + \theta) = \operatorname{cosec}\theta$

(ix) $\sin(360° - \theta) = -\sin\theta$, $\cos(360° - \theta) = \cos\theta$
$\tan(360° - \theta) = -\tan\theta$, $\operatorname{cosec}(360° - \theta) = -\operatorname{cosec}\theta$
$\sec(360° - \theta) = \sec\theta$, $\cot(360° - \theta) = -\cot\theta$

(x) Sine and Cosine functions and their reciprocals i.e. Cosecant and Secant functions are periodic functions with period 2π. Tangent and Cotangent functions are periodic with period π.

(xi)

Odd functions	*Even functions*
sine, tangent	cosine, secant
cotangent, cosecant	

CHAPTER 6

GRAPHS OF TRIGONOMETRIC FUNCTIONS

6.1 INTRODUCTION

We have already studied in the previous chapters that all trigonometric functions are periodic. For example, sine and cosine functions are periodic with period 2π, while tangent and cotangent functions are periodic with period π. We know that if $f(x)$ is a periodic function with period T and $a > 0$, then $f(ax+b)$ is periodic with period T/a. Therefore, $\sin(ax+b)$ and $\cos ax$ are periodic functions with period $\frac{2\pi}{a}$. If the graph of a periodic function with period T is to be drawn in a given interval, then it is sufficient to draw its graph only in an interval of length T. Because, once it is drawn in one such interval, it can be easily drawn completely by repeating it over the intervals of lengths T. The amplitude of a function is defined as the greatest numerical value which it can attain.

Using the knowledge acquired in the above discussion let us now draw the graphs of various trigonometric functions.

6.2 GRAPHS OF TRIGONOMETRIC FUNCTIONS

6.2.1 GRAPH OF $y = \sin x$

Since $\sin x$ is a periodic function with period 2π. So, we will draw the graph of $y = \sin x$ in the interval $[0, 2\pi]$. In order to draw the graph of $y = \sin x$ in the interval $[0, 2\pi]$, we first draw it in $[0, \pi/2]$. The values of $\sin x$ for different values of x in the interval $[0, \pi/2]$ are given in the following table.

x	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin x$	0	$1/2 = 0.5$	$1/\sqrt{2} = 0.707$	$\sqrt{3}/2 = 0.866$	1

Using the above table and the fact that $\sin x$ is an increasing function we obtain the graph of $y = \sin x$ in the interval $[0, \pi/2]$ as shown in Fig. 6.1. In the interval $[\pi/2, \pi]$, we draw the graph of $y = \sin x$ by using the fact that $\sin(\pi - x) = \sin x$. Finally, we draw it in the interval $[\pi, 2\pi]$, using the fact that $\sin(\pi + x) = -\sin x$ which means that the graph of $y = \sin x$ in $[\pi, 2\pi]$ is the mirror image of the graph of $y = \sin x$ in $[0, \pi]$. The graph is now sketched in Fig. 6.1.

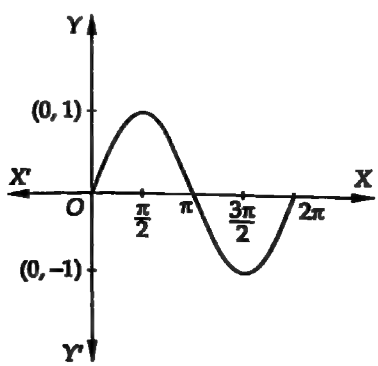


Fig. 6.1 Graph of $y = \sin x$

Since $\sin x$ is a periodic function with period 2π. Therefore, we use the following algorithm to draw the graphs of $y = c \sin ax$.

ALGORITHM

STEP I *Obtain the values of a and c.*

STEP II *Draw the graph of $y = \sin x$ and mark the points where it crosses x-axis.*

STEP III *Divide the x-coordinates of the points where $y = \sin x$ crosses x-axis by a and mark maximum and minimum values of $y = c \sin ax$ as c and $-c$ on y-axis.*

The graph so obtained is the graph of $y = c \sin ax$ as shown in Fig. 6.2.

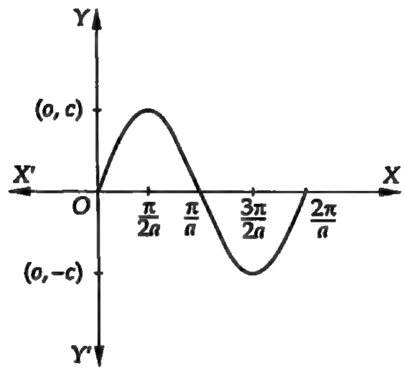


Fig. 6.2 Graph of $y = c \sin ax$

EXAMPLE 1 *Sketch the graph of $y = 3 \sin 2x$.*

SOLUTION To obtain the graph of $y = 3 \sin 2x$ we first draw the graph of $y = \sin x$ in the interval $[0, 2\pi]$ and then divide the x-coordinates of the points where it crosses x-axis by 2. The maximum and minimum values are 3 and -3 respectively as shown in Fig. 6.3.

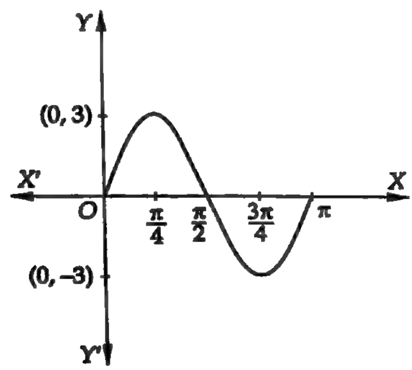


Fig. 6.3 Graph of $y = 3 \sin 2x$

EXAMPLE 2 *Sketch the graph of $y = 3 \sin (2x - 1)$.*

SOLUTION We have,

$$y = 3 \sin (2x - 1)$$

$$\Rightarrow \quad (y - 0) = 3 \sin 2\left(x - \frac{1}{2}\right) \quad \text{...(i)}$$

Shifting the origin at $(1/2, 0)$, we obtain

$$x = X + \frac{1}{2} \text{ and } y = Y + 0$$

Substituting these values in (i), we get

$$Y = 3 \sin 2X$$

Thus, if we draw the graph of $Y = 3 \sin 2X$ and shift it by 1/2 unit to the right, we get the required graph as shown in Fig. 6.4.

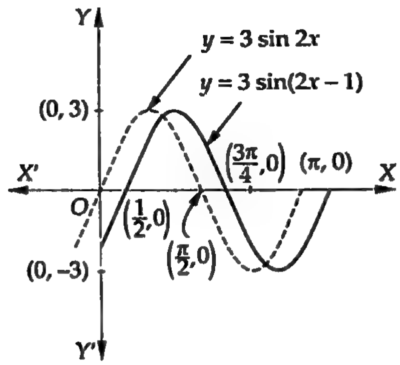


Fig. 6.4 Graph of $y = 3 \sin(2x - 1)$

EXAMPLE 3 *Sketch the graph of $y = \sin x$ and $y = \sin 2x$ on the same axes.*

SOLUTION Clearly $\sin 2x$ is a periodic function with period $2\pi/2 = \pi$ whereas $\sin x$ is periodic with period 2π. The graphs of $y = \sin x$ and $y = \sin 2x$ on different axes are shown in Figs. 6.5 and 6.6 respectively.

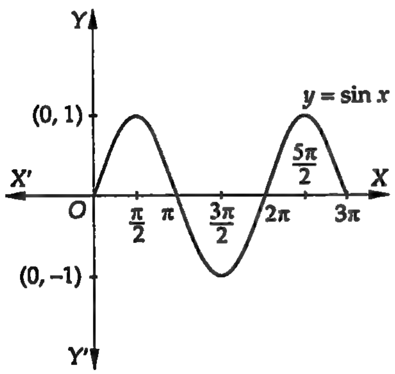


Fig. 6.5 Graph of $y = \sin x$

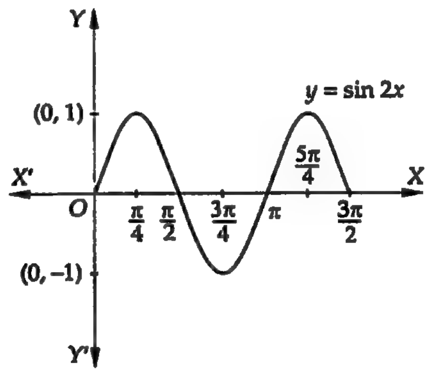


Fig. 6.6 Graph of $y = \sin 2x$

If these two graphs are drawn on the same axes, then the graphs are drawn in Fig. 6.7.

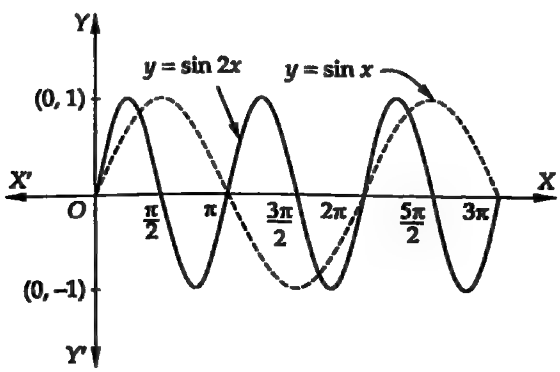


Fig. 6.7 Graphs of $y = \sin x$ and $y = \sin 2x$ on the same scale

EXERCISE 6.1

1. Sketch the following graphs:

   (i) $y=2\sin 2x$ (ii) $y=3\sin x$ (iii) $y=2\sin\left(x-\frac{\pi}{4}\right)$

   (iv) $y=2\sin(2x-1)$ (v) $y=3\sin(3x+1)$ (vi) $y=3\sin\left(2x-\frac{\pi}{4}\right)$

2. Sketch the graph of the following pairs of functions on the same axes :

   (i) $y=\sin x,\ y=\sin\left(x+\frac{\pi}{4}\right)$ (ii) $y=\sin x,\ y=\sin 3x$

6.2.2 GRAPH OF $y=\cos x$

Since $\cos x$ is a periodic function with period 2π. So, it is sufficient to draw the graph in the interval $[0, 2\pi]$. We first draw the graph in the interval $[0, \pi/2]$. For this we use the table given below and the fact that $\cos x$ is a decreasing function in this interval.

x	$0°$	$30°$	$45°$	$60°$	$90°$
$\cos x$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0

Now, to draw the graph of $y=\cos x$ in the interval $[\pi/2, \pi]$ we use the relation $\cos(\pi-x)=-\cos x$. If we give values of x between 0 and $\pi/2$, then this relation shows that the values of $\cos x$ between $\pi/2$ and π are negative of its values in the interval $[0, \pi/2]$. Thus, the graph of $y=\cos x$ between $\pi/2$ and π is below x-axis. Now, $\cos(\pi+x)=-\cos x$ shows that the graph of $y=\cos x$ between π and 2π is the mirror image in x-axis of its graph between 0 and π. Making use of all these results we obtain the sketch of the graph of $y=\cos x$ as shown in Fig. 6.8.

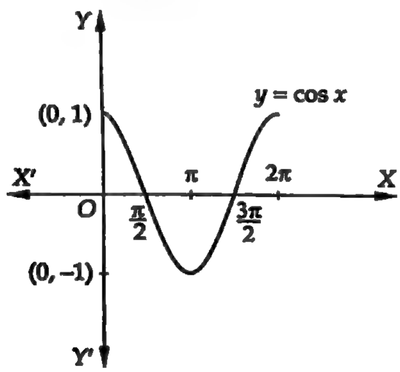


Fig. 6.8 Graph of $y=\cos x$

In order to draw the graph of $y=c\cos ax$ from the graph of $y=\cos x$, we may use the following algorithm.

ALGORITHM

STEP I *Obtain the values of a and c.*

STEP II *Draw the graph of $y=\cos x$ and mark the points where it crosses x-axis.*

STEP III *Divide the x-coordinates of the points where $y=\cos x$ meets x-axis by a and also mark maximum and minimum values of $y=c\cos ax$ as c and $-c$ on y-axis.*

The graph so obtained is the graph of $y=c\cos ax$ as shown in Fig. 6.9.

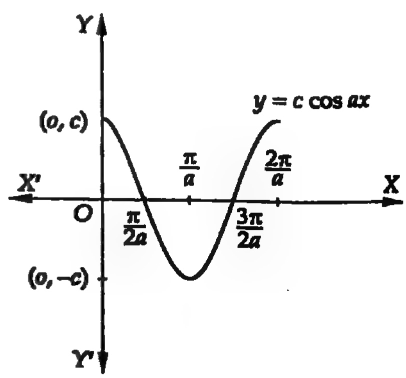


Fig. 6.9 Graph of $y=c\cos ax$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Draw the sketch of the graph of $y = 3\cos 2x$.*

SOLUTION Replacing c by 3 and a by 2 in the graph of $y = c\cos ax$, we obtain the graph of $y = 3\cos 2x$ as shown in Fig. 6.10.

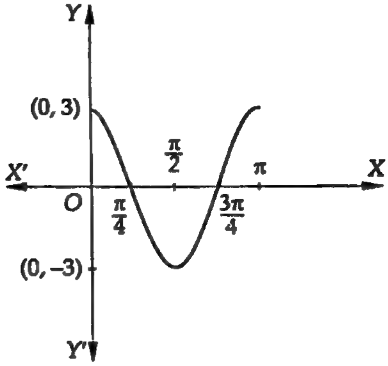


Fig. 6.10 Graph of $y = 3\cos 2x$

EXAMPLE 2 *Sketch the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$.*

SOLUTION We have,

$$y = \cos\left(x - \frac{\pi}{4}\right) \Rightarrow y - 0 = \cos\left(x - \frac{\pi}{4}\right) \qquad ...(i)$$

Shifting the origin at $(\pi/4, 0)$, we obtain

$$x = X + \frac{\pi}{4}, \; y = Y + 0$$

Substituting these values in (i), we get

$$Y = \cos X.$$

Thus, to draw the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$ we first draw the graph of $y = \cos x$ and then shift it through $\pi/4$ units to the right. The graph is drawn in Fig. 6.11.

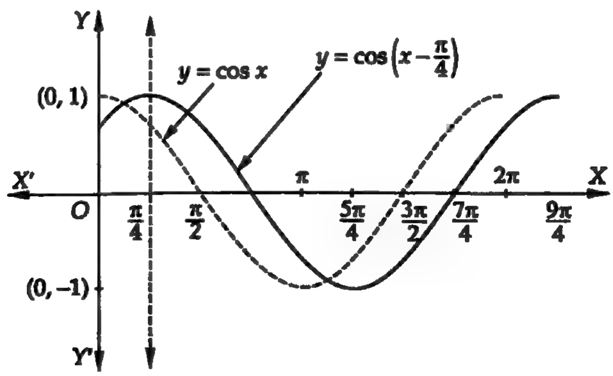


Fig. 6.11 Graph of $y = \cos\left(x - \frac{\pi}{4}\right)$

EXAMPLE 3 *Sketch the graphs of $y = \cos 2x$ and $y = \cos\left(2x - \frac{\pi}{4}\right)$ on the same scale.*

SOLUTION We have,

$$y = \cos\left(2x - \frac{\pi}{4}\right) = \cos 2\left(x - \frac{\pi}{8}\right) \Rightarrow y - 0 = \cos 2\left(x - \frac{\pi}{8}\right)$$

Shifting the origin at $(\pi/8, 0)$, we have

$$x = X + \frac{\pi}{8} \text{ and } y = Y + 0$$

Using these relations $y - 0 = \cos 2\left(x - \frac{\pi}{8}\right)$ reduces to $Y = \cos 2X$.

It follows from the above discussion that the graph of $y = \cos\left(2x - \frac{\pi}{4}\right)$ is similar to the graph of $y = \cos 2x$ but it lags the graph of $y = \cos 2x$ by $\pi/8$. Thus, if we draw the graph of $y = \cos 2x$ and shift it through $\pi/8$ units to the right, we obtain the graph of $y = \cos(2x - \pi/4)$ as shown in Fig. 6.12.

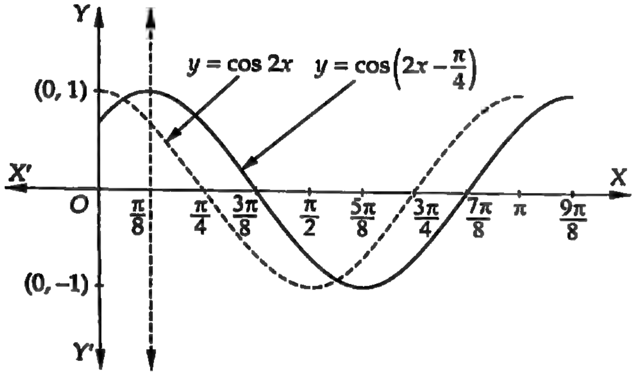


Fig. 6.12 Graph of $y = \cos 2x$ and $y = \cos\left(2x - \frac{\pi}{4}\right)$ on the same scale

EXERCISE 6.2

1. Sketch the following graphs:

   (i) $y = \cos\left(x + \frac{\pi}{4}\right)$ (ii) $y = \cos\left(x - \frac{\pi}{4}\right)$ (iii) $y = 3\cos(2x - 1)$ (iv) $y = 2\cos\left(x - \frac{\pi}{2}\right)$

2. Sketch the graphs of the following functions on the same scale.

   (i) $y = \cos x$ and $y = \cos\left(x - \frac{\pi}{4}\right)$ (ii) $y = \cos 2x$ and $y = \cos 2\left(x - \frac{\pi}{4}\right)$

   (iii) $y = \cos x$ and $y = \cos\left(\frac{x}{2}\right)$

6.2.3 GRAPH OF $y = \tan x$

Since $\tan x$ is a periodic function with period π. So, it is sufficient to draw the graph over an interval of length π, in particular $[-\pi/2, \pi/2]$. First we will draw the graph in the interval $[0, \pi/2]$. For this we use the table given below and the fact that $\tan x$ is increasing in this interval. Also, as $x \to \pi/2$, $\tan x \to \infty$. So, the graph gets closer and closer to the line $x = \pi/2$ as $x \to \pi/2$. But it never touches the line $x = \pi/2$.

x	0°	30°	45°	60°	90°
$\tan x$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

Since $\tan(-x) = -\tan x$, therefore, if $(x, \tan x)$ is any point on the curve $y = \tan x$, then $(-x, -\tan x)$ will also be a point on it. This means that the graph is symmetric in opposite quadrants.

Using all the above points we obtain the sketch of the curve $y = \tan x$ as given in Fig. 6.13.

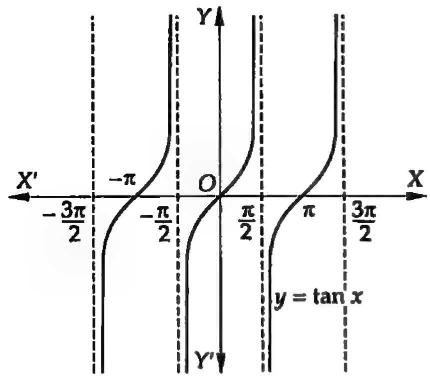


Fig. 6.13 Graph of $y = \tan x$

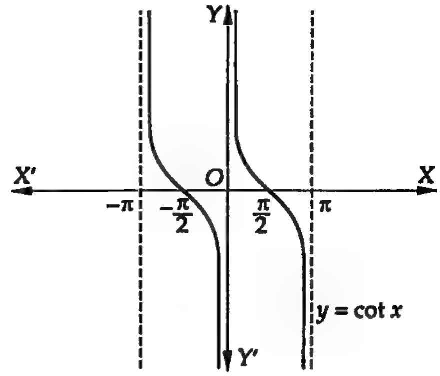


Fig. 6.14 Graph of $y = \cot x$

Proceeding as above we obtain the graphs of other trigonometric functions as shown in Figures 6.14, 6.15, and 6.16.

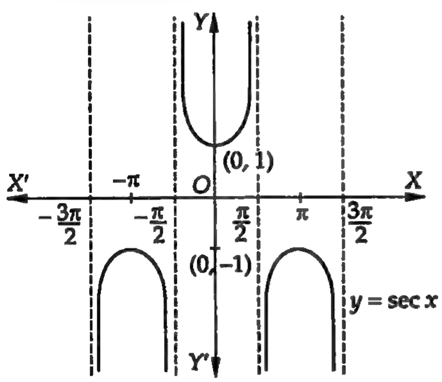


Fig. 6.15 Graph of $y = \sec x$

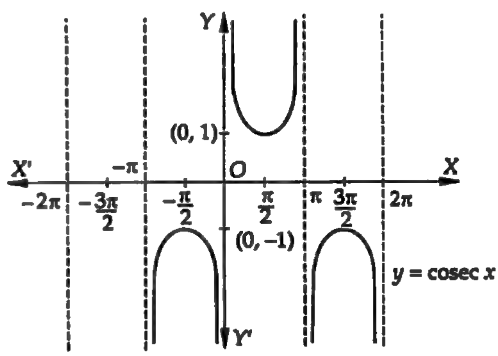


Fig. 6.16 Graph of $y = \operatorname{cosec} x$

It should be noted that cosec x and sec x are also periodic functions with period 2π while cot x is periodic with period π. As $x \to \pm \pi/2$, $\sec x \to \infty$. So, the curve comes closer and closer to ∞ as $x \to \pm \pi/2$. These lines are known as the asymptotes to the curve. Similarly, $x = 0$, $x = \pi$ etc. are asymptotes of $y = \operatorname{cosec} x$.

EXERCISE 6.3

Sketch the graphs of the following functions:

1. $y = \sin^2 x$
2. $y = \cos^2 x$
3. $y = \sin^2\left(x - \frac{\pi}{4}\right)$
4. $y = \tan 2x$
5. $y = 2\tan 3x$
6. $y = 2\cot 2x$

Sketch the graphs of the following functions on the same scale:

7. $y = \cos 2x$, $y = \cos\left(2x - \frac{\pi}{3}\right)$
8. $y = \sin^2 x$, $y = \sin x$
9. $y = \tan x$, $y = \tan^2 x$
10. $y = \tan 2x$, $y = \tan x$

SUMMARY

1. The curve $y = \sin x$ is symmetric in opposite quadrants and $-1 \le y \le 1$.
2. The curve $y = \cos x$ is symmetric about y-axis and $-1 \le y \le 1$.
3. The curve $y = \tan x$ is symmetric in opposite quadrants and $-\infty < y < \infty$.
4. The curve $y = \sec x$ is symmetric about y-axis and $y \ge 1$ or $y \le -1$. The values of y do not exist for $x = (2x + 1)\dfrac{\pi}{2}, n \in Z$.
5. The curve $y = \text{cosec } x$ is symmetric in opposite quadrants such that $y \ge 1$ or $y \le -1$. The values of y do not exist for $x = n\pi, n \in Z$.
6. The curve $y = \cot x$ is symmetric in opposite quadrants such that $-\infty < y < \infty$. The values of y do not exist for $x = n\pi, n \in Z$.

CHAPTER 7

TRIGONOMETRIC RATIOS OF COMPOUND ANGLES

7.1 INTRODUCTION

The algebraic sums of two or more angles are generally called compound angles and the angles are known as the constituent angles.

For example, if A, B, C are three angles, then $A \pm B$, $A+B+C$, $A-B+C$ etc. are compound angles.

In this chapter, we shall derive formulae which will express the trigonometric ratios of compound angles in terms of trigonometric ratios of constituent angles.

7.2 TRIGONOMETRIC RATIOS OF SUM AND DIFFERENCE OF TWO ANGLES

7.2.1 COSINE OF THE DIFFERENCE AND SUM OF TWO ANGLES

THEOREM *For all values of angle A and B, prove that*

(i) $\cos(A-B) = \cos A \cos B + \sin A \sin B$ (ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

PROOF (i) Let $X'OX$ and YOY' be the coordinate axes. Consider a unit circle with O as the centre.

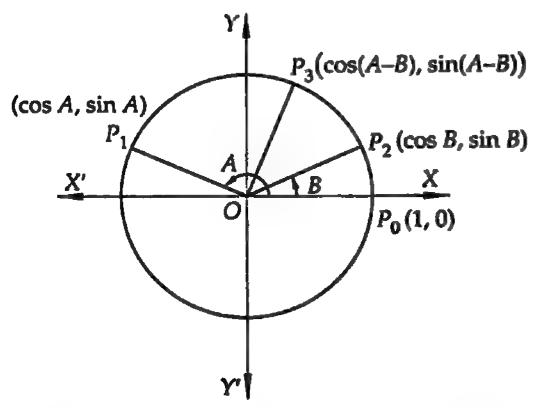


Fig. 7.1

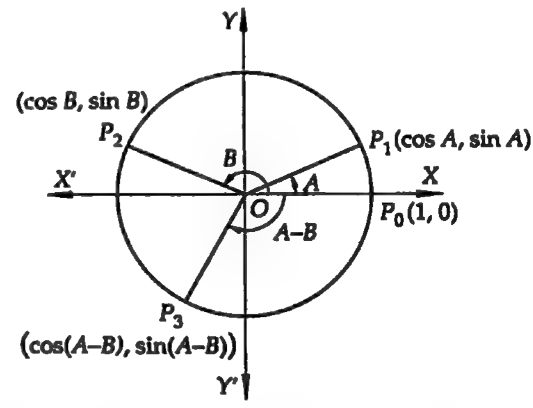


Fig. 7.2

Let P_1, P_2 and P_3 be three points on the circles such that $\angle XOP_1 = A$, $\angle XOP_2 = B$ and $\angle XOP_3 = A-B$. As we have seen in section 5.5 that the terminal side of any angle intersects the circle with centre at O and unit radius at a point whose coordinates are respectively the cosine and sine of the angle. Therefore, coordinates of P_1, P_2 and P_3 are $(\cos A, \sin A)$, $(\cos B, \sin B)$ and $(\cos(A-B), \sin(A-B))$ respectively.

We know that equal chords of a circle make equal angles at its centre and chords $P_0 P_3$ and $P_1 P_2$ subtend equal angles at O. Therefore,

$$\text{Chord } P_0 P_3 = \text{Chord } P_1 P_2$$

$$\Rightarrow \quad \sqrt{\{\cos(A-B)-1\}^2 + \{\sin(A-B)-0\}^2} = \sqrt{(\cos B - \cos A)^2 + (\sin B - \sin A)^2}$$

$$\Rightarrow \quad \{\cos(A-B)-1\}^2+\sin^2(A-B) = (\cos B-\cos A)^2+(\sin B-\sin A)^2$$

$$\Rightarrow \quad \cos^2(A-B)-2\cos(A-B)+1+\sin^2(A-B) = \cos^2 B+\cos^2 A-2\cos A\cos B + \sin^2 B+\sin^2 A-2\sin A\sin B$$

$$\Rightarrow \quad 2-2\cos(A-B) = 2-2\cos A\cos B-2\sin A\sin B$$

$$\Rightarrow \quad \cos(A-B) = \cos A\cos B+\sin A\sin B$$

Hence, $\cos(A-B) = \cos A\cos B+\sin A\sin B$

(ii) Clearly,

$$\cos(A+B) = \cos(A-(-B))$$

$$\Rightarrow \quad \cos(A+B) = \cos A\cos(-B)+\sin A\sin(-B) \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \cos(A+B) = \cos A\cos B-\sin A\sin B \qquad [\because \cos(-B)=\cos B,\ \sin(-B)=-\sin B]$$

Hence, $\cos(A+B) = \cos A\cos B-\sin A\sin B$

Q.E.D.

REMARK *This method of proof of the above formula is true for all values of angles A and B whether positive, zero or negative.*

7.2.2 SINE OF THE DIFFERENCE AND SUM OF TWO ANGLES

THEOREM *For all values of angles A and B, prove that*

(i) $\sin(A-B) = \sin A\cos B-\cos A\sin B$ (ii) $\sin(A+B) = \sin A\cos B+\cos A\sin B$

PROOF (i) We have,

$$\sin(A-B) = \cos(90^\circ-(A-B)) \qquad [\because \cos(90^\circ-\theta)=\sin\theta]$$

$$\Rightarrow \quad \sin(A-B) = \cos((90^\circ-A)+B)$$

$$\Rightarrow \quad \sin(A-B) = \cos(90^\circ-A)\cos B-\sin(90^\circ-A)\sin B$$

$$[\because \cos(A+B)=\cos A\cos B-\sin A\sin B]$$

$$\Rightarrow \quad \sin(A-B) = \sin A\cos B-\cos A\sin B$$

(ii) Clearly,

$$\sin(A+B) = \sin(A-(-B))$$

$$\Rightarrow \quad \sin(A+B) = \sin A\cos(-B)-\cos A\sin(-B) \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \sin(A+B) = \sin A\cos B+\cos A\sin B \qquad [\because \sin(-B)=-\sin B]$$

Q.E.D.

7.2.3 TANGENT OF THE DIFFERENCE AND SUM OF TWO ANGLES

THEOREM *For those values of angles A and B for which both sides are defined, prove that*

(i) $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\tan B}$ (ii) $\tan(A-B) = \dfrac{\tan A-\tan B}{1+\tan A\tan B}$

PROOF (i) We have,

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$\Rightarrow \quad \tan(A+B) = \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$\Rightarrow \quad \tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B} \qquad \left[\begin{array}{l}\text{On dividing the numerator and denominator}\\ \text{by } \cos A\cos B\end{array}\right]$$

(ii) We have,

$$\tan(A-B) = \tan(A+(-B))$$

$$\Rightarrow \quad \tan(A-B) = \frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B} \qquad \textbf{Q.E.D.}$$

Similarly, it can be proved that

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A} \text{ and, } \cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

7.3 MORE USEFUL RESULTS

THEOREM *Prove that:*

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(iii) $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

(iv) $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(v) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

PROOF (i) $\sin(A+B)\sin(A-B)$

$= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$

$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$

$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B$

$= \sin^2 A - \sin^2 B = (1-\cos^2 A) - (1-\cos^2 B) = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B)$

$= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$

$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B = \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$

$= \cos^2 A - \sin^2 B = (1-\sin^2 A) - (1-\cos^2 B) = \cos^2 B - \sin^2 A$

(iii) $\sin(A+B+C)$

$= \sin((A+B)+C) = \sin(A+B)\cos C + \cos(A+B)\sin C$

$= (\sin A \cos B + \cos A \sin B)\cos C + (\cos A \cos B - \sin A \sin B)\sin C$

$= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

(iv) $\cos(A+B+C)$

$= \cos((A+B)+C) = \cos(A+B)\cos C - \sin(A+B)\sin C$

$= (\cos A \cos B - \sin A \sin B)\cos C - (\sin A \cos B + \cos A \sin B)\sin C$

$= \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$

(v) $\tan(A+B+C) = \tan((A+B)+C)$

$$\Rightarrow \quad \tan(A+B+C) = \frac{\tan(A+B) + \tan C}{1-\tan(A+B)\tan C}$$

$$\Rightarrow \quad \tan(A+B+C) = \frac{\left(\dfrac{\tan A + \tan B}{1-\tan A \tan B}\right) + \tan C}{1-\left(\dfrac{\tan A + \tan B}{1-\tan A \tan B}\right)\tan C}$$

$$\Rightarrow \quad \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1-\tan A \tan B - \tan B \tan C - \tan C \tan A}$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE VALUES OF** $\sin(A \pm B)$, $\cos(A \pm B)$ **AND** $\tan(A \pm B)$ **WHEN ONE OF THE TRIGONOMETRIC RATIOS OF EACH OF ANGLES A AND B Is GIVEN**

EXAMPLE 1 *If* $\sin A = \frac{3}{5}$ *and* $\cos B = \frac{9}{41}$, $0 < A < \frac{\pi}{2}$, $0 < B < \frac{\pi}{2}$, *find the values of the following:*

(i) $\sin(A - B)$ (ii) $\sin(A + B)$ (iii) $\cos(A - B)$ (iv) $\cos(A + B)$

SOLUTION We have,

$$\sin A = \frac{3}{5} \text{ and, } \cos B = \frac{9}{41}$$

$$\therefore \quad \cos A = \sqrt{1 - \sin^2 A} \text{ and, } \sin B = \sqrt{1 - \cos^2 B}$$

$$\Rightarrow \quad \cos A = \sqrt{1 - \frac{9}{25}} = \frac{4}{5} \text{ and, } \sin B = \sqrt{1 - \frac{81}{1681}} = \frac{40}{41}$$

(i) $\sin(A - B) = \sin A \cos B - \cos A \sin B = \frac{3}{5} \times \frac{9}{41} - \frac{4}{5} \times \frac{40}{41} = -\frac{133}{205}$

(ii) $\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{3}{5} \times \frac{9}{41} + \frac{4}{5} \times \frac{40}{41} = \frac{187}{205}$

(iii) $\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{4}{5} \times \frac{9}{41} + \frac{3}{5} \times \frac{40}{41} = \frac{156}{205}$

(iv) $\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \times \frac{9}{41} - \frac{3}{5} \times \frac{40}{41} = -\frac{84}{205}$

EXAMPLE 2 *If* $\sin A = \frac{3}{5}$, $0 < A < \frac{\pi}{2}$ *and* $\cos B = \frac{-12}{13}$, $\pi < B < \frac{3\pi}{2}$, *find the following:*

(i) $\sin(A - B)$ (ii) $\cos(A + B)$ (iii) $\tan(A - B)$

SOLUTION We have, $\sin A = \frac{3}{5}$, where $0 < A < \frac{\pi}{2}$.

$$\therefore \quad \cos A = \pm\sqrt{1 - \sin^2 A} \Rightarrow \cos A = +\sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

In the first quadrant tangent function is positive. Therefore, $\tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$

It is given that: $\cos B = -\frac{12}{13}$ and $\pi < B < \frac{3\pi}{2}$.

$$\therefore \quad \sin B = \pm\sqrt{1 - \cos^2 B}$$

$$\Rightarrow \quad \sin B = -\sqrt{1 - \cos^2 B} \qquad [\because \text{ Sine is negative in the third quadrant}]$$

$$\Rightarrow \quad \sin B = -\sqrt{1 - \left(\frac{-12}{13}\right)^2} = -\frac{5}{13}$$

In the III quadrant tangent function is positive. Therefore, $\tan B = \frac{\sin B}{\cos B} = \frac{5}{12}$.

Now,

(i) $\sin(A - B) = \sin A \cos B - \cos A \sin B = \frac{3}{5} \times \frac{-12}{13} - \frac{4}{5} \times \frac{-5}{13} = \frac{-16}{65}$

(ii) $\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \times \frac{-12}{13} - \frac{3}{5} \times \frac{-5}{13} = \frac{-33}{65}$

(iii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B} = \dfrac{\frac{3}{4}-\frac{5}{12}}{1+\frac{3}{4}\times\frac{5}{12}} = \dfrac{16}{63}$

EXAMPLE 3 *If* $\cos A = \frac{4}{5}$, $\cos B = \frac{12}{13}$, $\frac{3\pi}{2} < A,\ B < 2\pi$, *find the values of the following:*

(i) $\cos(A+B)$ (ii) $\sin(A-B)$

SOLUTION Since A and B both lie in the IV quadrant, it follows that $\sin A$ and $\sin B$ are negative. Therefore,

$$\sin A = -\sqrt{1-\cos^2 A} \Rightarrow \sin A = -\sqrt{1-\frac{16}{25}} = -\frac{3}{5}$$

and, $$\sin B = -\sqrt{1-\cos^2 B} \Rightarrow \sin B = -\sqrt{1-\frac{144}{169}} = -\frac{5}{13}$$

Now,

(i) $\cos(A+B) = \cos A \cos B - \sin A \sin B = \frac{4}{5}\times\frac{12}{13} - \left(\frac{-3}{5}\right)\left(\frac{-5}{13}\right) = \frac{33}{65}$

(ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B = \frac{-3}{5}\times\frac{12}{13} - \frac{4}{5}\times\frac{-5}{13} = \frac{-16}{65}$

EXAMPLE 4 *If* $\cot\alpha = \frac{1}{2}$, $\sec\beta = -\frac{5}{3}$, *where* $\pi < \alpha < \frac{3\pi}{2}$ *and* $\frac{\pi}{2} < \beta < \pi$. *Find the value of* $\tan(\alpha+\beta)$. *State the quadrant in which* $\alpha+\beta$ *terminates.*

SOLUTION We have,

$$\cot\alpha = \frac{1}{2} \Rightarrow \tan\alpha = 2$$

Since β lies in the second quadrant. Therefore, $\tan\beta$ is negative. Consequently,

$$1+\tan^2\beta = \sec^2\beta \Rightarrow \tan\beta = -\sqrt{\sec^2\beta - 1} = -\sqrt{\frac{25}{9}-1} = -\frac{4}{3}$$

Hence, $$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{2-\frac{4}{3}}{1-2\times\frac{-4}{3}} = \frac{2}{11}$$

Now, $$\pi < \alpha < \frac{3\pi}{2} \text{ and } \frac{\pi}{2} < \beta < \pi \Rightarrow \frac{3\pi}{2} < \alpha+\beta < \frac{5\pi}{2}$$

We know that tangent function is positive in I and III quadrants.

$\therefore$ $\frac{3\pi}{2} < \alpha+\beta < \frac{5\pi}{2}$ and $\tan(\alpha+\beta) = \frac{2}{11} > 0 \Rightarrow \alpha+\beta$ lies in I quadrant.

Type II **ON FINDING THE TRIGONOMETRIC RATIOS OF ANGLES WHICH ARE MULTIPLES OF 15°**

EXAMPLE 5 *Find the values of the following:*

(i) $\sin 75°$ [NCERT] (ii) $\cos 75°$ [NCERT] (iii) $\sin 15°$ (iv) $\cos 15°$

SOLUTION (i) $\sin 75° = \sin(45°+30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$

$$\Rightarrow \quad \sin 75° = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\cos 75° = \cos(45°+30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$

$$\Rightarrow \quad \cos 75° = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(iii) $\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$

$\Rightarrow \quad \sin 15° = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(iv) $\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$\Rightarrow \quad \cos 15° = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

EXAMPLE 6 *Find the values of the following:*

(i) $\tan 15°$ [NCERT] (ii) $\tan 75°$ (iii) $\tan 105°$ (iv) $\tan \frac{13\pi}{12}$ [NCERT]

SOLUTION (i) $\tan 15° = \tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

(ii) $\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

(iii) $\tan 105° = \tan(90° + 15°) = -\cot 15° = -\frac{1}{\tan 15°} = -\frac{\sqrt{3}+1}{\sqrt{3}-1}$ [Using (i)]

(iv) $\tan \frac{13\pi}{12} = \tan\left(\pi + \frac{\pi}{12}\right) = \tan \frac{\pi}{12} = \tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1}$ [Using (i)]

EXAMPLE 7 *Prove that:* $\tan 75° + \cot 75° = 4$

SOLUTION We have,

$\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ [See Ex. 6]

$\therefore \quad \cot 75° = \frac{1}{\tan 75°} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

Now, LHS $= \tan 75° + \cot 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)}$

$= \frac{(4+2\sqrt{3}) + (4-2\sqrt{3})}{3-1} = \frac{8}{2} = 4 =$ RHS

Type III **ON THE APPLICATIONS OF THE FOLLOWING FORMULAE:**

(i) $\sin A \cos B \pm \cos A \sin B = \sin(A \pm B)$ (ii) $\cos A \cos B \pm \sin A \sin B = \cos(A \mp B)$

EXAMPLE 8 *Evaluate the following:*

(i) $\sin \frac{7\pi}{12} \cos \frac{\pi}{4} - \cos \frac{7\pi}{12} \sin \frac{\pi}{4}$ (ii) $\sin \frac{\pi}{4} \cos \frac{\pi}{12} + \cos \frac{\pi}{4} \sin \frac{\pi}{12}$

(iii) $\cos \frac{2\pi}{3} \cos \frac{\pi}{4} - \sin \frac{2\pi}{3} \sin \frac{\pi}{4}$

SOLUTION (i) $\sin \frac{7\pi}{12} \cos \frac{\pi}{4} - \cos \frac{7\pi}{12} \sin \frac{\pi}{4} = \sin\left(\frac{7\pi}{12} - \frac{\pi}{4}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(ii) $\sin \frac{\pi}{4} \cos \frac{\pi}{12} + \cos \frac{\pi}{4} \sin \frac{\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = \sin \frac{4\pi}{12} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{4}-\sin\frac{2\pi}{3}\sin\frac{\pi}{4}=\cos\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)=\cos\frac{11\pi}{12}=\cos 165°$

$$=\cos(180°-15°)=-\cos 15° = -\frac{\sqrt{3}+1}{2\sqrt{2}}$$ [See Ex. 5 (iv)]

EXAMPLE 9 *Prove that:* $\cos\left(\frac{\pi}{4}-A\right)\cos\left(\frac{\pi}{4}-B\right)-\sin\left(\frac{\pi}{4}-A\right)\sin\left(\frac{\pi}{4}-B\right)=\sin(A+B)$ [NCERT]

SOLUTION We have,

$$\text{LHS} = \cos\left(\frac{\pi}{4}-A\right)\cos\left(\frac{\pi}{4}-B\right)-\sin\left(\frac{\pi}{4}-A\right)\sin\left(\frac{\pi}{4}-B\right)$$

$$=\cos\left\{\left(\frac{\pi}{4}-A\right)+\left(\frac{\pi}{4}-B\right)\right\}=\cos\left\{\frac{\pi}{2}-(A+B)\right\}=\sin(A+B)=\text{RHS}$$

EXAMPLE 10 *Prove that:* $\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A=\cos A$ [NCERT]

SOLUTION $\text{LHS} = \sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A$

$$=\cos(n+2)A\cos(n+1)A+\sin(n+2)A\sin(n+1)A$$

$$=\cos\{(n+2)A-(n+1)A\}=\cos A=\text{RHS}$$

Type IV **ON APPLICATIONS OF THE FORMULAE:**

(i) $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$ (ii) $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

(iii) $\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

EXAMPLE 11 *Prove that:*

(i) $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$ [NCERT]

(ii) $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=-\sqrt{2}\sin x$ [NCERT]

SOLUTION (i) We have,

$$\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$$

$$=\left(\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x\right)+\left(\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x\right)$$

$$=2\cos\frac{\pi}{4}\cos x=2\times\frac{1}{\sqrt{2}}\times\cos x=\sqrt{2}\cos x$$

(ii) We have,

$$\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=-\sqrt{2}\sin x$$

$$=\left(\cos\frac{3\pi}{4}\cos x-\sin\frac{3\pi}{4}\sin x\right)+\left(\cos\frac{3\pi}{4}\cos x+\sin\frac{3\pi}{4}\sin x\right)$$

$$=\left(-\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)-\left(-\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)$$

$$=-\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\sin x=-\frac{2}{\sqrt{2}}\sin x=-\sqrt{2}\sin x$$

EXAMPLE 12 *Prove that:* $\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{\tan x+\tan y}{\tan x-\tan y}$ [NCERT]

SOLUTION We have,

$$\text{LHS} = \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$$

$$= \frac{\dfrac{\sin x\cos y + \cos x\sin y}{\cos x\cos y}}{\dfrac{\sin x\cos y - \cos x\sin y}{\cos x\cos y}} \quad \text{[Dividing the numerator and denominator by } \cos x\cos y]$$

$$= \frac{\tan x + \tan y}{\tan x - \tan y} = \text{RHS}$$

EXAMPLE 13 *Prove that:* $\dfrac{\sin(B-C)}{\cos B\cos C} + \dfrac{\sin(C-A)}{\cos C\cos A} + \dfrac{\sin(A-B)}{\cos A\cos B} = 0$

SOLUTION We have,

$$\text{LHS} = \frac{\sin(B-C)}{\cos B\cos C} + \frac{\sin(C-A)}{\cos C\cos A} + \frac{\sin(A-B)}{\cos A\cos B}$$

$$= \frac{\sin B\cos C - \cos B\sin C}{\cos B\cos C} + \frac{\sin C\cos A - \cos C\sin A}{\cos C\cos A} + \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B}$$

$$= \frac{\sin B\cos C}{\cos B\cos C} - \frac{\cos B\sin C}{\cos B\cos C} + \frac{\sin C\cos A}{\cos C\cos A} - \frac{\cos C\sin A}{\cos C\cos A} + \frac{\sin A\cos B}{\cos A\cos B} - \frac{\cos A\sin B}{\cos A\cos B}$$

$$= \tan B - \tan C + \tan C - \tan A + \tan A - \tan B = 0 = \text{RHS}$$

EXAMPLE 14 *Prove that:* $\dfrac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)} = \left(\dfrac{1+\tan x}{1-\tan x}\right)^2$ [NCERT]

SOLUTION

$$\text{LHS} = \frac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)} = \frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x} \times \frac{1+\tan\frac{\pi}{4}\tan x}{\tan\frac{\pi}{4}-\tan x}$$

$$= \frac{1+\tan x}{1-\tan x} \times \frac{1+\tan x}{1-\tan x} = \left(\frac{1+\tan x}{1-\tan x}\right)^2 = \text{RHS}$$

EXAMPLE 15 *If* $\tan A - \tan B = x$ *and* $\cot B - \cot A = y$, *prove that* $\cot(A-B) = \dfrac{1}{x} + \dfrac{1}{y}$

[NCERT EXEMPAR]

SOLUTION We have, $\tan A - \tan B = x$ and $\cot B - \cot A = y$

Now,

$$\cot B - \cot A = y$$

$$\Rightarrow \quad \frac{1}{\tan B} - \frac{1}{\tan A} = y \Rightarrow \frac{\tan A - \tan B}{\tan A\tan B} = y \Rightarrow \frac{x}{\tan A\tan B} = y \Rightarrow \tan A\tan B = \frac{x}{y}$$

$$\therefore \quad \cot(A-B) = \frac{1}{\tan(A-B)} = \frac{1+\tan A\tan B}{\tan A - \tan B} = \frac{1+\frac{x}{y}}{x} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y}$$

EXAMPLE 16 *If* $\tan\alpha = \dfrac{1}{\sqrt{x(x^2+x+1)}}$, $\tan\beta = \dfrac{\sqrt{x}}{\sqrt{x^2+x+1}}$ *and* $\tan\gamma = \sqrt{x^{-3}+x^{-2}+x^{-1}}$, *prove that* $\alpha+\beta=\gamma$.

SOLUTION We have,

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{\dfrac{1}{\sqrt{x(x^2+x+1)}}+\dfrac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1-\dfrac{1}{\sqrt{x(x^2+x+1)}}\times\dfrac{\sqrt{x}}{\sqrt{x^2+x+1}}} = \frac{(x+1)\sqrt{x(x^2+x+1)}}{x(x^2+x+1)-x}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{(x+1)\sqrt{x(x^2+x+1)}}{x^2(x+1)} = \frac{\sqrt{x(x^2+x+1)}}{x^2} = \sqrt{\frac{x(x^2+x+1)}{x^4}}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \sqrt{x^{-3}+x^{-2}+x^{-1}} = \tan\gamma$$

$$\Rightarrow \quad \alpha+\beta = \gamma.$$

EXAMPLE 17 *If α and β are acute angles such that* $\tan\alpha = \dfrac{m}{m+1}$ and $\tan\beta = \dfrac{1}{2m+1}$, *prove that* $\alpha+\beta = \dfrac{\pi}{4}$.

SOLUTION We have,

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{\dfrac{m}{m+1}+\dfrac{1}{2m+1}}{1-\dfrac{m}{m+1}\times\dfrac{1}{2m+1}} = \frac{2m^2+m+m+1}{2m^2+3m+1-m} = \frac{2m^2+2m+1}{2m^2+2m+1} = 1 = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \alpha+\beta = \frac{\pi}{4}$$

EXAMPLE 18 *If $A+B=\dfrac{\pi}{4}$, prove that:*

(i) $(1+\tan A)(1+\tan B)=2$ (ii) $(\cot A-1)(\cot B-1)=2$

SOLUTION (i) We have,

$$A+B = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan(A+B) = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{\tan A+\tan B}{1-\tan A\tan B} = 1$$

$$\Rightarrow \quad \tan A+\tan B = 1-\tan A\tan B$$

$$\Rightarrow \quad \tan A+\tan B+\tan A\tan B = 1$$

$$\Rightarrow \quad 1+\tan A+\tan B+\tan A\tan B = 2$$

$$\Rightarrow \quad (1+\tan A)+\tan B(1+\tan A) = 2 \Rightarrow (1+\tan A)(1+\tan B) = 2$$

(ii) We have,

$$A+B=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan(A+B) = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{\tan A+\tan B}{1-\tan A\tan B} = 1$$

$$\Rightarrow \quad \tan A+\tan B = 1-\tan A\tan B$$

$\Rightarrow \quad \tan A + \tan B + \tan A \tan B = 1$

$$\Rightarrow \quad \frac{\tan A + \tan B + \tan A \tan B}{\tan A \tan B} = \frac{1}{\tan A \tan B}$$

$\Rightarrow \quad \cot B + \cot A + 1 = \cot A \cot B$

$\Rightarrow \quad \cot A \cot B - \cot A - \cot B = 1$

$\Rightarrow \quad \cot A \cot B - \cot A - \cot B + 1 = 2$

$\Rightarrow \quad \cot A (\cot B - 1) - (\cot B - 1) = 2$

$\Rightarrow \quad (\cot A - 1)(\cot B - 1) = 2$

EXAMPLE 19 *If* $\tan \beta = \dfrac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}$, *show that* $\tan(\alpha - \beta) = (1 - n) \tan \alpha$.

SOLUTION We have,

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\dfrac{\sin \alpha}{\cos \alpha} - \dfrac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}}{1 + \dfrac{\sin \alpha}{\cos \alpha} \times \dfrac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}} \qquad \left[\because \tan \beta = \frac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}\right]$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\sin \alpha - n \sin^3 \alpha - n \sin \alpha \cos^2 \alpha}{\cos \alpha (1 - n \sin^2 \alpha) + n \sin^2 \alpha \cos \alpha} \qquad \text{[On taking LCM]}$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\sin \alpha - n \sin^3 \alpha - n \sin \alpha (1 - \sin^2 \alpha)}{\cos \alpha - n \sin^2 \alpha \cos \alpha + n \sin^2 \alpha \cos \alpha}$$

$$\Rightarrow \quad \tan(\alpha - \beta) = \frac{\sin \alpha - n \sin \alpha}{\cos \alpha} = \frac{\sin \alpha}{\cos \alpha}(1 - n) = (1 - n) \tan \alpha$$

EXAMPLE 20 *Prove that:*

(i) $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$ [NCERT]

(ii) $\cot A \cot 2A - \cot 2A \cot 3A - \cot 3A \cot A = 1$

SOLUTION (i) Clearly,

$3A = 2A + A$

$\Rightarrow \quad \tan 3A = \tan(2A + A)$

$$\Rightarrow \quad \tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$\Rightarrow \quad \tan 3A (1 - \tan 2A \tan A) = \tan 2A + \tan A$

$\Rightarrow \quad \tan 3A - \tan 3A \tan 2A \tan A = \tan 2A + \tan A$

$\Rightarrow \quad \tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$

(ii) Dividing both sides by $\tan A \tan 2A \tan 3A$, we get

$$\frac{\tan 3A \tan 2A \tan A}{\tan 3A \tan 2A \tan A} = \frac{\tan 3A - \tan 2A - \tan A}{\tan 3A \tan 2A \tan A}$$

$$\Rightarrow \quad 1 = \frac{1}{\tan 2A \tan A} - \frac{1}{\tan 3A \tan A} - \frac{1}{\tan 3A \tan 2A}$$

$\Rightarrow \quad 1 = \cot A \cot 2A - \cot 3A \cot A - \cot 3A \cot 2A$

Type V **ON THE APPLICATIONS OF THE FORMULAE:**

(i) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$ (ii) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$

EXAMPLE 21 *Prove that:* $\dfrac{\tan(A+B)}{\cot(A-B)}=\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}$

SOLUTION LHS $=\dfrac{\tan(A+B)}{\cot(A-B)}=\dfrac{\sin(A+B)\sin(A-B)}{\cos(A+B)\cos(A-B)}=\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}=$ RHS

EXAMPLE 22 *Prove that :* $\sin^2 6x-\sin^2 4x=\sin 2x\sin 10x$ **[NCERT]**

SOLUTION We know that $\sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$

$\therefore$ $\sin^2 6x-\sin^2 4x=\sin(6x+4x)\sin(6x-4x)=\sin 10x\sin 2x$

EXAMPLE 23 *Prove that :* $\cos^2 2x-\cos^2 6x=\sin 4x\sin 8x$ **[NCERT]**

SOLUTION We have,

$$\begin{aligned}\therefore\quad \text{LHS}&=\cos^2 2x-\cos^2 6x\\&=(1-\sin^2 2x)-(1-\sin^2 6x)\\&=\sin^2 6x-\sin^2 2x=\sin(6x+2x)\sin(6x-2x)=\sin 8x\sin 4x=\text{RHS}\end{aligned}$$

EXAMPLE 24 *Prove that* $\dfrac{\cos^2 33°-\cos^2 57°}{\sin^2\frac{21°}{2}-\sin^2\frac{69°}{2}}=-\sqrt{2}$

SOLUTION We have,

$$\begin{aligned}\text{LHS}&=\frac{\cos^2 33°-\cos^2 57°}{\sin^2\frac{21°}{2}-\sin^2\frac{69°}{2}}\\&=\frac{(1-\sin^2 33°)-(1-\sin^2 57°)}{\sin^2\frac{21°}{2}-\sin^2\frac{69°}{2}}\\&=\frac{\sin^2 57°-\sin^2 33°}{\sin^2\frac{21°}{2}-\sin^2\frac{69°}{2}}\\&=\frac{\sin(57°+33°)\sin(57°-33°)}{\sin\left(\frac{21°}{2}+\frac{69°}{2}\right)\sin\left(\frac{21°}{2}-\frac{69°}{2}\right)}\\&=\frac{\sin 90°\sin 24°}{\sin 45°\sin(-24°)}=\frac{\sin 24°}{-\frac{1}{\sqrt{2}}\sin 24°}=-\sqrt{2}=\text{RHS}\end{aligned}$$

EXAMPLE 25 *Prove that:* $\sin^2\left(\dfrac{\pi}{8}+\dfrac{A}{2}\right)-\sin^2\left(\dfrac{\pi}{8}-\dfrac{A}{2}\right)=\dfrac{1}{\sqrt{2}}\sin A$

SOLUTION Using $\sin^2 A-\sin^2 B=\sin(A+B)-\sin(A-B)$, we obtain

$$\begin{aligned}\text{LHS}&=\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)\\&=\sin\left\{\left(\frac{\pi}{8}+\frac{A}{2}\right)+\left(\frac{\pi}{8}-\frac{A}{2}\right)\right\}\sin\left\{\left(\frac{\pi}{8}+\frac{A}{2}\right)-\left(\frac{\pi}{8}-\frac{A}{2}\right)\right\}\\&=\sin\frac{\pi}{4}\sin A=\frac{1}{\sqrt{2}}\sin A=\text{RHS}\end{aligned}$$

EXAMPLE 26 *Prove that:* $\cos 2\alpha\cos 2\beta+\sin^2(\alpha-\beta)-\sin^2(\alpha+\beta)=\cos 2(\alpha+\beta)$

SOLUTION We have,

$$\text{LHS}=\cos 2\alpha\cos 2\beta+\sin^2(\alpha-\beta)-\sin^2(\alpha+\beta)$$

$$= \cos 2\alpha \cos 2\beta + \sin(\alpha - \beta + \alpha + \beta)\sin(\alpha - \beta - \alpha - \beta) \qquad \left[\because \begin{array}{r} \sin^2 A - \sin^2 B \\ = \sin(A+B)\sin(A-B) \end{array}\right]$$

$$= \cos 2\alpha \cos 2\beta + \sin 2\alpha \sin(-2\beta)$$

$$= \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta = \cos(2\alpha + 2\beta)$$

EXAMPLE 27 *Prove that:* $\sin^2 A = \cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A \cos B$.

SOLUTION We have,

$$\begin{aligned} \text{RHS} &= \cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A\cos B \\ &= \cos^2 B + \cos^2(A-B) - 2\cos(A-B)\cos A\cos B \\ &= \cos^2 B + \cos(A-B)\{\cos(A-B) - 2\cos A\cos B\} \\ &= \cos^2 B + \cos(A-B)\{\cos A\cos B + \sin A\sin B - 2\cos A\cos B\} \\ &= \cos^2 B + \cos(A-B)\{\sin A\sin B - \cos A\cos B\} \\ &= \cos^2 B - \cos(A-B)(\cos A\cos B - \sin A\sin B) \\ &= \cos^2 B - \cos(A-B)\cos(A+B) \\ &= \cos^2 B - (\cos^2 A - \sin^2 B) \\ &= \cos^2 B + \sin^2 B - \cos^2 A = 1 - \cos^2 A = \sin^2 A = \text{LHS} \end{aligned}$$

LEVEL-2

EXAMPLE 28 *If* $3\tan A\tan B = 1$, *prove that* $2\cos(A+B) = \cos(A-B)$.

SOLUTION We have,

$$3\tan A\tan B = 1$$

$$\Rightarrow \quad \frac{3\sin A\sin B}{\cos A\cos B} = 1$$

$$\Rightarrow \quad \frac{\cos A\cos B}{\sin A\sin B} = \frac{3}{1}$$

$$\Rightarrow \quad \frac{\cos A\cos B + \sin A\sin B}{\cos A\cos B - \sin A\sin B} = \frac{3+1}{3-1} \qquad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \quad \frac{\cos(A-B)}{\cos(A+B)} = 2 \Rightarrow 2\cos(A+B) = \cos(A-B)$$

EXAMPLE 29 *If* $\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -\frac{3}{2}$, *prove that*

$$\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$$

SOLUTION We have,

$$\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -\frac{3}{2}$$

$$\Rightarrow \quad 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha = -3$$

$$\Rightarrow \quad (2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha) + (2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha) + 3 = 0$$

$$\Rightarrow \quad (2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha) + (2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha + (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) + (\cos^2\gamma + \sin^2\gamma) = 0$$

$$\Rightarrow \quad (\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha) + (\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha) = 0$$

$$\Rightarrow \quad (\cos\alpha+\cos\beta+\cos\gamma)^2+(\sin\alpha+\sin\beta+\sin\gamma)^2 = 0$$

$$\Rightarrow \quad \cos\alpha+\cos\beta+\cos\gamma = 0 \text{ and } \sin\alpha+\sin\beta+\sin\gamma = 0$$

EXAMPLE 30 *If* $\sin B = 3\sin(2A+B)$, *prove that* $2\tan A+\tan(A+B)=0$.

SOLUTION We have,

$$\sin B = 3\sin(2A+B)$$

$$\Rightarrow \quad \frac{\sin(2A+B)}{\sin B} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{\sin\{(A+B)+A\}}{\sin\{(A+B)-A\}} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{\sin\{A+B)+A\}+\sin\{(A+B)-A\}}{\sin\{(A+B)+A\}-\sin\{(A+B)-A\}} = \frac{1+3}{1-3} \quad \text{[Using componendo–dividendo]}$$

$$\Rightarrow \quad \frac{\{\sin(A+B)\cos A+\cos(A+B)\sin A\}+\{\sin(A+B)\cos A-\cos(A+B)\sin A\}}{\{\sin(A+B)\cos A+\cos(A+B)\sin A\}-\{\sin(A+B)\cos A-\cos(A+B)\sin A\}} = \frac{1+3}{1-3}$$

$$\Rightarrow \quad \frac{2\sin(A+B)\cos A}{2\cos(A+B)\sin A} = -2$$

$$\Rightarrow \quad \tan(A+B)\cot A = -2 \Rightarrow \tan(A+B) = -2\tan A \Rightarrow 2\tan A+\tan(A+B)=0$$

EXAMPLE 31 *If* $2\tan\beta+\cot\beta=\tan\alpha$, *prove that* $\cot\beta=2\tan(\alpha-\beta)$.

SOLUTION Clearly,

$$2\tan(\alpha-\beta) = 2\left\{\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right\}$$

$$= 2\left\{\frac{2\tan\beta+\cot\beta-\tan\beta}{1+(2\tan\beta+\cot\beta)\tan\beta}\right\} \quad [\text{Using}: \tan\alpha = 2\tan\beta+\cot\beta]$$

$$= 2\left\{\frac{\tan\beta+\cot\beta}{1+2\tan^2\beta+1}\right\}$$

$$= \frac{2(\tan\beta+\cot\beta)}{2+2\tan^2\beta} = \frac{2\left\{\tan\beta+\frac{1}{\tan\beta}\right\}}{2(1+\tan^2\beta)} = \frac{1}{\tan\beta} = \cot\beta$$

Hence, $\cot\beta = 2\tan(\alpha-\beta)$.

EXAMPLE 32 *If* $\cos(\alpha+\beta)\sin(\gamma+\delta)=\cos(\alpha-\beta)\sin(\gamma-\delta)$, *prove that* $\cot\alpha\cot\beta\cot\gamma=\cot\delta$.

SOLUTION We have,

$$\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$$

$$\Rightarrow \quad \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \frac{\sin(\gamma+\delta)}{\sin(\gamma-\delta)}$$

$$\Rightarrow \quad \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)} = \frac{\sin(\gamma+\delta)+\sin(\gamma-\delta)}{\sin(\gamma+\delta)-\sin(\gamma-\delta)} \quad \text{[Using componendo–dividendo]}$$

$$\Rightarrow \quad \frac{2\cos\alpha\cos\beta}{2\sin\alpha\sin\beta} = \frac{2\sin\gamma\cos\delta}{2\cos\gamma\sin\delta} \Rightarrow \cot\alpha\cot\beta = \tan\gamma\cot\delta \Rightarrow \cot\alpha\cot\beta\cot\gamma = \cot\delta$$

EXAMPLE 33 *Prove that:* $\dfrac{\sin(x+\theta)}{\sin(x+\phi)} = \cos(\theta-\phi)+\cot(x+\phi)\sin(\theta-\phi)$.

SOLUTION We have,

$$\frac{\sin(x+\theta)}{\sin(x+\phi)} = \frac{\sin\{(x+\phi)+(\theta-\phi)\}}{\sin(x+\phi)}$$

$$= \frac{\sin (x + \phi) \cos (\theta - \phi) + \cos (x + \phi) \sin (\theta - \phi)}{\sin (x + \phi)}$$

$$= \cos (\theta - \phi) + \cot (x + \phi) \sin (\theta - \phi)$$

EXAMPLE 34 *If* $\cos (\alpha + \beta) = \frac{4}{5}$, $\sin (\alpha - \beta) = \frac{5}{13}$ *and* α, β *lie between 0 and* $\frac{\pi}{4}$, *prove that* $\tan 2\alpha = \frac{56}{33}$ [NCERT EXEMPLAR]

SOLUTION It is given that α, β lie between 0 and $\pi/4$. Therefore, $-\pi/4 < \alpha - \beta < \pi/4$ and $0 < \alpha + \beta < \pi/2$.

So, $\cos (\alpha - \beta)$ and $\sin (\alpha + \beta)$ are positive.

Now, $\sin (\alpha + \beta) = \sqrt{1 - \cos^2 (\alpha + \beta)} \Rightarrow \sin (\alpha + \beta) = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

and, $\cos (\alpha - \beta) = \sqrt{1 - \sin^2 (\alpha - \beta)} \Rightarrow \cos (\alpha - \beta) = \sqrt{1 - \frac{25}{169}} = \frac{12}{13}$

$\therefore \quad \tan (\alpha + \beta) = \frac{\sin (\alpha + \beta)}{\cos (\alpha + \beta)} = \frac{3/5}{4/5} = \frac{3}{4}$ and, $\tan (\alpha - \beta) = \frac{\sin (\alpha - \beta)}{\cos (\alpha - \beta)} = \frac{5}{12}$

Now,

$$\tan 2\alpha = \tan \{(\alpha + \beta) + (\alpha - \beta)\} = \frac{\tan (\alpha + \beta) + \tan (\alpha - \beta)}{1 - \tan (\alpha + \beta) \tan (\alpha - \beta)} = \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}} = \frac{56}{33}$$

EXAMPLE 35 *Prove that:* $\tan 70° = \tan 20° + 2 \tan 50°$.

SOLUTION We have,

$$\tan A - \tan B = \tan (A - B)(1 + \tan A \tan B)$$

$$\therefore \quad \tan 70° - \tan 20° = \tan (70° - 20°)(1 + \tan 70° \tan 20°)$$
$$= \tan 50° \; (1 + \tan 70° \cot 70°)$$
$$= 2 \tan 50° \qquad [\because \; 1 + \tan 70° \cot 70° = 2]$$

EXAMPLE 36 *If* $\tan (\alpha + \theta) = n \tan (\alpha - \theta)$, *show that:* $(n + 1) \sin 2\theta = (n - 1) \sin 2\alpha$.

SOLUTION We have,

$$\tan (\alpha + \theta) = n \tan (\alpha - \theta)$$

$$\Rightarrow \quad \frac{\tan (\alpha + \theta)}{\tan (\alpha - \theta)} = \frac{n}{1}$$

$$\Rightarrow \quad \frac{\tan (\alpha + \theta) + \tan (\alpha - \theta)}{\tan (\alpha + \theta) - \tan (\alpha - \theta)} = \frac{n+1}{n-1} \qquad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \quad \frac{\sin (\alpha + \theta) \cos (\alpha - \theta) + \cos (\alpha + \theta) \sin (\alpha - \theta)}{\sin (\alpha + \theta) \cos (\alpha - \theta) - \cos (\alpha + \theta) \sin (\alpha - \theta)} = \frac{n+1}{n-1}$$

$$\Rightarrow \quad \frac{\sin (\alpha + \theta) + \alpha - \theta)}{\sin (\alpha + \theta) - (\alpha - \theta))} = \frac{n+1}{n-1}$$

$$\Rightarrow \quad \frac{\sin 2\alpha}{\sin 2\theta} = \frac{n+1}{n-1} \Rightarrow (n+1) \sin 2\theta = (n-1) \sin 2\alpha$$

EXAMPLE 37 *Prove that:* $\cot \theta \cot 2\theta + \cot 2\theta \cot 3\theta + 2 = \cot \theta (\cot \theta - \cot 3\theta)$

SOLUTION

$$\therefore \quad \text{LHS} = \cot \theta \cot 2\theta + \cot 2\theta \cot 3\theta + 2$$
$$= (\cot \theta \cot 2\theta + 1) + (\cot 2\theta \cot 3\theta + 1)$$
$$= \left(\frac{\cos \theta \cos 2\theta}{\sin \theta \sin 2\theta}\right) + \left(\frac{\cos 2\theta \cos 3\theta}{\sin 2\theta \sin 3\theta} + 1\right)$$

$$=\left(\frac{\cos 2\theta \cos\theta+\sin 2\theta \sin\theta}{\sin\theta \sin 2\theta}\right)+\left(\frac{\cos 3\theta \cos 2\theta+\sin 3\theta \sin 2\theta}{\sin 2\theta \sin 3\theta}\right)$$

$$=\frac{\cos(2\theta-\theta)}{\sin\theta \sin 2\theta}+\frac{\cos(3\theta-2\theta)}{\sin 2\theta \sin 3\theta}$$

$$=\frac{\cos\theta}{\sin\theta \sin 2\theta}+\frac{\cos\theta}{\sin 2\theta \sin 3\theta}$$

$$=\cos\theta\left\{\frac{1}{\sin\theta \sin 2\theta}+\frac{1}{\sin 2\theta \sin 3\theta}\right\}$$

$$=\frac{\cos\theta}{\sin\theta}\left\{\frac{\sin\theta}{\sin\theta \sin 2\theta}+\frac{\sin\theta}{\sin 2\theta \sin 3\theta}\right\}$$

$$=\cot\theta\left\{\frac{\sin(2\theta-\theta)}{\sin\theta \sin 2\theta}+\frac{\sin(3\theta-2\theta)}{\sin 2\theta \sin 3\theta}\right\}$$

$$=\cot\theta\left\{\frac{\sin 2\theta\cos\theta-\cos 2\theta\sin\theta}{\sin\theta\sin 2\theta}+\frac{\sin 3\theta\cos 2\theta-\cos 3\theta\sin 2\theta}{\sin 2\theta\sin 3\theta}\right\}$$

$$=\cot\theta\,\{\cot\theta-\cot 2\theta+\cot 2\theta-\cot 3\theta\}=\cot\theta\,(\cot\theta-\cot 3\theta)=\text{RHS}$$

EXAMPLE 38 *If* $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$, *prove that* $\cos\left(\theta-\frac{\pi}{4}\right)=\pm\frac{1}{2\sqrt{2}}$.

SOLUTION We have,

$$\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$$

$$\Rightarrow \frac{\sin(\pi\cos\theta)}{\cos(\pi\cos\theta)}=\frac{\cos(\pi\sin\theta)}{\sin(\pi\sin\theta)}$$

$$\Rightarrow \sin(\pi\cos\theta)\sin(\pi\sin\theta)=\cos(\pi\sin\theta)\cos(\pi\cos\theta)$$

$$\Rightarrow \cos(\pi\cos\theta)\cos(\pi\sin\theta)-\sin(\pi\cos\theta)\sin(\pi\sin\theta)=0$$

$$\Rightarrow \cos(\pi\cos\theta+\pi\sin\theta)=0$$

$$\Rightarrow \pi\cos\theta+\pi\sin\theta=\pm\frac{\pi}{2}\qquad\left[\because \cos\left(\pm\frac{\pi}{2}\right)=0\right]$$

$$\Rightarrow \cos\theta+\sin\theta=\pm\frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta=\pm\frac{1}{2\sqrt{2}}\qquad\left[\text{Multiplying both sides by } 1/\sqrt{2}\right]$$

$$\Rightarrow \cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}=\pm\frac{1}{2\sqrt{2}}$$

$$\Rightarrow \cos\left(\theta-\frac{\pi}{4}\right)=\pm\frac{1}{2\sqrt{2}}$$

EXAMPLE 39 *If* $a\tan\alpha+b\tan\beta=(a+b)\tan\left(\frac{\alpha+\beta}{2}\right)$, *where* $\alpha\neq\beta$, *prove that* $a\cos\beta=b\cos\alpha$.

SOLUTION We have,

$$a\tan\alpha+b\tan\beta=(a+b)\tan\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow a\left\{\tan\alpha-\tan\left(\frac{\alpha+\beta}{2}\right)\right\}=b\left\{\tan\left(\frac{\alpha+\beta}{2}\right)-\tan\beta\right\}$$

$$\Rightarrow \quad \frac{a \sin\left(\alpha - \frac{\alpha+\beta}{2}\right)}{\cos\alpha \cos\left(\frac{\alpha+\beta}{2}\right)} = \frac{b \sin\left(\frac{\alpha+\beta}{2} - \beta\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)\cos\beta} \qquad \left[\because \tan A - \tan B = \frac{\sin(A-B)}{\cos A \cos B}\right]$$

$$\Rightarrow \quad \frac{a \sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\alpha} = \frac{b \sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\beta}$$

$$\Rightarrow \quad a\cos\beta = b\cos\alpha \qquad \left[\because \alpha \neq \beta \therefore \sin\left(\frac{\alpha-\beta}{2}\right) \neq 0\right]$$

EXAMPLE 40 *If* $\sin\alpha + \sin\beta = a$ *and* $\cos\alpha + \cos\beta = b$, *show that*

(i) $\cos(\alpha+\beta) = \dfrac{b^2 - a^2}{b^2 + a^2}$ (ii) $\sin(\alpha+\beta) = \dfrac{2ab}{a^2+b^2}$

SOLUTION (i) We have,

$$b^2 + a^2 = (\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$$

$$\Rightarrow \quad b^2 + a^2 = (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

$$\Rightarrow \quad b^2 + a^2 = 1 + 1 + 2\cos(\alpha-\beta) = 2 + 2\cos(\alpha-\beta) \qquad \text{...(i)}$$

and, $$b^2 - a^2 = (\cos\alpha + \cos\beta)^2 - (\sin\alpha + \sin\beta)^2$$

$$\Rightarrow \quad b^2 - a^2 = \cos^2\alpha + \cos^2\beta - \sin^2\alpha - \sin^2\beta + 2(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$$

$$\Rightarrow \quad b^2 - a^2 = (\cos^2\alpha - \sin^2\beta) + (\cos^2\beta - \sin^2\alpha) + 2\cos(\alpha+\beta)$$

$$\Rightarrow \quad b^2 - a^2 = \cos(\alpha+\beta)\cos(\alpha-\beta) + \cos(\beta+\alpha)\cos(\beta-\alpha) + 2\cos(\alpha+\beta)$$

$$\Rightarrow \quad b^2 - a^2 = 2\cos(\alpha+\beta)\cos(\alpha-\beta) + 2\cos(\alpha+\beta)$$

$$[\because \cos(\beta-\alpha) = \cos\{-(\alpha-\beta)\} = \cos(\alpha-\beta)]$$

$$\Rightarrow \quad b^2 - a^2 = \cos(\alpha+\beta)\{2\cos(\alpha-\beta) + 2\}$$

$$\Rightarrow \quad b^2 - a^2 = \cos(\alpha+\beta)(b^2+a^2) \qquad [\text{Using (i)}]$$

Thus, $b^2 - a^2 = (b^2+a^2)\cos(\alpha+\beta) \Rightarrow \cos(\alpha+\beta) = \dfrac{b^2-a^2}{b^2+a^2}$

(ii) $$\sin(\alpha+\beta) = \sqrt{1 - \cos^2(\alpha+\beta)}$$

$$\Rightarrow \quad \sin(\alpha+\beta) = \sqrt{1 - \left(\frac{b^2-a^2}{b^2+a^2}\right)^2} = \sqrt{\frac{4a^2b^2}{(a^2+b^2)^2}} = \frac{2ab}{b^2+a^2}$$

EXAMPLE 41 *If* α *and* β *are the solutions of the equation* $a\tan\theta + b\sec\theta = c$, *then show that* $\tan(\alpha+\beta) = \dfrac{2ac}{a^2-c^2}$ [NCERT EXEMPLAR]

SOLUTION We have,

$$a\tan\theta + b\sec\theta = c \qquad \text{...(i)}$$

$$\Rightarrow \quad c - a\tan\theta = b\sec\theta$$

$$\Rightarrow \quad (c - a\tan\theta)^2 = b^2\sec^2\theta$$

$$\Rightarrow \quad c^2 + a^2\tan^2\theta - 2ac\tan\theta = b^2(1+\tan^2\theta)$$

$$\Rightarrow \quad \tan^2\theta(a^2-b^2) - 2ac\tan\theta + (c^2-b^2) = 0 \qquad \text{...(ii)}$$

It is given that α & β are the solutions of (i). Therefore, $\tan\alpha$ and $\tan\beta$ are roots of equation (ii).

$$\therefore \quad \tan\alpha+\tan\beta=\frac{2ac}{a^2-b^2} \text{ and } \tan\alpha\tan\beta=\frac{c^2-b^2}{a^2-b^2}$$

$$\Rightarrow \quad \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{2ac}{(a^2-b^2)}}{1-\frac{c^2-b^2}{a^2-b^2}}=\frac{2ac}{a^2-c^2}$$

EXAMPLE 42 *If α and β are the solutions of $a\cos\theta+b\sin\theta=c$, then show that*

(i) $\cos(\alpha+\beta)=\dfrac{a^2-b^2}{a^2+b^2}$ (ii) $\cos(\alpha-\beta)=\dfrac{2c^2-(a^2+b^2)}{a^2+b^2}$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$a\cos\theta+b\sin\theta=c \qquad \text{...(i)}$$

$$\Rightarrow \quad a\cos\theta=c-b\sin\theta$$

$$\Rightarrow \quad a^2\cos^2\theta=(c-b\sin\theta)^2$$

$$\Rightarrow \quad a^2(1-\sin^2\theta)=c^2-2bc\sin\theta+b^2\sin^2\theta$$

$$\Rightarrow \quad (a^2+b^2)\sin^2\theta-2bc\sin\theta+(c^2-a^2)=0 \qquad \text{...(ii)}$$

Since α, β are roots of equation (i). Therefore, $\sin\alpha$ and $\sin\beta$ are roots of equation (ii).

$$\therefore \quad \sin\alpha\sin\beta=\frac{c^2-a^2}{a^2+b^2} \qquad \text{...(iii)}$$

Again, $a\cos\theta+b\sin\theta=c$

$$\Rightarrow \quad b\sin\theta=c-a\cos\theta$$

$$\Rightarrow \quad b^2\sin^2\theta=(c-a\cos\theta)^2$$

$$\Rightarrow \quad b^2(1-\cos^2\theta)=(c-a\cos\theta)^2$$

$$\Rightarrow \quad (a^2+b^2)\cos^2\theta-2ac\cos\theta+c^2-b^2=0 \qquad \text{...(iv)}$$

It is given that α, β are the roots of equation (i). So, $\cos\alpha$, $\cos\beta$ are the roots of equation (iv).

$$\therefore \quad \cos\alpha\cos\beta=\frac{c^2-b^2}{a^2+b^2} \qquad \text{...(v)}$$

Now, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$$\Rightarrow \quad \cos(\alpha+\beta)=\frac{c^2-b^2}{a^2+b^2}-\frac{c^2-a^2}{a^2+b^2}=\frac{a^2-b^2}{a^2+b^2}$$

and, $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$$\Rightarrow \quad \cos(\alpha-\beta)=\frac{c^2-b^2}{a^2+b^2}+\frac{c^2-a^2}{a^2+b^2}=\frac{2c^2-(a^2+b^2)}{a^2+b^2}$$

EXAMPLE 43 *Prove that:*

$$\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\sin 9x}{\cos 27x}=\frac{1}{2}(\tan 27x-\tan x)$$

SOLUTION We have,

$$\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\sin 9x}{\cos 27x}$$

$$= \frac{1}{2}\left\{\frac{2\sin x\cos x}{\cos 3x\cos x}+\frac{2\sin 3x\cos 3x}{\cos 9x\cos 3x}+\frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(x+x)}{\cos 3x\cos x}+\frac{\sin(3x+3x)}{\cos 9x\cos 3x}+\frac{\sin(9x+9x)}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 2x}{\cos 3x\cos x}+\frac{\sin 6x}{\cos 9x\cos 3x}+\frac{\sin 18x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(3x-x)}{\cos 3x\cos x}+\frac{\sin(9x-3x)}{\cos 9x\cos 3x}+\frac{\sin(27x-9x)}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 3x\cos x-\cos 3x\sin x}{\cos 3x\cos x}+\frac{\sin 9x\cos 3x-\cos 9x\sin 3x}{\cos 9x\cos 3x}+\frac{\sin 27x\cos 9x-\cos 27x\sin 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 3x\cos x}{\cos 3x\cos x}-\frac{\cos 3x\sin x}{\cos 3x\cos x}+\frac{\sin 9x\cos 3x}{\cos 9x\cos 3x}-\frac{\cos 9x\sin 3x}{\cos 9x\cos 3x}+\frac{\sin 27x\cos 9x}{\cos 27x\cos 9x}-\frac{\cos 27x\sin 9x}{\cos 27x\cos 9x}\right\}$$

$$= \frac{1}{2}\{(\tan 3x-\tan x)+(\tan 9x-\tan 3x)+(\tan 27x-\tan 9x)\} = \frac{1}{2}(\tan 27x-\tan x)$$

EXERCISE 7.1

LEVEL-1

1. If $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$, where $0 < A,\ B < \frac{\pi}{2}$, find the values of the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$ (iii) $\sin(A-B)$ (iv) $\cos(A-B)$
2. (a) If $\sin A = \frac{12}{13}$ and $\sin B = \frac{4}{5}$, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, find the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$
   (b) If $\sin A = \frac{3}{5}$, $\cos B = -\frac{12}{13}$, where A and B both lie in second quadrant, find the value of $\sin(A+B)$. [NCERT]
3. If $\cos A = -\frac{24}{25}$ and $\cos B = \frac{3}{5}$, where $\pi < A < \frac{3\pi}{2}$ and $\frac{3\pi}{2} < B < 2\pi$, find the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$
4. If $\tan A = \frac{3}{4}$, $\cos B = \frac{9}{41}$, where $\pi < A < \frac{3\pi}{2}$ and $0 < B < \frac{\pi}{2}$, find $\tan(A+B)$.
5. If $\sin A = \frac{1}{2}$, $\cos B = \frac{12}{13}$, where $\frac{\pi}{2} < A < \pi$ and $\frac{3\pi}{2} < B < 2\pi$, find $\tan(A-B)$.
6. If $\sin A = \frac{1}{2}$, $\cos B = \frac{\sqrt{3}}{2}$, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, find the following:
   (i) $\tan(A+B)$ (ii) $\tan(A-B)$
7. Evaluate the following:
   (i) $\sin 78°\cos 18° - \cos 78°\sin 18°$ (ii) $\cos 47°\cos 13° - \sin 47°\sin 13°$
   (iii) $\sin 36°\cos 9° + \cos 36°\sin 9°$ (iv) $\cos 80°\cos 20° + \sin 80°\sin 20°$
8. If $\cos A = -\frac{12}{13}$ and $\cot B = \frac{24}{7}$, where A lies in the second quadrant and B in the third quadrant, find the values of the following:
   (i) $\sin(A+B)$ (ii) $\cos(A+B)$ (iii) $\tan(A+B)$
9. Prove that: $\cos 105° + \cos 15° = \sin 75° - \sin 15°$
10. Prove that: $\frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin(A+B)}{\sin(A-B)}$

11. Prove that:

(i) $\dfrac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \tan 56^\circ$ (ii) $\dfrac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ} = \tan 54^\circ$

(iii) $\dfrac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \tan 37^\circ$

12. Prove that:

(i) $\sin(60^\circ - \theta)\cos(30^\circ + \theta) + \cos(60^\circ - \theta)\sin(30^\circ + \theta) = 1$

(ii) $\sin\left(\dfrac{4\pi}{9}+7\right)\cos\left(\dfrac{\pi}{9}+7\right) - \cos\left(\dfrac{4\pi}{9}+7\right)\sin\left(\dfrac{\pi}{9}+7\right) = \dfrac{\sqrt{3}}{2}$

(iii) $\sin\left(\dfrac{3\pi}{8}-5\right)\cos\left(\dfrac{\pi}{8}+5\right) + \cos\left(\dfrac{3\pi}{8}-5\right)\sin\left(\dfrac{\pi}{8}+5\right) = 1$

13. Prove that: $\dfrac{\tan 69^\circ + \tan 66^\circ}{1 - \tan 69^\circ \tan 66^\circ} = -1$

14. (i) If $\tan A = \dfrac{5}{6}$ and $\tan B = \dfrac{1}{11}$, prove that $A + B = \dfrac{\pi}{4}$

(ii) If $\tan A = \dfrac{m}{m-1}$ and $\tan B = \dfrac{1}{2m-1}$, then prove that $A - B = \dfrac{\pi}{4}$

15. Prove that:

(i) $\cos^2 45^\circ - \sin^2 15^\circ = \dfrac{\sqrt{3}}{4}$

(ii) $\sin^2(n+1)A - \sin^2 nA = \sin(2n+1)A \sin A$

16. Prove that:

(i) $\dfrac{\sin(A+B) + \sin(A-B)}{\cos(A+B) + \cos(A-B)} = \tan A$

(ii) $\dfrac{\sin(A-B)}{\cos A \cos B} + \dfrac{\sin(B-C)}{\cos B \cos C} + \dfrac{\sin(C-A)}{\cos C \cos A} = 0$

(iii) $\dfrac{\sin(A-B)}{\sin A \sin B} + \dfrac{\sin(B-C)}{\sin B \sin C} + \dfrac{\sin(C-A)}{\sin C \sin A} = 0$

(iv) $\sin^2 B = \sin^2 A + \sin^2(A-B) - 2\sin A \cos B \sin(A-B)$

(v) $\cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A+B) = \sin^2(A+B)$

(vi) $\dfrac{\tan(A+B)}{\cot(A-B)} = \dfrac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B}$

17. Prove that:

(i) $\tan 8\theta - \tan 6\theta - \tan 2\theta = \tan 8\theta \tan 6\theta \tan 2\theta$

(ii) $\tan 15^\circ + \tan 30^\circ + \tan 15^\circ \tan 30^\circ = 1$

(iii) $\tan 36^\circ + \tan 9^\circ + \tan 36^\circ \tan 9^\circ = 1$

(iv) $\tan 13\theta - \tan 9\theta - \tan 4\theta = \tan 13\theta \tan 9\theta \tan 4\theta$

18. Prove that: $\dfrac{\tan^2 2\theta - \tan^2 \theta}{1 - \tan^2 2\theta \tan^2 \theta} = \tan 3\theta \tan \theta$

19. If $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{a+b}{a-b}$, show that $\dfrac{\tan x}{\tan y} = \dfrac{a}{b}$. [NCERT EXAMPLAR]

20. If $\tan A = x \tan B$, prove that $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{x-1}{x+1}$

21. If $\tan(A+B)=x$ and $\tan(A-B)=y$, find the values of $\tan 2A$ and $\tan 2B$.

22. If $\cos A+\sin B=m$ and $\sin A+\cos B=n$, prove that $2\sin(A+B)=m^2+n^2-2$.

LEVEL-2

23. If $\tan A+\tan B=a$ and $\cot A+\cot B=b$, prove that: $\cot(A+B)=\frac{1}{a}-\frac{1}{b}$.

24. If θ lies in the first quadrant and $\cos\theta=\frac{8}{17}$, then prove that

$$\cos\left(\frac{\pi}{6}+\theta\right)+\cos\left(\frac{\pi}{4}-\theta\right)+\cos\left(\frac{2\pi}{3}-\theta\right)=\left(\frac{\sqrt{3}-1}{2}+\frac{1}{\sqrt{2}}\right)\frac{23}{17}$$ [NCERT EXEMPLAR]

25. If $\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right)=3$, then prove that $\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}=1$.

26. If $\sin(\alpha+\beta)=1$ and $\sin(\alpha-\beta)=\frac{1}{2}$, where $0\le\alpha,\ \beta\le\frac{\pi}{2}$, then find the values of $\tan(\alpha+2\beta)$ and $\tan(2\alpha+\beta)$.

27. If α,β are two different values of θ lying between 0 and 2π which satisfy the equation $6\cos\theta+8\sin\theta=9$, find the value of $\sin(\alpha+\beta)$.

28. If $\sin\alpha+\sin\beta=a$ and $\cos\alpha+\cos\beta=b$, show that

(i) $\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}$ (ii) $\cos(\alpha+\beta)=\frac{b^2-a^2}{b^2+a^2}$

29. Prove that :

(i) $\frac{1}{\sin(x-a)\sin(x-b)}=\frac{\cot(x-a)-\cot(x-b)}{\sin(a-b)}$

(ii) $\frac{1}{\sin(x-a)\cos(x-b)}=\frac{\cot(x-a)+\tan(x-b)}{\cos(a-b)}$

(iii) $\frac{1}{\cos(x-a)\cos(x-b)}=\frac{\tan(x-b)-\tan(x-a)}{\sin(a-b)}$

30. If $\sin\alpha\sin\beta-\cos\alpha\cos\beta+1=0$, prove that $1+\cot\alpha\tan\beta=0$.

31. If $\tan\alpha=x+1$, $\tan\beta=x-1$, show that $2\cot(\alpha-\beta)=x^2$.

32. If angle θ is divided into two parts such that the tangents of one part is λ times the tangent of other, and ϕ is their difference, then show that $\sin\theta=\frac{\lambda+1}{\lambda-1}\sin\phi$.

[NCERT EXEMPLAR]

33. If $\tan\theta=\frac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}$, then show that $\sin\alpha+\cos\alpha=\sqrt{2}\cos\theta$

[NCERT EXEMPLAR]

ANSWERS

1. (i) $\frac{56}{65}$ (ii) $\frac{33}{65}$ (iii) $\frac{16}{65}$ (iv) $\frac{63}{65}$

[illegible] $\frac{5\sqrt{3}-12}{5+12\sqrt{3}}$ [illegible]

7. (i) $\frac{\sqrt{3}}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{\sqrt{2}}$ (iv) $\frac{1}{2}$

8. (i) $\frac{-36}{325}$ (ii) $\frac{323}{325}$ (iii) $-\frac{36}{323}$ 18. $\frac{x+y}{1-xy}, \frac{x-y}{1+xy}$

26. $-\sqrt{3}, -\frac{1}{\sqrt{3}}$ 27. $\frac{24}{25}$

HINTS TO SELECTED PROBLEMS

9. LHS $= \cos(90° + 15°) + \cos(90° - 75°) = -\sin 15° + \sin 75° =$ RHS

11. LHS $= \dfrac{\dfrac{\cos 11° + \sin 11°}{\cos 11°}}{\dfrac{\cos 11° - \sin 11°}{\cos 11°}} = \dfrac{1 + \tan 11°}{1 - \tan 11°} = \dfrac{\tan 45° + \tan 11°}{1 - \tan 45° \tan 11°} = \tan(45° + 11°) = \tan 56° =$ RHS

29. (i) We have,

$$\frac{1}{\sin(x-a)\sin(x-b)} = \frac{1}{\sin(a-b)}\left\{\frac{\sin(a-b)}{\sin(x-a)\sin(x-b)}\right\}$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}[\cot(x-a) - \cot(x-b)]$$

Similarly, we can prove other two parts.

33. $\tan\theta = \dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$

$$\tan\theta = \frac{\dfrac{\sin\alpha - \cos\theta}{\cos\alpha}}{\dfrac{\sin\alpha + \cos\alpha}{\cos\alpha}}$$ [Dividing numerator and denominator by $\cos\alpha$]

$$\Rightarrow \quad \tan\theta = \frac{\tan\alpha - 1}{\tan\alpha + 1}$$

$$\Rightarrow \quad \tan\theta = \tan\left(\alpha - \frac{\pi}{4}\right) \Rightarrow \theta = \alpha - \frac{\pi}{4} \Rightarrow \alpha = \theta + \frac{\pi}{4}$$

$$\therefore \quad \sin\alpha + \cos\alpha = \sin\left(\theta + \frac{\pi}{4}\right) + \cos\left(\theta + \frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}(\sin\theta + \cos\theta) + \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta) = \sqrt{2}\cos\theta$$

7.4 MAXIMUM AND MINIMUM VALUES OF TRIGONOMETRICAL EXPRESSIONS

We have learnt that for those values of θ for which trigonometrical functions are define[illegible] have

$$-1 \le \sin\theta \le 1, \ -1 \le \cos\theta \le 1, \ -\infty < \tan\theta < \infty \ \text{cosec}\,\theta \ge 1 \text{ or } \text{cosec}\,\theta \le -1$$
$$\sec\theta \ge 1 \text{ or } \sec\theta \le -1 \text{ and } -\infty < \cot\theta < \infty$$

In this section, we will find the maximum and minimum values of trigonometrical expressions of the form $a \cos\theta + b \sin\theta$ for varying values of θ.

Let $f(\theta) = a\cos\theta + b\sin\theta$ Further, let $a = r\sin\alpha$ and $b = r\cos\alpha$. Then,

$$a^2+b^2 = r^2\sin^2\alpha + r^2\cos^2\alpha \text{ and, } \frac{a}{b} = \frac{r\sin\alpha}{r\cos\alpha}$$

$$\Rightarrow \quad a^2+b^2 = r^2(\sin^2\alpha+\cos^2\alpha) \text{ and, } \frac{a}{b} = \tan\alpha$$

$$\Rightarrow \quad r = \sqrt{a^2+b^2} \text{ and, } \tan\alpha = \frac{a}{b}$$

Substituting the values of a and b in $f(\theta)$, we obtain

$$f(\theta) = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta = r(\sin\alpha\cos\theta+\cos\alpha\sin\theta) = r\sin(\alpha+\theta)$$

We know that

$$-1 \le \sin(\alpha+\theta) \le 1 \quad \text{for all } \theta$$

$$\Rightarrow \quad -r \le r\sin(\alpha+\theta) \le r \quad \text{for all } \theta \qquad \text{[Multiplying throughout by } r]$$

$$\Rightarrow \quad -\sqrt{a^2+b^2} \le f(\theta) \le \sqrt{a^2+b^2} \quad \text{for all } \theta$$

$$\Rightarrow \quad -\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2} \quad \text{for all } \theta$$

Hence, maximum and minimum values of $a\cos\theta + b\sin\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ respectively.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the maximum and minimum values of* $7\cos\theta + 24\sin\theta$.

SOLUTION We know that the maximum and minimum values of $a\cos\theta + b\sin\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ respectively. Hence, the maximum and minimum values of $7\cos\theta + 24\sin\theta$ are $\sqrt{7^2+24^2} = 25$ and $-\sqrt{7^2+24^2} = -25$ respectively.

EXAMPLE 2 *Find the maximum and minimum values of the following expressions:*

(i) $3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$ (ii) $4\sin\theta - 3\cos\theta + 7$

SOLUTION (i) Let $f(\theta) = 3\cos\theta + 5\sin\left(\theta-\frac{\pi}{6}\right)$. Then,

$$f(\theta) = 3\cos\theta + 5\left\{\sin\theta\cos\frac{\pi}{6} - \cos\theta\sin\frac{\pi}{6}\right\} = 3\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta - \frac{5}{2}\cos\theta$$

$$\Rightarrow \quad f(\theta) = \frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta$$

$$\because \quad -\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2} \le \frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2} \quad \text{for all } \theta.$$

$$\therefore \quad -\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2} \le f(\theta) \le \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2} \quad \text{for all } \theta$$

$$\Rightarrow \quad -\sqrt{\frac{1}{4}+\frac{75}{4}} \le 3\cos\theta + 5\sin\left(\theta-\frac{\pi}{6}\right) \le \sqrt{\frac{1}{4}+\frac{75}{4}} \quad \text{for all } \theta$$

$$\Rightarrow \quad -\sqrt{19} \le 3\cos\theta + 5\sin\left(\theta-\frac{\pi}{6}\right) \le \sqrt{19} \quad \text{for all } \theta$$

Hence, $-\sqrt{19}$ and $\sqrt{19}$ are respectively the minimum and the maximum values of $3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$.

(ii) Let $f(\theta) = 4\sin\theta - 3\cos\theta + 7$

We know that

$$-\sqrt{4^2 + (-3)^2} \le 4\sin\theta - 3\cos\theta \le \sqrt{4^2 + (-3)^2} \text{ for all } \theta$$

$$\Rightarrow \quad -5 \le 4\sin\theta - 3\cos\theta \le 5 \text{ for all } \theta$$

$$\Rightarrow \quad -5 + 7 \le 4\sin\theta - 3\cos\theta + 7 \le 5 + 7 \text{ for all } \theta$$

$$\Rightarrow \quad 2 \le f(\theta) \le 12 \text{ for all } \theta$$

Hence, minimum and maximum values of $4\sin\theta - 3\cos\theta + 7$ are 2 and 12 respectively.

EXAMPLE 3 *Prove that* $5\cos\theta + 3\cos\left(\theta \div \frac{\pi}{3}\right) + 3$ *lies between* -4 *and* 10.

SOLUTION Let $f(\theta) = 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$. Then,

$$f(\theta) = 5\cos\theta + 3\left(\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right) + 3 = 5\cos\theta + \frac{3}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

$$\Rightarrow \quad f(\theta) = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3 \qquad \text{...(i)}$$

$$\because \quad -\sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \quad -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7 \text{ for all } \theta$$

$$\Rightarrow \quad -7 + 3 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3 \le 7 + 3 \text{ for all } \theta$$

$$\Rightarrow \quad -4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10 \text{ for all } \theta \qquad \text{[Using (i)]}$$

EXAMPLE 4 *Find* a *and* b *such that the following inequality holds good for all* θ:

$$a \le 3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right) \le b$$

SOLUTION Let $f(\theta) = 3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$. Then,

$$f(\theta) = 3\cos\theta + 5\left(\sin\theta\cos\frac{\pi}{6} - \cos\theta\sin\frac{\pi}{6}\right) = 3\cos\theta + 5\left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right)$$

$$\Rightarrow \quad f(\theta) = \frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta$$

We have,

$$\because \quad \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} \le \frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} \text{ for all } \theta$$

$$\Rightarrow \quad -\sqrt{19} \le 3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right) \le \sqrt{19} \text{ for all } \theta$$

Hence, $a = -\sqrt{19}$ and $b = \sqrt{19}$

7.5 TO EXPRESS $a\cos\theta + b\sin\theta$ IN THE FORM $r\sin(\theta \pm \alpha)$ OR $r\cos(\theta \pm \alpha)$

Sometimes we need to express trigonometrical expressions of the form $a\cos\theta + b\sin\theta$ in terms of sine or cosine of single term. We may use the following algorithm to do so.

ALGORITHM

STEP I Multiply and divide $f(\theta) = a\cos\theta + b\sin\theta$ by $\sqrt{a^2+b^2}$ to get

$$f(\theta) = \sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right\}$$

STEP II In order to express $f(\theta)$ in terms of sine of some term, replace $\frac{a}{\sqrt{a^2+b^2}}$ i.e. coefficient of $\cos\theta$ by $\sin\alpha$ and $\frac{b}{\sqrt{a^2+b^2}}$ i.e. coefficient of $\sin\theta$ by $\cos\alpha$. This gives the following :

$$f(\theta) = \sqrt{a^2+b^2}\ \{\sin\alpha\cos\theta + \cos\alpha\sin\theta\} = \sqrt{a^2+b^2}\ \sin(\theta+\alpha)$$

To express $f(\theta)$ in terms of cosine of some term, replace coefficient of $\cos\theta$ i.e. $\frac{a}{\sqrt{a^2+b^2}}$ by $\cos\alpha$ and coefficient of $\sin\theta$ i.e. $\frac{b}{\sqrt{a^2+b^2}}$ by $\sin\alpha$. This gives the following:

$$f(\theta) = \sqrt{a^2+b^2}\ \{\cos\alpha\cos\theta + \sin\alpha\sin\theta\} = \sqrt{a^2+b^2}\ \cos(\theta-\alpha).$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Reduce* $\sqrt{3}\sin\theta + \cos\theta$ *as a single term consisting (i) sine only (ii) cosine only.*

SOLUTION Let $f(\theta) = \sqrt{3}\sin\theta + \cos\theta$. Then,

$$f(\theta) = \sqrt{3}\sin\theta + \cos\theta$$

$$\Rightarrow\quad f(\theta) = 2\left\{\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right\}\quad \left[\text{Multiplying and dividing by } \sqrt{(\sqrt{3})^2+1^2} \text{ i.e. by } 2\right]$$

$$\Rightarrow\quad f(\theta) = 2\left\{\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right\}\quad \left[\because \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \text{ and } \frac{1}{2} = \sin\frac{\pi}{6}\right]$$

$$\Rightarrow\quad f(\theta) = 2\sin\left(\theta+\frac{\pi}{6}\right)$$

Again,

$$f(\theta) = 2\left\{\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right\} = 2\left\{\sin\frac{\pi}{3}\sin\theta + \cos\frac{\pi}{3}\cos\theta\right\}$$

$$\Rightarrow\quad f(\theta) = 2\left\{\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right\} = 2\cos\left(\theta-\frac{\pi}{3}\right).$$

EXAMPLE 2 *Express* $3\cos\theta - 4\sin\theta$ *as sines and cosines of a single expression.*

SOLUTION Let $f(\theta) = 3\cos\theta - 4\sin\theta$. Multiplying and dividing by $\sqrt{3^2+(-4)^2}$ i.e. by 5, we get

$$f(\theta) = \sqrt{3^2+(-4)^2}\left\{\frac{3}{\sqrt{3^2+(-4)^2}}\cos\theta - \frac{4}{\sqrt{3^2+(-4)^2}}\sin\theta\right\}$$

$$\Rightarrow\quad f(\theta) = 5\left(\frac{3}{5}\cos\theta - \frac{4}{5}\sin\theta\right)$$

$$\Rightarrow\quad f(\theta) = 5(\sin\alpha\cos\theta - \cos\alpha\sin\theta),\text{ where } \sin\alpha = \frac{3}{5} \text{ and } \cos\alpha = \frac{4}{5}$$

$$\Rightarrow\quad f(\theta) = 5\sin(\alpha-\theta),\text{ where } \tan\alpha = \frac{3}{4}$$

Again,

$$f(\theta) = 5\left(\frac{3}{5}\cos\theta - \frac{4}{5}\sin\theta\right)$$

$\Rightarrow$ $f(\theta) = 5(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$, where $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$

$\Rightarrow$ $f(\theta) = 5\cos(\alpha+\theta)$, where $\tan\alpha = \frac{4}{3}$

EXAMPLE 3 *Find the sign of the expression* $\sin 100° + \cos 100°$.

SOLUTION We have,

$$\begin{aligned}\sin 100° + \cos 100° &= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin 100° + \frac{1}{\sqrt{2}}\cos 100°\right)\\ &= \sqrt{2}\,(\cos 45°\sin 100° + \sin 45°\cos 100°)\\ &= \sqrt{2}\sin(100° + 45°)\\ &= \sqrt{2}\sin 145°, \text{ which is a positive real number.} \quad [\because \sin 145° \text{ is positive}]\end{aligned}$$

EXERCISE 7.2

LEVEL-1

1. Find the maximum and minimum values of each of the following trigonometrical expressions:
   (i) $12\sin\theta - 5\cos\theta$ (ii) $12\cos\theta + 5\sin\theta + 4$
   (iii) $5\cos\theta + 3\sin\left(\frac{\pi}{6} - \theta\right) + 4$ (iv) $\sin\theta - \cos\theta + 1$
2. Reduce each of the following expressions to the sine and cosine of a single expression:
   (i) $\sqrt{3}\sin\theta - \cos\theta$ (ii) $\cos\theta - \sin\theta$ (iii) $24\cos\theta + 7\sin\theta$
3. Show that $\sin 100° - \sin 10°$ is positive.
4. Prove that $(2\sqrt{3} + 3)\sin\theta + 2\sqrt{3}\cos\theta$ lies between $-(2\sqrt{3} + \sqrt{15})$ and $(2\sqrt{3} + \sqrt{15})$.

ANSWERS

1.

	Minimum	Maximum
(i)	-13	13
(ii)	-9	17
(iii)	-3	11
(iv)	$1-\sqrt{2}$	$1+\sqrt{2}$

2. (i) $2\sin\left(\theta - \frac{\pi}{6}\right), -2\cos\left(\frac{\pi}{3} + \theta\right)$ (ii) $\sqrt{2}\sin\left(\frac{\pi}{4} - \theta\right), \sqrt{2}\cos\left(\frac{\pi}{4} + \theta\right)$
   (iii) $25\sin(\alpha + \theta)$, where $\tan\alpha = \frac{24}{7}$, $25\cos(\theta - \alpha)$, where $\tan\alpha = \frac{7}{24}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $\alpha + \beta - \gamma = \pi$, and $\sin^2\alpha + \sin^2\beta - \sin^2\gamma = \lambda\sin\alpha\sin\beta\cos\gamma$, then write the value of λ.
2. If $x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right) = z\cos\left(\theta + \frac{4\pi}{3}\right)$, then write the value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.
3. Write the maximum and minimum values of $3\cos x + 4\sin x + 5$.
4. Write the maximum value of $12\sin\theta - 9\sin^2\theta$.
5. If $12\sin\theta - 9\sin^2\theta$ attains its maximum value at $\theta = \alpha$, then write the value of $\sin\alpha$.

6. Write the interval in which the values of $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ lie.

7. If $\tan(A+B) = p$ and $\tan(A-B) = q$, then write the value of $\tan 2B$.

8. If $\frac{\cos(x-y)}{\cos(x+y)} = \frac{m}{n}$, then write the value of $\tan x \tan y$.

9. If $a = b\cos\frac{2\pi}{3} = c\cos\frac{4\pi}{3}$, then write the value of $ab + bc + ca$.

10. If $A + B = C$, then write the value of $\tan A \tan B \tan C$.

11. If $\sin\alpha - \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, then write the value of $\cos(\alpha+\beta)$.

12. If $\tan\alpha = \frac{1}{1+2^{-x}}$ and $\tan\beta = \frac{1}{1+2^{x+1}}$, then write the value of $\alpha+\beta$ lying in the interval $(0, \pi/2)$.

ANSWERS

1. 2 2. 0 3. Maximum = 10, Minimum = 0 4. 4 5. $\frac{2}{3}$

6. $[-4, 10]$ 7. $\frac{p-q}{1+pq}$ 8. $\frac{m-n}{m+n}$ 9. 0 10. $\tan C - \tan A - \tan B$

11. $\frac{a^2+b^2-2}{2}$ 12. $\frac{\pi}{4}$

MULTIPLE CHOICE QUATIONS (MCQs)

Mark the correct alternative in each of the following:

1. The value of $\sin^2 75° - \sin^2 15°$ is
   (a) $1/2$ (b) $\sqrt{3}/2$ (c) 1 (d) 0

2. If $A+B+C = 180°$, then $\sec A(\cos B\cos C - \sin B\sin C)$ is equal to
   (a) 0 (b) -1 (c) 1 (d) none of these

3. $\tan 20° + \tan 40° + \sqrt{3}\tan 20°\tan 40°$ is equal to
   (a) $\frac{\sqrt{3}}{4}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\sqrt{3}$ (d) 1

4. If $\tan A = \frac{a}{a+1}$ and $\tan B = \frac{1}{2a+1}$, then the value of $A+B$ is
   (a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

5. If $3\sin\theta + 4\cos\theta = 5$, then $4\sin\theta - 3\cos\theta =$
   (a) 0 (b) 5 (c) 1 (d) none of these

6. If in a ΔABC, $\tan A + \tan B + \tan C = 6$, then $\cot A\cot B\cot C =$
   (a) 6 (b) 1 (c) $1/6$ (d) none of these

7. $\tan 3A - \tan 2A - \tan A$ is equal to
   (a) $\tan 3A\tan 2A\tan A$
   (b) $-\tan 3A\tan 2A\tan A$
   (c) $\tan A\tan 2A - \tan 2A\tan 3A - \tan 3A\tan A$
   (d) none of these.

8. If $A+B+C = 180°$, then $\frac{\tan A + \tan B + \tan C}{\tan A\ \tan B\ \tan C}$ is equal to
   (a) $\tan A\tan B\tan C$
   (b) 0
   (c) 1
   (d) none of these.

9. If $\cos P = \frac{1}{7}$ and $\cos Q = \frac{13}{14}$, where P and Q both are acute angles. Then, the value of $P - Q$ is

(a) $30°$ (b) $60°$ (c) $45°$ (d) $75°$

10. If $\cot(\alpha + \beta) = 0$, then $\sin(\alpha + 2\beta)$ is equal to

(a) $\sin \alpha$ (b) $\cos 2\beta$ (c) $\cos \alpha$ (d) $\sin 2\alpha$

11. $\frac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°}$ is equal to

(a) $\tan 55°$ (b) $\cot 55°$ (c) $-\tan 35°$ (d) $-\cot 35°$

12. The value of $\cos^2\left(\frac{\pi}{6} + \theta\right) - \sin^2\left(\frac{\pi}{6} - \theta\right)$ is

(a) $\frac{1}{2}\cos 2\theta$ (b) 0 (c) $-\frac{1}{2}\cos 2\theta$ (d) $\frac{1}{2}$

13. If $\tan \theta_1 \tan \theta_2 = k$, then $\frac{\cos(\theta_1 - \theta_2)}{\cos(\theta_1 + \theta_2)} =$

(a) $\frac{1+k}{1-k}$ (b) $\frac{1-k}{1+k}$ (c) $\frac{k+1}{k-1}$ (d) $\frac{k-1}{k+1}$

14. If $\sin(\pi \cos \theta) = \cos(\pi \sin \theta)$, then $\sin 2\theta =$

(a) $\pm\frac{3}{4}$ (b) $\pm\frac{4}{3}$ (c) $\pm\frac{1}{3}$ (d) none of these

15. If $\tan \theta = \frac{1}{2}$ and $\tan \phi = \frac{1}{3}$, then the value of $\theta + \phi$ is

(a) $\frac{\pi}{6}$ (b) π (c) 0 (d) $\frac{\pi}{4}$

16. The value of $\cos(36° - A)\cos(36° + A) + \cos(54° + A)\cos(54° - A)$ is

(a) $\sin 2A$ (b) $\cos 2A$ (c) $\cos 3A$ (d) $\sin 3A$

17. If $\tan(\pi/4 + \theta) + \tan(\pi/4 - \theta) = a$, then $\tan^2(\pi/4 + \theta) + \tan^2(\pi/4 - \theta) =$

(a) $a^2 + 1$ (b) $a^2 + 2$ (c) $a^2 - 2$ (d) none of these

18. If $\tan(A - B) = 1$, $\sec(A + B) = \frac{2}{\sqrt{3}}$, then the smallest positive value of B is

(a) $\frac{25\pi}{24}$ (b) $\frac{19\pi}{24}$ (c) $\frac{13\pi}{24}$ (d) $\frac{11\pi}{24}$

19. If $A - B = \pi/4$, then $(1 + \tan A)(1 - \tan B)$ is equal to

(a) 2 (b) 1 (c) 0 (d) 3

20. The maximum value of $\sin^2(120° + \theta) + \sin^2(120° - \theta)$ is

(a) $1/2$ (b) $3/2$ (c) $1/4$ (d) $3/4$

21. If $\cos(A - B) = \frac{3}{5}$ and $\tan A \tan B = 2$, then

(a) $\cos A \cos B = \frac{1}{5}$ (b) $\cos A \cos B = -\frac{1}{5}$

(c) $\sin A \sin B = -\frac{1}{5}$ (d) $\sin A \sin B = -\frac{1}{5}$

22. If $\tan 69° + \tan 66° - \tan 69° \tan 66° = 2k$, then $k =$

(a) -1 (b) $1/2$ (c) $-1/2$ (d) none of these

23. If $\tan \alpha = \frac{x}{x+1}$ and $\tan \beta = \frac{1}{2x+1}$, then $\alpha + \beta$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

ANSWERS

1. (b)	2. (b)	3. (c)	4. (d)	5. (a)	6. (c)	7. (a)	8. (c)
9. (b)	10. (a)	11. (a)	12. (a)	13. (a)	14. (a)	15. (d)	16. (b)
17. (c)	18. (b)	19. (a)	20. (b)	21. (a)	22. (c)	23. (d)	

SUMMARY

1. (i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

   (ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

   (iii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

   (iv) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

   (v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

   (vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

   (vii) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

   (viii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

2. (i) $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

   (ii) $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

   (iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

3. (i) If $A+B = \pi$, then $\sin A = \sin B$, $\cos A = -\cos B$ and $\tan A = -\tan B$

   (ii) If $A+B = 2\pi$, then $\sin A = -\sin B$, $\cos A = \cos B$ and $\tan A = -\tan B$

CHAPTER 8

TRANSFORMATION FORMULAE

8.1 INTRODUCTION

In this chapter, we will establish two sets of transformation formulae: One to transform the products of two sines or two cosines or one sine and one cosine into the sum or difference of two sines or two cosines and the other to convert the sum or difference of two sines or two cosines in the product of two sines or two cosines or one sine and one cosine. These two sets of formulae are of fundamental importance and one should have thorough acquaintance with these formulae.

8.2 FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE

In the previous chapter we have derived the following formulae:

$$\sin A \cos B + \cos A \sin B = \sin (A + B) \qquad ...(i)$$

$$\sin A \cos B - \cos A \sin B = \sin (A - B) \qquad ...(ii)$$

$$\cos A \cos B - \sin A \sin B = \cos (A + B) \qquad ...(iii)$$

$$\cos A \cos B + \sin A \sin B = \cos (A - B) \qquad ...(iv)$$

Adding (i) and (ii), we obtain

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

Subtracting (ii) from (i), we get

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$$

Adding (iii) and (iv), we get

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

Subtracting (iii) from (iv), we get

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$

Thus, we obtain the following formulae :

(a) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$ (b) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

(c) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$ (d) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

These four formulae convert the product of two sines or two cosines or one sine and one cosine into the sum or difference of two sines or two cosines.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Convert each of the following products into the sum or difference of sines and cosines:*

(i) $2 \sin 5\theta \cos \theta$ (ii) $2 \cos 4\theta \cos 3\theta$ (iii) $2 \sin 3\theta \sin \theta$

(iv) $\sin 75° \cos 15°$ (v) $\cos 75° \cos 15°$

SOLUTION (i) Using $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$, we obtain

$$2 \sin 5\theta \cos \theta = \sin (5\theta + \theta) + \sin (5\theta - \theta) = \sin 6\theta + \sin 4\theta$$

(ii) Using $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$, we obtain

$$2 \cos 4\theta \cos 3\theta = \cos (4\theta + 3\theta) + \cos (4\theta - 3\theta) = \cos 7\theta + \cos \theta$$

(iii) Using $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$, we obtain

$$2\sin 3\theta \sin\theta = \cos(3\theta-\theta)-\cos(3\theta+\theta)=\cos 2\theta-\cos 4\theta$$

(iv) $\sin 75^\circ \cos 15^\circ = \frac{1}{2}(2\sin 75^\circ \cos 15^\circ)$

$$= \frac{1}{2}\{\sin(75^\circ+15^\circ)+\sin(75^\circ-15^\circ)\} = \frac{1}{2}(\sin 90^\circ+\sin 60^\circ)$$

(v) $\cos 75^\circ \cos 15^\circ = \frac{1}{2}(2\cos 75^\circ \cos 15^\circ)$

$$= \frac{1}{2}\{\cos(75^\circ+15^\circ)+\cos(75^\circ-15^\circ)\} = \frac{1}{2}(\cos 90^\circ+\cos 60^\circ)$$

EXAMPLE 2 *Prove that:* $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$ [NCERT]

SOLUTION We have,

$$\begin{aligned}
\text{LHS} &= 2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}\\
&= \cos\left(\frac{9\pi}{13}+\frac{\pi}{13}\right)+\cos\left(\frac{9\pi}{13}-\frac{\pi}{13}\right)+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}\\
&= \cos\frac{10\pi}{13}+\cos\frac{8\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}\\
&= \cos\left(\pi-\frac{3\pi}{13}\right)+\cos\left(\pi-\frac{5\pi}{13}\right)+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}\\
&= -\cos\frac{3\pi}{13}-\cos\frac{5\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13} = 0 = \text{RHS} \quad [\because \cos(\pi-\theta)=-\cos\theta]
\end{aligned}$$

EXAMPLE 3 *Prove that:* $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}$

SOLUTION We have,

$$\text{LHS} = \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$$

$$\Rightarrow \text{LHS} = \cos 60^\circ(\cos 20^\circ \cos 40^\circ)\cos 80^\circ$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\times\frac{1}{2}(2\cos 20^\circ\cos 40^\circ)\cos 80^\circ \quad \left[\because \cos 60^\circ=\frac{1}{2}\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left[\{\cos(40^\circ+20^\circ)+\cos(40^\circ-20^\circ)\}\cos 80^\circ\right]$$

$$[\because 2\cos A\cos B=\cos(A+B)+\cos(A-B)]$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{(\cos 60^\circ+\cos 20^\circ)\cos 80^\circ\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{\left(\frac{1}{2}+\cos 20^\circ\right)\cos 80^\circ\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{\frac{1}{2}\cos 80^\circ+\cos 80^\circ\cos 20^\circ\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{8}\left\{\cos 80^\circ+2\cos 80^\circ\cos 20^\circ\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{8}\left[\cos 80^\circ+\left\{\cos(80^\circ+20^\circ)+\cos(80^\circ-20^\circ)\right\}\right]$$

$$[\because 2\cos A\cos B=\cos(A+B)+\cos(A-B)]$$

$$\Rightarrow \text{LHS} = \frac{1}{8}\{\cos 80^\circ+\cos 100^\circ+\cos 60^\circ\}$$

$$\Rightarrow \text{LHS} = \frac{1}{8}\{\cos 80^\circ+\cos(180^\circ-80^\circ)+\cos 60^\circ\}$$

$\Rightarrow \quad \text{LHS} = \frac{1}{8}\{\cos 80° - \cos 80° + \cos 60°\}$ $\quad \left[\begin{array}{l}\because \cos(180° - \theta) = -\cos\theta \\ \therefore \cos(180° - 80°) = -\cos 80°\end{array}\right]$

$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left\{\cos 80° - \cos 80° + \frac{1}{2}\right\} = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} = \text{RHS}$

EXAMPLE 4 *Prove that:* $\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}$.

SOLUTION We have,

$$\text{LHS} = \sin 30° (\sin 10° \sin 50°) \sin 70°$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{2}(\sin 50° \sin 10°) \sin 70°$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{2} \times \frac{1}{2}(2 \sin 50° \sin 10°) \sin 70°$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{(2 \sin 50° \sin 10°) \sin 70°\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{\{\cos(50° - 10°) - \cos(50° + 10°)\} \sin 70°\right\}$$

$[\because 2 \sin A \sin B = \cos(A - B) - \cos(A + B)]$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left[(\cos 40° - \cos 60°) \sin 70°\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left[\sin 70° \cos 40° - \sin 70° \cos 60°\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left[\sin 70° \cos 40° - \frac{1}{2} \sin 70°\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left[2 \sin 70° \cos 40° - \sin 70°\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left[\sin(70° + 40°) + \sin(70° - 40°) - \sin 70°\right]$$

$[\because 2 \sin A \cos B = \sin(A + B) + \sin(A - B)]$

$$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left\{\sin 110° + \sin 30° - \sin 70°\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left\{\sin(180° - 70°) + \sin 30° - \sin 70°\right\}$$

$\Rightarrow \quad \text{LHS} = \frac{1}{8}\left\{\sin 70° + \frac{1}{2} - \sin 70°\right\}$ $\quad [\because \sin(180 - \theta) = \sin\theta \therefore \sin(180° - 70°) = \sin 70°]$

$$\Rightarrow \quad \text{LHS} = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} = \text{RHS}$$

<u>ALITER</u> We have,

$$\text{LHS} = \sin 10° \sin 30° \sin 50° \sin 70°$$

$$\Rightarrow \quad \text{LHS} = \sin(90° - 80°) \sin(90° - 60°) \sin(90° - 40°) \sin(90° - 20°)$$

$$\Rightarrow \quad \text{LHS} = \cos 80° \cos 60° \cos 40° \cos 20°$$

$\Rightarrow \quad \text{LHS} = \cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16} = \text{RHS}$ [See Ex. 3]

EXAMPLE 5 *Prove that:* $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

SOLUTION We have,

$$\text{LHS} = \sin 20° \sin 40° \sin 60° \sin 80°$$

$$\Rightarrow \quad \text{LHS} = \sin 60° (\sin 20° \sin 40°) \sin 80°$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{2} \times \frac{1}{2} (2 \sin 20° \sin 40°) \sin 80°$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{4} \left[\left\{ \cos (40° - 20°) - \cos (40° + 20°) \right\} \sin 80° \right]$$

$[\because \ 2 \sin A \sin B = \cos (A - B) - \cos (A + B)]$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{4} \left[(\cos 20° - \cos 60°) \sin 80° \right]$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{4} \left[\left(\cos 20° - \frac{1}{2} \right) \sin 80° \right]$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{8} \left[2 \cos 20° \sin 80° - \sin 80° \right]$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{8} \left[\sin (80°+20°) + \sin (80°-20°) - \sin 80° \right]$$

$[\because \ 2 \sin A \cos B = \sin (A + B) + \sin (A - B)]$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{8} \left\{ \sin 100° + \sin 60° - \sin 80° \right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{8} \left\{ \sin (180° - 80°) + \frac{\sqrt{3}}{2} - \sin 80° \right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{8} \left\{ \sin 80° + \frac{\sqrt{3}}{2} - \sin 80° \right\} \qquad [\because \ \sin (180° - 80°) = \sin 80°]$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2} = \frac{3}{16} = \text{RHS}$$

EXAMPLE 6 *Prove that:* $4 \cos 12° \cos 48° \cos 72° = \cos 36°$

SOLUTION We have,

$$\text{LHS} = 4 \cos 12° \cos 48° \cos 72°$$

$$\Rightarrow \quad \text{LHS} = 2 (2 \cos 12° \cos 48°) \cos 72°$$

$$\Rightarrow \quad \text{LHS} = 2 (\cos 60° + \cos 36°) \cos 72°$$

$$\Rightarrow \quad \text{LHS} = 2 \cos 60° \cos 72° + 2 \cos 36° \cos 72°$$

$$\Rightarrow \quad \text{LHS} = \cos 72° + \cos 108° + \cos 36°$$

$$\Rightarrow \quad \text{LHS} = \cos 72° + \cos (180° - 72°) + \cos 36°$$

$$\Rightarrow \quad \text{LHS} = \cos 72° - \cos 72° + \cos 36°$$

$$\Rightarrow \quad \text{LHS} = \cos 36° = \text{RHS}$$

EXAMPLE 7 *Prove that:* $\tan 20° \tan 40° \tan 80° = \tan 60°$

SOLUTION We have,

$$\text{LHS} = \tan 20° \tan 40° \tan 80° = \frac{\sin 20° \sin 40° \sin 80°}{\cos 20° \cos 40° \cos 80°}$$

$$\Rightarrow \quad \text{LHS} = \frac{(2 \sin 20° \sin 40°) \sin 80°}{(2 \cos 20° \cos 40°) \cos 80°} = \frac{(\cos 20° - \cos 60°) \sin 80°}{(\cos 60° + \cos 20°) \cos 80°}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin 80° \cos 20° - (1/2) \sin 80°}{(1/2) \cos 80° + \cos 80° \cos 20°} = \frac{2 \sin 80° \cos 20° - \sin 80°}{\cos 80° + 2 \cos 80° \cos 20°}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin 100^\circ + \sin 60^\circ - \sin 80^\circ}{\cos 80^\circ + \cos 100^\circ + \cos 60^\circ} = \frac{\sin (180^\circ - 80^\circ) + \sin 60^\circ - \sin 80^\circ}{\cos 80^\circ + \cos (180^\circ - 80^\circ) + \cos 60^\circ}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin 80^\circ + \sin 60^\circ - \sin 80^\circ}{\cos 80^\circ - \cos 80^\circ + \cos 60^\circ} = \frac{\sin 60^\circ}{\cos 60^\circ} = \tan 60^\circ = \text{RHS}$$

EXAMPLE 8 *Prove that:* $\sin A \sin (60^\circ - A) \sin (60^\circ + A) = \frac{1}{4} \sin 3A$

SOLUTION We have,

$$\begin{aligned}
\text{LHS} &= \sin A \sin (60^\circ - A) \sin (60^\circ + A)\\
&= \frac{1}{2} \sin A \left\{ 2 \sin (60^\circ - A) \sin(60^\circ + A) \right\}\\
&= \frac{1}{2} \sin A \left[\cos \left\{ (60^\circ - A) - (60^\circ + A) \right\} - \cos \left\{ (60^\circ - A) + (60^\circ + A) \right\} \right]\\
&= \frac{1}{2} \sin A \left\{ \cos (-2A) - \cos 120^\circ \right\}\\
&= \frac{1}{2} \sin A \left\{ \cos 2A + \frac{1}{2} \right\}\\
&= \frac{1}{2} \sin A \cos 2A + \frac{1}{4} \sin A\\
&= \frac{1}{4} (2 \sin A \cos 2A) + \frac{1}{4} \sin A\\
&= \frac{1}{4} \left\{ \sin (A + 2A) + \sin (A - 2A) \right\} + \frac{1}{4} \sin A\\
&= \frac{1}{4} \left\{ \sin 3A + \sin (-A) \right\} + \frac{1}{4} \sin A\\
&= \frac{1}{4} \sin 3A - \frac{1}{4} \sin A + \frac{1}{4} \sin A = \frac{1}{4} \sin 3A = \text{RHS}
\end{aligned}$$

EXAMPLE 9 *Prove that:* $\cos A \cos (60^\circ - A) \cos (60^\circ + A) = \frac{1}{4} \cos 3A.$

SOLUTION We have,

$$\begin{aligned}
\text{LHS} &= \cos A \cos (60^\circ - A) \cos (60^\circ + A)\\
&= \frac{1}{2} \cos A \{2 \cos (60^\circ - A) \cos (60^\circ + A)\}\\
&= \frac{1}{2} \cos A \left[\cos \left\{ (60^\circ - A) + (60^\circ + A) \right\} + \cos \left\{ (60^\circ - A) - (60^\circ + A) \right\} \right]\\
&= \frac{1}{2} \cos A \left\{ \cos 120^\circ + \cos (-2A) \right\}\\
&= \frac{1}{2} \cos A \left\{ -\frac{1}{2} + \cos 2A \right\}\\
&= -\frac{1}{4} \cos A + \frac{1}{2} \cos A \cos 2A\\
&= -\frac{1}{4} \cos A + \frac{1}{4} (2 \cos 2A \cos A)\\
&= -\frac{1}{4} \cos A + \frac{1}{4} \left\{ \cos (2A + A) + \cos (2A - A) \right\}
\end{aligned}$$

$$= -\frac{1}{4}\cos A + \frac{1}{4}(\cos 3A + \cos A)$$

$$= \frac{1}{4}\cos 3A = \text{RHS.}$$

LEVEL-2

EXAMPLE 10 *Prove that:* $4 \sin \theta \sin\left(\frac{\pi}{3}+\theta\right)\sin\left(\frac{2\pi}{3}+\theta\right) = \sin 3\theta$

SOLUTION We have,

$$\text{LHS} = 4 \sin \theta \sin\left(\frac{\pi}{3}+\theta\right)\sin\left(\frac{2\pi}{3}+\theta\right)$$

$$= 2 \sin \theta\left\{2 \sin\left(\frac{2\pi}{3}+\theta\right)\sin\left(\frac{\pi}{3}+\theta\right)\right\}$$

$$= 2 \sin \theta\left[\cos\left\{\left(\frac{2\pi}{3}+\theta\right)-\left(\frac{\pi}{3}+\theta\right)\right\} - \cos\left\{\left(\frac{2\pi}{3}+\theta\right)+\left(\frac{\pi}{3}+\theta\right)\right\}\right]$$

$$= 2 \sin \theta\left\{\cos\frac{\pi}{3} - \cos(\pi + 2\theta)\right\}$$

$$= 2 \sin \theta\left\{\frac{1}{2} + \cos 2\theta\right\}$$

$$= \sin \theta + 2 \sin \theta \cos 2\theta$$

$$= \sin \theta + \{\sin(\theta + 2\theta) + \sin(\theta - 2\theta)\}$$

$$= \sin \theta + \sin 3\theta + \sin(-\theta) = \sin \theta + \sin 3\theta - \sin \theta = \sin 3\theta = \text{RHS}$$

EXAMPLE 11 *Show that:* $\tan(60° + \theta)\tan(60° - \theta) = \dfrac{2\cos 2\theta + 1}{2\cos 2\theta - 1}$

SOLUTION We have

LHS

$60° - \theta)\}$

$-(60° - \theta)\}$

$$= \frac{\cos 2\theta - \cos 120°}{\cos 120° + \cos 2\theta} = \frac{\cos 2\theta + \frac{1}{2}}{-\frac{1}{2} + \cos 2\theta} = \frac{2\cos 2\theta + 1}{2\cos 2\theta - 1} = \text{RHS}$$

EXAMPLE 12 *If* $\alpha + \beta = 90°$, *find the maximum and minimum values of* $\sin \alpha \sin \beta$.

SOLUTION Let $y = \sin \alpha \sin \beta$. Then,

$$y = \frac{1}{2}(2 \sin \alpha \sin \beta)$$

$$\Rightarrow \quad y = \frac{1}{2}\{\cos(\alpha - \beta) - \cos(\alpha + \beta)\} = \frac{1}{2}\{\cos(\alpha - \beta) - \cos 90°\} = \frac{1}{2}\cos(\alpha - \beta)$$

We know that

$$-1 \le \cos(\alpha - \beta) \le 1$$

$$\Rightarrow \quad \frac{-1}{2} \le \frac{1}{2}\cos(\alpha - \beta) \le \frac{1}{2}$$

$\Rightarrow \quad \dfrac{-1}{2} \le y \le \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{-1}{2} \le \sin \alpha \sin \beta \le \dfrac{1}{2}$

Hence, $-\dfrac{1}{2}$ and $\dfrac{1}{2}$ are respectively the minimum and maximum values of $\sin \alpha \sin \beta$.

EXERCISE 8.1

LEVEL-1

1. Express each of the following as the sum or difference of sines and cosines:
   (i) $2 \sin 3\theta \cos \theta$ (ii) $2 \cos 3\theta \sin 2\theta$
   (iii) $2 \sin 4\theta \sin 3\theta$ (iv) $2 \cos 7\theta \cos 3\theta$
2. Prove that:
   (i) $2 \sin \dfrac{5\pi}{12} \sin \dfrac{\pi}{12} = \dfrac{1}{2}$ (ii) $2 \cos \dfrac{5\pi}{12} \cos \dfrac{\pi}{12} = \dfrac{1}{2}$
   (iii) $2 \sin \dfrac{5\pi}{12} \cos \dfrac{\pi}{12} = \dfrac{\sqrt{3}+2}{2}$
3. Show that:
   (i) $\sin 50° \cos 85° = \dfrac{1-\sqrt{2} \sin 35°}{2\sqrt{2}}$ (ii) $\sin 25° \cos 115° = \dfrac{1}{2}(\sin 140° - 1)$
4. Prove that: $4 \cos \theta \cos\left(\dfrac{\pi}{3}+\theta\right)\cos\left(\dfrac{\pi}{3}-\theta\right) = \cos 3\theta$
5. Prove that :
   (i) $\cos 10° \cos 30° \cos 50° \cos 70° = \dfrac{3}{16}$ (ii) $\cos 40° \cos 80° \cos 160° = -\dfrac{1}{8}$
   (iii) $\sin 20° \sin 40° \sin 80° = \dfrac{\sqrt{3}}{8}$ (iv) $\cos 20° \cos 40° \cos 80° = \dfrac{1}{8}$
   (v) $\tan 20° \tan 40° \tan 60° \tan 80° = 3$ (vi) $\tan 20° \tan 30° \tan 40° \tan 80° = 1$
   (vii) $\sin 10° \sin 50° \sin 60° \sin 70° = \dfrac{\sqrt{3}}{16}$ (viii) $\sin 20° \sin 40° \sin 60° \sin 80° = \dfrac{3}{16}$
6. Show that :
   (i) $\sin A \sin(B-C) + \sin B \sin(C-A) + \sin C \sin(A-B) = 0$
   (ii) $\sin(B-C)\cos(A-D) + \sin(C-A)\cos(B-D) + \sin(A-B)\cos(C-D) = 0$

LEVEL-2

7. Prove that : $\tan \theta \tan(60° - \theta) \tan(60° + \theta) = \tan 3\theta$.
8. If $\alpha + \beta = 90°$, show that the maximum value of $\cos \alpha \cos \beta$ is $\dfrac{1}{2}$.

ANSWERS

1. (i) $\sin 4\theta + \sin 2\theta$ (ii) $\sin 5\theta - \sin \theta$ (iii) $\cos \theta - \cos 7\theta$ (iv) $\cos 10\theta + \cos 4\theta$

HINTS TO SELECTED PROBLEM

2. (i) $\text{LHS} = \cos\left(\dfrac{5\pi}{12}-\dfrac{\pi}{12}\right) - \cos\left(\dfrac{5\pi}{12}+\dfrac{\pi}{12}\right) = \cos\dfrac{\pi}{3} - \cos\dfrac{\pi}{2} = \dfrac{1}{2} = \text{RHS}$

   (ii) $\text{LHS} = \cos\left(\dfrac{5\pi}{12}+\dfrac{\pi}{12}\right) + \cos\left(\dfrac{5\pi}{12}-\dfrac{\pi}{12}\right) = \cos\dfrac{\pi}{2} + \cos\dfrac{\pi}{3} = \dfrac{1}{2} = \text{RHS}$

   (iii) $\text{LHS} = \sin\left(\dfrac{5\pi}{12}+\dfrac{\pi}{12}\right) + \sin\left(\dfrac{5\pi}{12}-\dfrac{\pi}{12}\right) = \sin\dfrac{\pi}{2} + \sin\dfrac{\pi}{3} = 1 + \dfrac{\sqrt{3}}{2} = \text{RHS}$

8.3 FORMULAE TO TRANSFORM THE SUM OR DIFFERENCE INTO PRODUCT

In the previous section, we have used the following formulae:

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B, \quad \sin(A+B)-\sin(A-B) = 2\cos A\sin B$$

$$\cos(A+B)+\cos(A-B) = 2\cos A\cos B \text{ and, } \cos(A-B)-\cos(A+B) = 2\sin A\sin B.$$

Let $A+B=C$ and $A-B=D$. Then, $A=\dfrac{C+D}{2}$ and $B=\dfrac{C-D}{2}$.

Substituting the values of A, B, C and D in the above formulae, we get

$$\sin C+\sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \quad \text{...(i)}$$

$$\sin C-\sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right) \quad \text{...(ii)}$$

$$\cos C+\cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \quad \text{...(iii)}$$

$$\cos D-\cos C = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

or, $$\cos C-\cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \quad \text{...(iv)}$$

or, $$\cos C-\cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$$

These four formulae are used to convert the sum or difference of two sines or two cosines into the product of sines and cosines.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Express each of the following as a product:*

(i) $\sin 4\theta+\sin 2\theta$ (ii) $\sin 6\theta-\sin 2\theta$

(iii) $\cos 4\theta+\cos 8\theta$ (iv) $\cos 6\theta-\cos 8\theta$

SOLUTION (i) $\sin 4\theta+\sin 2\theta$

$$= 2\sin\left(\frac{4\theta+2\theta}{2}\right)\cos\left(\frac{4\theta-2\theta}{2}\right) \quad \left[\because \sin C+\sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$= 2\sin 3\theta\cos\theta$

(ii) $\sin 6\theta-\sin 2\theta$

$$= 2\sin\left(\frac{6\theta-2\theta}{2}\right)\cos\left(\frac{6\theta+2\theta}{2}\right) \quad \left[\because \sin C-\sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)\right]$$

$= 2\sin 2\theta\cos 4\theta$

(iii) $\cos 4\theta+\cos 8\theta$

$$= 2\cos\left(\frac{8\theta+4\theta}{2}\right)\cos\left(\frac{8\theta-4\theta}{2}\right) \quad \left[\because \cos C+\cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$= 2\cos 6\theta\cos 2\theta$

(iv) $\cos 6\theta-\cos 8\theta$

$$= 2\sin\left(\frac{6\theta+8\theta}{2}\right)\sin\left(\frac{8\theta-6\theta}{2}\right) \quad \left[\because \cos C-\cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\right]$$

$= 2\sin 7\theta\sin\theta$

EXAMPLE 2 *Prove that:* $\cos 18° - \sin 18° = \sqrt{2} \sin 27°$

SOLUTION We have,

$$\text{LHS} = \cos 18° - \sin 18°$$
$$= \cos 18° - \cos 72° \quad [\because \sin 18° = \sin(90° - 72°) = \cos 72°]$$
$$= 2 \sin\left(\frac{18° + 72°}{2}\right) \sin\left(\frac{72° - 18°}{2}\right) = 2 \sin 45° \sin 27° = \sqrt{2} \sin 27° = \text{RHS}$$

EXAMPLE 3 *Prove that:*

(i) $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \tan A$ (ii) $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$ [NCERT]

(iii) $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A+B}{2}\right)$ (iv) $\dfrac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$ [NCERT]

SOLUTION (i) $\text{LHS} = \dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A}$

$$= \frac{2 \sin\left(\frac{5A - 3A}{2}\right) \cos\left(\frac{5A + 3A}{2}\right)}{2 \cos\left(\frac{5A + 3A}{2}\right) \cos\left(\frac{5A - 3A}{2}\right)} = \frac{2 \sin A \cos 4A}{2 \cos 4A \cos A} = \tan A = \text{RHS}$$

(ii) $\text{LHS} = \dfrac{\sin 3A + \sin A}{\cos 3A + \cos A}$

$$= \frac{2 \sin\left(\frac{3A + A}{2}\right) \cos\left(\frac{3A - A}{2}\right)}{2 \cos\left(\frac{3A + A}{2}\right) \cos\left(\frac{3A - A}{2}\right)} = \frac{\sin 2A \cos A}{\cos 2A \cos A} = \tan 2A = \text{RHS}$$

(iii) $$\text{LHS} = \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)}{2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)} = \tan\left(\frac{A+B}{2}\right) = \text{RHS}$$

(iv) $$\text{LHS} = \frac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \frac{2 \cos\left(\frac{7A+5A}{2}\right) \cos\left(\frac{7A-5A}{2}\right)}{2 \sin\left(\frac{7A-5A}{2}\right) \cos\left(\frac{7A+5A}{2}\right)} = \frac{2 \cos 6A \cos A}{2 \sin A \cos 6A} = \cot A = \text{RHS}$$

EXAMPLE 4 *Prove that:*

(i) $\dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\dfrac{\sin 2x}{\cos 10x}$ [NCERT] (ii) $\dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$ [NCERT]

(iii) $(\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x = 0$ [NCERT]

SOLUTION (i) $\text{LHS} = \dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x}$

$$= \frac{-2 \sin\left(\frac{9x + 5x}{2}\right) \sin\left(\frac{9x - 5x}{2}\right)}{2 \sin\left(\frac{17x - 3x}{2}\right) \cos\left(\frac{17x + 3x}{2}\right)} = \frac{-2 \sin 7x \sin 2x}{2 \sin 7x \cos 10x} = \frac{-\sin 2x}{\cos 10x} = \text{RHS}$$

(ii) $\text{LHS} = \dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x}$

$$= \frac{2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)}{2\cos\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)} = \frac{2\sin 4x\cos x}{2\cos 4x\cos x} = \tan 4x = \text{RHS}$$

(iii) $\text{LHS} = (\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x$

$$= \left\{2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)\right\}\sin x + \left\{-2\sin\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)\right\}\cos x$$

$$= 2\sin 2x\cos x\sin x - 2\sin 2x\sin x\cos x = 0 = \text{RHS}$$

EXAMPLE 5 *Prove that:* $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$ [NCERT]

SOLUTION $\text{LHS} = \cot 4x(\sin 5x + \sin 3x)$

$$= \cot 4x \times 2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)$$

$$= \frac{\cos 4x}{\sin 4x} \times 2\sin 4x\cos x = 2\cos 4x\cos x \qquad \text{...(i)}$$

$\text{RHS} = \cot x(\sin 5x - \sin 3x)$

$$= \cot x \times 2\sin\left(\frac{5x-3x}{2}\right)\cos\left(\frac{5x+3x}{2}\right)$$

$$= \frac{\cos x}{\sin x} \times 2\sin x\cos 4x = 2\cos 4x\cos x \qquad \text{...(ii)}$$

From (i) and (ii), we obtain that LHS = RHS.

EXAMPLE 6 *Prove that:* $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x\cos 2x\sin 4x$ [NCERT]

SOLUTION $\text{LHS} = \sin x + \sin 3x + \sin 5x + \sin 7x$

$$= (\sin 7x + \sin x) + (\sin 5x + \sin 3x)$$

$$= 2\sin\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right) + 2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)$$

$$= 2\sin 4x\cos 3x + 2\sin 4x\cos x$$

$$= 2\sin 4x(\cos 3x + \cos x)$$

$$= 2\sin 4x \times 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$$

$$= 2\sin 4x \times 2\cos 2x\cos x = 4\cos x\cos 2x\sin 4x = \text{RHS}$$

EXAMPLE 7 *Prove that:* $1 + \cos 2x + \cos 4x + \cos 6x = 4\cos x\cos 2x\cos 3x$

SOLUTION $\text{LHS} = 1 + \cos 2x + \cos 4x + \cos 6x$

$$= (\cos 0x + \cos 2x) + (\cos 4x + \cos 6x)$$

$$= 2\cos x\cos x + 2\cos 5x\cos x$$

$$= 2\cos x(\cos x + \cos 5x)$$

$$= 2\cos x(2\cos 3x\cos 2x) = 4\cos x\cos 2x\cos 3x = \text{RHS}$$

EXAMPLE 8 *Prove that:*

(i) $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$

(ii) $\cos 2\theta\cos\frac{\theta}{2} - \cos 3\theta\cos\frac{9\theta}{2} = \sin 5\theta\sin\frac{5\theta}{2}$ [NCERT]

(iii) $\sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) = 0$

SOLUTION (i) We have,

$\text{LHS} = (\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A$

$$= \left\{2\sin\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)\right\}\sin A + \left\{-2\sin\left(\frac{3A+A}{2}\right)\sin\left(\frac{3A-A}{2}\right)\right\}\cos A$$

$$= 2\sin 2A\cos A\sin A - 2\sin 2A\sin A\cos A = 0 = \text{RHS}$$

(ii) $\text{LHS} = \cos 2\theta\cos\frac{\theta}{2} - \cos 3\theta\cos\frac{9\theta}{2}$

$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left\{2\cos 2\theta\cos\frac{\theta}{2} - 2\cos 3\theta\cos\frac{9\theta}{2}\right\}$

$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left[\cos\left\{\left(2\theta+\frac{\theta}{2}\right)+\cos\left(2\theta-\frac{\theta}{2}\right)\right\} - \left\{\cos\left(3\theta+\frac{9\theta}{2}\right)+\cos\left(\frac{9\theta}{2}-3\theta\right)\right\}\right]$

[Using: $2\cos A\cos B = \cos(A+B)+\cos(A-B)$]

$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left\{\cos\frac{5\theta}{2}+\cos\frac{3\theta}{2}-\cos\frac{15\theta}{2}-\cos\frac{3\theta}{2}\right\}$

$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left\{\cos\frac{5\theta}{2}-\cos\frac{15\theta}{2}\right\}$

$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left\{2\sin\left(\frac{\frac{5\theta}{2}+\frac{15\theta}{2}}{2}\right)\sin\left(\frac{\frac{15\theta}{2}-\frac{5\theta}{2}}{2}\right)\right\}$ $\left[\because \cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\right]$

$\Rightarrow \quad \text{LHS} = \sin 5\theta\sin\frac{5\theta}{2} = \text{RHS}$

(iii) $\text{LHS} = \sin\alpha + \sin\left(\alpha+\frac{2\pi}{3}\right)+\sin\left(\alpha+\frac{4\pi}{3}\right)$

$\Rightarrow \quad \text{LHS} = \sin\alpha + \left[\sin\left(\alpha+\frac{2\pi}{3}\right)+\sin\left(\alpha+\frac{4\pi}{3}\right)\right]$

$\Rightarrow \quad \text{LHS} = \sin\alpha + \left[2\sin\left(\frac{\alpha+\frac{2\pi}{3}+\alpha+\frac{4\pi}{3}}{2}\right)\cos\left(\frac{\alpha+\frac{4\pi}{3}-\alpha-\frac{2\pi}{3}}{2}\right)\right]$

$\Rightarrow \quad \text{LHS} = \sin\alpha + 2\sin(\alpha+\pi)\cos\frac{\pi}{3} = \sin\alpha + 2(-\sin\alpha)\left(\frac{1}{2}\right) = \sin\alpha - \sin\alpha = 0 = \text{RHS}$

EXAMPLE 9 *Prove that:*

(i) $(\cos\alpha+\cos\beta)^2 + (\sin\alpha+\sin\beta)^2 = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$ [NCERT]

(ii) $(\cos\alpha-\cos\beta)^2 + (\sin\alpha-\sin\beta)^2 = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$ [NCERT]

(iii) $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$

SOLUTION (i) $\text{LHS} = (\cos\alpha+\cos\beta)^2 + (\sin\alpha+\sin\beta)^2$

$\Rightarrow \quad \text{LHS} = \left\{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2$

$$\Rightarrow \quad \text{LHS} = 4\cos^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right) + 4\sin^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left\{\cos^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\left(\frac{\alpha+\beta}{2}\right)\right\} = 4\cos^2\left(\frac{\alpha-\beta}{2}\right) = \text{RHS}$$

(ii) $\quad \text{LHS} = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$$\Rightarrow \quad \text{LHS} = \left\{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\right\}^2$$

$$\Rightarrow \quad \text{LHS} = 4\sin^2\left(\frac{\alpha+\beta}{2}\right)\sin^2\left(\frac{\alpha-\beta}{2}\right) + 4\sin^2\left(\frac{\alpha-\beta}{2}\right)\cos^2\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)\left\{\sin^2\left(\frac{\alpha+\beta}{2}\right) + \cos^2\left(\frac{\alpha+\beta}{2}\right)\right\}$$

$$\Rightarrow \quad \text{LHS} = 4\sin^2\left(\frac{\alpha-\beta}{2}\right) = \text{RHS} \qquad \left[\because \sin^2\left(\frac{\alpha+\beta}{2}\right) + \cos^2\left(\frac{\alpha+\beta}{2}\right) = 1\right]$$

(iii) $\quad \text{LHS} = \cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma)$

$$\Rightarrow \quad \text{LHS} = (\cos\alpha + \cos\beta) + [\cos\gamma + \cos(\alpha+\beta+\gamma)]$$

$$\Rightarrow \quad \text{LHS} = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta+\gamma+\gamma}{2}\right)\cos\left(\frac{\alpha+\beta+\gamma-\gamma}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right\}$$

$$\Rightarrow \quad \text{LHS} = 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2\gamma}{2}}{2}\right)\cos\left(\frac{\frac{\alpha+\beta+2\gamma}{2}-\frac{\alpha-\beta}{2}}{2}\right)\right\}$$

$$\Rightarrow \quad \text{LHS} = 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\}$$

$$\Rightarrow \quad \text{LHS} = 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right) = \text{RHS}$$

EXAMPLE 10 *Prove that:* $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$ [NCERT]

SOLUTION $\text{LHS} = \dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x}$

$$\Rightarrow \quad \text{LHS} = \frac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \cos 3x}{2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \sin 3x} = \frac{2\cos 3x\cos x + \cos 3x}{2\sin 3x\cos x + \sin 3x}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos 3x(2\cos x + 1)}{\sin 3x(2\cos x + 1)} = \frac{\cos 3x}{\sin 3x} = \cot 3x = \text{RHS}$$

EXAMPLE 11 *Prove that:* $\dfrac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A$

SOLUTION We have,

$$\text{LHS}=\frac{(\sin 7A+\sin A)+(\sin 5A+\sin 3A)}{(\cos 7A+\cos A)+(\cos 5A+\cos 3A)}$$

$$=\frac{2\sin\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right)+2\sin\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}{2\cos\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right)+2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}$$

$$=\frac{\sin 4A\cos 3A+\sin 4A\cos A}{\cos 4A\cos 3A+\cos 4A\cos A}=\frac{\sin 4A(\cos 3A+\cos A)}{\cos 4A(\cos 3A+\cos A)}=\tan 4A=\text{RHS}$$

EXAMPLE 12 *Prove that:* $\dfrac{\cos 8A\cos 5A-\cos 12A\cos 9A}{\sin 8A\cos 5A+\cos 12A\sin 9A}=\tan 4A$

SOLUTION We have,

$$\text{LHS}=\frac{2\cos 8A\cos 5A-2\cos 12A\cos 9A}{2\sin 8A\cos 5A+2\cos 12A\sin 9A}$$

$$\Rightarrow\quad \text{LHS}=\frac{\{\cos(8A+5A)+\cos(8A-5A)\}-\{\cos(12A+9A)+\cos(12A-9A)\}}{\{\sin(8A+5A)+\sin(8A-5A)\}+\{\sin(9A+12A)+\sin(9A-12A)\}}$$

$$\Rightarrow\quad \text{LHS}=\frac{\{\cos 13A+\cos 3A\}-\{\cos 21A+\cos 3A\}}{\{\sin 13A+\sin 3A\}+\{\sin 21A+\sin(-3A)\}}$$

$$\Rightarrow\quad \text{LHS}=\frac{(\cos 13A+\cos 3A)-(\cos 21A+\cos 3A)}{(\sin 13A+\sin 3A)+(\sin 21A-\sin 3A)}=\frac{\cos 13A-\cos 21A}{\sin 13A+\sin 21A}$$

$$\Rightarrow\quad \text{LHS}=\frac{2\sin\left(\frac{13A+21A}{2}\right)\sin\left(\frac{21A-13A}{2}\right)}{2\sin\left(\frac{3A+21A}{2}\right)\cos\left(\frac{21A-13A}{2}\right)}=\frac{\sin 17A\sin 4A}{\sin 17A\cos 4A}=\tan 4A=\text{RHS}$$

EXAMPLE 13 *Prove that:* $\dfrac{\cos 2A\cos 3A-\cos 2A\cos 7A+\cos A\cos 10A}{\sin 4A\sin 3A-\sin 2A\sin 5A+\sin 4A\sin 7A}=\cot 6A\cot 5A$

SOLUTION We have,

$$\text{LHS}=\frac{2\cos 3A\cos 2A-2\cos 7A\cos 2A+2\cos 10A\cos A}{2\sin 4A\sin 3A-2\sin 5A\sin 2A+2\sin 7A\sin 4A}$$

$$=\frac{(\cos 5A+\cos A)-(\cos 9A+\cos 5A)+(\cos 11A+\cos 9A)}{(\cos A-\cos 7A)-(\cos 3A-\cos 7A)+(\cos 3A-\cos 11A)}$$

$$=\frac{\cos A+\cos 11A}{\cos A-\cos 11A}=\frac{2\cos\left(\frac{11A+A}{2}\right)\cos\left(\frac{11A-A}{2}\right)}{2\sin\left(\frac{A+11A}{2}\right)\sin\left(\frac{11A-A}{2}\right)}$$

$$=\frac{\cos 6A\cos 5A}{\sin 6A\sin 5A}=\cot 6A\cot 5A=\text{RHS}$$

EXAMPLE 14 *Prove that:* $\dfrac{\sin(A-C)+2\sin A+\sin(A+C)}{\sin(B-C)+2\sin B+\sin(B+C)}=\dfrac{\sin A}{\sin B}$

SOLUTION We have,

$$\text{LHS} = \frac{\sin(A-C)+\sin(A+C)+2\sin A}{\sin(B-C)+\sin(B+C)+2\sin B}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin\left(\frac{A-C+A+C}{2}\right)\cos\left(\frac{A+C-A+C}{2}\right)+2\sin A}{2\sin\left(\frac{B+C+B-C}{2}\right)\cos\left(\frac{B+C-B+C}{2}\right)+2\sin B}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin A\cos C+2\sin A}{2\sin B\cos C+2\sin B} = \frac{2\sin A(\cos C+1)}{2\sin B(\cos C+1)} = \frac{\sin A}{\sin B} = \text{RHS}$$

LEVEL-2

EXAMPLE 15 *If* $\sin\theta = n\sin(\theta+2\alpha)$, *prove that* $\tan(\theta+\alpha) = \frac{1+n}{1-n}\tan\alpha$.

SOLUTION We have,

$$\sin\theta = n\sin(\theta+2\alpha)$$

$$\Rightarrow \quad \frac{\sin(\theta+2\alpha)}{\sin\theta} = \frac{1}{n}$$

$$\Rightarrow \quad \frac{\sin(\theta+2\alpha)+\sin\theta}{\sin(\theta+2\alpha)-\sin\theta} = \frac{1+n}{1-n}$$ [Applying componendo-dividendo]

$$\Rightarrow \quad \frac{2\sin(\theta+\alpha)\cos\alpha}{2\sin\alpha\cos(\theta+\alpha)} = \frac{1+n}{1-n}$$

$$\Rightarrow \quad \tan(\theta+\alpha) = \frac{1+n}{1-n}\tan\alpha$$

EXAMPLE 16 *Prove that:*

$$\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n + \left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n = \begin{cases} 2\cot^n\left(\frac{A-B}{2}\right) & , \text{ if } n \text{ is even} \\ 0 & , \text{ if } n \text{ is odd} \end{cases}$$

SOLUTION We have,

$$\text{LHS} = \left(\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}\right)^n + \left(\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}}\right)^n$$

$$\Rightarrow \quad \text{LHS} = \left\{\cot\left(\frac{A-B}{2}\right)\right\}^n + \left\{-\cot\left(\frac{A-B}{2}\right)\right\}^n$$

$$\Rightarrow \quad \text{LHS} = \cot^n\left(\frac{A-B}{2}\right) + (-1)^n\cot^n\left(\frac{A-B}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = \cot^n\left(\frac{A-B}{2}\right)^n\left\{1+(-1)^n\right\} = \begin{cases} 2\cot^n\left(\frac{A-B}{2}\right) & , \text{ if } n \text{ is even} \\ 0 & , \text{ if } n \text{ is odd} \end{cases}$$

EXAMPLE 17 *If three angles A, B and C are in A.P., prove that:* $\cot B = \frac{\sin A - \sin C}{\cos C - \cos A}$.

SOLUTION We have,

$$\text{RHS} = \frac{2\sin\frac{A-C}{2}\cos\frac{A+C}{2}}{2\sin\frac{A+C}{2}\sin\frac{A-C}{2}} = \cot\left(\frac{A+C}{2}\right) = \cot B = \text{LHS} \qquad \left[\begin{array}{l}\because A, B, C \text{ are in A.P.}\\ \therefore 2B = A + C\end{array}\right]$$

EXAMPLE 18 *If* $\sin\theta + \sin\phi = \sqrt{3}(\cos\phi - \cos\theta)$, *prove that* $\sin 3\theta + \sin 3\phi = 0$ [NCERT EXEMPLAR]

SOLUTION We have,

$$\sin\theta + \sin\phi = \sqrt{3}(\cos\phi - \cos\theta)$$

$$\Rightarrow \quad 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = 2\sqrt{3}\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

$$\Rightarrow \quad \left\{\cos\frac{\theta-\phi}{2} - \sqrt{3}\sin\frac{\theta-\phi}{2}\right\}\sin\left(\frac{\theta+\phi}{2}\right) = 0$$

$$\Rightarrow \quad \sin\left(\frac{\theta+\phi}{2}\right) = 0 \text{ or, } \cos\left(\frac{\theta-\phi}{2}\right) - \sqrt{3}\sin\left(\frac{\theta-\phi}{2}\right) = 0$$

$$\Rightarrow \quad \sin\left(\frac{\theta+\phi}{2}\right) = 0 \text{ or, } \tan\left(\frac{\theta-\phi}{2}\right) = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$$

$$\Rightarrow \quad \frac{\theta+\phi}{2} = 0 \quad \text{or,} \quad \frac{\theta-\phi}{2} = \frac{\pi}{6}$$

$$\Rightarrow \quad \theta = -\phi \quad \text{or,} \quad \theta - \phi = \frac{\pi}{3}$$

CASE I When $\theta = -\phi$: In this case, we have

$$\sin 3\theta + \sin 3\phi = \sin 3(-\phi) + \sin 3\phi = -\sin 3\phi + \sin 3\phi = 0$$

CASE II When $\theta - \phi = \frac{\pi}{3}$: In this case, we have

$$\theta - \phi = \frac{\pi}{3} \Rightarrow 3\theta - 3\phi = \pi \Rightarrow 3\theta = \pi + 3\phi$$

$$\therefore \quad \sin 3\theta + \sin 3\phi = \sin(\pi + 3\phi) + \sin 3\phi = -\sin 3\phi + \sin 3\phi = 0$$

EXAMPLE 19 *If* $\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)} = \frac{1-m}{1+m}$, *prove that* $\tan\left(\frac{\pi}{4} - \theta\right)\tan\left(\frac{\pi}{4} - \alpha\right) = m$

SOLUTION We have,

$$\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)} = \frac{1-m}{1+m}$$

$$\Rightarrow \quad \frac{\sin(\theta+\alpha) + \cos(\theta-\alpha)}{\sin(\theta+\alpha) - \cos(\theta-\alpha)} = \frac{2}{-2m} \qquad \text{[Using componendo-dividendo]}$$

$$\Rightarrow \quad \frac{\sin(\theta+\alpha) + \sin\left\{\frac{\pi}{2} - (\theta-\alpha)\right\}}{\sin(\theta+\alpha) - \sin\left\{\frac{\pi}{2} - (\theta-\alpha)\right\}} = -\frac{1}{m}$$

$$\Rightarrow \quad \frac{2\sin\left(\frac{\theta+\alpha+\frac{\pi}{2}-\theta+\alpha}{2}\right)\cos\left(\frac{\theta+\alpha-\frac{\pi}{2}+\theta-\alpha}{2}\right)}{2\sin\left(\frac{\theta+\alpha-\frac{\pi}{2}+\theta-\alpha}{2}\right)\cos\left(\frac{\theta+\alpha+\frac{\pi}{2}-\theta+\alpha}{2}\right)} = -\frac{1}{m}$$

$$\Rightarrow \quad \frac{\sin\left(\frac{\pi}{4}+\alpha\right)\cos\left(-\frac{\pi}{4}+\theta\right)}{\sin\left(-\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}+\alpha\right)} = -\frac{1}{m}$$

$$\Rightarrow \quad \frac{\sin\left(\frac{\pi}{4}+\alpha\right)}{\cos\left(\frac{\pi}{4}+\alpha\right)} \cdot \frac{\cos\left(\frac{\pi}{4}-\theta\right)}{\sin\left(\frac{\pi}{4}-\theta\right)} = \frac{1}{m}$$

$$\Rightarrow \quad \tan\left(\frac{\pi}{4}+\alpha\right)\cot\left(\frac{\pi}{4}-\theta\right) = \frac{1}{m}$$

$$\Rightarrow \quad m = \cot\left(\frac{\pi}{4}+\alpha\right)\tan\left(\frac{\pi}{4}-\theta\right)$$

$$\Rightarrow \quad m = \tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}+\alpha\right)\right\}\tan\left(\frac{\pi}{4}-\theta\right)$$

$$\Rightarrow \quad m = \tan\left(\frac{\pi}{4}-\alpha\right)\tan\left(\frac{\pi}{4}-\theta\right)$$

EXAMPLE 20 *If* $a \sin\theta = b\sin\left(\theta+\frac{2\pi}{3}\right) = c\sin\left(\theta+\frac{4\pi}{3}\right)$, *prove that* $ab+bc+ca=0$.

SOLUTION We have,

$$a\sin\theta = b\sin\left(\theta+\frac{2\pi}{3}\right) = c\sin\left(\theta+\frac{4\pi}{3}\right) = \lambda \text{ (say)}$$

$$\Rightarrow \quad \frac{\lambda}{a} = \sin\theta,\ \frac{\lambda}{b} = \sin\left(\theta+\frac{2\pi}{3}\right) \text{and } \frac{\lambda}{c} = \sin\left(\theta+\frac{4\pi}{3}\right)$$

$$\Rightarrow \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = \sin\theta+\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta+\frac{4\pi}{3}\right)$$

$$\Rightarrow \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = \left\{\sin\left(\theta+\frac{4\pi}{3}\right)+\sin\theta\right\}+\sin\left(\theta+\frac{2\pi}{3}\right)$$

$$\Rightarrow \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = 2\sin\left(\theta+\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}+\sin\left(\theta+\frac{2\pi}{3}\right)$$

$$\Rightarrow \quad \frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c} = -\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta+\frac{2\pi}{3}\right) = 0$$

$$\Rightarrow \quad \lambda\left(\frac{bc+ca+ab}{abc}\right) = 0 \Rightarrow ab+bc+ca = 0$$

EXAMPLE 21 *If* $\sin(y+z-x)$, $\sin(z+x-y)$, $\sin(x+y-z)$ *are in A.P., prove that* $\tan x$, $\tan y$, $\tan z$ *are also in A.P.*

SOLUTION It is given that $\sin(y+z-x)$, $\sin(z+x-y)$ and $\sin(x+y-z)$ are in A.P.

$\therefore \quad \sin(z+x-y)-\sin(y+z-x) = \sin(x+y-z)-\sin(z+x-y)$

$\Rightarrow \quad 2\sin(x-y)\cos z = 2\sin(y-z)\cos x$

$\Rightarrow \quad \sin(x-y)\cos z = \sin(y-z)\cos x$

$\Rightarrow \quad \sin x\cos y\cos z-\cos x\sin y\cos z = \sin y\cos z\cos x-\cos y\sin z\cos x$

$\Rightarrow \quad 2\sin y\cos x\cos z = \sin x\cos y\cos z+\cos x\cos y\sin z$

$\Rightarrow \quad 2\tan y = \tan x+\tan z$ [Dividing throughout by $\cos x\cos y\cos z$]

$\Rightarrow \quad \tan x, \tan y, \tan z$ are in A.P.

EXAMPLE 22 *If* $\frac{\tan(\theta+\alpha)}{a} = \frac{\tan(\theta+\beta)}{b} = \frac{\tan(\theta+\gamma)}{c}$, *prove that*

$$\frac{a+b}{a-b}\sin^2(\alpha-\beta)+\frac{b+c}{b-c}\sin^2(\beta-\gamma)+\frac{c+a}{c-a}\sin^2(\gamma-\alpha)=0$$

SOLUTION We have,

$$\frac{\tan(\theta+\alpha)}{a} = \frac{\tan(\theta+\beta)}{b}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{\tan(\theta+\alpha)}{\tan(\theta+\beta)}$$

$$\Rightarrow \quad \frac{a+b}{a-b} = \frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)} \qquad \text{[Applying Componendo-dividendo]}$$

$$\Rightarrow \quad \frac{a+b}{a-b} = \frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)} \qquad \left[\because \frac{\tan A+\tan B}{\tan A-\tan B} = \frac{\sin(A+B)}{\sin(A-B)}\right]$$

$$\Rightarrow \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) = \sin(2\theta+\alpha+\beta)\sin(\alpha-\beta)$$

$$\Rightarrow \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) = \frac{1}{2}\left\{2\sin(2\theta+\alpha+\beta)\sin(\alpha-\beta)\right\}$$

$$\Rightarrow \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta) = \frac{1}{2}\left\{\cos(2\theta+2\beta)-\cos(2\theta+2\alpha)\right\}$$

Similarly, we obtain

$$\frac{b+c}{b-c}\sin^2(\beta-\gamma) = \frac{1}{2}\left\{\cos(2\theta+2\gamma)-\cos(2\theta+2\beta)\right\}$$

and, $$\frac{c+a}{c-a}\sin^2(\gamma-\alpha) = \frac{1}{2}\left\{\cos(2\theta+2\alpha)-\cos(2\theta+2\gamma)\right\}$$

$$\therefore \quad \frac{a+b}{a-b}\sin^2(\alpha-\beta)+\frac{b+c}{b-c}\sin^2(\beta-\gamma)+\frac{c+a}{c-a}\sin^2(\gamma-\alpha)$$

$$=\frac{1}{2}\left\{\cos(2\theta+2\beta)-\cos(2\theta+2\alpha)+\cos(2\theta+2\gamma)-\cos(2\theta+2\beta)+\cos(2\theta+2\alpha)-\cos(2\theta+2\gamma)\right\}$$

$$=\frac{1}{2}\times 0 = 0$$

EXAMPLE 23 *Prove that :* $\frac{\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10}{\cos 5\theta+5\cos 3\theta+10\cos\theta} = 2\cos\theta$

SOLUTION We have,

$$\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10$$
$$=(\cos 6\theta+\cos 4\theta)+(5\cos 4\theta+5\cos 2\theta)+(10\cos 2\theta+10)$$
$$=(\cos 6\theta+\cos 4\theta)+5(\cos 4\theta+\cos 2\theta)+10(\cos 2\theta+\cos 0\theta)$$
$$=2\cos 5\theta\cos\theta+5\times 2\cos 3\theta\cos\theta+10\times 2\cos\theta\cos\theta$$
$$=2\cos\theta\,(\cos 5\theta+5\cos 3\theta+10\cos\theta)$$

$$\therefore \quad \text{LHS} = \frac{\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10}{\cos 5\theta+5\cos 3\theta+10\cos\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\cos\theta(\cos 5\theta+5\cos 3\theta+10\cos\theta)}{\cos 5\theta+5\cos 3\theta+10\cos\theta} = 2\cos\theta = \text{RHS}$$

EXERCISE 8.2

LEVEL-1

1. Express each of the following as the product of sines and cosines:
   (i) $\sin 12\theta + \sin 4\theta$ (ii) $\sin 5\theta - \sin \theta$ (iii) $\cos 12\theta + \cos 8\theta$
   (iv) $\cos 12\theta - \cos 4\theta$ (v) $\sin 2\theta + \cos 4\theta$

2. Prove that:
   (i) $\sin 38° + \sin 22° = \sin 82°$ (ii) $\cos 100° + \cos 20° = \cos 40°$
   (iii) $\sin 50° + \sin 10° = \cos 20°$ (iv) $\sin 23° + \sin 37° = \cos 7°$
   (v) $\sin 105° + \cos 105° = \cos 45°$ (vi) $\sin 40° + \sin 20° = \cos 10°$

3. Prove that:
   (i) $\cos 55° + \cos 65° + \cos 175° = 0$ (ii) $\sin 50° - \sin 70° + \sin 10° = 0$
   (iii) $\cos 80° + \cos 40° - \cos 20° = 0$ (iv) $\cos 20° + \cos 100° + \cos 140° = 0$
   (v) $\sin \frac{5\pi}{18} - \cos \frac{4\pi}{9} = \sqrt{3} \sin \frac{\pi}{9}$ (vi) $\cos \frac{\pi}{12} - \sin \frac{\pi}{12} = \frac{1}{\sqrt{2}}$
   (vii) $\sin 80° - \cos 70° = \cos 50°$ (viii) $\sin 51° + \cos 81° = \cos 21°$

4. Prove that:
   (i) $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2} \sin x$ [NCERT]
   (ii) $\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2} \cos x$ [NCERT]

5. Prove that :
   (i) $\sin 65° + \cos 65° = \sqrt{2} \cos 20°$ (ii) $\sin 47° + \cos 77° = \cos 17°$

6. Prove that :
   (i) $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4 \cos 4A \cos 5A \cos 6A$
   (ii) $\cos A + \cos 3A + \cos 5A + \cos 7A = 4 \cos A \cos 2A \cos 4A$
   (iii) $\sin A + \sin 2A + \sin 4A + \sin 5A = 4 \cos \frac{A}{2} \cos \frac{3A}{2} \sin 3A$
   (iv) $\sin 3A + \sin 2A - \sin A = 4 \sin A \cos \frac{A}{2} \cos \frac{3A}{2}$
   (v) $\cos 20° \cos 100° + \cos 100° \cos 140° - \cos 140° \cos 200° = -\frac{3}{4}$
   (vi) $\sin \frac{\theta}{2} \sin \frac{7\theta}{2} + \sin \frac{3\theta}{2} \sin \frac{11\theta}{2} = \sin 2\theta \sin 5\theta.$
   (vii) $\cos \theta \cos \frac{\theta}{2} - \cos 3\theta \cos \frac{9\theta}{2} = \sin 7\theta \sin 8\theta.$ [NCERT EXEMPLAR]

7. Prove that:
   (i) $\frac{\sin A + \sin 3A}{\cos A - \cos 3A} = \cot A$
   (ii) $\frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A} = \cot 8A$
   (iii) $\frac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{A-B}{2}$

(iv) $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan\left(\dfrac{A+B}{2}\right)\cot\left(\dfrac{A-B}{2}\right)$

(v) $\dfrac{\cos A + \cos B}{\cos B - \cos A} = \cot\left(\dfrac{A+B}{2}\right)\cot\left(\dfrac{A-B}{2}\right)$

8. Prove that:

(i) $\dfrac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$

(ii) $\dfrac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \dfrac{\cos 5A}{\cos 3A}$

(iii) $\dfrac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \cot 3A$ [NCERT]

(iv) $\dfrac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$

(v) $\dfrac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \cot 6A$

(vi) $\dfrac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \tan A$

(vii) $\dfrac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$

(viii) $\dfrac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A} = \tan 2A$

(ix) $\dfrac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \tan 5A$

(x) $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}$

(xi) $\dfrac{\sin(\theta+\phi) - 2\sin\theta + \sin(\theta-\phi)}{\cos(\theta+\phi) - 2\cos\theta + \cos(\theta-\phi)} = \tan\theta$

9. Prove that:

(i) $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\beta+\gamma}{2}\right)\sin\left(\dfrac{\gamma+\alpha}{2}\right)$

(ii) $\cos(A+B+C) + \cos(A-B+C) + \cos(A+B-C) + \cos(-A+B+C)$
$= 4\cos A \cos B \cos C$

LEVEL-2

10. If $\cos A + \cos B = \dfrac{1}{2}$ and $\sin A + \sin B = \dfrac{1}{4}$, prove that: $\tan\left(\dfrac{A+B}{2}\right) = \dfrac{1}{2}$

11. If $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$, prove that: $\tan A \tan B = \cot\dfrac{A+B}{2}$

12. If $\sin 2A = \lambda \sin 2B$, prove that: $\dfrac{\tan(A+B)}{\tan(A-B)} = \dfrac{\lambda+1}{\lambda-1}$

13. Prove that:

(i) $\dfrac{\cos(A+B+C)+\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)}{\sin(A+B+C)+\sin(-A+B+C)+\sin(A-B+C)-\sin(A+B-C)} = \cot C$

(ii) $\sin(B-C)\cos(A-D)+\sin(C-A)\cos(B-D)+\sin(A-B)\cos(C-D) = 0$

14. If $\dfrac{\cos(A-B)}{\cos(A+B)} + \dfrac{\cos(C+D)}{\cos(C-D)} = 0$, prove that $\tan A \tan B \tan C \tan D = -1$

15. If $\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$, prove that $\cot\alpha\cot\beta\cot\gamma = \cot\delta$

16. If $y \sin\phi = x\sin(2\theta+\phi)$, prove that $(x+y)\cot(\theta+\phi) = (y-x)\cot\theta$

17. If $\cos(A+B)\sin(C-D) = \cos(A-B)\sin(C+D)$, prove that $\tan A\tan B\tan C + \tan D = 0$

18. If $x\cos\theta = y\cos\left(\theta+\dfrac{2\pi}{3}\right) = z\cos\left(\theta+\dfrac{4\pi}{3}\right)$, prove that $xy+yz+zx=0$.

[NCERT EXEMPLAR]

19. If $m\sin\theta = n\sin(\theta+2\alpha)$, prove that $\tan(\theta+\alpha)\cot\alpha = \dfrac{m+n}{m-n}$. [NCERT EXEMPLAR]

ANSWERS

1. (i) $2\sin 8\theta\cos 4\theta$ (ii) $2\sin 2\theta\cos 3\theta$ (iii) $2\cos 10\theta\cos 2\theta$

(iv) $-2\sin 8\theta\sin 4\theta$ (v) $2\cos\left(\dfrac{\pi}{4}+\theta\right)\cos\left(\dfrac{\pi}{4}-3\theta\right)$

HINTS TO NCERT & SELECTED PROBLEMS

4. (i) LHS $=\cos\left(\dfrac{3\pi}{4}+x\right)-\cos\left(\dfrac{3\pi}{4}-x\right)$

$$=-2\sin\left\{\frac{\left(\frac{3\pi}{4}+x\right)+\left(\frac{3\pi}{4}-x\right)}{2}\right\}\sin\left\{\frac{\left(\frac{3\pi}{4}+x\right)-\left(\frac{3\pi}{4}-x\right)}{2}\right\}$$

$$=-2\sin\frac{3\pi}{4}\sin x = -2\times\frac{1}{\sqrt{2}}\sin x = -\sqrt{2}\sin x = \text{RHS}$$

(ii) LHS $=\cos\left(\dfrac{\pi}{4}+x\right)+\cos\left(\dfrac{\pi}{4}-x\right)$

$$=2\cos\left\{\frac{\left(\frac{\pi}{4}+x\right)+\left(\frac{\pi}{4}-x\right)}{2}\right\}\cos\left\{\frac{\left(\frac{\pi}{4}+x\right)-\left(\frac{\pi}{4}-x\right)}{2}\right\}$$

$$=2\cos\frac{\pi}{4}\cos x = 2\times\frac{1}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{RHS}$$

8. (iii) LHS $=\dfrac{\cos 4A+\cos 3A+\cos 2A}{\sin 4A+\sin 3A+\sin 2A}$

$$=\frac{(\cos 4A+\cos 2A)+\cos 3A}{(\sin 4A+\sin 2A)+\sin 3A}$$

$$=\frac{2\cos 3A\cos A+\cos 3A}{2\sin 3A\cos A+\sin 3A}=\frac{\cos 3A(2\cos A+1)}{\sin 3A(2\cos A+1)}=\cot 3A=\text{RHS}$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = \lambda \cos^2\left(\frac{\alpha-\beta}{2}\right)$, write the value of λ.
2. Write the value of $\sin\frac{\pi}{12}\sin\frac{5\pi}{12}$.
3. If $\sin A + \sin B = \alpha$ and $\cos A + \cos B = \beta$, then write the value of $\tan\left(\frac{A+B}{2}\right)$.
4. If $\cos A = m\cos B$, then write the value of $\cot\frac{A+B}{2}\cot\frac{A-B}{2}$.
5. Write the value of the expression $\frac{1-4\sin 10^\circ \sin 70^\circ}{2\sin 10^\circ}$.
6. If $A + B = \frac{\pi}{3}$ and $\cos A + \cos B = 1$, then find the value of $\cos\frac{A-B}{2}$.
7. Write the value of $\sin 12^\circ \sin 48^\circ \sin 54^\circ$.
8. If $\sin 2A = \lambda \sin 2B$, then write the value of $\frac{\lambda+1}{\lambda-1}$.
9. Write the value of $\frac{\sin A + \sin 3A}{\cos A + \cos 3A}$.
10. If $\cos(A+B)\sin(C-D) = \cos(A-B)\sin(C+D)$, then write the value $\tan A \tan B \tan C$.

ANSWERS

1. 4 2. $\frac{1}{2}$ 3. $\frac{\alpha}{\beta}$ 4. $\frac{1+m}{1-m}$ 5. 1 6. $\frac{1}{\sqrt{3}}$

7. $\frac{1}{8}$ 8. $\frac{\tan(A+B)}{\tan(A-B)}$ 9. $\tan 2A$ 10. $-\tan D$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. $\cos 40^\circ + \cos 80^\circ + \cos 160^\circ + \cos 240^\circ =$
   (a) 0 (b) 1 (c) 1/2 (d) −1/2
2. $\sin 163^\circ \cos 347^\circ + \sin 73^\circ \sin 167^\circ =$
   (a) 0 (b) 1/2 (c) 1 (d) none of these
3. If $\sin 2\theta + \sin 2\phi = \frac{1}{2}$ and $\cos 2\theta + \cos 2\phi = \frac{3}{2}$, then $\cos^2(\theta-\phi) =$
   (a) 3/8 (b) 5/8 (c) 3/4 (d) 5/4
4. The value of $\cos 52^\circ + \cos 68^\circ + \cos 172^\circ$ is
   (a) 0 (b) 1 (c) 2 (d) 3/2
5. The value of $\sin 78^\circ - \sin 66^\circ - \sin 42^\circ + \sin 6^\circ$ is
   (a) 1/2 (b) −1/2 (c) −1 (d) none of these
6. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha - \cos\beta = b$, then $\tan\frac{\alpha-\beta}{2} =$

(a) $-\frac{a}{b}$ (b) $-\frac{b}{a}$ (c) $\sqrt{a^2+b^2}$ (d) none of these

7. $\cos 35° + \cos 85° + \cos 155° =$

(a) 0 (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\cos 275°$

8. The value of $\sin 50° - \sin 70° + \sin 10°$ is equal to

(a) 1 (b) 0 (c) 1/2 (d) 2

9. $\sin 47° + \sin 61° - \sin 11° - \sin 25°$ is equal to

(a) $\sin 36°$ (b) $\cos 36°$ (c) $\sin 7°$ (d) $\cos 7°$

10. If $\cos A = m \cos B$, then $\cot \frac{A+B}{2} \cot \frac{B-A}{2} =$

(a) $\frac{m-1}{m+1}$ (b) $\frac{m+2}{m-2}$ (c) $\frac{m+1}{m-1}$ (d) none of these

11. If A, B, C are in A.P., then $\frac{\sin A - \sin C}{\cos C - \cos A} =$

(a) $\tan B$ (b) $\cot B$ (c) $\tan 2B$ (d) none of these

12. If $\sin(B+C-A)$, $\sin(C+A-B)$, $\sin(A+B-C)$ are in A.P., then $\cot A$, $\cot B$, $\cot C$ are in

(a) GP (b) HP (c) AP (d) none of these

13. If $\sin x + \sin y = \sqrt{3}(\cos y - \cos x)$, then $\sin 3x + \sin 3y =$

(a) $2 \sin 3x$ (b) 0 (c) 1 (d) none of these

14. If $\tan \alpha = \frac{x}{x+1}$ and $\tan \beta = \frac{1}{2x+1}$, then $\alpha + \beta$ is equal to

(a) $\pi/2$ (b) $\pi/3$ (c) $\pi/6$ (d) $\pi/4$

ANSWERS

1. (d) 2. (b) 3. (b) 4. (a) 5. (b) 6. (b) 7. (a) 8. (b)
9. (d) 10. (c) 11. (b) 12. (b) 13. (b) 14. (d)

SUMMARY

1. (i) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

(ii) $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

(iii) $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

(iv) $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

2. (i) $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$

(ii) $\sin C - \sin D = 2 \sin \frac{C-D}{2} \cos \frac{C+D}{2}$

(iii) $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

(iv) $\cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}$

CHAPTER 9

TRIGONOMETRIC RATIOS OF MULTIPLE AND SUBMULTIPLE ANGLES

9.1 INTRODUCTION

In this chapter, we intend to express the trigonometric ratios of multiple angles $2A, 3A, 4A, \ldots$ etc. in terms of the trigonometric ratios of angle A and the trigonometric ratios of angle A in terms of the trigonometric ratios of sub-multiple angles $A/2, A/3, A/4 \ldots$ etc. These results will be used to find the trigonometric ratios of some important angles viz. $18°, 36°, 54°, 7\frac{1}{2}°, 11\frac{1}{4}°$ etc.

9.2 TRIGONOMETRIC RATIOS OF ANGLE $2A$ IN TERMS OF THAT OF ANGLE A

THEOREM 1 *For the values of angle A for which the two sides are meaningful prove that:*

(i) $\sin 2A = 2 \sin A \cos A$ (ii) $\cos 2A = \cos^2 A - \sin^2 A$

(iii) $\cos 2A = 2\cos^2 A - 1$ or, $1 + \cos 2A = 2\cos^2 A$

(iv) $\cos 2A = 1 - 2\sin^2 A$ or, $1 - \cos 2A = 2\sin^2 A$

(v) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$ (vi) $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$

(vii) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

PROOF (i) We know that

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$\Rightarrow \sin 2A = \sin A \cos A + \cos A \sin A$ [Replacing B by A]

$\Rightarrow \sin 2A = 2 \sin A \cos A$

(ii) We know that

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$\Rightarrow \cos 2A = \cos A \cos A - \sin A \sin A$ [Replacing B by A]

$\Rightarrow \cos 2A = \cos^2 A - \sin^2 A$

(iii) We have,

$$\cos 2A = \cos^2 A - \sin^2 A$$

$\Rightarrow \cos 2A = \cos^2 A - (1 - \cos^2 A)$

$\Rightarrow \cos 2A = 2\cos^2 A - 1$

Again, $\cos 2A = 2\cos^2 A - 1$

$\Rightarrow 1 + \cos 2A = 2\cos^2 A$

(iv) We have,

$$\cos 2A = \cos^2 A - \sin^2 A$$

$\Rightarrow \cos 2A = (1 - \sin^2 A) - \sin^2 A$

$\Rightarrow \quad \cos 2A = 1 - 2\sin^2 A$

Again, $\cos 2A = 1 - 2\sin^2 A$

$\Rightarrow \quad 1 - \cos 2A = 2\sin^2 A$

(v) We know that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\Rightarrow \quad \tan 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A} \qquad \text{[Replacing } B \text{ by } A]$$

$$\Rightarrow \quad \tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

(vi) We have,

$$\sin 2A = 2\sin A \cos A$$

$$\Rightarrow \quad \sin 2A = \frac{2\sin A \cos A}{1}$$

$$\Rightarrow \quad \sin 2A = \frac{2\sin A \cos A}{\sin^2 A + \cos^2 A} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$\Rightarrow \quad \sin 2A = \frac{\dfrac{2\sin A \cos A}{\cos^2 A}}{\dfrac{\sin^2 A + \cos^2 A}{\cos^2 A}} \qquad \text{[Dividing Numerator and Denominator by } \cos^2 A]$$

$$\Rightarrow \quad \sin 2A = \frac{\dfrac{2\sin A}{\cos A}}{\dfrac{\sin^2 A}{\cos^2 A} + \dfrac{\cos^2 A}{\cos^2 A}}$$

$$\Rightarrow \quad \sin 2A = \frac{2\tan A}{1 + \tan^2 A}$$

(vii) We have,

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\Rightarrow \quad \cos 2A = \frac{\cos^2 A - \sin^2 A}{1}$$

$$\Rightarrow \quad \cos 2A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} \qquad \text{[Replacing 1 by } \cos^2 A + \sin^2 A]$$

$$\Rightarrow \quad \cos 2A = \frac{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}} \qquad \text{[Dividing Numerator and Denominator by } \cos^2 A]$$

$$\Rightarrow \quad \cos 2A = \frac{\dfrac{\cos^2 A}{\cos^2 A} - \dfrac{\sin^2 A}{\cos^2 A}}{\dfrac{\cos^2 A}{\cos^2 A} + \dfrac{\sin^2 A}{\cos^2 A}}$$

$$\Rightarrow \quad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

REMARK *In the above formulae it should be noted that the angle on the RHS is half of the angle on LHS.*

$$\therefore \quad \sin 60° = 2 \sin 30° \cos 30°, \cos 120° = \cos^2 60 - \sin^2 60°, \tan 60° = \frac{2 \tan 30°}{1 - \tan^2 30} \text{ etc.}$$

9.3 TRIGONOMETRIC RATIOS OF THE ANGLE A IN TERMS OF THAT OF ANGLE $\frac{A}{2}$

The relations in section 9.2 are true for all values of the angle A for which the two sides are meaningful. Replacing A by $A/2$ in the above relations, we obtain the following relations:

(i) $\sin A = 2 \sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$ (ii) $\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$

(iii) $\cos A = 2\cos^2\left(\frac{A}{2}\right) - 1$ or, $1 + \cos A = 2\cos^2\left(\frac{A}{2}\right)$

(iv) $\cos A = 1 - 2\sin^2\left(\frac{A}{2}\right)$ or, $1 - \cos A = 2\sin^2\left(\frac{A}{2}\right)$

(v) $\tan A = \dfrac{2\tan\left(\frac{A}{2}\right)}{1 - \tan^2\left(\frac{A}{2}\right)}$ (vi) $\sin A = \dfrac{2\tan\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)}$

(vii) $\cos A = \dfrac{1 - \tan^2\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)}$

9.4 TRIGONOMETRIC RATIOS OF THE ANGLE $A/2$ IN TERMS OF COS A

We have,

$$\cos A = 2\cos^2\frac{A}{2} - 1 \Rightarrow 2\cos^2\frac{A}{2} = 1 + \cos A \Rightarrow \cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$$

The sign on RHS depends upon the quadrant in which angle $\frac{A}{2}$ lies.

Also,

$$\cos A = 1 - 2\sin^2\frac{A}{2} \Rightarrow 2\sin^2\frac{A}{2} = 1 - \cos A \Rightarrow \sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$$

The sign on RHS depends upon the quadrant in which angle $\frac{A}{2}$ lies.

Now, $$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \pm\frac{\sqrt{\frac{1 - \cos A}{2}}}{\sqrt{\frac{1 + \cos A}{2}}} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

The sign on RHS depends upon the quadrant in which angle $\frac{A}{2}$ lies.

REMARK *These relations are very useful to find the trigonometric ratios of the angles* $22\frac{1}{2}^\circ$, $7\frac{1}{2}^\circ$, $11\frac{1}{2}^\circ$ *etc.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE VALUES OF $\sin 2A, \cos 2A, \tan 2A$ ETC WHEN VALUES OF $\sin A$ OR $\cos A$ OR $\tan A$ ARE GIVEN**

EXAMPLE 1 *If* $\sin A = \frac{3}{5}$, *where* $0° < A < 90°$, *find the values of* $\sin 2A$, $\cos 2A$, $\tan 2A$ *and* $\sin 4A$.

SOLUTION We have, $\sin A = \frac{3}{5}$, where $0° < A < 90°$.

$$\therefore \quad \cos^2 A = 1 - \sin^2 A$$

$$\Rightarrow \quad \cos A = +\sqrt{1-\sin^2 A} = \sqrt{1-\frac{9}{25}} = \frac{4}{5} \qquad [\because \cos A > 0 \text{ for } 0 < A < 90°]$$

$$\therefore \quad \tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$$

Now,

$$\sin 2A = 2\sin A\cos A = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$$

$$\cos 2A = 1 - 2\sin^2 A = 1 - 2 \times \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2 \times \frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{6}{4}}{1-\frac{9}{16}} = \frac{24}{7} \qquad \left[\because \tan A = \frac{3}{4}\right]$$

and, $$\sin 4A = 2\sin 2A\cos 2A = 2 \times \frac{24}{25} \times \frac{7}{25} = \frac{336}{625} \qquad \left[\because \sin 2A = \frac{24}{25} \text{ and } \cos 2A = \frac{7}{25}\right]$$

EXAMPLE 2 *If* $\tan \alpha = \frac{1}{7}$, $\sin \beta = \frac{1}{\sqrt{10}}$. *Prove that* $\alpha + 2\beta = \frac{\pi}{4}$, *where* $0 < \alpha < \frac{\pi}{2}$ *and* $0 < \beta < \frac{\pi}{2}$.

SOLUTION In order to prove that $\alpha + 2\beta = \frac{\pi}{4}$, it is sufficient to prove that $\tan(\alpha + 2\beta) = \tan\frac{\pi}{4} = 1$.

We have, $\sin \beta = \frac{1}{\sqrt{10}}$, where $0 < \beta < \frac{\pi}{2}$.

$$\therefore \quad \cos \beta = \sqrt{1-\sin^2 \beta} = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}} \text{ and, } \tan \beta = \frac{\sin \beta}{\cos \beta} = \frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}} = \frac{1}{3}$$

In order to find the value of $\tan(\alpha + 2\beta)$, we require the values of $\tan \alpha$ and $\tan 2\beta$. The values of $\tan \alpha$ is given. So, let us find $\tan 2\beta$.

Now, $$\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta}$$

$$\Rightarrow \quad \tan 2\beta = \frac{2 \times \frac{1}{3}}{1-\frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2 \times 9}{3 \times 8} = \frac{3}{4}$$

Thus, we have

$$\tan \alpha = \frac{1}{7} \text{ and } \tan 2\beta = \frac{3}{4}$$

$$\therefore \quad \tan(\alpha + 2\beta) = \frac{\tan\alpha + \tan 2\beta}{1 - \tan\alpha \tan 2\beta} = \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7} \times \frac{3}{4}} = \frac{4 + 21}{28 - 3} = 1$$

$$\Rightarrow \quad \alpha + 2\beta = \frac{\pi}{4}$$

Type II **ON PROVING RESULTS AND IDENTITIES BASED UPON THE FOLLOWING FORMULAE:**

$$\sin 2\theta = 2\sin\theta\cos\theta,\ 1 + \cos 2\theta = 2\cos^2\theta,\ 1 - \cos 2\theta = 2\sin^2\theta$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},\ 1 + \cos\theta = 2\cos^2\frac{\theta}{2},\ 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

EXAMPLE 3 *Prove that :*

(i) $\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$ (ii) $\dfrac{\sin 2\theta}{1 - \cos 2\theta} = \cot\theta$

(iii) $\dfrac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} = \cot\theta$ (iv) $\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \tan\dfrac{\theta}{2}$

(v) $\dfrac{\cos 2\theta}{1 + \sin 2\theta} = \tan\left(\dfrac{\pi}{4} - \theta\right)$ (vi) $\dfrac{\cos\theta}{1 + \sin\theta} = \tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$

SOLUTION (i) We have, $\sin 2\theta = 2\sin\theta\cos\theta$ and $1 + \cos 2\theta = 2\cos^2\theta$.

$$\therefore \quad \text{LHS} = \frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \tan\theta = \text{RHS}$$

(ii)
$$\text{LHS} = \frac{\sin 2\theta}{1 - \cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\sin^2\theta} = \cot\theta = \text{RHS}$$

(iii)
$$\text{LHS} = \frac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} = \frac{(1 + \cos 2\theta) + \sin 2\theta}{(1 - \cos 2\theta) + \sin 2\theta}$$

$$= \frac{2\cos^2\theta + 2\sin\theta\cos\theta}{2\sin^2\theta + 2\sin\theta\cos\theta} = \frac{2\cos\theta(\cos\theta + \sin\theta)}{2\sin\theta(\cos\theta + \sin\theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta = \text{RHS}$$

(iv)
$$\text{LHS} = \frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \frac{(1 - \cos\theta) + \sin\theta}{(1 + \cos\theta) + \sin\theta}$$

$$= \frac{2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)} = \tan\frac{\theta}{2} = \text{RHS}$$

(v)
$$\text{LHS} = \frac{\cos 2\theta}{1 + \sin 2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin\left(\frac{\pi}{2} - 2\theta\right)}{1 + \cos\left(\frac{\pi}{2} - 2\theta\right)} \qquad \left[\because \cos A = \sin\left(\frac{\pi}{2} - A\right),\ \sin A = \cos\left(\frac{\pi}{2} - A\right)\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin\left(\frac{\pi}{4} - \theta\right)\cos\left(\frac{\pi}{4} - \theta\right)}{2\cos^2\left(\frac{\pi}{4} - \theta\right)} \qquad \left[\because \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\ \&\ 1 + \cos A = 2\cos^2\frac{A}{2}\right]$$

$$\Rightarrow \quad \text{LHS} = \tan\left(\frac{\pi}{4}-\theta\right) = \text{RHS}$$

(vi) $$\text{LHS} = \frac{\cos\theta}{1+\sin\theta} = \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)} = \frac{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)} = \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \text{RHS}$$

EXAMPLE 4 *Show that:* $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta,\ 0<\theta<\frac{\pi}{8}$

SOLUTION We have,

$$\text{LHS} = \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}} \qquad \left[\because 1+\cos 8\theta = 2\cos^2\frac{8\theta}{2} = 2\cos^2 4\theta\right]$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+\sqrt{2+\sqrt{(4\cos^2 4\theta)}}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+\sqrt{2+2\cos 4\theta}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+\sqrt{2(1+\cos 4\theta)}}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+\sqrt{2(2\cos^2 2\theta)}} \qquad [\because 1+\cos 4\theta = 2\cos^2 2\theta]$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2+2\cos 2\theta} = \sqrt{2(1+\cos 2\theta)} = \sqrt{2(2\cos^2\theta)} = 2\cos\theta = \text{RHS}$$

EXAMPLE 5 *Prove that:* $\cos 4x = 1-8\sin^2 x\cos^2 x$. [NCERT]

SOLUTION We have,

$$\text{LHS} = \cos 4x = \cos 2(2x) = 1-2\sin^2 2x = 1-2(\sin 2x)^2$$
$$= 1-2(2\sin x\cos x)^2 = 1-8\sin^2 x\cos^2 x = \text{RHS}$$

EXAMPLE 6 *Prove that:* $(\cos A+\cos B)^2+(\sin A-\sin B)^2 = 4\cos^2\left(\frac{A+B}{2}\right)$.

SOLUTION We have,

$$\text{LHS} = (\cos A+\cos B)^2+(\sin A-\sin B)^2$$

$$\Rightarrow \quad \text{LHS} = (\cos^2 A+\cos^2 B+2\cos A\cos B)+(\sin^2 A+\sin^2 B-2\sin A\sin B)$$

$$\Rightarrow \quad \text{LHS} = (\cos^2 A+\sin^2 A)+(\cos^2 B+\sin^2 B)+2(\cos A\cos B-\sin A\sin B)$$

$$\Rightarrow \quad \text{LHS} = 1+1+2\cos(A+B)$$

$$\Rightarrow \quad \text{LHS} = 2+2\cos(A+B)$$

$$\Rightarrow \quad \text{LHS} = 2\{1+\cos(A+B)\}$$

$$\Rightarrow \quad \text{LHS} = 2\times 2\cos^2\left(\frac{A+B}{2}\right) \qquad \left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right]$$

$$\Rightarrow \quad \text{LHS} = 4\cos^2\left(\frac{A+B}{2}\right) = \text{RHS}$$

EXAMPLE 7 *Prove that:* $\frac{\sec 8\theta-1}{\sec 4\theta-1} = \frac{\tan 8\theta}{\tan 2\theta}$ [NCERT EXEMPLAR]

SOLUTION We have,

$$\text{LHS} = \frac{\sec 8\theta - 1}{\sec 4\theta - 1}$$

$$\Rightarrow \text{LHS} = \frac{\frac{1}{\cos 8\theta} - 1}{\frac{1}{\cos 4\theta} - 1} = \frac{1 - \cos 8\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{1 - \cos 4\theta}$$

$$\Rightarrow \text{LHS} = \frac{2\sin^2 4\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{2\sin^2 2\theta} \qquad \left[\begin{array}{l} \because \quad 1 - \cos 8\theta = 2\sin^2 \frac{8\theta}{2} = 2\sin^2 4\theta \\ \text{and, } 1 - \cos 4\theta = 2\sin^2 \frac{4\theta}{2} = 2\sin^2 2\theta \end{array}\right]$$

$$\Rightarrow \text{LHS} = \frac{(2\sin 4\theta \cos 4\theta)}{\cos 8\theta} \times \frac{\sin 4\theta}{2\sin^2 2\theta}$$

$$\Rightarrow \text{LHS} = \left(\frac{2\sin 4\theta \cos 4\theta}{\cos 8\theta}\right) \times \left(\frac{2\sin 2\theta \cos 2\theta}{2\sin^2 2\theta}\right)$$

$$\Rightarrow \text{LHS} = \left(\frac{\sin 2(4\theta)}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right) = \left(\frac{\sin 8\theta}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right) = \tan 8\theta \cot 2\theta = \frac{\tan 8\theta}{\tan 2\theta} = \text{RHS}$$

EXAMPLE 8 *Prove that:* $\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right) = \frac{1}{8}$

[NCERT EXEMPLAR]

SOLUTION We observe that

$$\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8} \quad \text{and,} \quad \cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$$

$$\therefore \text{LHS} = \left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right)$$

$$\Rightarrow \text{LHS} = \left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)$$

$$\Rightarrow \text{LHS} = \left\{\left(1 + \cos\frac{\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)\right\}\left\{\left(1 + \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{3\pi}{8}\right)\right\}$$

$$\Rightarrow \text{LHS} = \left(1 - \cos^2\frac{\pi}{8}\right)\left(1 - \cos^2\frac{3\pi}{8}\right)$$

$$\Rightarrow \text{LHS} = \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left(2\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right)$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{\left(1 - \cos\frac{\pi}{4}\right)\left(1 - \cos\frac{3\pi}{4}\right)\right\} \qquad [\because 2\sin^2\theta = 1 - \cos 2\theta]$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}}\right)\right\} = \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8} = \text{RHS}$$

EXAMPLE 9 *Prove that:*

(i) $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}$ **[NCERT EXEMPLAR]**

(ii) $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \frac{3}{2}$ [illegible]

SOLUTION (i) We know that

$$\frac{7\pi}{8} = \pi - \frac{\pi}{8} \text{ and } \frac{5\pi}{8} = \pi - \frac{3\pi}{8}$$

$$\therefore \cos\frac{7\pi}{8} = -\cos\frac{\pi}{8} \text{ and } \cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8} \qquad [\because \cos(\pi-\theta) = -\cos\theta]$$

$$\Rightarrow \cos^4\frac{7\pi}{8} = \cos^4\frac{\pi}{8} \text{ and } \cos^4\frac{5\pi}{8} = \cos^4\frac{3\pi}{8}$$

$$\therefore \text{ LHS} = \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$$

$$\Rightarrow \text{LHS} = \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{\pi}{8}$$

$$\Rightarrow \text{LHS} = 2\cos^4\frac{\pi}{8} + 2\cos^4\frac{3\pi}{8}$$

$$\Rightarrow \text{LHS} = 2\left\{\left(\cos^2\frac{\pi}{8}\right)^2 + \left(\cos^2\frac{3\pi}{8}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = 2\left[\left\{\frac{1+\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1+\cos\frac{3\pi}{4}}{2}\right\}^2\right] \qquad \left[\because \frac{1+\cos 2\theta}{2} = \cos^2\theta\right]$$

$$\Rightarrow \text{LHS} = \frac{2}{4}\left\{\left(1+\cos\frac{\pi}{4}\right)^2 + \left(1+\cos\frac{3\pi}{4}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = \frac{2}{4}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right\} = \frac{1}{2}\left\{\left(1+\frac{1}{2}+\sqrt{2}\right)+\left(1+\frac{1}{2}-\sqrt{2}\right)\right\} = \frac{3}{2} = \text{RHS}$$

(ii) We observe that

$$\sin^4\frac{7\pi}{8} = \sin^4\left(\pi-\frac{\pi}{8}\right) = \sin^4\frac{\pi}{8} \text{ and, } \sin^4\frac{5\pi}{8} = \sin^4\left(\pi-\frac{3\pi}{8}\right) = \sin^4\frac{3\pi}{8}$$

$$\therefore \text{ LHS} = \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8}$$

$$\Rightarrow \text{LHS} = \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{\pi}{8}$$

$$\Rightarrow \text{LHS} = 2\left\{\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8}\right\}$$

$$\Rightarrow \text{LHS} = 2\left\{\left(\sin^2\frac{\pi}{8}\right)^2 + \left(\sin^2\frac{3\pi}{8}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = 2\left[\left\{\frac{1-\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1-\cos\frac{3\pi}{4}}{2}\right\}^2\right] \qquad \left[\because \frac{1-\cos 2\theta}{2} = \sin^2\theta\right]$$

$$\Rightarrow \text{LHS} = \frac{2}{4}\left\{\left(1-\cos\frac{\pi}{4}\right)^2 + \left(1-\cos\frac{3\pi}{4}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2\right\} = \frac{1}{2}\left\{\left(1+\frac{1}{2}-\sqrt{2}\right)+\left(1+\frac{1}{2}+\sqrt{2}\right)\right\} = \frac{3}{2} = \text{RHS}$$

EXAMPLE 10 *Prove that:*

(i) $\cos^2 A + \cos^2\left(A + \frac{2\pi}{3}\right) + \cos^2\left(A - \frac{2\pi}{3}\right) = \frac{3}{2}$

(ii) $\cos^2 A + \cos^2\left(A + \frac{\pi}{3}\right) + \cos^2\left(A - \frac{\pi}{3}\right) = \frac{3}{2}$ **[NCERT EXEMPLAR]**

SOLUTION (i) We have,

$$\text{LHS} = \cos^2 A + \cos^2\left(A + \frac{2\pi}{3}\right) + \cos^2\left(A - \frac{2\pi}{3}\right)$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{2\cos^2 A + 2\cos^2\left(A + \frac{2\pi}{3}\right) + 2\cos^2\left(A - \frac{2\pi}{3}\right)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[1 + \cos 2A + \left\{1 + \cos 2\left(A + \frac{2\pi}{3}\right)\right\} + \left\{1 + \cos 2\left(A - \frac{2\pi}{3}\right)\right\}\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[1 + \cos 2A + 1 + \cos\left(2A + \frac{4\pi}{3}\right) + 1 + \cos\left(2A - \frac{4\pi}{3}\right)\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[3 + \cos 2A + \left\{\cos\left(2A + \frac{4\pi}{3}\right) + \cos\left(2A - \frac{4\pi}{3}\right)\right\}\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[3 + \cos 2A + 2\cos 2A \cos\frac{4\pi}{3}\right] \qquad [\because \cos(A+B) + \cos(A-B) = 2\cos A\cos B]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[3 + \cos 2A + 2(\cos 2A)\left(-\frac{1}{2}\right)\right] = \frac{1}{2}(3 + \cos 2A - \cos 2A) = \frac{3}{2} = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \cos^2 A + \cos^2\left(A + \frac{\pi}{3}\right) + \cos^2\left(A - \frac{\pi}{3}\right)$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{2\cos^2 A + 2\cos^2\left(A + \frac{\pi}{3}\right) + 2\cos^2\left(A - \frac{\pi}{3}\right)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{(1 + \cos 2A) + 1 + \cos\left(2A + \frac{2\pi}{3}\right) + 1 + \cos\left(2A - \frac{2\pi}{3}\right)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[3 + \cos 2A + \left\{\cos\left(2A + \frac{2\pi}{3}\right) + \cos\left(2A - \frac{2\pi}{3}\right)\right\}\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{3 + \cos 2A + 2\cos 2A \cos\frac{2\pi}{3}\right\} \qquad [\because \cos(A+B) + \cos(A-B) = 2\cos A\cos B]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{3 + \cos 2A + 2(\cos 2A) \times -\frac{1}{2}\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\{3 + \cos 2A - \cos 2A\} = \frac{3}{2} = \text{RHS}$$

EXAMPLE 11 *Prove that:*

(i) $\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$ **[NCERT]**

(ii) $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$ **[NCERT]**

(iii) $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$ **[NCERT]**

SOLUTION (i) We have,

$$\text{LHS} = \frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\sin 5x + \sin x) - 2\sin 3x}{\cos 5x - \cos x}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) - 2\sin 3x}{-2\sin\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right)} = \frac{2\sin 3x\cos 2x - 2\sin 3x}{-2\sin 3x\sin 2x}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin 3x(1-\cos 2x)}{-2\sin 3x\sin 2x} = \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin\cos x} = \tan x = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \sin 2x + 2\sin 4x + \sin 6x$$

$$\Rightarrow \quad \text{LHS} = (\sin 6x + \sin 2x) + 2\sin 4x$$

$$\Rightarrow \quad \text{LHS} = 2\sin\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right) + 2\sin 4x$$

$$\Rightarrow \quad \text{LHS} = 2\sin 4x\cos 2x + 2\sin 4x$$

$$\Rightarrow \quad \text{LHS} = 2\sin 4x(\cos 2x + 1) = 2\sin 4x \times 2\cos^2 x = 4\cos^2 x\sin 4x = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = \frac{2\sin\left(\frac{x-3x}{2}\right)\cos\left(\frac{x+3x}{2}\right)}{-(\cos^2 x - \sin^2 x)}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin(-x)\cos 2x}{-\cos 2x} = \frac{-2\sin x\cos 2x}{-\cos 2x} = 2\sin x = \text{RHS}$$

EXAMPLE 12 *Show that* : $2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta) = \cos 2\alpha$ [NCERT EXEMPLAR]

SOLUTION We have,

$$\text{LHS} = 2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$$

$$\Rightarrow \quad \text{LHS} = 2\sin^2\beta + 2\cos(\alpha+\beta)(2\sin\alpha\sin\beta) + \cos 2(\alpha+\beta)$$

$$\Rightarrow \quad \text{LHS} = 2\sin^2\beta + 2\cos(\alpha+\beta)\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\} + \cos 2(\alpha+\beta)$$

$$\Rightarrow \quad \text{LHS} = 2\sin^2\beta + 2\cos(\alpha+\beta)\cos(\alpha-\beta) - 2\cos^2(\alpha+\beta) + \cos 2(\alpha+\beta)$$

$$\Rightarrow \quad \text{LHS} = 2\sin^2\beta + 2(\cos^2\alpha - \sin^2\beta) - 2\cos^2(\alpha+\beta) + 2\cos^2(\alpha+\beta) - 1$$

$$\Rightarrow \quad \text{LHS} = 2\sin^2\beta + 2\cos^2\alpha - 2\sin^2\beta - 2\cos^2(\alpha+\beta) + 2\cos^2(\alpha+\beta) - 1$$

$$\Rightarrow \quad \text{LHS} = 2\cos^2\alpha - 1 = \cos 2\alpha = \text{RHS}$$

EXAMPLE 13 *Show that* $\sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ = 4$ [NCERT EXEMPLAR]

SOLUTION We have,

$$\text{LHS} = \sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ$$

$\Rightarrow$ LHS = [illegible]

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\left\{\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right\}}{\sin 20^\circ \cos 20^\circ}$$

$$\Rightarrow \quad \text{LHS} = \frac{2(\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ)}{\sin 20^\circ \cos 20^\circ}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sin(60^\circ - 20^\circ)}{\sin 20^\circ \cos 20^\circ} = \frac{2\sin 40^\circ}{\sin 20^\circ \cos 20^\circ} = \frac{4\sin 40^\circ}{2\sin 20^\circ \cos 20^\circ} = \frac{4\sin 40^\circ}{\sin 40^\circ} = 4 = \text{RHS}$$

EXAMPLE 14 *Prove that:* $\tan 4\theta = \dfrac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}$ [NCERT]

SOLUTION We have,

$$\text{LHS} = \tan 4\theta = \tan\Big(2(2\theta)\Big)$$

$$\Rightarrow \quad \text{LHS} = \frac{2\tan 2\theta}{1-\tan^2 2\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)}{1-\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)^2}$$

$$\Rightarrow \quad \text{LHS} = \frac{4\tan\theta(1-\tan^2\theta)}{(1-\tan^2\theta)^2 - 4\tan^2\theta} = \frac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta} = \text{RHS}$$

Type III ON FINDING THE VALUES OF $\sin\frac{x}{2}, \cos\frac{x}{2}$ AND $\tan\frac{x}{2}$ WHEN VALUES OF $\sin x$ OR $\cos x$ OR $\tan x$ ARE GIVEN

EXAMPLE 15 *If* $0 \le x \le 2\pi$, *find* $\sin\frac{x}{2}$, $\cos\frac{x}{2}$, *and* $\tan\frac{x}{2}$, *when:*

(i) $\tan x = -\frac{4}{3}$, *x lies in quadrant II* (ii) $\cos x = -\frac{1}{3}$, *x lies in quadrant III*

(iii) $\sin x = -\frac{1}{2}$, *x lies in quadrant IV.*

SOLUTION (i) It is given that x lies in IInd quadrant in which $\cos x$ is negative.

$$\therefore \quad \cos x = -\frac{1}{\sqrt{1+\tan^2 x}} = -\frac{1}{\sqrt{1+16/9}} = -\frac{3}{5}$$

It is given that x lies in IInd quadrant.

i.e. $\quad \frac{\pi}{2} < x < \pi$

$\Rightarrow \quad \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

$\Rightarrow \quad \frac{x}{2}$ lies in first quadrant $\Rightarrow \sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are positive

$$\therefore \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \Rightarrow \cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} = \sqrt{\frac{1-3/5}{2}} = \frac{1}{\sqrt{5}}$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \Rightarrow \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1+3/5}{2}} = \frac{2}{\sqrt{5}}$$

and, $$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \Rightarrow \tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{1+3/5}{1-3/5}} = 2$$

(ii) It is given that x lies in the III quadrant.

i.e. $\pi < x < \frac{3\pi}{2}$

$\Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$

$\Rightarrow \frac{x}{2}$ lies in IInd quadrant $\Rightarrow \cos\frac{x}{2} < 0$, $\sin\frac{x}{2} > 0$ and $\tan\frac{x}{2} < 0$

$$\therefore \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\Rightarrow \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} \qquad \left[\because \cos\frac{x}{2} \text{ is} - \text{ve}\right]$$

$$\Rightarrow \quad \cos\frac{x}{2} = -\sqrt{\frac{1-1/3}{2}} = -\frac{1}{\sqrt{3}} \qquad \left[\because \cos x = -\frac{1}{3}\right]$$

and, $$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\Rightarrow \quad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} \qquad \left[\because \sin\frac{x}{2} > 0\right]$$

$$\Rightarrow \quad \sin\frac{x}{2} = \sqrt{\frac{1+1/3}{2}} = \sqrt{\frac{2}{3}} \qquad \left[\because \cos x = \frac{1}{3}\right]$$

and, $$\tan\frac{x}{2} = \frac{\sin x/2}{\cos x/2} = \sqrt{\frac{2}{3}} \times -\sqrt{3} = -\sqrt{2}$$

(iii) It is given that x lies in IVth quadrant in which $\cos x$ is positive.

$$\therefore \quad \sin x = -\frac{1}{2} \Rightarrow \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$$

Now,

x lies in IVth quadrant

$\Rightarrow \frac{3\pi}{2} < x < 2\pi$

$\Rightarrow \frac{3\pi}{4} < \frac{x}{2} < \pi$

$\Rightarrow \frac{x}{2}$ lies in IInd quadrant $\Rightarrow \cos\frac{x}{2} < 0$, $\sin\frac{x}{2} > 0$ and $\tan\frac{x}{2} < 0$

$$\therefore \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\Rightarrow \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} \qquad \left[\because \cos\frac{x}{2} < 0\right]$$

$$\Rightarrow \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\sqrt{3}/2}{2}} = -\frac{\sqrt{2+\sqrt{3}}}{2} \qquad \left[\because \cos x = \frac{\sqrt{3}}{2}\right]$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\Rightarrow \quad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} \qquad \left[\because \sin\frac{x}{2} > 0\right]$$

$$\Rightarrow \quad \sin\frac{x}{2} = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2} \qquad \left[\because \cos x = \frac{\sqrt{3}}{2}\right]$$

and, $$\tan\frac{x}{2} = \frac{\sin(x/2)}{\cos(x/2)} = \frac{\sqrt{2-\sqrt{3}}}{2} \times \frac{-2}{\sqrt{2+\sqrt{3}}} = -\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$$

EXAMPLE 16 *If* $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$, *find the values of* $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$. [NCERT]

SOLUTION It is given that

$$\tan x = \frac{3}{4} \text{ and, } \pi < x < \frac{3\pi}{2}$$

$$\therefore \quad \cos x = -\frac{1}{\sqrt{1+\tan^2 x}} \qquad \left[\because \pi < x < \frac{3\pi}{2} \therefore \cos x \text{ is negative}\right]$$

$$\Rightarrow \quad \cos x = -\frac{1}{\sqrt{1+\frac{9}{16}}} = -\frac{4}{5}$$

$$\because \quad \pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \Rightarrow \cos x < 0 \text{ and } \sin x > 0$$

$$\therefore \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} \text{ and, } \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$$

$$\Rightarrow \quad \cos\frac{x}{2} = -\sqrt{\frac{1-4/5}{2}} \text{ and, } \sin\frac{x}{2} = \sqrt{\frac{1+\frac{4}{5}}{2}}$$

$$\Rightarrow \quad \cos\frac{x}{2} = -\frac{1}{\sqrt{10}} \text{ and, } \sin\frac{x}{2} = \frac{3}{\sqrt{10}}$$

$$\therefore \quad \tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{3/\sqrt{10}}{-1/\sqrt{10}} = -3$$

Hence, $\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$, $\sin\frac{x}{2} = \frac{3}{\sqrt{10}}$ and $\tan\frac{x}{2} = -3$

Type IV ON FINDING THE VALUES TRIGONOMETRICAL FUNCTIONS FOR $\frac{\pi}{24}, \frac{\pi}{16}, \frac{\pi}{8}$

Formulae: $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$, $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$

EXAMPLE 17 *Find the values of*

(i) $\cos\frac{\pi}{8}$ (ii) $\sin\frac{\pi}{8}$ (iii) $\tan\frac{\pi}{8}$ [NCERT] (iv) $\sin\frac{\pi}{24}$ (v) $\cos\frac{\pi}{24}$

SOLUTION (i) We know that $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$

Putting $A = \frac{\pi}{4}$, we get

$$\cos\frac{\pi}{8} = \sqrt{\frac{1+\cos\pi/4}{2}} \qquad \left[\because \cos\frac{\pi}{8} \text{ is} + \text{ve}\right]$$

$$\Rightarrow \quad \cos\frac{\pi}{8} = \sqrt{\frac{1+1/\sqrt{2}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

(ii) We have,

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

Putting $A = \frac{\pi}{4}$, we get

$$\sin\frac{\pi}{8} = \sqrt{\frac{1-\cos\pi/4}{2}} \qquad \left[\because \sin\frac{\pi}{8} \text{ is} + \text{ve}\right]$$

$$\Rightarrow \quad \sin\frac{\pi}{8} = \sqrt{\frac{1-1/\sqrt{2}}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

(iii) We have,

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

Putting $A = \frac{\pi}{4}$, we get

$$\tan\frac{\pi}{8} = \sqrt{\frac{1-\cos\pi/4}{1+\cos\pi/4}} \qquad \left[\because \tan\frac{\pi}{8} \text{ is} + \text{ve}\right]$$

$$\Rightarrow \quad \tan\frac{\pi}{8} = \sqrt{\frac{1-1/\sqrt{2}}{1+1/\sqrt{2}}} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}} = \sqrt{2}-1$$

(iv) We observe that

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6}+\sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Putting $A = \frac{\pi}{12}$ in $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, we get

$$\sin\frac{\pi}{24} = \sqrt{\frac{1-\cos\frac{\pi}{12}}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{2}} = \sqrt{\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}} = \sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{8}} = \frac{\sqrt{4-\sqrt{6}-\sqrt{2}}}{2\sqrt{2}}$$

(v) Putting $A = \frac{\pi}{12}$ in $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$, we get

$$\cos\frac{\pi}{24} = \sqrt{\frac{1+\cos\frac{\pi}{12}}{2}}$$

$$\Rightarrow \quad \cos\frac{\pi}{24} = \sqrt{\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{2}} \qquad \left[\because \cos\frac{\pi}{24} = \cos 7\frac{1}{2}^{\circ} \text{ is} + \text{ve}\right]$$

$$\Rightarrow \quad \cos\frac{\pi}{24} = \sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}} = \sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}} = \frac{\sqrt{4+\sqrt{6}+\sqrt{2}}}{2\sqrt{2}}$$

EXAMPLE 18 *Prove that:*

(i) $\cot \dfrac{\pi}{24} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$ (ii) $\tan \dfrac{\pi}{16} = \sqrt{4 + 2\sqrt{2}} - (\sqrt{2} + 1)$

(iii) $\tan 142 \dfrac{1^\circ}{2} = 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}$

SOLUTION (i) We have,

$$\text{LHS} = \cot \frac{\pi}{24}$$

$$\Rightarrow \quad \text{LHS} = \frac{\cos \frac{\pi}{24}}{\sin \frac{\pi}{24}} = \frac{2 \cos \frac{\pi}{24} \cos \frac{\pi}{24}}{2 \sin \frac{\pi}{24} \cos \frac{\pi}{24}} = \frac{2 \cos^2 \frac{\pi}{24}}{2 \sin \frac{\pi}{24} \cos \frac{\pi}{24}} = \frac{1 + \cos \frac{\pi}{12}}{\sin \frac{\pi}{12}}$$

$$\Rightarrow \quad \text{LHS} = \frac{1 + \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)} = \frac{1 + \frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1} = \frac{(2\sqrt{2} + \sqrt{3} + 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$$

$$\Rightarrow \quad \text{LHS} = \frac{2\sqrt{6} + 2\sqrt{2} + 3 + \sqrt{3} + \sqrt{3} + 1}{3 - 1} = \frac{2\sqrt{6} + 2\sqrt{2} + 2\sqrt{3} + 4}{2}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{2} + \sqrt{3} + 2 + \sqrt{6} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6} = \text{RHS}$$

(ii)
$$\text{LHS} = \tan \frac{\pi}{16} = \frac{\sin \frac{\pi}{16}}{\cos \frac{\pi}{16}} = \frac{\sin \frac{\pi}{16}}{\cos \frac{\pi}{16}} \times \frac{2 \sin \frac{\pi}{16}}{2 \sin \frac{\pi}{16}} = \frac{2 \sin^2 \frac{\pi}{16}}{2 \sin \frac{\pi}{16} \cos \frac{\pi}{16}} = \frac{1 - \cos \frac{\pi}{8}}{\sin \frac{\pi}{8}}$$

$$\Rightarrow \quad \text{LHS} = \frac{1 - \sqrt{\frac{1 + \cos \frac{\pi}{4}}{2}}}{\sqrt{\frac{1 - \cos \frac{\pi}{4}}{2}}} \qquad \left[\because \cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos\theta}{2}} \text{ and } \sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{2}}\right]$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{2} - \sqrt{1 + \cos \frac{\pi}{4}}}{\sqrt{1 - \cos \frac{\pi}{4}}} = \frac{\sqrt{2} - \sqrt{1 + \frac{1}{\sqrt{2}}}}{\sqrt{1 - \frac{1}{\sqrt{2}}}} = \frac{\sqrt{2} - \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}}}}{\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}}}}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{2\sqrt{2}} - \sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}-1}} = \frac{\sqrt{2\sqrt{2}} - \sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}-1}} \times \frac{\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}+1}}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{2\sqrt{2}}\sqrt{\sqrt{2}+1} - \sqrt{(\sqrt{2}+1)^2}}{\sqrt{(\sqrt{2}+1)(\sqrt{2}-1)}} = \frac{\sqrt{2\sqrt{2}(\sqrt{2}+1)} - (\sqrt{2}+1)}{2 - 1}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{4 + 2\sqrt{2}} - (\sqrt{2} + 1) = \text{RHS}$$

(iii)
$$\text{LHS} = \tan 142 \frac{1}{2}^\circ = \tan\left(180^\circ = 37 \frac{1}{2}^\circ\right) = \tan 37 \frac{1}{2}^\circ = -\tan \frac{5\pi}{24}$$

$$\Rightarrow \quad \text{LHS} = -\frac{\sin\frac{5\pi}{24}}{\cos\frac{5\pi}{24}} = -\frac{2\sin^2\frac{5\pi}{24}}{2\sin\frac{5\pi}{24}\cos\frac{5\pi}{24}} = -\frac{1-\cos\frac{5\pi}{12}}{\sin\frac{5\pi}{12}}$$

$$\Rightarrow \quad \text{LHS} = -\frac{1-\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)} = -\frac{1-\left(\cos\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\sin\frac{\pi}{6}\right)}{\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}}$$

$$\Rightarrow \quad \text{LHS} = \left(\frac{1-\frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}\right) = -\left(\frac{2\sqrt{2}-\sqrt{3}+1}{\sqrt{3}+1}\right) = -\frac{(2\sqrt{2}-\sqrt{3}+1)}{(\sqrt{3}+1)} \times \frac{(\sqrt{3}-1)}{\sqrt{3}-1}$$

$$\Rightarrow \quad \text{LHS} = -\left\{\frac{(2\sqrt{2}-\sqrt{3}+1)(\sqrt{3}-1)}{3-1}\right\} = -\left\{\frac{2\sqrt{2}(\sqrt{3}-1)-(\sqrt{3}-1)^2}{2}\right\}$$

$$\Rightarrow \quad \text{LHS} = -\left\{\frac{2\sqrt{2}(\sqrt{3}-1)-(3+1-2\sqrt{3})}{2}\right\} = -\left\{\sqrt{2}(\sqrt{3}-1) - (2-\sqrt{3})\right\}$$

$$\Rightarrow \quad \text{LHS} = -\sqrt{6}+\sqrt{2}+2-\sqrt{3} = 2+\sqrt{2}-\sqrt{3}-\sqrt{6} = \text{RHS}$$

ILLUSTRATIVE EXAMPLES

LEVEL-2

AN IMPORTANT RESULT *Prove that:* $\cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A = \dfrac{\sin 2^n A}{2^n \sin A}$

<u>PROOF</u> $\text{LHS} = \cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A$

$$= \frac{1}{2\sin A}\left\{(2\sin A\cos A)\cos 2A\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1}A\right\}$$

$$= \frac{1}{2\sin A}\left\{\sin 2A\cos 2A\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1}A\right\}$$

$$= \frac{1}{2^2\sin A}\left\{(2\sin 2A\cos 2A)\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1}A\right\}$$

$$= \frac{1}{2^2\sin A}\left\{\sin 2(2A)\cos 2^2 A\cos 2^3 A\ldots\cos 2^{n-1}A\right\}$$

$$= \frac{1}{2^3\sin A}\left\{(2\sin 2^2 A\cos 2^2 A)\cos 2^3 A\ldots\cos 2^{n-1}A\right\}$$

$$= \frac{1}{2^3\sin A}\left\{\sin(2\times 2^2 A)\cos 2^3 A\ldots\cos 2^{n-1}A\right\}$$

$$\ldots A\cos 2^3 A\cos 2^4 A\ldots\cos 2^{n-1}A\}$$

$$\ldots \sin 2^{n-1}A\cos 2^{n-1}A\} = \frac{1}{2^n\sin A}\left\{2\sin 2^{n-1}A\cos 2^{n-1}A\right\}$$

$$\ldots = \text{RHS}$$

Type V PROBLEMS BASED ON THE FORMULA

$$\cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$$

EXAMPLE 19 *If* $\theta = \dfrac{\pi}{2^n + 1}$, *prove that:* $2^n \cos\theta \cos 2\theta \cos 2^2 \theta \ldots \cos 2^{n-1}\theta = 1.$

SOLUTION We have,

$$\theta = \frac{\pi}{2^n + 1} \Rightarrow 2^n \theta + \theta = \pi \quad \Rightarrow 2^n \theta = \pi - \theta$$

$$\therefore \quad 2^n \cos\theta \cos 2\theta \cos 2^2 \theta \ldots \cos 2^{n-1}\theta$$

$$= 2^n \left\{\frac{\sin 2^n \theta}{2^n \sin\theta}\right\} = \frac{\sin 2^n \theta}{\sin\theta} = \frac{\sin(\pi - \theta)}{\sin\theta} = 1 \qquad [\because\ 2^n\theta = \pi - \theta]$$

EXAMPLE 20 *Prove that:* $\cos 20° \cos 40° \cos 60° \cos 80° = \dfrac{1}{16}$

SOLUTION We have,

$$\text{LHS} = \cos 20° \cos 40° \cos 60° \cos 80°$$

$$\Rightarrow \text{LHS} = \frac{1}{2}(\cos 20° \cos 40° \cos 80°)$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left(\cos A \cos 2A \cos 2^2 A\right), \quad \text{where} \quad A = 20°$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left(\frac{\sin 2^3 A}{2^3 \sin A}\right) = \frac{1}{2^4}\frac{\sin 8A}{\sin A}$$

$$\Rightarrow \text{LHS} = \frac{1}{2^4}\frac{\sin 160°}{\sin 20°} = \frac{1}{2^4}\frac{\sin(180° - 20°)}{\sin 20°} = \frac{1}{2^4}\frac{\sin 20°}{\sin 20°} = \frac{1}{2^4} = \frac{1}{16} = \text{RHS}$$

EXAMPLE 21 *Prove that:* $\cos\dfrac{\pi}{7}\cos\dfrac{2\pi}{7}\cos\dfrac{4\pi}{7} = -\dfrac{1}{8}.$

SOLUTION We have,

$$\text{LHS} = \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$$

$$\Rightarrow \text{LHS} = \cos A \cos 2A \cos 2^2 A, \text{ where } A = \frac{\pi}{7}$$

$$\Rightarrow \text{LHS} = \frac{\sin 2^3 A}{2^3 \sin A}$$

$$\Rightarrow \text{LHS} = \frac{\sin 8\frac{\pi}{7}}{8\sin\frac{\pi}{7}} \; \frac{\sin\left(\pi + \frac{\pi}{7}\right)}{8\sin\frac{\pi}{7}} = -\frac{\sin\frac{\pi}{7}}{8\sin\frac{\pi}{7}} = -\frac{1}{8} = \text{RHS}$$

EXAMPLE 22 *Prove that:* $\cos\dfrac{\pi}{7}\cos\dfrac{2\pi}{7}\cos\dfrac{3\pi}{7}\cos\dfrac{3\pi}{7} = \dfrac{1}{8}$

SOLUTION We have,

$$\text{LHS} = \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$$

$$\Rightarrow \text{LHS} = \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\left(\pi - \frac{4\pi}{7}\right)$$

$\Rightarrow$ LHS $= -\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\left(-\frac{1}{8}\right)$ [See Example 21]

$\Rightarrow$ LHS $= \frac{1}{8} =$ RHS

EXAMPLE 23 *Prove that:* $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15} = \frac{1}{16}$

SOLUTION We have,

$$\text{LHS} = \cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\left(\pi-\frac{\pi}{15}\right)$$

$$\Rightarrow \text{LHS} = \left(\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right)\left(-\cos\frac{\pi}{15}\right) \quad [\because \cos(\pi-\theta) = -\cos\theta]$$

$$\Rightarrow \text{LHS} = -\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}$$

$$\Rightarrow \text{LHS} = -\cos A\cos 2A\cos 2^2 A\cos 2^3 A, \quad \text{where } A = \frac{\pi}{15}$$

$$\Rightarrow \text{LHS} = -\frac{\sin 2^4 A}{2^4\sin A} = -\frac{\sin 16A}{2^4\sin A}$$

$$\text{LHS} = -\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}} = -\frac{\sin\left(\pi+\frac{\pi}{15}\right)}{16\sin\frac{\pi}{15}} = \frac{\sin\frac{\pi}{15}}{16\sin\frac{\pi}{15}} = \frac{1}{16} = \text{RHS}$$

EXAMPLE 24 *Prove that:*

$$\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14} = \frac{1}{64}$$

SOLUTION We know that

$$A+B=\pi \Rightarrow \sin A = \sin(\pi-B) = \sin B$$

$$\because \frac{\pi}{14}+\frac{13\pi}{14} = \pi, \quad \frac{3\pi}{14}+\frac{11\pi}{14} = \pi, \quad \frac{5\pi}{14}+\frac{9\pi}{14} = \pi$$

$$\therefore \sin\frac{\pi}{14} = \sin\frac{13\pi}{14}, \quad \sin\frac{3\pi}{14} = \sin\frac{11\pi}{14}, \quad \sin\frac{5\pi}{14} = \sin\frac{9\pi}{14}$$

$$\therefore \text{LHS} = \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$$

$$\Rightarrow \text{LHS LHS} = \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{3\pi}{14}\sin\frac{\pi}{14}$$

$$\Rightarrow \text{LHS} = \left\{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right\}^2 \times \sin\frac{7\pi}{14}$$

$$\Rightarrow \text{LHS} = \left\{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right\}^2 \times 1$$

$$\Rightarrow \text{LHS} = \left\{\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)\right\}^2$$

$$\Rightarrow \text{LHS} = \left\{\cos\frac{6\pi}{14}\cos\frac{4\pi}{14}\cos\frac{2\pi}{14}\right\}^2 = \left\{\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right\}^2 = \left(\frac{1}{8}\right)^2 = \frac{1}{64} \quad \text{[See Example 21]}$$

EXAMPLE 25 *Find the value of* $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$.

SOLUTION We have,

$$\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$$
$$=\cos\left(\frac{\pi}{2}-\frac{\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{7\pi}{18}\right)$$
$$=\cos\frac{4\pi}{9}\cos\frac{2\pi}{9}\cos\frac{\pi}{9}$$
$$=\frac{\sin\left(2^3\times\frac{\pi}{9}\right)}{2^3\sin\frac{\pi}{9}}=\frac{\sin\frac{8\pi}{9}}{8\sin\frac{\pi}{9}}=\frac{\sin\left(\pi-\frac{\pi}{9}\right)}{8\sin\frac{\pi}{9}}=\frac{\sin\frac{\pi}{9}}{8\sin\frac{\pi}{9}}=\frac{1}{8}$$

EXAMPLE 26 *Prove that:*

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{128}$$

SOLUTION We have,

$$\text{LHS}=\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$$
$$=\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}\cos\frac{5\pi}{15}$$
$$=\frac{1}{2}\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}\qquad\left[\because\cos\frac{5\pi}{15}=\cos\frac{\pi}{3}=\frac{1}{2}\right]$$
$$=-\frac{1}{2}\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}\left[\because\cos\frac{7\pi}{15}=\cos\left(\pi-\frac{8\pi}{15}\right)=-\cos\frac{8\pi}{15}\right]$$
$$=-\frac{1}{2}\left\{\frac{\sin\frac{2^4\pi}{15}}{2^4\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\left(2^2\times\frac{3\pi}{15}\right)}{2^2\sin\frac{3\pi}{15}}\right\}$$
$$=-\frac{1}{2}\left\{\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\frac{12\pi}{15}}{4\sin\frac{3\pi}{15}}\right\}=-\frac{1}{2}\left\{\frac{\sin\left(\pi+\frac{\pi}{15}\right)}{16\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\left(\pi-\frac{3\pi}{15}\right)}{4\sin\frac{3\pi}{15}}\right\}$$
$$=-\frac{1}{2}\left\{\frac{-\sin\frac{\pi}{15}}{16\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\frac{3\pi}{15}}{4\sin\frac{3\pi}{15}}\right\}=-\frac{1}{2}\times-\frac{1}{16}\times\frac{1}{4}=\frac{1}{128}=\text{RHS}$$

EXAMPLE 27 *Prove that:* $(1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta)\ldots(1+\sec 2^n\theta)=\tan 2^n\theta\cot\theta, n\in N$.

SOLUTION We have,

$$\text{LHS}=(1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta)\ldots(1+\sec 2^n\theta)$$
$$\Rightarrow \text{LHS}=\frac{(1+\cos 2\theta)(1+\cos 4\theta)(1+\cos 8\theta)\ldots(1+\cos 2^n\theta)}{\cos 2\theta\cos 4\theta\cos 8\theta\ldots\cos 2^n\theta}$$
$$\Rightarrow \text{LHS}=\frac{(2\cos^2\theta)(2\cos^2 2\theta)(2\cos^2 2^2\theta)\ldots(2\cos^2 2^{n-1}\theta)}{\cos 2\theta\cos 2^2\theta\cos 2^3\theta\ldots\cos 2^n\theta}$$

$$\Rightarrow \quad \text{LHS} = \frac{2^n \cos\theta}{\cos 2^n\theta}\left\{\cos\theta\cos 2\theta\cos 2^2\theta \ldots \cos 2^{n-1}\theta\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{2^n \cos\theta}{\cos 2^n\theta}\left\{\frac{\sin 2^n\theta}{2^n \sin\theta}\right\} = \tan 2^n\theta\cot\theta = \text{RHS}$$

Type VI **MISCELLANEOUS PROBLEMS BASED UPON FOLLOWING FORMULAE**

$\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\cos 2\theta = 2\cos^2\theta - 1$,

$\cos 2\theta = 1 - 2\sin^2\theta$, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$, $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

EXAMPLE 28 *If* $\tan^2\theta = 2\tan^2\phi + 1$, *prove that* $\cos 2\theta + \sin^2\phi = 0$.

SOLUTION We have,

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\therefore \quad \cos 2\theta + \sin^2\phi = \frac{1-\tan^2\theta}{1+\tan^2\theta} + \sin^2\phi$$

$$\Rightarrow \quad \cos 2\theta + \sin^2\phi = \frac{1-(2\tan^2\phi+1)}{1+(2\tan^2\phi+1)} + \sin^2\phi \qquad [\because \tan^2\theta = 2\tan^2\phi + 1]$$

$$\Rightarrow \quad \cos 2\theta + \sin^2\phi = \frac{-2\tan^2\phi}{2+2\tan^2\phi} + \sin^2\phi = \frac{-\tan^2\phi}{\sec^2\phi} + \sin^2\phi = -\sin^2\phi + \sin^2\phi = 0$$

EXAMPLE 29 *Prove that:*

(i) $\dfrac{1+\cos 4x}{\cot x - \tan x} = \dfrac{1}{2}\sin 4x$ (ii) $\dfrac{\cos 5x + \cos 4x}{1 - 2\cos 3x} = -\cos 2x - \cos x$

(iii) $\dfrac{\cos 7x - \cos 8x}{1 + 2\cos 5x} = \cos 2x - \cos 3x$

SOLUTION (i) We have,

$$\text{LHS} = \frac{1+\cos 4x}{\cot x - \tan x} = \frac{2\cos^2 2x \times \cos x \sin x}{\cos^2 x - \sin^2 x} = \frac{2\cos^2 2x \times 2\sin x\cos x}{2\cos 2x}$$

$$= \cos 2x \sin 2x = \frac{1}{2}(2\sin 2x\cos 2x) = \frac{1}{2}\sin 4x = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sin 3x(\cos 5x + \cos 4x)}{\sin 3x(1-2\cos 3x)} \qquad \text{[Multiplying and dividing by } \sin 3x]$$

$$\Rightarrow \quad \text{LHS} = \frac{\left\{2\sin\frac{3x}{2}\cos\frac{3x}{2}\right\}\left\{2\cos\frac{9x}{2}\cos\frac{x}{2}\right\}}{\sin 3x - 2\sin 3x\cos 3x}$$

$$\Rightarrow \quad \text{LHS} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{\sin 3x - \sin 6x} = \frac{4\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{2\sin\left(\frac{3x-6x}{2}\right)\cos\left(\frac{3x+6x}{2}\right)}$$

$$\Rightarrow \text{LHS} = \frac{4 \sin \frac{3x}{2} \cos \frac{3x}{2} \cos \frac{9x}{2} \cos \frac{x}{2}}{2 \sin\left(-\frac{3x}{2}\right) \cos \frac{9x}{2}}$$

$$\Rightarrow \text{LHS} = \frac{4 \sin \frac{3x}{2} \cos \frac{3x}{2} \cos \frac{9x}{2} \cos \frac{x}{2}}{-2 \sin \frac{3x}{2} \cos \frac{9x}{2}} = -2 \cos \frac{3x}{2} \cos \frac{x}{2} = -(\cos 2x + \cos x) = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \frac{\cos 7x - \cos 8x}{1 + 2 \cos 5x}$$

$$\Rightarrow \text{LHS} = \frac{2 \sin \frac{5x}{2} (\cos 7x - \cos 8x)}{2 \sin \frac{5x}{2} (1 + 2 \cos 5x)} \quad \left[\text{Multiplying numerator and denominator by } 2 \sin \frac{5x}{2}\right]$$

$$\Rightarrow \text{LHS} = \frac{2 \sin \frac{5x}{2} \cos 7x - 2 \sin \frac{5x}{2} \cos 8x}{2 \sin \frac{5x}{2} + 4 \sin \frac{5x}{2} \cos 5x} = \frac{2\sin \frac{5x}{2} \cos 7x - 2\sin \frac{5x}{2} \cos 8x}{2\left\{\sin \frac{5x}{2} + 2\sin \frac{5x}{2} \cos 5x\right\}}$$

$$\Rightarrow \text{LHS} = \frac{\left(\sin \frac{19x}{2} - \sin \frac{9x}{2}\right) - \left(\sin \frac{21x}{2} - \sin \frac{11x}{2}\right)}{2\left\{\sin \frac{5x}{2} + \sin \frac{15x}{2} - \sin \frac{5x}{2}\right\}}$$

$$\Rightarrow \text{LHS} = \frac{\left(\sin \frac{19x}{2} + \sin \frac{11x}{2}\right) - \left(\sin \frac{9x}{2} + \sin \frac{21x}{2}\right)}{2 \sin \frac{15x}{2}}$$

$$\Rightarrow \text{LHS} = \frac{2 \sin \left(\frac{\frac{19x}{2} + \frac{11x}{2}}{2}\right) \cos \left(\frac{\frac{19x}{2} - \frac{11x}{2}}{2}\right) - 2 \sin \left(\frac{\frac{9x}{2} + \frac{21x}{2}}{2}\right) \cos \left(\frac{\frac{9x}{2} - \frac{21x}{2}}{2}\right)}{2 \sin \frac{15x}{2}}$$

$$\Rightarrow \text{LHS} = \frac{2 \sin \frac{15x}{2} \cos 2x - 2 \sin \frac{15x}{2} \cos (-3x)}{2 \sin \frac{15x}{2}} = \cos 2x - \cos 3x = \text{RHS}$$

EXAMPLE 30 *If* $\tan \alpha = \frac{p}{q}$, *where* $\alpha = 6\beta$, α *being an acute angle, prove that*

$$\frac{1}{2}\left\{p \operatorname{cosec} 2\beta - q \sec 2\beta\right\} = \sqrt{p^2 + q^2}$$

SOLUTION We have, $\tan \alpha = \frac{p}{q}$

$$\therefore \quad \sin \alpha = \frac{p}{\sqrt{p^2 + q^2}} \text{ and, } \cos \alpha = \frac{q}{\sqrt{p^2 + q^2}}$$

Now,

$$\text{LHS} = \frac{1}{2}\left\{p \operatorname{cosec} 2\beta - q \sec 2\beta\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{p^2+q^2}}{2}\left\{\frac{p}{\sqrt{p^2+q^2}} \operatorname{cosec} 2\beta - \frac{q}{\sqrt{p^2+q^2}} \sec 2\beta\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{p^2+q^2}}{2}\left\{\sin\alpha \operatorname{cosec} 2\beta - \cos\alpha \sec 2\beta\right\} = \frac{\sqrt{p^2+q^2}}{2}\left\{\frac{\sin\alpha}{\sin 2\beta} - \frac{\cos\alpha}{\cos 2\beta}\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{p^2+q^2}}{2}\left\{\frac{\sin\alpha\cos 2\beta - \cos\alpha\sin 2\beta}{\sin 2\beta\cos 2\beta}\right\}$$

$$\Rightarrow \quad \text{LHS} = \sqrt{p^2+q^2}\left\{\frac{\sin(\alpha-2\beta)}{2\sin 2\beta\cos 2\beta}\right\} = \sqrt{p^2+q^2}\left\{\frac{\sin(6\beta-2\beta)}{\sin 4\beta}\right\} \qquad [\because \alpha = 6\beta]$$

$$\Rightarrow \quad \text{LHS} = \sqrt{p^2+q^2} = \text{RHS}.$$

EXAMPLE 31 *Prove that:* $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$

SOLUTION We have,

$$\cot\theta - \tan\theta = \frac{1}{\tan\theta} - \tan\theta = \frac{1-\tan^2\theta}{\tan\theta} = 2\left\{\frac{1-\tan^2\theta}{2\tan\theta}\right\} = \frac{2}{\tan 2\theta}$$

$$\Rightarrow \quad \cot\theta - \tan\theta = 2\cot 2\theta \qquad \text{...(i)}$$

Now,

$$\text{LHS} = \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \cot\alpha + \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{\cot\alpha - \tan\alpha - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha\}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{(\cot\alpha - \tan\alpha) - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha\}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{2\cot 2\alpha - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha\}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{2(\cot 2\alpha - \tan 2\alpha) - 4\tan 4\alpha - 8\cot 8\alpha\} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{2\times 2\cot 2(2\alpha) - 4\tan 4\alpha - 8\cot 8\alpha\} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{4(\cot 4\alpha - \tan 4\alpha) - 8\cot 8\alpha\} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - \{4\times 2\cot 2(4\alpha) - 8\cot 8\alpha\}$$

$$\Rightarrow \quad \text{LHS} = \cot\alpha - (8\cot 8\alpha - 8\cot 8\alpha) = \cot\alpha = \text{RHS}$$

EXAMPLE 32 *If* $\tan\beta = \dfrac{\tan\alpha + \tan\gamma}{1+\tan\alpha\tan\gamma}$, *prove that* $\sin 2\beta = \dfrac{\sin 2\alpha + \sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$

SOLUTION It is given that

$$\tan\beta = \frac{\tan\alpha + \tan\gamma}{1+\tan\alpha\tan\gamma}$$

$$\Rightarrow \quad \tan\beta = \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\gamma}{\cos\gamma}}{1 + \dfrac{\sin\alpha}{\cos\alpha}\times\dfrac{\sin\gamma}{\cos\gamma}} = \frac{\sin\alpha\cos\gamma + \cos\alpha\sin\gamma}{\cos\alpha\cos\gamma + \sin\alpha\sin\gamma} = \frac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)} \qquad \text{...(i)}$$

Now, $\quad \sin 2\beta = \dfrac{2\tan\beta}{1+\tan^2\beta}$

$$\Rightarrow \sin 2\beta = \frac{\dfrac{2\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}}{1+\dfrac{\sin^2(\alpha+\gamma)}{\cos^2(\alpha-\gamma)}} \qquad \text{[Using (i)]}$$

$$\Rightarrow \sin 2\beta = \frac{2\sin(\alpha+\gamma)\cos(\alpha-\gamma)}{\cos^2(\alpha-\gamma)+\sin^2(\alpha+\gamma)}$$

$$\Rightarrow \sin 2\beta = \frac{2(\sin 2\alpha+\sin 2\gamma)}{2\cos^2(\alpha-\gamma)+2\sin^2(\alpha+\gamma)}$$

$$\Rightarrow \sin 2\beta = \frac{2(\sin 2\alpha+\sin 2\gamma)}{1+\cos(2\alpha-2\gamma)+1-\cos(2\alpha+2\gamma)} = \frac{2(\sin 2\alpha+\sin 2\gamma)}{2+\cos(2\alpha-2\gamma)-\cos(2\alpha+2\gamma)}$$

$$\Rightarrow \sin 2\beta = \frac{2(\sin 2\alpha+\sin 2\gamma)}{2+2\sin 2\alpha\sin 2\gamma} = \frac{\sin 2\alpha+\sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$$

EXAMPLE 33 *If* $\sin(\theta+\alpha)=a$ *and* $\sin(\theta+\beta)=b$, *prove that* $\cos 2(\alpha-\beta)-4ab\cos(\alpha-\beta)=1-2a^2-2b^2$

SOLUTION We have, [NCERT EXEMPLAR]

$\sin(\theta+\alpha)=a$ and $\sin(\theta+\beta)=b$

$\therefore\ \cos(\theta+\alpha)=\sqrt{1-\sin^2(\theta+\alpha)}=\sqrt{1-a^2}$ and $\cos(\theta+\beta)=\sqrt{1-\sin^2(\theta+\beta)}=\sqrt{1-b^2}$

Now,

$$\cos(\alpha-\beta)=\cos\{(\theta+\alpha)-(\theta+\beta)\}$$

$$\Rightarrow \cos(\alpha-\beta)=\cos(\theta+\alpha)\cos(\theta+\beta)+\sin(\theta+\alpha)\sin(\theta+\beta)$$

$$\Rightarrow \cos(\alpha-\beta)=\sqrt{1-a^2}\,\sqrt{1-b^2}+ab$$

$$\Rightarrow \cos(\alpha-\beta)=ab+\sqrt{1-a^2-b^2+a^2b^2}$$

$$\therefore\ \cos 2(\alpha-\beta)-4ab\cos(\alpha-\beta)$$

$$=2\cos^2(\alpha-\beta)-1-4ab\cos(\alpha-\beta)$$

$$=2\left\{ab+\sqrt{1-a^2-b^2+a^2b^2}\right\}^2-1-4ab\left\{ab+\sqrt{1-a^2-b^2+a^2b^2}\right\}$$

$$=2\left\{a^2b^2+2ab\sqrt{1-a^2-b^2+a^2b^2}+1-a^2-b^2+a^2b^2\right\}-1-4a^2b^2-4ab\sqrt{1-a^2-b^2+a^2b^2}$$

$$=2-2a^2-2b^2-1=1-2a^2-2b^2.$$

EXAMPLE 34 *If* $\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)=\tan^3\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)$, *prove that* $\sin\theta=\dfrac{3\sin\alpha+\sin^3\alpha}{1+3\sin^2\alpha}$.

SOLUTION We have,

$$\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\tan^3\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$\Rightarrow \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}=\left\{\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right\}^3$$

$$\Rightarrow \frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}=\left\{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right\}^3$$

$$\Rightarrow \left\{\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right\}^2 = \left[\left\{\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right\}^2\right]^3 \quad \text{[On squaring both sides]}$$

$$\Rightarrow \frac{1+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \left\{\frac{1+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right\}^3$$

$$\Rightarrow \frac{1+\sin\theta}{1-\sin\theta} = \left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)^3$$

$$\Rightarrow \frac{(1+\sin\theta)-(1-\sin\theta)}{(1+\sin\theta)+(1-\sin\theta)} = \frac{(1+\sin\alpha)^3-(1-\sin\alpha)^3}{(1+\sin\alpha)^3+(1-\sin\alpha)^3} \quad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \frac{2\sin\theta}{2} = \frac{6\sin\alpha+2\sin^3\alpha}{2+6\sin^2\alpha}$$

$$\Rightarrow \sin\theta = \frac{3\sin\alpha+\sin^3\alpha}{1+3\sin^2\alpha}$$

EXAMPLE 35 *If* $\tan\frac{\theta}{2}=\sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2}$, *prove that* $\cos\phi=\dfrac{\cos\theta-e}{1-e\cos\theta}$.

SOLUTION We have,

$$\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}$$

$$\Rightarrow \cos\phi = \frac{1-\frac{1+e}{1-e}\tan^2\frac{\theta}{2}}{1+\frac{1+e}{1-e}\tan^2\frac{\theta}{2}} \quad \left[\because \tan\frac{\theta}{2}=\sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2} \Rightarrow \tan\frac{\phi}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{\theta}{2}\right]$$

$$\Rightarrow \cos\phi = \frac{(1-e)-(1+e)\tan^2\frac{\theta}{2}}{(1-e)+(1+e)\tan^2\frac{\theta}{2}}$$

$$\Rightarrow \cos\phi = \frac{(1-\tan^2\theta/2)-e(1+\tan^2\theta/2)}{(1+\tan^2\theta/2)-e(1-\tan^2\theta/2)}$$

$$\Rightarrow \cos\phi = \frac{\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}-e}{1-e\,\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}} \quad \left[\text{Dividing numerator and denominator by } 1+\tan^2\frac{\theta}{2}\right]$$

$$\Rightarrow \cos\phi = \frac{\cos\theta-e}{1-e\cos\theta} \quad \left[\because \cos\theta=\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right]$$

EXAMPLE 36 *Prove that:*

$$\frac{2\cos 2^n\theta+1}{2\cos\theta+1}=(2\cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$$

SOLUTION RHS

$$=(2\cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$$

$$=\frac{1}{(2\cos\theta+1)}\left\{(2\cos\theta+1)(2\cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1\right\}$$

$$=\frac{1}{(2\cos\theta+1)}\left\{(4\cos^2\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)\right\}$$

$$=\frac{1}{(2\cos\theta+1)}\left\{\{2(1+\cos 2\theta)-1\}(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)\right\}$$

$$=\frac{1}{(2\cos\theta+1)}\left\{(2\cos 2\theta+1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)\right\}$$

$$=\frac{1}{(2\cos\theta+1)}\left\{(4\cos^2 2\theta-1)(2\cos 2^2\theta-1)(2\cos 2^3\theta-1)\ldots(2\cos 2^{n-1}\theta-1)\right\}$$

$$=\frac{1}{(2\cos\theta+1)}\left\{\left\{2(\cos 4\theta+1)-1\right\}(2\cos 2^2\theta-1)(2\cos 2^3\theta-1)\ldots(2\cos 2^{n-1}\theta-1)\right\}$$

$$=\frac{1}{(2\cos\theta+1)}\left\{(2\cos 2^2\theta+1)(2\cos 2^2\theta-1)(2\cos 2^3\theta-1)\ldots\ldots(2\cos 2^{n-1}\theta-1)\right\}$$

$$=\frac{1}{(2\cos\theta+1)}\left\{(4\cos^2 2^2\theta-1)(2\cos 2^3\theta-1)\ldots\ldots(2\cos 2^{n-1}\theta-1)\right\}$$

$$=\frac{1}{(2\cos q+1)}\left\{\left\{2(1+\cos 2^3\theta)-1\right\}(2\cos 2^3\theta-1)\ldots(2\cos 2^{n-1}\theta-1)\right\}$$

$$=\frac{1}{(2\cos\theta+1)}\left\{(2\cos 2^3\theta+1)(2\cos 2^3\theta-1)\ldots(2\cos 2^{n-1}\theta-1)\right\}$$

....

$$=\frac{1}{(2\cos\theta+1)}(2\cos 2^{n-1}\theta+1)(2\cos 2^{n-1}\theta-1)$$

$$=\frac{1}{(2\cos\theta+1)}(4\cos^2 2^{n-1}\theta-1)$$

$$=\frac{1}{(2\cos\theta+1)}\left\{2(\cos 2.2^{n-1}\theta+1)-1\right\}$$

$$=\frac{1}{(2\cos\theta+1)}(2\cos 2^n\theta+2-1)=\frac{2\cos 2^n\theta+1}{2\cos\theta+1}=\text{LHS}$$

Type VII **ON CONDITIONAL IDENTITIES**

EXAMPLE 37 *If* $\tan\frac{\theta}{2}=\sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}$, *prove that* $\cos\theta=\frac{a\cos\phi+b}{a+b\cos\phi}$.

SOLUTION We have,

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$\Rightarrow \quad \cos\theta = \frac{1-\frac{a-b}{a+b}\tan^2\frac{\phi}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{\phi}{2}} \qquad \left[\text{Putting } \tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}\right]$$

$$\Rightarrow \quad \cos\theta = \frac{(a+b)-(a-b)\tan^2\frac{\phi}{2}}{(a+b)+(a-b)\tan^2\frac{\phi}{2}}$$

$$\Rightarrow \quad \cos\theta = \frac{a\left(1-\tan^2\frac{\phi}{2}\right)+b\left(1+\tan^2\frac{\phi}{2}\right)}{a\left(1+\tan^2\frac{\phi}{2}\right)+b\left(1-\tan^2\frac{\phi}{2}\right)}$$

$$\Rightarrow \quad \cos\theta = \frac{a\left(\frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\right)+b}{a+b\left(\frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\right)} \qquad \left[\text{Dividing numerator and denominator } by\ 1+\tan^2\frac{\phi}{2}\right]$$

$$\Rightarrow \quad \cos\theta = \frac{a\cos\phi+b}{a+b\cos\phi}$$

EXAMPLE 38 *If* $\cos\theta = \cos\alpha\cos\beta$, *prove that* $\tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2} = \tan^2\frac{\beta}{2}$.

SOLUTION We have,

$$\cos\theta = \cos\alpha\cos\beta$$

$$\Rightarrow \quad \cos\beta = \frac{\cos\theta}{\cos\alpha}$$

$$\Rightarrow \quad \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} = \frac{\cos\theta}{\cos\alpha}$$

$$\Rightarrow \quad \frac{\left(1-\tan^2\frac{\beta}{2}\right)+\left(1+\tan^2\frac{\beta}{2}\right)}{\left(1-\tan^2\frac{\beta}{2}\right)-\left(1+\tan^2\frac{\beta}{2}\right)} = \frac{\cos\theta+\cos\alpha}{\cos\theta-\cos\alpha} \qquad [\text{Applying componendo - dividendo}]$$

$$\Rightarrow \quad \frac{2}{-2\tan^2\frac{\beta}{2}} = \frac{2\cos\frac{\theta+\alpha}{2}\cos\frac{\theta-\alpha}{2}}{-2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}}$$

$$\Rightarrow \quad \frac{1}{\tan^2\frac{\beta}{2}} = \frac{1}{\tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}} \Rightarrow \tan^2\frac{\beta}{2} = \tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}$$

EXAMPLE 39 *If* $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$, *prove that* $\tan\dfrac{\theta}{2} = \pm\tan\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$.

SOLUTION We have,

$$\cos\theta = \frac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$$

$$\Rightarrow \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \frac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$$

$$\Rightarrow \frac{\left(1 - \tan^2\frac{\theta}{2}\right) + \left(1 + \tan^2\frac{\theta}{2}\right)}{\left(1 - \tan^2\frac{\theta}{2}\right) - \left(1 + \tan^2\frac{\theta}{2}\right)} = \frac{(\cos\alpha - \cos\beta) + (1 - \cos\alpha\cos\beta)}{(\cos\alpha - \cos\beta) - (1 - \cos\alpha\cos\beta)}$$

$$\Rightarrow \frac{2}{-2\tan^2\frac{\theta}{2}} = \frac{1 + \cos\alpha - \cos\beta - \cos\alpha\cos\beta}{-\{1 - \cos\alpha + \cos\beta - \cos\alpha\cos\beta\}}$$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}} = \frac{(1 + \cos\alpha)(1 - \cos\beta)}{(1 - \cos\alpha)(1 + \cos\beta)}$$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}} = \frac{2\cos^2\frac{\alpha}{2} \times 2\sin^2\frac{\beta}{2}}{2\sin^2\frac{\alpha}{2} \times 2\cos^2\frac{\beta}{2}} \Rightarrow \tan^2\frac{\theta}{2} = \tan^2\frac{\alpha}{2}\cot^2\frac{\beta}{2} \Rightarrow \tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$$

EXAMPLE 40 *If* $\cos\theta = \dfrac{\cos\alpha\cos\beta}{1 - \sin\alpha\sin\beta}$, *prove that one value of* $\tan\dfrac{\theta}{2} = \dfrac{\tan\frac{\alpha}{2} - \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$.

SOLUTION We have, $\cos\theta = \dfrac{\cos\alpha\cos\beta}{1 - \sin\alpha\sin\beta}$

Now, $\tan^2\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{1 + \cos\theta}$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1 - \dfrac{\cos\alpha\cos\beta}{1 - \sin\alpha\sin\beta}}{1 + \dfrac{\cos\alpha\cos\beta}{1 - \sin\alpha\sin\beta}}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1 - \sin\alpha\sin\beta - \cos\alpha\cos\beta}{1 - \sin\alpha\sin\beta + \cos\alpha\cos\beta}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1 - (\cos\alpha\cos\beta + \sin\alpha\sin\beta)}{1 + (\cos\alpha\cos\beta - \sin\alpha\sin\beta)}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{1 - \cos(\alpha - \beta)}{1 + \cos(\alpha + \beta)}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = \frac{2\sin^2\left(\frac{\alpha - \beta}{2}\right)}{2\cos^2\left(\frac{\alpha + \beta}{2}\right)}$$

$$\Rightarrow \quad \tan\frac{\theta}{2} = \pm\frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$\Rightarrow \quad \tan\frac{\theta}{2} = \pm\frac{\sin\frac{\alpha}{2}\cos\frac{\beta}{2}-\cos\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$\Rightarrow \quad \tan\frac{\theta}{2} = \pm\frac{\tan\frac{\alpha}{2}-\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} \quad \left[\text{Dividing numerator and denominator by } \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\right]$$

EXAMPLE 41 *If* $\sin\alpha+\sin\beta = a$ *and* $\cos\alpha+\cos\beta = b$, *prove that*

(i) $\cos(\alpha-\beta) = \dfrac{a^2+b^2-2}{2}$ (ii) $\tan\dfrac{\alpha-\beta}{2} = \pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$

SOLUTION (i) We have,

$$\sin\alpha+\sin\beta = a \text{ and } \cos\alpha+\cos\beta = b$$

$$\Rightarrow \quad (\sin\alpha+\sin\beta)^2+(\cos\alpha+\cos\beta)^2 = a^2+b^2$$

$$\Rightarrow \quad (\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)+2(\cos\alpha\cos\beta+\sin\alpha\sin\beta) = a^2+b^2$$

$$\Rightarrow \quad 2+2\cos(\alpha-\beta) = a^2+b^2$$

$$\Rightarrow \quad \cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}$$

(ii) Now, $\tan^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta}$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1-\cos(\alpha-\beta)}{1+\cos(\alpha-\beta)}$$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1-\frac{a^2+b^2-2}{2}}{1+\frac{a^2+b^2-2}{2}}$$

$$\Rightarrow \quad \tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{4-a^2-b^2}{a^2+b^2}$$

$$\Rightarrow \quad \tan\left(\frac{\alpha-\beta}{2}\right) = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

EXAMPLE 42 *If* α *and* β *are distinct roots of* $a\cos\theta+b\sin\theta = c$, *prove that* $\sin(\alpha+\beta) = \dfrac{2ab}{a^2+b^2}$.

SOLUTION It is given that α and β are distinct roots of $a\cos\theta+b\sin\theta = c$

$$\therefore \quad a\cos\alpha+b\sin\alpha = c \text{ and } a\cos\beta+b\sin\beta = c$$

$$\Rightarrow \quad (a\cos\alpha+b\sin\alpha)-(a\cos\beta+b\sin\beta) = c-c$$

$$\Rightarrow \quad a(\cos\alpha-\cos\beta)+(b\sin\alpha-\sin\beta) = 0$$

$\Rightarrow \quad -2a \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2} + 2b \sin \frac{\alpha-\beta}{2} \cos \frac{\alpha+\beta}{2} = 0$

$\Rightarrow \quad 2a \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2} = 2b \sin \frac{\alpha-\beta}{2} \cos \frac{\alpha+\beta}{2}$

$\Rightarrow \quad \tan \frac{\alpha+\beta}{2} = \frac{b}{a}$ $\left[\because \alpha \neq \beta \therefore \sin \frac{\alpha-\beta}{2} \neq 0\right]$

$\therefore \quad \sin(\alpha+\beta) = \frac{2 \tan \frac{\alpha+\beta}{2}}{1+\tan^2 \frac{\alpha+\beta}{2}} \Rightarrow \sin(\alpha+\beta) = \frac{\frac{2b}{a}}{1+\frac{b^2}{a^2}} = \frac{2ab}{a^2+b^2}$

<u>ALITER</u> We have,

$a\cos\theta + b\sin\theta = c$ ...(i)

$\Rightarrow \quad a\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right) + b\left(\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right) = c$

$\Rightarrow \quad a\left(1-\tan^2\frac{\theta}{2}\right) + 2b\tan\frac{\theta}{2} = c\left(1+\tan^2\frac{\theta}{2}\right)$

$\Rightarrow \quad (c+a)\tan^2\frac{\theta}{2} - 2b\tan\frac{\theta}{2} + (c-a) = 0$ ...(ii)

It is given that α and β are roots of the equation (i). Therefore, $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are roots of equation (ii).

$\therefore \quad \tan\frac{\alpha}{2} + \tan\frac{\beta}{2} = \frac{2b}{c+a}$ and, $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{c-a}{c+a}$ ...(iii)

Now, $\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) = \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$

$\Rightarrow \quad \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{2b/c+a}{1-\frac{c-a}{c+a}} = \frac{b}{a}$ [Using (iii)]

$\therefore \quad \sin(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{2\frac{b}{a}}{1+\frac{b^2}{a^2}} = \frac{2ab}{a^2+b^2}$

EXERCISE 9.1

LEVEL-1

Prove that: (1–27)

1. $\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \tan\theta$

2. $\frac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$

3. $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

4. $\sqrt{2+\sqrt{2+2\cos 4\theta}} = 2\cos\theta$

5. $\dfrac{1-\cos 2\theta+\sin 2\theta}{1+\cos 2\theta+\sin 2\theta} = \tan\theta$

6. $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

7. $\dfrac{\cos 2\theta}{1+\sin 2\theta} = \tan\left(\dfrac{\pi}{4}-\theta\right)$

8. $\dfrac{\cos\theta}{1-\sin\theta} = \tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$

9. $\cos^2\dfrac{\pi}{8}+\cos^2\dfrac{3\pi}{8}+\cos^2\dfrac{5\pi}{8}+\cos^2\dfrac{7\pi}{8} = 2$

10. $\sin^2\dfrac{\pi}{8}+\sin^2\dfrac{3\pi}{8}+\sin^2\dfrac{5\pi}{8}+\sin^2\dfrac{7\pi}{8} = 2$

11. $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2 = 4\cos^2\left(\dfrac{\alpha-\beta}{2}\right)$

12. $\sin^2\left(\dfrac{\pi}{8}+\dfrac{A}{2}\right)-\sin^2\left(\dfrac{\pi}{8}-\dfrac{A}{2}\right)=\dfrac{1}{\sqrt{2}}\sin A$

13. $1+\cos^2 2\theta = 2(\cos^4\theta+\sin^4\theta)$

14. $\cos^3 2\theta+3\cos 2\theta = 4(\cos^6\theta-\sin^6\theta)$

15. $(\sin 3A+\sin A)\sin A+(\cos 3A-\cos A)\cos A = 0$

16. $\cos^2\left(\dfrac{\pi}{4}-\theta\right)-\sin^2\left(\dfrac{\pi}{4}-\theta\right) = \sin 2\theta$

17. $\cos 4A = 1-8\cos^2 A+8\cos^4 A$

18. $\sin 4A = 4\sin A\cos^3 A-4\cos A\sin^3 A$ [NCERT EXEMPLAR]

19. $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x) = 13$

20. $2(\sin^6 x+\cos^6 x)-3(\sin^4 x+\cos^4 x)+1 = 0$

21. $\cos^6 A-\sin^6 A = \cos 2A\left(1-\dfrac{1}{4}\sin^2 2A\right)$

22. $\tan\left(\dfrac{\pi}{4}+\theta\right)+\tan\left(\dfrac{\pi}{4}-\theta\right) = 2\sec 2\theta$

23. $\cot^2 A-\tan^2 A = 4\cot 2A\,\mathrm{cosec}\,2A$

24. $\cos 4\theta-\cos 4\alpha = 8(\cos\theta-\cos\alpha)(\cos\theta+\cos\alpha)(\cos\theta-\sin\alpha)(\cos\theta+\sin\alpha)$

25. $\sin 3x+\sin 2x-\sin x = 4\sin x\cos\dfrac{x}{2}\cos\dfrac{3x}{2}$ [NCERT]

26. $\tan 82\dfrac{1}{2}^\circ = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1) = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

27. $\cot 22\dfrac{1}{2}^\circ = \sqrt{2}+1$

28. (i) If $\cos x = -\frac{3}{5}$ and x lies in the IIIrd quadrant, find the values of $\cos\frac{x}{2}$, $\sin\frac{x}{2}$ and, $\sin 2x$.

(ii) If $\cos x = -\frac{3}{5}$ and x lies in IInd quadrant, find the values of $\sin 2x$ and $\sin\frac{x}{2}$.

29. If $\sin x = \frac{\sqrt{5}}{3}$ and x lies in IInd quadrant, find the values of $\cos\frac{x}{2}$, $\sin\frac{x}{2}$ and $\tan\frac{x}{2}$.

30. (i) If $0 \le x \le \pi$ and x lies in the IInd quadrant such that $\sin x = \frac{1}{4}$. Find the values of $\cos\frac{x}{2}$, $\sin\frac{x}{2}$ and $\tan\frac{x}{2}$.

(ii) If $\cos\theta = \frac{4}{5}$ and θ is acute, find $\tan 2\theta$

(iii) If $\sin\theta = \frac{4}{5}$ and $0 < \theta < \frac{\pi}{2}$, find the value of $\sin 4\theta$

31. If $\tan x = \frac{b}{a}$, then find the value of $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$. [NCERT]

32. If $\tan A = \frac{1}{7}$ and $\tan B = \frac{1}{3}$, show that $\cos 2A = \sin 4B$.

LEVEL-2

33. Prove that: $\cos 7^\circ \cos 14^\circ \cos 28^\circ \cos 56^\circ = \frac{\sin 68^\circ}{16 \cos 83^\circ}$

34. Prove that: $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} = \frac{1}{16}$

35. Prove that: $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5} = \frac{-1}{16}$

36. Prove that: $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65} = \frac{1}{64}$

37. If $2\tan\alpha = 3\tan\beta$, prove that $\tan(\alpha-\beta) = \frac{\sin 2\beta}{5-\cos 2\beta}$.

38. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, prove that

(i) $\sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}$ (ii) $\cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}$

39. If $2\tan\frac{\alpha}{2} = \tan\frac{\beta}{2}$, prove that $\cos\alpha = \frac{3+5\cos\beta}{5+3\cos\beta}$

40. If $\cos\theta = \frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$, prove that $\tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$

41. If $\sec(\theta+\alpha) + \sec(\theta-\alpha) = 2\sec\theta$, prove that $\cos\theta = \pm\sqrt{2}\cos\frac{\alpha}{2}$

42. If $\cos\alpha + \cos\beta = \frac{1}{3}$ and $\sin\alpha + \sin\beta = \frac{1}{4}$, prove that $\cos\frac{\alpha-\beta}{2} = \pm\frac{5}{24}$.

43. If $\sin\alpha = \frac{4}{5}$ and $\cos\beta = \frac{5}{13}$, prove that $\cos\frac{\alpha-\beta}{2} = \frac{8}{\sqrt{65}}$.

44. If $a\cos 2\theta + b\sin 2\theta = c$ has α and β as its roots, then prove that

(i) $\tan\alpha + \tan\beta = \dfrac{2b}{a+c}$ [NCERT EXEMPLAR] (ii) $\tan\alpha\tan\beta = \dfrac{c-a}{c+a}$

(iii) $\tan(\alpha+\beta) = \dfrac{b}{a}$ [NCERT EXEMPLAR]

45. If $\cos\alpha + \cos\beta = 0 = \sin\alpha + \sin\beta$, then prove that $\cos 2\alpha + \cos 2\beta = -2\cos(\alpha+\beta)$.

[NCERT EXEMPLAR]

ANSWERS

28. (i) $-\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}, \dfrac{24}{25}$ (ii) $-\dfrac{24}{25}, \dfrac{2}{\sqrt{5}}$ 29. $\dfrac{1}{\sqrt{6}}, \sqrt{\dfrac{5}{6}}, \sqrt{5}$

30. (i) $\sqrt{\dfrac{4-\sqrt{15}}{8}}, \sqrt{\dfrac{4+\sqrt{15}}{8}}, 4+\sqrt{15}$ (ii) $\dfrac{24}{7}$ (iii) $-\dfrac{336}{625}$ 31. $\dfrac{2\cos x}{\sqrt{\cos 2x}}$

HINTS TO NCERT & SELECTED PROBLEMS

25. $$\begin{aligned}\text{LHS} &= \sin 3x + \sin 2x - \sin x\\ &= (\sin 3x - \sin x) + \sin 2x\\ &= 2\sin x\cos 2x + \sin 2x\\ &= 2\sin x\cos 2x + 2\sin x\cos x\\ &= 2\sin x(\cos 2x + \cos x) = 2\sin x\left(2\cos\frac{3x}{2}\cos\frac{x}{2}\right) = 4\sin x\cos\frac{x}{2}\cos\frac{3x}{2} = \text{RHS}\end{aligned}$$

32. Use: $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$, $\sin 4B = \dfrac{2\tan 2B}{1+\tan^2 2B}$, where $\tan 2B = \dfrac{2\tan B}{1-\tan^2 B}$

33. Let $A = 7°$. Then,

$$\cos 7°\cos 14°\cos 28°\cos 56° = \cos A\cos 2A\cos 2^2 A\cos 2^3 A$$

$$= \frac{\sin 2^4 A}{2^4\sin A} = \frac{\sin 16A}{16\sin A} = \frac{\sin 112°}{2\sin 7°} = \frac{\sin 68°}{2\cos 83°}$$

34. Let $A = \dfrac{2\pi}{15}$. Then,

$$\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}$$

$$= \cos A\cos 2A\cos 2^2 A\cos 2^3 A$$

$$= \frac{\sin 2^4 A}{2^4\sin A} = \frac{\sin 16A}{16\sin A} = \frac{\sin\frac{32\pi}{15}}{16\sin\frac{2\pi}{15}} = \frac{\sin\left(2\pi+\frac{2\pi}{15}\right)}{16\sin\frac{2\pi}{15}} = \frac{\sin\frac{2\pi}{15}}{16\sin\frac{2\pi}{15}} = \frac{1}{16} = \text{RHS}$$

44. We have,

$$a\cos 2\theta + b\sin 2\theta = c \quad \ldots\text{(i)}$$

$$\Rightarrow\quad a\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + \frac{2b\tan\theta}{1+\tan^2\theta} = c$$

$$\Rightarrow\quad (c+a)\tan^2\theta - 2b\tan\theta + (c-a) = 0 \quad \ldots\text{(ii)}$$

It is given that α, β are roots of equation (i). Therefore, $\tan\alpha, \tan\beta$ are roots of equation (ii).

$$\therefore\quad \tan\alpha + \tan\beta = \frac{2b}{c+a} \text{ and, } \tan\alpha\tan\beta = \frac{c-a}{c+a}$$

Now,

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \Rightarrow \tan(\alpha+\beta)=\frac{2b/c+a}{1-\frac{c-a}{c+a}}=\frac{b}{a}.$$

45. We have, $\cos\alpha+\cos\beta=0$ and $\sin\alpha+\sin\beta=0$

$\therefore$ $(\cos\alpha+\cos\beta)^2-(\sin\alpha+\sin\beta)^2=0^2-0^2$

$\Rightarrow$ $(\cos^2\alpha-\sin^2\alpha)+(\cos^2\beta-\sin^2\beta)+2(\cos\alpha\cos\beta-\sin\alpha\sin\beta)=0$

$\Rightarrow$ $\cos 2\alpha+\cos 2\beta+2\cos(\alpha+\beta)=0$

$\Rightarrow$ $\cos 2\alpha+\cos 2\beta=-2\cos(\alpha+\beta)$

9.5 TRIGONOMETRIC RATIOS OF ANGLE 3A IN TERMS OF ANGLE A

THEOREM *For the values of angle A, for which the two sides are meaningful prove that:*

(i) $\sin 3A = 3\sin A-4\sin^3 A$ (ii) $\cos 3A = 4\cos^3 A-3\cos A$

(iii) $\tan 3A = \dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

<u>PROOF</u> (i) We know that

$\sin(A+B) = \sin A\cos B+\cos A\sin B.$

$\Rightarrow$ $\sin(A+2A) = \sin A\cos 2A+\cos A\sin 2A$ [Replacing B by $2A$]

$\Rightarrow$ $\sin 3A = \sin A(1-2\sin^2 A)+\cos A(2\sin A\cos A)$

[$\because$ $\cos 2A=1-2\sin^2 A$ & $\sin 2A=2\sin A\cos A$]

$\Rightarrow$ $\sin 3A = \sin A-2\sin^3 A+2\sin A(1-\sin^2 A)$

$\Rightarrow$ $\sin 3A = 3\sin A-4\sin^3 A$

Hence, $\sin 3A = 3\sin A-4\sin^3 A$

(ii) We know that

$\cos(A+B) = \cos A\cos B-\sin A\sin B$

$\Rightarrow$ $\cos(A+2A) = \cos A\cos 2A-\sin A\sin 2A$ [Replacing B by $2A$]

$\Rightarrow$ $\cos 3A = \cos A\cos 2A-\sin A(2\sin A\cos A)$ [$\because$ $\sin 2A=2\sin A\cos A$]

$\Rightarrow$ $\cos 3A = \cos A(2\cos^2 A-1)-2\cos A(1-\cos^2 A)$ [$\because$ $\cos 2A=2\cos^2 A-1$]

$\Rightarrow$ $\cos 3A = 4\cos^3 A-3\cos A$

Hence, $\cos 3A = 4\cos^3 A-3\cos A$

(iii) We know that

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\Rightarrow \tan(A+2A)=\frac{\tan A+\tan 2A}{1-\tan A\tan 2A} \quad \text{[Replacing } B \text{ by } 2A]$$

$$\Rightarrow \tan 3A=\frac{\tan A+\dfrac{2\tan A}{1-\tan^2 A}}{1-\tan A\times\dfrac{2\tan A}{1-\tan^2 A}} \quad \left[\because \tan 2A=\frac{2\tan A}{1-\tan^2 A}\right]$$

$$\Rightarrow \tan 3A=\frac{\tan A(1-\tan^2 A)+2\tan A}{1-\tan^2 A-2\tan^2 A}=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$$

Hence, $\tan 3A = \dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$ Q.E.D.

<u>REMARK</u> *It should be noted the angle on the RHS of these formulae is one third of the angle on LHS.*

$\therefore$ $\sin 60° = 3\sin 20°-4\sin^3 20°$, $\sin 30° = 3\sin 10°-4\sin^3 10°$,

$\cos 120° = 4\cos^3 40°-3\cos 40°$ etc.

9.6 TRIGONOMETRIC RATIOS OF ANGLE A IN TERMS OF ANGLE $A/3$

Replacing A by $A/3$ in the formulas in the above section, we obtain the following formulae:

(i) $\sin A = 3\sin\left(\frac{A}{3}\right) - 4\sin^3\left(\frac{A}{3}\right)$

(ii) $\cos A = 4\cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right)$

(iii) $\tan A = \dfrac{3\tan\left(\frac{A}{3}\right) - \tan^3\left(\frac{A}{3}\right)}{1 - 3\tan^2\left(\frac{A}{3}\right)}$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove that:* $8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} = 1$

SOLUTION We have,

$$\text{LHS} = 2\left(4\cos^3\frac{\pi}{9} - 3\cos\frac{\pi}{9}\right) = 2\cos\left(3\times\frac{\pi}{9}\right) = 2\cos\frac{\pi}{3} = 1 = \text{RHS}$$

EXAMPLE 2 *Prove that:* $108\sin\frac{\pi}{18} - 144\sin^3\frac{\pi}{18} = 18$

SOLUTION We have,

$$\text{LHS} = 108\sin\frac{\pi}{18} - 144\sin^3\frac{\pi}{18} = 36\left(3\sin\frac{\pi}{18} - 4\sin^3\frac{\pi}{18}\right)$$

$$= 36\sin\left(3\times\frac{\pi}{18}\right) = 36\sin\frac{\pi}{6} = 36\times\frac{1}{2} = 18 = \text{RHS}$$

EXAMPLE 3 *Prove that:* $15\sin\frac{5\pi}{12} + 15\cos\frac{5\pi}{12} - 20\sin^3\frac{5\pi}{12} - 20\cos^3\frac{5\pi}{12} = 0$

SOLUTION We have,

$$\text{LHS} = 15\sin\frac{5\pi}{12} + 15\cos\frac{5\pi}{12} - 20\sin^3\frac{5\pi}{12} - 20\cos^3\frac{5\pi}{12}$$

$$= \left(15\sin\frac{5\pi}{12} - 20\sin^3\frac{5\pi}{12}\right) - \left(20\cos^3\frac{5\pi}{12} - 15\cos\frac{5\pi}{12}\right)$$

$$= 5\left(3\sin\frac{5\pi}{12} - 4\sin^3\frac{5\pi}{12}\right) - 5\left(4\cos^3\frac{5\pi}{12} - 3\cos\frac{5\pi}{12}\right)$$

$$= 5\sin\left(3\times\frac{5\pi}{12}\right) - 5\cos\left(3\times\frac{5\pi}{12}\right) = 5\sin\frac{5\pi}{4} - 5\cos\frac{5\pi}{4} = -5\sin\frac{\pi}{4} + 5\cos\frac{\pi}{4} = 0$$

EXAMPLE 4 *Prove that:* $\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1$ [NCERT]

SOLUTION We have,

$\text{LHS} = \cos 6A$

$\Rightarrow \text{LHS} = 2\cos^2 3A - 1$ [$\because \cos 2\theta = 2\cos^2\theta - 1$]

$\Rightarrow \text{LHS} = 2(4\cos^3 A - 3\cos A)^2 - 1$

$\Rightarrow \text{LHS} = 2(16\cos^6 A + 9\cos^2 A - 24\cos^4 A) - 1$

$\Rightarrow \text{LHS} = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1 = \text{RHS}$

EXAMPLE 5 *Prove that:* $\cos A \cos (60 - A) \cos (60 + A) = \frac{1}{4} \cos 3A$

SOLUTION We have,

$$\text{LHS} = \cos A \cos (60 - A) \cos (60 + A)$$

$$\Rightarrow \quad \text{LHS} = \cos A (\cos^2 60° - \sin^2 A) \qquad [\because \cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B]$$

$$\Rightarrow \quad \text{LHS} = \cos A \left(\frac{1}{4} - \sin^2 A\right) = \cos A \left\{\frac{1}{4} - (1 - \cos^2 A)\right\} = \cos A \left(-\frac{3}{4} + \cos^2 A\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4} \cos A (-3 + 4 \cos^2 A) = \frac{1}{4}(4 \cos^3 A - 3 \cos A) = \frac{1}{4} \cos 3A = \text{RHS}$$

EXAMPLE 6 *Prove that:* $\sin A \sin (60 - A) \sin (60 + A) = \frac{1}{4} \sin 3A$

SOLUTION We have,

$$\text{LHS} = \sin A \sin (60 - A) \sin (60 + A)$$

$$\Rightarrow \quad \text{LHS} = \sin A (\sin^2 60° - \sin^2 A) \qquad [\because \sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B]$$

$$\Rightarrow \quad \text{LHS} = \sin A \left(\frac{3}{4} - \sin^2 A\right)$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4} \sin A (3 - 4 \sin^2 A)$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}(3 \sin A - 4 \sin^3 A) = \frac{1}{4} \sin 3A.$$

<u>NOTE</u> *Reader is advised to learn the results derived in the above two examples as standard results. The following example is an application of the above results.*

EXAMPLE 7 *Prove that:* $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

SOLUTION We have,

$$\text{LHS} = \frac{\sqrt{3}}{2}\left\{\sin 20° \sin (60° - 20°) \sin (60 + 20°)\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{2}\left\{(\sin A \sin (60° - A) \sin (60° + A)\right\}, \text{ where } A = 20°$$

$$\Rightarrow \quad \text{LHS} = \frac{\sqrt{3}}{2} \times \frac{1}{4} \sin 3A = \frac{\sqrt{3}}{8} \times \sin 60° = \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2} = \frac{3}{16} = \text{RHS}$$

EXAMPLE 8 *Prove that:*

(i) $\tan A + \tan (60° + A) - \tan (60° - A) = 3 \tan 3A$

(ii) $\cot A + \cot (60° + A) - \cot (60° - A) = 3 \cot 3A$

SOLUTION (i) We have,

$$\text{LHS} = \tan A + \tan (60° + A) - \tan (60° - A)$$

$$\Rightarrow \quad \text{LHS} = \tan A + \frac{\sqrt{3} + \tan A}{1 - \sqrt{3} \tan A} - \frac{\sqrt{3} - \tan A}{1 + \sqrt{3} \tan A}$$

$$\Rightarrow \quad \text{LHS} = \tan A + \frac{(\sqrt{3} + \tan A)(1 + \sqrt{3} \tan A) - (\sqrt{3} - \tan A)(1 - \sqrt{3} \tan A)}{(1 - \sqrt{3} \tan A)(1 + \sqrt{3} \tan A)}$$

$$\Rightarrow \quad \text{LHS} = \tan A + \frac{8 \tan A}{1 - 3 \tan^2 A}$$

$$\Rightarrow \quad \text{LHS} = \frac{9 \tan A - 3 \tan^3 A}{1 - 3 \tan^2 A} = 3\left(\frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}\right) = 3 \tan 3A = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \cot A + \cot(60^\circ + A) - \cot(60^\circ - A)$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\tan A} + \frac{1}{\tan(60^\circ + A)} - \frac{1}{\tan(60^\circ - A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\tan A} + \frac{1 - \sqrt{3}\tan A}{\sqrt{3} + \tan A} - \frac{1 + \sqrt{3}\tan A}{\sqrt{3} - \tan A}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\tan A} + \frac{(1 - \sqrt{3}\tan A)(\sqrt{3} - \tan A) - (1 + \sqrt{3}\tan A)(\sqrt{3} + \tan A)}{(\sqrt{3} + \tan A)(\sqrt{3} - \tan A)}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{\tan A} - \frac{8\tan A}{3 - \tan^2 A}$$

$$\Rightarrow \quad \text{LHS} = \frac{3 - 9\tan^2 A}{3\tan A - \tan^3 A} = 3\left(\frac{1 - 3\tan^2 A}{3\tan A - \tan^3 A}\right) = \frac{3}{\tan 3A} = 3\cot 3A = \text{RHS}$$

LEVEL-2

EXAMPLE 9 *If* $\cos\alpha + \cos\beta + \cos\gamma = 0$, *then prove that*

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 12\cos\alpha\cos\beta\cos\gamma$$

SOLUTION $\cos 3\alpha + \cos 3\beta + \cos 3\gamma$

$$= (4\cos^3\alpha - 3\cos\alpha) + (4\cos^3\beta - 3\cos\beta) + (4\cos^3\gamma - 3\cos\gamma)$$

$$= 4(\cos^3\alpha + \cos^3\beta + \cos^3\gamma) - 3(\cos\alpha + \cos\beta + \cos\gamma)$$

$$= 4(\cos^3\alpha + \cos^3\beta + \cos^3\gamma) - 3\times 0$$

$$= 4\times 3\cos\alpha\cos\beta\cos\gamma \qquad [\because a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc]$$

$$= 12\cos\alpha\cos\beta\cos\gamma$$

EXAMPLE 10 *Prove that:* $\sin 3A\sin^3 A + \cos 3A\cos^3 A = \cos^3 2A$

SOLUTION We know that

$$\sin 3A = 3\sin A - 4\sin^3 A \Rightarrow \sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

Similarly,

$$\cos 3A = 4\cos^3 A - 3\cos A \Rightarrow \cos^3 A = \frac{\cos 3A + 3\cos A}{4}$$

$$\therefore \quad \text{LHS} = \sin 3A\sin^3 A + \cos 3A\cos^3 A$$

$$\Rightarrow \quad \text{LHS} = \sin 3A\left\{\frac{3\sin A - \sin 3A}{4}\right\} + \cos 3A\left\{\frac{\cos 3A + 3\cos A}{4}\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{3(\cos A\cos 3A + \sin A\sin 3A) + (\cos^2 3A - \sin^2 3A)\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{3\cos(3A - A) + \cos 2(3A)\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{3\cos 2A + \cos 3(2A)\right\}$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{4}\left\{3\cos 2A + (4\cos^3 2A - 3\cos 2A)\right\} = \cos^3 2A = \text{RHS}$$

EXAMPLE 11 *Prove that:* $\cos^3 A + \cos^3(120^\circ + A) + \cos^3(240^\circ + A) = \frac{3}{4}\cos 3A$

SOLUTION We know that $\cos 3A = 4\cos^3 A - 3\cos A$

$\therefore \cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$

Now,

$$\text{LHS} = \frac{1}{4}\left\{\cos 3A + 3\cos A\right\} + \frac{1}{4}\left\{\cos(360^\circ + 3A) + 3\cos(120^\circ + A)\right\} + \frac{1}{4}\left\{\cos(720^\circ + 3A) + 3\cos(240^\circ + A)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left\{\cos 3A + 3\cos A\right\} + \frac{1}{4}\left\{\cos 3A + 3\cos 120^\circ + A)\right\} + \frac{1}{4}\left\{\cos 3A + 3\cos(240^\circ + A)\right\}$$

$$\Rightarrow \text{LHS} = \frac{3}{4}\cos 3A + \frac{3}{4}\left\{\cos A + \cos(120^\circ + A) + \cos(240^\circ + A)\right\}$$

$$\Rightarrow \text{LHS} = \frac{3}{4}\cos 3A + \frac{3}{4}\left\{\cos A + 2\cos(180^\circ + A)\cos 60^\circ\right\}$$

$$\Rightarrow \text{LHS} = \frac{3}{4}\cos 3A + \frac{3}{4}\left\{\cos A - 2\cos A \times \frac{1}{2}\right\} = \frac{3}{4}\cos 3A = \text{RHS}$$

<u>ALITER</u> We have,

$$\cos A + \cos(120^\circ + A) + \cos(240^\circ + A)$$

$$= \cos A + 2\cos\left(\frac{240^\circ + A + 120^\circ + A}{2}\right)\cos\left(\frac{240^\circ + A - 120^\circ - A}{2}\right)$$

$$= \cos A + 2\cos(180^\circ + A)\cos 60^\circ = \cos A - 2(\cos A) \times \frac{1}{2}$$

$$= \cos A - \cos A = 0$$

$$\therefore \cos^3 A + \cos^3(120^\circ + A) + \cos^3(240^\circ + A)$$

$$= 3\cos A\cos(120^\circ + A)\cos(240^\circ + A) \qquad [\because a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc]$$

$$= 3\cos A\cos(180^\circ - 60^\circ + A)\cos(180^\circ + 60^\circ + A)$$

$$= 3\cos A\cos\{180^\circ - (60^\circ - A)\}\cos\{180^\circ + (60^\circ + A)\}$$

$$= (3\cos A)\{-\cos(60^\circ - A)\}\{-\cos(60^\circ + A)\}$$

$$= 3\cos A\cos(60^\circ - A)\cos(60^\circ + A) = 3 \times \frac{1}{4}\cos 3A = \frac{3}{4}\cos 3A$$

EXAMPLE 12 *Prove that* $\frac{\tan 3x}{\tan x}$ *never lies between* $\frac{1}{3}$ *and 3.*

SOLUTION Let $y = \frac{\tan 3x}{\tan x}$. Then,

$$y = \frac{3\tan x - \tan^3 x}{\tan x(1 - 3\tan^2 x)}$$

$$\Rightarrow y = \frac{3 - \tan^2 x}{1 - 3\tan^2 x} \Rightarrow (3y - 1)\tan^2 x = y - 3 \Rightarrow \tan^2 x = \frac{y - 3}{3y - 1}$$

But, $\tan^2 x \geq 0$ for all x

Fig. 9.1 Signs of $\frac{y-3}{3y-1}$ for different values of y

$\therefore \quad \dfrac{y-3}{3y-1} \geq 0$

$\Rightarrow \quad y < \dfrac{1}{3}$ or, $y \geq 3$

$\Rightarrow \quad y$ does not lie between 1/3 and 3.

Hence, $\dfrac{\tan 3x}{\tan x}$ never lies between $\dfrac{1}{3}$ and 3.

EXAMPLE 13 *Prove that:* $\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$

SOLUTION We have,

$$\cos 5A = \cos(3A + 2A) = \cos 3A \cos 2A - \sin 3A \sin 2A$$

$$\Rightarrow \cos 5A = (4\cos^3 A - 3\cos A)(2\cos^2 A - 1) - (3\sin A - 4\sin^3 A)(2\sin A\cos A)$$

$$\Rightarrow \cos 5A = (4\cos^3 A - 3\cos A)(2\cos^2 A - 1) - (3 - 4\sin^2 A)(2\sin^2 A\cos A)$$

$$\Rightarrow \cos 5A = (4\cos^3 A - 3\cos A)(2\cos^2 A - 1) - \{3 - 4(1 - \cos^2 A)\}\, 2(1 - \cos^2 A)\cos A$$

$$\Rightarrow \cos 5A = (8\cos^5 A - 10\cos^3 A + 3\cos A) - 2\cos A(1 - \cos^2 A)(4\cos^2 A - 1)$$

$$\Rightarrow \cos 5A = (8\cos^5 A - 10\cos^3 A + 3\cos A) - 2\cos A(5\cos^2 A - 4\cos^4 A - 1)$$

$$\Rightarrow \cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A = \text{RHS}$$

EXERCISE 9.2

LEVEL-1

Prove the following identities (1– 8)

1. $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$
2. $4(\cos^3 10° + \sin^3 20°) = 3(\cos 10° + \sin 20°)$
3. $\cos^3\theta \sin 3\theta + \sin^3\theta\cos 3\theta = \dfrac{3}{4}\sin 4\theta$
4. $\sin 5A = 5\cos^4 A\sin A - 10\cos^2 A\sin^3 A + \sin^5 A$
5. $\tan\theta\tan(\theta + 60°) + \tan\theta\tan(\theta - 60°) + \tan(\theta + 60°)\tan(\theta - 60°) = -3$
6. $\tan A + \tan(60° + A) - \tan(60° - A) = 3\tan 3A$
7. $\cot A + \cot(60° + A) - \cot(60° - A) = 3\cot 3A$
8. $\cot A + \cot(60° + A) + \cot(120° + A) = 3\cot 3A$

LEVEL-2

9. Prove that: $\sin^3 A + \sin^3\left(\dfrac{2\pi}{3} + A\right) + \sin^3\left(\dfrac{4\pi}{3} + A\right) = -\dfrac{3}{4}\sin 3A.$
10. Prove that: $|\sin\theta\sin(60 - \theta)\sin(60 + \theta)| \leq \dfrac{1}{4}$ for all values of θ.
11. Prove that: $|\cos\theta\cos(60° - \theta)\cos(60° + \theta)| \leq \dfrac{1}{4}$ for all values of θ.

HINTS TO SELECTED PROBLEMS

10. We have, $|\sin\theta\sin(60 - \theta)\sin(60 + \theta)| = \dfrac{1}{4}|\sin 3\theta| \leq \dfrac{1}{4}$ $\quad [\because |\sin 3\theta| \leq 1]$
11. We have, $|\cos\theta\cos(60° - \theta)\cos(60° + \theta)| = \left|\dfrac{1}{4}\cos 3\theta\right| \leq \dfrac{1}{4}$

9.7 TRIGONOMETRICAL RATIOS OF SOME IMPORTANT ANGLES

By using the formulae introduced in the previous sections we can now find the trigonometrical ratios of some important angles of degree measures 18°, 36°, 54° etc.

THEOREM 1 *Prove that:* $\sin 18° = \frac{\sqrt{5}-1}{4}$.

PROOF Let $\theta = 18°$. Then,

$$5\theta = 90°$$
$$\Rightarrow \quad 2\theta + 3\theta = 90°$$
$$\Rightarrow \quad 2\theta = 90° - 3\theta$$
$$\Rightarrow \quad \sin 2\theta = \sin(90° - 3\theta)$$
$$\Rightarrow \quad \sin 2\theta = \cos 3\theta$$
$$\Rightarrow \quad 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$$
$$\Rightarrow \quad \cos\theta(2\sin\theta - 4\cos^2\theta + 3) = 0$$
$$\Rightarrow \quad 2\sin\theta - 4\cos^2\theta + 3 = 0 \qquad [\because \cos\theta = \cos 18° \neq 0]$$
$$\Rightarrow \quad 2\sin\theta - 4(1-\sin^2\theta) + 3 = 0$$
$$\Rightarrow \quad 4\sin^2\theta + 2\sin\theta - 1 = 0$$
$$\Rightarrow \quad \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8}$$
$$\Rightarrow \quad \sin\theta = \frac{-1 \pm \sqrt{5}}{4}$$
$$\Rightarrow \quad \sin\theta = \frac{-1+\sqrt{5}}{4} = \frac{\sqrt{5}-1}{4} \qquad [\because \theta \text{ lies in Ist quadrant } \therefore \sin\theta > 0]$$

Hence, $\sin 18° = \frac{\sqrt{5}-1}{4}$

Q.E.D.

THEOREM 2 *Prove that:* $\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$.

PROOF Putting $\theta = 18°$ in $\cos\theta = \sqrt{1-\sin^2\theta}$, we get

$$\cos 18° = \sqrt{1-\sin^2 18°} = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2} = \sqrt{\frac{16-(5+1-2\sqrt{5})}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

Hence, $\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$

Q.E.D.

REMARK *The complement of* 18° is 72°.

$$\therefore \quad \sin 72° = \sin(90° - 18°) = \cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

and, $$\cos 72° = \cos(90° - 18°) = \sin 18° = \frac{\sqrt{5}-1}{4}$$

The remaining trigonometrical ratios of 18° *may be obtained from the above values.*

THEOREM 3 Prove that: $\cos 36° = \frac{\sqrt{5}+1}{4}$.

PROOF We have, $\cos 2\theta = 1 - 2\sin^2\theta$

$$\therefore \quad \cos 36° = 1 - 2\sin^2 18° \qquad [\text{Putting } \theta = 18°]$$

$$\Rightarrow \quad \cos 36° = 1-2\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1-2\left(\frac{6-2\sqrt{5}}{16}\right) = 1-\left(\frac{3-\sqrt{5}}{4}\right) = \frac{\sqrt{5}+1}{4}$$

Hence, $\cos 36° = \dfrac{\sqrt{5}+1}{4}$ **Q.E.D.**

THEOREM 4 *Prove that:* $\sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$.

<u>PROOF</u> Putting $\theta = 36°$ in $\sin\theta = \sqrt{1-\cos^2\theta}$, we obtain

$$\sin 36° = \sqrt{1-\cos^2 36°} = \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2} = \sqrt{\frac{16-(6+2\sqrt{5})}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

Hence, $\sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$ **Q.E.D.**

<u>REMARK</u> *The complement of 36° is 54°.*

$\therefore\ \sin 54° = \sin(90°-36°) = \cos 36° = \dfrac{\sqrt{5}+1}{4}$ *and,* $\cos 54° = \cos(90°-36°) = \sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

The other trigonometrical ratios of 36° may be obtained from the above values.

<u>ILLUSTRATIVE EXAMPLES</u>

LEVEL-1

EXAMPLE 1 *Prove that:*

(i) $\sin^2 72° - \sin^2 60° = \dfrac{\sqrt{5}-1}{8}$ (ii) $\cos^2 48° - \sin^2 12° = \dfrac{\sqrt{5}+1}{8}$

(iii) $\sin\dfrac{\pi}{10} + \sin\dfrac{13\pi}{10} = -\dfrac{1}{2}$ (iv) $\sin\dfrac{\pi}{10}\sin\dfrac{13\pi}{10} = -\dfrac{1}{4}$

(v) $\sin^2 24° - \sin^2 6° = \dfrac{\sqrt{5}-1}{8}$

SOLUTION (i) We have,

$$\text{LHS} = \sin^2 72° - \sin^2 60°$$

$$\Rightarrow \quad \text{LHS} = \cos^2 18° - \sin^2 60° \qquad [\because \sin 72° = \cos 18°]$$

$$\Rightarrow \quad \text{LHS} = \left\{\frac{\sqrt{10+2\sqrt{5}}}{4}\right\}^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{10+2\sqrt{5}}{16} - \frac{3}{4} = \frac{2\sqrt{5}-2}{16} = \frac{\sqrt{5}-1}{8} = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \cos^2 48° - \sin^2 12°$$

$$\Rightarrow \quad \text{LHS} = \cos(48°+12°)\cos(48°-12°) \qquad [\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)]$$

$$\Rightarrow \quad \text{LHS} = \cos 60° \cos 36° = \frac{1}{2}\times\frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}+1}{8} = \text{RHS}$$

(iii) We have,

$$\text{LHS} = \sin\frac{\pi}{10} + \sin\frac{13\pi}{10}$$

$$\Rightarrow \quad \text{LHS} = \sin 18° + \sin 234° = \sin 18° + \sin(270°-36°)$$

$$\Rightarrow \quad \text{LHS} = \sin 18° - \cos 36° = \frac{\sqrt{5}-1}{4} - \frac{\sqrt{5}+1}{4} = -\frac{1}{2}$$

(iv) We have,

$$\Rightarrow \quad \text{LHS} = \sin\frac{\pi}{10}\sin\frac{13\pi}{10}$$

$$\Rightarrow \quad \text{LHS} = \sin 18° \sin 234° = -\sin 18° \cos 36° = -\frac{\sqrt{5}-1}{4}\times\frac{\sqrt{5}+1}{4} = -\left(\frac{5-1}{16}\right) = -\frac{1}{4} = \text{RHS}$$

(v) We have,

$$\text{LHS} = \sin^2 24° - \sin^2 6°$$

$$\Rightarrow \quad \text{LHS} = \sin(24°+6°)\sin(24°-6°) \qquad [\because \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B]$$

$$\Rightarrow \quad \text{LHS} = \sin 30° \sin 18° = \frac{1}{2}\times\frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}-1}{8} = \text{RHS}$$

EXAMPLE 2 *Prove that:* $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$

SOLUTION If $A + B = \pi$, then $A = \pi - B \Rightarrow \sin A = \sin(\pi - B) \Rightarrow \sin A = \sin B$

$$\therefore \quad \frac{\pi}{5}+\frac{4\pi}{5} = \pi \Rightarrow \sin\frac{\pi}{5} = \sin\frac{4\pi}{5} \text{ and } \frac{2\pi}{5}+\frac{3\pi}{5} = \pi \Rightarrow \sin\frac{2\pi}{5} = \sin\frac{3\pi}{5}$$

Using these values, we obtain

$$\text{LHS} = \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}$$

$$\Rightarrow \quad \text{LHS} = \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{2\pi}{5}\sin\frac{\pi}{5}$$

$$\Rightarrow \quad \text{LHS} = \left(\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\right)^2 = (\sin 36° \sin 72°)^2 = (\sin 36° \cos 18°)^2$$

$$\Rightarrow \quad \text{LHS} = \left\{\frac{\sqrt{10-2\sqrt{5}}}{4}\times\frac{\sqrt{10+2\sqrt{5}}}{4}\right\}^2 = \frac{10-2\sqrt{5}}{16}\times\frac{10+2\sqrt{5}}{16} = \frac{100-20}{256} = \frac{80}{256} = \frac{5}{16} = \text{RHS}$$

EXAMPLE 3 *Prove that:* $16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15} = 1$

SOLUTION We have,

$$\text{LHS} = 16\cos 24° \cos 48° \cos 96° \cos 168°$$

$$\Rightarrow \quad \text{LHS} = 4(2\cos 24° \cos 96°)(2\cos 48° \cos 168°)$$

$$\Rightarrow \quad \text{LHS} = 4(\cos 120° + \cos 72°)(\cos 216° + \cos 120°)$$

$$\Rightarrow \quad \text{LHS} = 4(-\sin 30° + \sin 18°)(-\cos 36° - \sin 30°)$$

$$\Rightarrow \quad \text{LHS} = 4\left(-\frac{1}{2}+\frac{\sqrt{5}-1}{4}\right)\left(-\frac{\sqrt{5}+1}{4}-\frac{1}{2}\right)$$

$$\Rightarrow \quad \text{LHS} = 4\left(\frac{\sqrt{5}-3}{4}\right)\left(\frac{-\sqrt{5}-3}{4}\right) = 4\left(\frac{3-\sqrt{5}}{4}\right)\left(\frac{3+\sqrt{5}}{4}\right) = \left(\frac{9-5}{4}\right) = 1 = \text{RHS}$$

EXAMPLE 4 *Prove that:* $\sin 12° \sin 48° \sin 54° = \frac{1}{8}$.

SOLUTION We have,

$$\text{LHS} = \frac{1}{2}(2\sin 48° \sin 12°)\sin 54°$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{2}(\cos 36° - \cos 60°)\cos 36° \qquad [\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$$

$$\Rightarrow \quad \text{LHS} = \frac{1}{2}\left(\frac{\sqrt{5}+1}{4}-\frac{1}{2}\right)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{1}{2}\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{1}{8} = \text{RHS}$$

EXAMPLE 5 *Prove that:* $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)=\frac{1}{16}$.

SOLUTION If $A+B=\pi$, then $\cos A=\cos(\pi-B)=-\cos B$

$$\therefore \quad \frac{\pi}{10}+\frac{9\pi}{10}=\pi \Rightarrow \cos\frac{9\pi}{10}=-\cos\frac{\pi}{10} \text{ and, } \frac{3\pi}{10}+\frac{7\pi}{10}=\pi \Rightarrow \cos\frac{7\pi}{10}=-\cos\frac{3\pi}{10}$$

Using these values, we obtain

$$\text{LHS} = \left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{\pi}{10}\right)$$

$$\Rightarrow \quad \text{LHS} = \left(1-\cos^2\frac{\pi}{10}\right)\left(1-\cos^2\frac{3\pi}{10}\right)$$

$$\Rightarrow \quad \text{LHS} = (1-\cos^2 18^\circ)(1-\cos^2 54^\circ) = \sin^2 18^\circ \sin^2 54 = \sin^2 18^\circ \cos^2 36^\circ$$

$$\Rightarrow \quad \text{LHS} = (\sin 18^\circ \cos 36^\circ)^2 = \left(\frac{\sqrt{5}-1}{4}\times\frac{\sqrt{5}+1}{4}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16} = \text{RHS}$$

EXAMPLE 6 *Prove that:* $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1$.

SOLUTION We have,

$$\text{LHS} = \frac{\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ}{\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ} = \frac{(2\sin 66^\circ \sin 6^\circ)(2\sin 78^\circ \sin 42^\circ)}{(2\cos 66^\circ \cos 6^\circ)(2\cos 78^\circ \cos 42^\circ)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\cos 60^\circ - \cos 72^\circ)(\cos 36^\circ - \cos 120^\circ)}{(\cos 60^\circ + \cos 72^\circ)(\cos 36^\circ + \cos 120^\circ)}$$

$$\Rightarrow \quad \text{LHS} = \frac{(\cos 60^\circ - \sin 18^\circ)(\cos 36^\circ + \sin 30^\circ)}{(\cos 60^\circ + \sin 18^\circ)(\cos 36^\circ - \sin 30^\circ)}$$

$$\Rightarrow \quad \text{LHS} = \frac{\left(\frac{1}{2}-\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}+\frac{1}{2}\right)}{\left(\frac{1}{2}+\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}-1}{4}-\frac{1}{2}\right)} = \frac{(3-\sqrt{5})(3+\sqrt{5})}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{9-5}{5-1} = 1 = \text{RHS}$$

LEVEL-2

EXAMPLE 7 *Prove that :* $4\sin 27^\circ = \sqrt{(5+\sqrt{5})} - \sqrt{(3-\sqrt{5})}$.

SOLUTION We have,

$$16\sin^2 27^\circ = 8(2\sin^2 27^\circ) = 8(1-\cos 54^\circ) = 8(1-\sin 36^\circ)$$

$$= 8\left\{1-\frac{\sqrt{10-2\sqrt{5}}}{4}\right\} = 2\left\{4-\sqrt{10-2\sqrt{5}}\right\} = 8-2\sqrt{10-2\sqrt{5}}$$

$$= (5+\sqrt{5})+(3-\sqrt{5})-2\sqrt{(5+\sqrt{5})(3-\sqrt{5})}$$

$$= \left\{\sqrt{5+\sqrt{5}}\right\}^2 + \left\{\sqrt{3-\sqrt{5}}\right\}^2 - 2\sqrt{(5+\sqrt{5})(3-\sqrt{5})}$$

$$= \left\{\sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}\right\}^2$$

Taking square roots of both sides, we obtain

$$\therefore \quad 4\sin 27^\circ = \sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}} \qquad [\because \sin 27^\circ \text{ is positive}]$$

EXAMPLE 8 *Find the value of* $\tan 9° - \tan 27° - \tan 63° + \tan 81°$.

SOLUTION $\tan 9° - \tan 27° - \tan 63° + \tan 81°$

$$= (\tan 9° + \tan 81°) - (\tan 27° + \tan 63°)$$

$$= (\tan 9° + \cot 9°) - (\tan 27° + \cot 27°) \quad \left[\because \begin{array}{l} \tan 81° = \tan(90°-9°) = \cot 9° \\ \tan 63° = \tan(90°-27°) = \cot 27° \end{array}\right]$$

$$= \frac{1}{\sin 9° \cos 9°} - \frac{1}{\sin 27° \cos 27°} \quad \left[\because \tan\theta + \cot\theta = \frac{1}{\sin\theta\cos\theta}\right]$$

$$= \frac{2}{\sin 18°} - \frac{2}{\sin 54°} = \frac{2}{\sin 18°} - \frac{2}{\cos 36°} = \frac{8}{\sqrt{5}-1} - \frac{8}{\sqrt{5}+1} = \frac{8 \times 2}{5-1} = 4$$

EXERCISE 9.3

LEVEL-1

Prove that:

1. $\sin^2 72° - \sin^2 60° = \dfrac{\sqrt{5}-1}{8}$
2. $\sin^2 24° - \sin^2 6° = \dfrac{\sqrt{5}-1}{8}$
3. $\sin^2 42° - \cos^2 78° = \dfrac{\sqrt{5}+1}{8}$
4. $\cos 78° \cos 42° \cos 36° = \dfrac{1}{8}$
5. $\cos\dfrac{\pi}{15}\cos\dfrac{2\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{7\pi}{15} = \dfrac{1}{16}$
6. $\cos\dfrac{\pi}{15}\cos\dfrac{2\pi}{15}\cos\dfrac{3\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{5\pi}{15}\cos\dfrac{6\pi}{15}\cos\dfrac{7\pi}{15} = \dfrac{1}{128}$
7. $\cos 6° \cos 42° \cos 66° \cos 78° = \dfrac{1}{16}$
8. $\sin 6° \sin 42° \sin 66° \sin 78° = \dfrac{1}{16}$
9. $\cos 36° \cos 42° \cos 60° \cos 78° = \dfrac{1}{16}$
10. $\sin 36° \sin 72° \sin 108° \sin 144° = \dfrac{5}{16}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $\cos 4x = 1 + k \sin^2 x \cos^2 x$, then write the value of k.
2. If $\tan\dfrac{x}{2} = \dfrac{m}{n}$, then write the value of $m \sin x + n \cos x$.
3. If $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$, then write the value of $\sqrt{\dfrac{1+\cos 2\theta}{2}}$.
4. If $\dfrac{\pi}{2} < \theta < \pi$, then write the value of $\sqrt{2+\sqrt{2+2\cos 2\theta}}$ in the simplest form.
5. If $\dfrac{\pi}{2} < \theta < \pi$, then write the value of $\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$.

6. If $\pi < \theta < \frac{3\pi}{2}$, then write the value of $\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$.

7. In a right angled triangle ABC, write the value of $\sin^2 A + \sin^2 B + \sin^2 C$.

8. Write the value of $\cos^2 76° + \cos^2 16° - \cos 76° \cos 16°$.

9. If $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, then write the value of $\sqrt{1-\sin 2\theta}$.

10. Write the value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$.

11. If $\tan A = \frac{1-\cos B}{\sin B}$, then find the value of $\tan 2A$.

12. If $\sin x + \cos x = a$, find the value of $\sin^6 x + \cos^6 x$.

13. If $\sin x + \cos x = a$, find the value of $|\sin x - \cos x|$.

ANSWERS

1. -8 2. n 3. $-\cos\theta$ 4. $2\sin\frac{\theta}{2}$ 5. $-\tan\theta$ 6. $\tan\theta$ 7. 2
8. $\frac{3}{4}$ 9. $\sin\theta - \cos\theta$ 10. $-\frac{1}{8}$ 11. $\tan B$ 12. $\frac{1}{4}\{4-3(a^2-1)^2\}$ 13. $\sqrt{2-a^2}$.

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. $8\sin\frac{x}{8}\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}$ is equal to
   (a) $8\cos x$ (b) $\cos x$ (c) $8\sin x$ (d) $\sin x$

2. $\frac{\sec 8A - 1}{\sec 4A - 1}$ is equal to
   (a) $\frac{\tan 2A}{\tan 8A}$ (b) $\frac{\tan 8A}{\tan 2A}$ (c) $\frac{\cot 8A}{\cot 2A}$ (d) none of these

3. The value of $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}$ is
   (a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $\frac{1}{32}$ (d) none of these.

4. If $\cos 2x + 2\cos x = 1$ then, $(2-\cos^2 x)\sin^2 x$ is equal to
   (a) 1 (b) -1 (c) $-\sqrt{5}$ (d) $\sqrt{5}$

5. For all real values of x, $\cot x - 2\cot 2x$ is equal to
   (a) $\tan 2x$ (b) $\tan x$ (c) $-\cot 3x$ (d) none of these

6. The value of $2\tan\frac{\pi}{10} + 3\sec\frac{\pi}{10} - 4\cos\frac{\pi}{10}$ is
   (a) 0 (b) $\sqrt{5}$ (c) 1 (d) none of these

7. If in a ΔABC, $\tan A + \tan B + \tan C = 0$, then $\cot A \cot B \cot C =$
   (a) 6 (b) 1 (c) $\frac{1}{6}$ (d) none of these

8. If $\cos\theta = \frac{1}{2}\left(a + \frac{1}{a}\right)$, and $\cos 3\theta = \lambda\left(a^3 + \frac{1}{a^3}\right)$, then $\lambda =$

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) none of these

9. If $2 \tan \alpha = 3 \tan \beta$, then $\tan(\alpha - \beta) =$

(a) $\frac{\sin 2\beta}{5 - \cos 2\beta}$ (b) $\frac{\cos 2\beta}{5 - \cos 2\beta}$ (c) $\frac{\sin 2\beta}{5 + \cos 2\beta}$ (d) none of these

10. If $\tan \alpha = \frac{1 - \cos \beta}{\sin \beta}$, then

(a) $\tan 3\alpha = \tan 2\beta$ (b) $\tan 2\alpha = \tan \beta$
(c) $\tan 2\beta = \tan \alpha$ (d) none of these

11. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha - \cos \beta = b$, then $\tan \frac{\alpha - \beta}{2} =$

(a) $-\frac{a}{b}$ (b) $-\frac{b}{a}$ (c) $\sqrt{a^2 + b^2}$ (d) none of these

12. The value of $\left(\cot \frac{x}{2} - \tan \frac{x}{2}\right)^2 (1 - 2 \tan x \cot 2x)$ is

(a) 1 (b) 2 (c) 3 (d) 4

13. The value of $\tan \theta \sin\left(\frac{\pi}{2} + \theta\right) \cos\left(\frac{\pi}{2} - \theta\right)$ is

(a) 1 (b) −1 (c) $\frac{1}{2} \sin 2\theta$ (d) none of these

14. The value of $\sin^2\left(\frac{\pi}{18}\right) + \sin^2\left(\frac{\pi}{9}\right) + \sin^2\left(\frac{7\pi}{18}\right) + \sin^2\left(\frac{4\pi}{9}\right)$ is

(a) 1 (b) 2 (c) 4 (d) none of these

15. If $5 \sin \alpha = 3 \sin(\alpha + 2\beta) \neq 0$, then $\tan(\alpha + \beta)$ is equal to

(a) $2 \tan \beta$ (b) $3 \tan \beta$ (c) $4 \tan \beta$ (d) $6 \tan \beta$

16. The value of $2 \cos \theta - \cos 3\theta - \cos 5\theta - 16 \cos^3 \theta \sin^2 \theta$ is

(a) 2 (b) 1 (c) 0 (d) -1

17. If $A = 2 \sin^2 \theta - \cos 2\theta$, then A lies in the interval

(a) $[-1, 3]$ (b) $[1, 2]$ (c) $[-2, 4]$ (d) none of these

18. The value of $\frac{\cos 3\theta}{2 \cos 2\theta - 1}$ is equal to

(a) $\cos \theta$ (b) $\sin \theta$ (c) $\tan \theta$ (d) none of these

19. If $\tan(\pi/4 + \theta) + \tan(\pi/4 - \theta) = \lambda \sec 2\theta$, them

(a) 3 (b) 4 (c) 1 (d) 2

20. The value of $\cos^2\left(\frac{\pi}{6} + \theta\right) - \sin^2\left(\frac{\pi}{6} - \theta\right)$ is

(a) $\frac{1}{2} \cos 2\theta$ (b) 0 (c) $-\frac{1}{2} \cos 2\theta$ (d) $\frac{1}{2}$

21. $\frac{\sin 3\theta}{1 + 2 \cos 2\theta}$ is equal to

(a) $\cos \theta$ (b) $\sin \theta$ (c) $-\cos \theta$ (d) $\sin \theta$

22. The value of $2 \sin^2 B + 4 \cos(A + B) \sin A \sin B + \cos 2(A + B)$ is

(a) 0 (b) $\cos 3A$ (c) $\cos 2A$ (d) none of these

CHAPTER 10

SINE AND COSINE FORMULAE AND THEIR APPLICATIONS

10.1 INTRODUCTION

In any triangle the three sides and the three angles are generally called the elements of the triangle. A triangle which does not contain a right angle is called an *oblique triangle.*

In any triangle ABC, the measures of the angles $\angle BAC$, $\angle CBA$ and $\angle ACB$ are denoted by the letters A, B and C respectively. The sides BC, CA and AB opposite to the angles A, B and C respectively are denoted by a, b and c. These six elements of a triangle are not independent and are connected by the relations: (i) $A + B + C = \pi$ (ii) $a + b > c$; $b + c > a$; $c + a > b$. In addition to these relations, the elements of a triangle are connected by some trigonometric relations. We intend to discuss those relations in the sections to follow of this chapter.

10.2 THE LAW OF SINES OR SINE RULE

THEOREM *The sides of a triangle are proportional to the sines of the angles opposite to them i.e. in a* ΔABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

PROOF The following cases arise:

CASE I *When ΔABC is an acute angled triangle:*

Draw AD perpendicular from A to the opposite side BC meeting it in the point D.

In the triangle ABD, we have

$$\sin B = \frac{AD}{AB} \Rightarrow \sin B = \frac{AD}{c} \Rightarrow AD = c \sin B \qquad \text{...(i)}$$

In the triangle ACD, we have

$$\sin C = \frac{AD}{AC} \Rightarrow \sin C = \frac{AD}{b} \Rightarrow AD = b \sin C \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$c \sin B = b \sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$$

In a similar manner, by drawing a perpendicular from B on AC, we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Hence, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

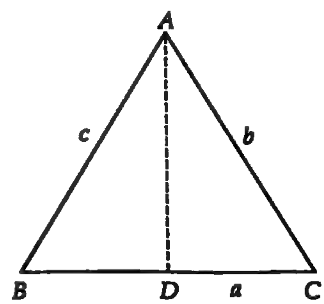


Fig. 10.1

CASE II *When ΔABC is an obtuse angled triangle.*

Draw AD perpendicular from A on CB produced meeting it in D.

In ΔADB, we have

$$\sin \angle ABD = \frac{AD}{AB} \Rightarrow \sin (180 - B) = \frac{AD}{c} \Rightarrow \sin B = \frac{AD}{c} \Rightarrow AD = c \sin B \qquad \text{...(i)}$$

In ΔACD, we have

$$\sin C = \frac{AD}{AC} \Rightarrow \sin C = \frac{AD}{b} \Rightarrow AD = b \sin C \quad \text{...(ii)}$$

From (i) and (ii), we obtain

$$c \sin B = b \sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly, by drawing perpendicular from B on AC, we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Hence, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

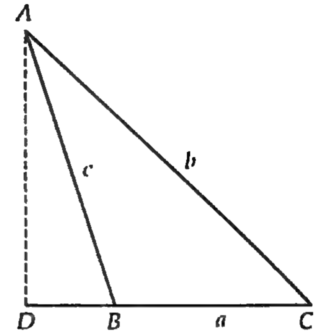


Fig. 10.2

CASE III *When ΔABC is a right angled triangle:*
In ΔABC, we have

$$\sin C = \sin \frac{\pi}{2} = 1,\ \sin A = \frac{BC}{AB} = \frac{a}{c} \text{ and, } \sin B = \frac{AC}{AB} = \frac{b}{c}$$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = c$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{1}$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \left[\because \sin C = \sin \frac{\pi}{2} = 1\right]$$

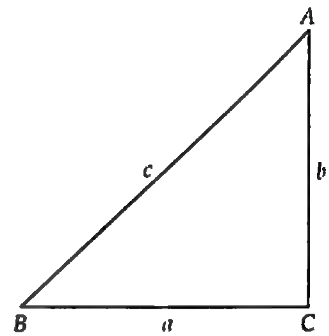


Fig. 10.3

Hence, in all the cases, we obtain

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Q.E.D.

REMARK 1 *The above rule may also be expressed as* $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

REMARK 2 *The sine rule is a very useful tool to express sides of a triangle in terms of the sines of angles and vice-versa in the following manner.*

Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (say) Then, $a = k \sin A$, $b = k \sin B$, $c = k \sin C$.

Similarly,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \lambda \text{ (say)} \Rightarrow \sin A = \lambda a,\ \sin B = \lambda b \text{ and } \sin C = \lambda c.$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS BASED ON SINE RULE**

RESULTS In a ΔABC, we have

$$\sin(B+C) = \sin A,\ \sin(C+A) = \sin B,\ \sin(A+B) = \sin C$$
$$\cos(B+C) = -\cos A,\ \cos(C+A) = -\cos B,\ \cos(A+B) = -\cos C$$
$$\tan(B+C) = -\tan A,\ \tan(C+A) = -\tan B,\ \tan(A+B) = -\tan C$$

EXAMPLE 1 *In a ΔABC, if $a = 2$, $b = 3$ and $\sin A = \dfrac{2}{3}$, find $\angle B$.*

SOLUTION We have,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{2}{(2/3)} = \frac{3}{\sin B} \Rightarrow 3 = \frac{3}{\sin B} \Rightarrow \sin B = 1 \Rightarrow \angle B = 90°$$

EXAMPLE 2 *If in any triangle the angles be to one another as 1 : 2 : 3, prove that the corresponding sides are 1 : $\sqrt{3}$: 2.*

SOLUTION Let the measures of the angles be x, $2x$ and $3x$. Then,

$$x + 2x + 3x = 180° \Rightarrow 6x = 180° \Rightarrow x = 30°$$

So, the angles are 30°, 60° and 90°

Let a, b, c be the lengths of the sides apposite to these angles. Then,

$$\frac{a}{\sin 30°} = \frac{b}{\sin 60°} = \frac{c}{\sin 90°}$$

$$\Rightarrow \quad a : b : c = \sin 30° : \sin 60° : \sin 90°$$

$$\Rightarrow \quad a : b : c : = \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 \Rightarrow a : b : c = 1 : \sqrt{3} : 2$$

EXAMPLE 3 *The angles of a triangle ABC are in A. P. and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, find $\angle A$.*

SOLUTION It is given that the angles $\angle A$, $\angle B$, $\angle C$ are in A.P.

$$\therefore \quad 2\angle B = \angle A + \angle C \Rightarrow 3\angle B = \angle A + \angle B + \angle C \Rightarrow 3\angle B = 180° \Rightarrow \angle B = 60°$$

Now, $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$$\Rightarrow \quad \frac{b}{c} = \frac{\sin B}{\sin C}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sin 60°}{\sin C} \qquad [\because b : c = \sqrt{3} : \sqrt{2}]$$

$$\Rightarrow \quad \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}/2}{\sin C}$$

$$\Rightarrow \quad \sin C = \frac{1}{\sqrt{2}} \Rightarrow \angle C = 45°$$

$$\therefore \quad \angle A = 180° - (\angle B + \angle C) = 180° - (60° + 45°) = 75°$$

EXAMPLE 4 *In any triangle ABC, prove that:*

(i) $\dfrac{\sin (B-C)}{\sin (B+C)} = \dfrac{b^2 - c^2}{a^2}$ [NCERT]

(ii) $a \sin (B-C) + b \sin (C-A) + c \sin (A-B) = 0$ [NCERT]

(iii) $a^3 \sin (B-C) + b^3 \sin (C-A) + c^3 \sin (A-B) = 0$

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then,

$$a = k \sin A,\ b = k \sin B \text{ and } c = k \sin C. \qquad \text{...(i)}$$

(i) $$\text{RHS} = \frac{b^2 - c^2}{a^2} = \frac{k^2 \sin^2 B - k^2 \sin^2 C}{k^2 \sin^2 A} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \text{RHS} = \frac{\sin^2 B - \sin^2 C}{\sin^2 A} = \frac{\sin (B+C) \sin (B-C)}{\sin^2 A}$$

$$\Rightarrow \quad \text{RHS} = \frac{\sin (\pi - A) \sin (B-C)}{\sin^2 A} \qquad [\because A + B + C = \pi \Rightarrow B + C = \pi - A]$$

$$\Rightarrow \quad \text{RHS} = \frac{\sin A \sin (B-C)}{\sin^2 A} = \frac{\sin (B-C)}{\sin A} = \frac{\sin (B-C)}{\sin (B+C)} = \text{LHS}$$

(ii) $\text{LHS} = a \sin (B-C) + b \sin (C-A) + c \sin (A-B)$

$= k \sin A \sin (B-C) + k \sin B \sin (C-A) + k \sin C \sin (A-B)$ [Using (i)]

$= k \left\{ \sin (B+C) \sin (B-C) + \sin (C+A) \sin (C-A) + \sin (A+B) \sin (A-B) \right\}$

$= k \left\{ \sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B \right\} = k(0) = \text{RHS}$

(iii) $\text{LHS} = k^3 \sin^3 A \sin (B-C) + k^3 \sin^3 B \sin (C-A) + k^3 \sin^3 C \sin (A-B)$

$= k^3 \left\{ \sin^2 A \sin A \sin (B-C) + \sin^2 B \sin B \sin (C-A) + \sin^2 C \sin C \sin (A-B) \right\}$

$= k^3 \left\{ \sin^2 A \sin (B+C) \sin (B-C) + \sin^2 B \sin (C+A) \sin (C-A) + \sin^2 C \sin (A+B) \sin (A-B) \right\}$

$= k^3 \left\{ \sin^2 A (\sin^2 B - \sin^2 C) + \sin^2 B (\sin^2 C - \sin^2 A) + \sin^2 C (\sin^2 A - \sin^2 B) \right\}$

$= k^3 \left\{ \sin^2 A \sin^2 B - \sin^2 A \sin^2 C + \sin^2 B \sin^2 C - \sin^2 B \sin^2 A + \sin^2 C \sin^2 A - \sin^2 C \sin^2 B \right\}$

$= k^3 \times 0 = 0 = \text{RHS}$

EXAMPLE 5 *In any triangle ABC, prove that:*

$$\frac{a^2 \sin (B-C)}{\sin B + \sin C} + \frac{b^2 \sin (C-A)}{\sin C + \sin A} + \frac{c^2 \sin (A-B)}{\sin A + \sin B} = 0$$

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k \sin A, b = k \sin B, c = k \sin C$

Now,

$$\text{LHS} = \frac{a^2 \sin (B-C)}{\sin B + \sin C} + \frac{b^2 \sin (C-A)}{\sin C + \sin A} + \frac{c^2 \sin (A-B)}{\sin A + \sin B}$$

$$= \frac{k^2 \sin^2 A \sin (B-C)}{\sin B + \sin C} + \frac{k^2 \sin^2 B \sin (C-A)}{\sin C + \sin A} + \frac{k^2 \sin^2 C \sin (A-B)}{\sin A + \sin B}$$

$$= k^2 \left\{ \frac{\sin A \sin (B+C) \sin (B-C)}{\sin B + \sin C} + \frac{\sin B \sin (C+A) \sin (C-A)}{\sin C + \sin A} + \frac{\sin C \sin (A+B) \sin (A-B)}{\sin A + \sin B} \right\}$$

$$= k^2 \left\{ \frac{\sin A (\sin^2 B - \sin^2 C)}{\sin B + \sin C} + \frac{\sin B (\sin^2 C - \sin^2 A)}{\sin C + \sin A} + \frac{\sin C (\sin^2 A - \sin^2 B)}{\sin A + \sin B} \right\}$$

$= k^2 \{\sin A (\sin B - \sin C) + \sin B (\sin C - \sin A) + \sin C (\sin A - \sin B)\}$

$= k^2 \times 0 = 0 = \text{RHS}$

EXAMPLE 6 *In any triangle ABC, prove that:*

(i) $$\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2}$$

(ii) $$\frac{b^2-c^2}{\cos B+\cos C}+\frac{c^2-a^2}{\cos C+\cos A}+\frac{a^2-b^2}{\cos A+\cos B}=0$$

SOLUTION Let $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k$. Then, $a=k\sin A, b=k\sin B, c=k\sin C$

(i) $$\frac{a\sin(B-C)}{b^2-c^2}=\frac{k\sin A\sin(B-C)}{k^2\sin^2 B-k^2\sin^2 C}=\frac{k\sin(B+C)\sin(B-C)}{k^2(\sin^2 B-\sin^2 C)}$$

$$=\frac{k(\sin^2 B-\sin^2 C)}{k^2(\sin^2 B-\sin^2 C)}=\frac{1}{k}$$

and, $$\frac{b\sin(C-A)}{c^2-a^2}=\frac{k\sin B\sin(C-A)}{k^2\sin^2 C-k^2\sin^2 A}=\frac{k\sin(C+A)\sin(C-A)}{k^2(\sin^2 C-\sin^2 A)}$$

$$=\frac{k(\sin^2 C-\sin^2 A)}{k^2(\sin^2 C-\sin^2 A)}=\frac{1}{k}$$

Similarly, it can be shown that $\dfrac{c\sin(A-B)}{a^2-b^2}=\dfrac{1}{k}$

Hence, $$\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2}$$

(ii) LHS $$=\frac{k^2(\sin^2 B-\sin^2 C)}{\cos B+\cos C}+\frac{k^2(\sin^2 C-\sin^2 A)}{\cos C+\cos A}+\frac{k^2(\sin^2 A-\sin^2 B)}{\cos A+\cos B}$$ [Using (i)]

$$=\frac{k^2\{(1-\cos^2 B)-(1-\cos^2 C)\}}{\cos B+\cos C}+\frac{k^2\{(1-\cos^2 C)-(1-\cos^2 A)\}}{\cos C+\cos A}$$

$$+\frac{k^2\{(1-\cos^2 A)-(1-\cos^2 B)\}}{\cos A+\cos B}$$

$$=k^2\left\{\frac{(\cos^2 C-\cos^2 B)}{\cos B+\cos C}+\frac{(\cos^2 A-\cos^2 C)}{\cos C+\cos A}+\frac{(\cos^2 B-\cos^2 A)}{\cos A+\cos B}\right\}$$

$$=k^2\{(\cos C-\cos B)+(\cos A-\cos C)+(\cos B-\cos A)\}=k\times 0=0=\text{RHS}$$

EXAMPLE 7 *In any triangle ABC, prove that:*

(i) $$\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\frac{a^2+b^2}{a^2+c^2}$$

(ii) $a\cos A+b\cos B+c\cos C=2a\sin B\sin C$ **[NCERT]**

SOLUTION Let $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k$. Then, $a=k\sin A, b=k\sin B, c=k\sin C$

(i) LHS $$=\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\frac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)}=\frac{1-(\cos^2 A-\sin^2 B)}{1-(\cos^2 A-\sin^2 C)}$$

$$=\frac{1-\cos^2 A+\sin^2 B}{1-\cos^2 A+\sin^2 C}=\frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C}=\frac{a^2/k^2+b^2/k^2}{a^2/k^2+c^2/k^2}=\frac{a^2+b^2}{a^2+c^2}=\text{RHS}$$

(ii) $$\begin{aligned}\text{LHS} &= a\cos A + b\cos B + c\cos C\\ &= k\sin A\cos A + k\sin B\cos B + k\sin C\cos C\\ &= \frac{k}{2}\{\sin 2A + \sin 2B + \sin 2C\} = \frac{k}{2}(4\sin A\sin B\sin C)\\ &= 2k\sin A\sin B\sin C = 2a\sin B\sin C = \text{RHS} \qquad [\because k\sin A = a]\end{aligned}$$

EXAMPLE 8 *In any triangle ABC, prove that:*

(i) $\sin\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{a}\right)\cos\frac{A}{2}$ [NCERT] (ii) $a\cos\left(\frac{B-C}{2}\right) = (b+c)\sin\frac{A}{2}$ [NCERT]

(iii) $$\frac{b-c}{b+c} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$$

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k\sin A, b = k\sin B, c = k\sin C$

(i) $$\begin{aligned}\text{RHS} &= \left(\frac{b-c}{a}\right)\cos\frac{A}{2} = \left\{\frac{k\sin B - k\sin C}{k\sin A}\right\}\cos\frac{A}{2} = \left\{\frac{\sin B - \sin C}{\sin A}\right\}\cos\frac{A}{2}\\ &= \frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}}\cos\frac{A}{2} = \frac{\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{\pi-A}{2}\right)}{\sin\frac{A}{2}}\\ &= \frac{\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)}{\sin\frac{A}{2}} = \frac{\sin\left(\frac{B-C}{2}\right)\sin\frac{A}{2}}{\sin\frac{A}{2}} = \sin\left(\frac{B-C}{2}\right) = \text{LHS}\end{aligned}$$

(ii) $$\begin{aligned}\text{RHS} &= (b+c)\sin\frac{A}{2} = (k\sin B + k\sin C)\sin\frac{A}{2} = k(\sin B + \sin C)\sin\frac{A}{2}\\ &= k\times 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\sin\frac{A}{2} = 2k\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\sin\frac{A}{2}\\ &= 2k\cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right)\sin\frac{A}{2} = k\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\\ &= k\sin A\cos\left(\frac{B-C}{2}\right) = a\cos\left(\frac{B-C}{2}\right) = \text{LHS} \qquad [\because k\sin A = a]\end{aligned}$$

(iii) $$\begin{aligned}\text{LHS} &= \frac{b-c}{b+c} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C} = \frac{\sin B - \sin C}{\sin B + \sin C}\\ &= \frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)} = \text{RHS}\end{aligned}$$

EXAMPLE 9 *In a triangle ABC, if $a\cos A = b\cos B$, show that the triangle is either isosceles or right angled.*

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k\sin A$, $b = k\sin B$, $c = k\sin C$

Now, $a\cos A = b\cos B$

$\Rightarrow \quad k \sin A \cos A = k \sin B \cos B$

$\Rightarrow \quad 2 \sin A \cos A = 2 \sin B \cos B$

$\Rightarrow \quad \sin 2A = \sin 2B$

$\Rightarrow \quad 2A = 2B$ or, $2A = \pi - 2B$

$\Rightarrow \quad A = B$ or, $A + B = \pi/2$

$\Rightarrow \quad A = B$ or, $C = \dfrac{\pi}{2}$

$\Rightarrow \quad BC = CA$ or, $C = \dfrac{\pi}{2}$

$\Rightarrow \quad \Delta ABC$ is either isosceles or right angled.

EXAMPLE 10 *If in a* ΔABC, $\dfrac{\sin A}{\sin C} = \dfrac{\sin(A-B)}{\sin(B-C)}$, *prove that* a^2, b^2, c^2 *are in A.P.*

SOLUTION Let $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k$. Then, $\sin A = ak$, $\sin B = bk$, $\sin C = ck$

Now, $\dfrac{\sin A}{\sin C} = \dfrac{\sin(A-B)}{\sin(B-C)}$

$\Rightarrow \quad \dfrac{\sin(B+C)}{\sin(A+B)} = \dfrac{\sin(A-B)}{\sin(B-C)}$ $\quad$ [$\because$ $\sin A = \sin(B+C)$ and $\sin C = \sin(A+B)$]

$\Rightarrow \quad \sin(B+C)\sin(B-C) = \sin(A+B)\sin(A-B)$

$\Rightarrow \quad \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$

$\Rightarrow \quad k^2 b^2 - k^2 c^2 = k^2 a^2 - k^2 b^2$

$\Rightarrow \quad b^2 - c^2 = a^2 - b^2 \Rightarrow 2b^2 = a^2 + c^2 \Rightarrow a^2, b^2, c^2$ are in A.P.

Type II APPLICATIONS OF SINE FORMULA IN PROBLEMS ON HEIGHTS AND DISTANCES

EXAMPLE 11 *A tree stands vertically on a hill side which makes an angle of* 15° *with the horizontal. From a point on the ground* 35 m *down the hill from the base of the tree, the angle of elevation of the top of the tree is* 60°. *Find the height of the tree.* **[NCERT]**

SOLUTION Let PQ be the tree on the hill which makes an angel of 15° with the horizontal AR, where A is a point on the ground 35 m down the hill from the base P of the tree.

In ΔARQ, we have

$\angle RAQ = 60°$ and $\angle ARQ = 90°$

$\therefore \quad \angle AQP = 30°$

In ΔAPQ, we have

$\angle PAQ = 45°$ and $\angle AQP = 30°$

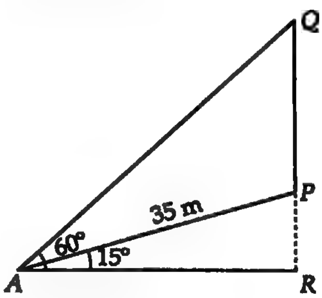


Fig. 10.4

Using Sine rule in ΔAPQ, we get

$$\frac{AP}{\sin \angle AQP} = \frac{PQ}{\sin \angle PAQ}$$

$\Rightarrow \quad \dfrac{35}{\sin 30°} = \dfrac{PQ}{\sin 45°} \Rightarrow \dfrac{35}{1/2} = \dfrac{PQ}{1/\sqrt{2}} \Rightarrow PQ = \dfrac{70}{\sqrt{2}} \Rightarrow PQ = 35\sqrt{2}$ m.

Hence, height of the tree is $35\sqrt{2}$ m.

EXAMPLE 12 *A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is* 60°, *when he retreats* 20 m *from the bank, he finds the angle to be* 30°. *Find the height of the tree and the breadth of the river.*

SOLUTION Let AB be the tree at one bank of the river and let the P be the position of the person at the other bank of the river. After retreating 20 m from the bank, let the man be at Q. It is given that $\angle APB = 60°$ and $\angle AQB = 30°$.

Now, $\angle APB = 60° \Rightarrow \angle BPQ = 120°$ and $\angle PBA = 30°$

In $\Delta\, BPQ$, we have

$$\angle PQB = 30°\,,\ \angle BPQ = 120° \text{ and, } \angle PBQ = 30°$$

Using Sine rule in triangles BPQ and PAB, we get

$$\frac{BP}{\sin \angle PQB} = \frac{PQ}{\sin \angle PBQ} \text{ and } \frac{BP}{\sin \angle PAB} = \frac{AP}{\sin \angle ABP} = \frac{AB}{\sin \angle APB}$$

$$\Rightarrow \quad \frac{BP}{\sin 30°} = \frac{PQ}{\sin 30°} \text{ and } \frac{BP}{\sin 90°} = \frac{AP}{\sin 30°} = \frac{AB}{\sin 60°}$$

$$\Rightarrow \quad \frac{BP}{1/2} = \frac{20}{1/2} \text{ and } BP = 2AP = \frac{2AB}{\sqrt{3}}$$

$$\Rightarrow \quad BP = 20 \text{ and } BP = 2AP = \frac{2AB}{\sqrt{3}} \qquad [\because PQ = 20 \text{ m}]$$

$$\Rightarrow \quad 20 = 2AP = \frac{2AB}{\sqrt{3}}$$

$$\Rightarrow \quad AP = 10 \text{ and } AB = 10\sqrt{3}$$

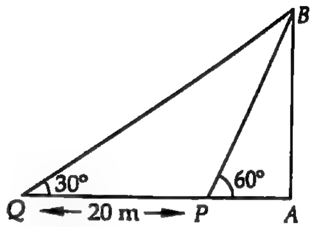


Fig. 10.5

Hence, the breadth of the river is 10 m and height of the tree is $10\sqrt{3}$ m.

EXAMPLE 13 *The angle of elevation of the top point P of the vertical tower PQ of heigth h from a point A is 45° and from a point B, the angle of elevation is 60°, where B is a point at a distance d from the point A measured along the line AB which makes an angle 30° with AQ. Prove that $d = (\sqrt{3} - 1)\, h$.* [NCERT]

SOLUTION It is given that $\angle PAQ = 45°$ and $\angle BAQ = 30°$. Therefore, $\angle BAP = 15°$.

In $\Delta\, AQP$, we have

$$\angle PAQ = 45° \text{ and } \angle PQA = 90°$$

$$\therefore \quad \angle APQ = 45°$$

In ΔBRP, we have

$$\angle PBR = 60° \text{ and } \angle PRB = 90°$$

$$\therefore \quad \angle BPR = 30°$$

Now, $\angle APQ = 45°$ and $\angle BPR = 30° \Rightarrow \angle BPA = 15°$

In $\Delta\, ABP$, we have

$$\angle PAB = 15° \text{ and } \angle BPA = 15°.$$

$$\therefore \quad \angle ABP = 150°$$

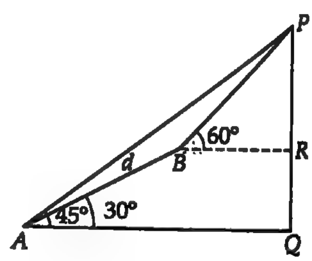


Fig. 10.6

Using Sine rule in triangle ABP, we get

$$\frac{AB}{\sin \angle APB} = \frac{BP}{\angle PAB} = \frac{AP}{\sin \angle ABP}$$

$$\Rightarrow \quad \frac{d}{\sin 15°} = \frac{BP}{\sin 15°} = \frac{AP}{\sin 150°}$$

$$\Rightarrow \quad \frac{d}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{AP}{\frac{1}{2}} \Rightarrow AP = \frac{\sqrt{2}\, d}{\sqrt{3}-1} \qquad \text{...(i)}$$

Using Sine rule in $\Delta\, AQP$, we get

$$\frac{AP}{\sin \angle AQP} = \frac{AQ}{\sin \angle APQ} = \frac{PQ}{\sin \angle PAQ}$$

$$\Rightarrow \quad \frac{AP}{\sin 90^\circ} = \frac{PQ}{\sin 45^\circ} \Rightarrow AP = \sqrt{2}\ PQ \Rightarrow AP = \sqrt{2}\,h \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\sqrt{2}h = \frac{\sqrt{2}\,d}{\sqrt{3}-1} \Rightarrow d = (\sqrt{3}-1)\,h$$

LEVEL-2

EXAMPLE 14 *If in a* ΔABC, $\frac{a^2-b^2}{a^2+b^2} = \frac{\sin(A-B)}{\sin(A+B)}$, *prove that it is either a right angled or an isosceles triangle.*

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$, (say). Then, $a = k\sin A, b = k\sin B, c = k\sin C$

$$\therefore \quad \frac{a^2-b^2}{a^2+b^2} = \frac{\sin(A-B)}{\sin(A+B)}$$

$$\Rightarrow \quad \frac{k^2\sin^2 A - k^2\sin^2 B}{k^2\sin^2 A + k^2\sin^2 B} = \frac{\sin(A-B)}{\sin(A+B)}$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin(\pi-C)} = \frac{\sin(A+B)\sin(A-B)}{\sin^2 A + \sin^2 B}$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin C} = \frac{\sin(\pi-C)\sin(A-B)}{\sin^2 A + \sin^2 B}$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin C} = \frac{\sin C\sin(A-B)}{\sin^2 A + \sin^2 B}$$

$$\Rightarrow \quad \sin(A-B)\left\{\frac{1}{\sin C} - \frac{\sin C}{\sin^2 A + \sin^2 B}\right\} = 0$$

$$\Rightarrow \quad \text{either } \sin(A-B) = 0 \quad \text{or,} \quad \frac{1}{\sin C} - \frac{\sin C}{\sin^2 A + \sin^2 B} = 0$$

$$\Rightarrow \quad \text{either } A - B = 0 \quad \text{or,} \quad \sin^2 A + \sin^2 B - \sin^2 C = 0$$

$$\Rightarrow \quad \text{either } A = B \quad \text{or,} \quad \frac{a^2}{k^2} + \frac{b^2}{k^2} - \frac{c^2}{k^2} = 0 \qquad [\because\ a = k\sin A, b = k\sin B, c = k\sin C]$$

$$\Rightarrow \quad \text{either } A = B \quad \text{or,} \quad a^2 + b^2 = c^2$$

$\Rightarrow$ either the triangle is isosceles or it is right angled.

EXAMPLE 15 *In any triangle ABC, prove that:*

$$(b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2} = 0$$

SOLUTION We have,

$$\text{LHS} = (b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2}$$

$$= k(\sin B - \sin C)\cot\frac{A}{2} + k(\sin C - \sin A)\cot\frac{B}{2} + k(\sin A - \sin B)\cot\frac{C}{2}$$

$$= k\left[2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)\cot\frac{A}{2} + 2\sin\left(\frac{C-A}{2}\right)\cos\left(\frac{C+A}{2}\right)\cot\frac{B}{2}\right.$$
$$\left. + 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\cot\frac{C}{2}\right]$$

$$= k\left[2\sin\left(\frac{B-C}{2}\right)\sin\frac{A}{2}\cot\frac{A}{2} + 2\sin\left(\frac{C-A}{2}\right)\sin\frac{B}{2}\cot\frac{B}{2} + 2\sin\left(\frac{A-B}{2}\right)\sin\frac{C}{2}\cot\frac{C}{2}\right]$$

$$= k\left[2\cos\frac{A}{2}\sin\left(\frac{B-C}{2}\right) + 2\cos\frac{B}{2}\sin\left(\frac{C-A}{2}\right) + 2\cos\frac{C}{2}\sin\left(\frac{A-B}{2}\right)\right]$$

$$= 2k\left[\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right) + \sin\left(\frac{C+A}{2}\right)\sin\left(\frac{C-A}{2}\right) + \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$\left[\because \cos\frac{A}{2} = \sin\left(\frac{B+C}{2}\right) \text{etc.}\right]$$

$$= 2k\left\{\sin^2\frac{B}{2} - \sin^2\frac{C}{2} + \sin^2\frac{C}{2} - \sin^2\frac{A}{2} + \sin^2\frac{A}{2} - \sin^2\frac{B}{2}\right\} = 2k \times 0 = 0 = \text{RHS}$$

EXAMPLE 16 *Let O be a point inside a triangle ABC such that $\angle OAB = \angle OBC = \angle OCA = \omega$, then show that:*

(i) $\cot\omega = \cot A + \cot B + \cot C$ (ii) $\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$

SOLUTION (i) In ΔOBC,

$\angle OCB = \angle C - \omega$ and, $\angle BOC = 180 - \omega - (C - \omega) = 180° - C$

Similarly, we obtain $\angle AOB = 180° - B$

Applying sine rule in ΔOAB, we obtain

$$\frac{OB}{\sin\angle OAB} = \frac{AB}{\sin\angle AOB}$$

$$\frac{OB}{\sin\omega} = \frac{AB}{\sin(180° - B)}$$

$$\Rightarrow \quad \frac{OB}{\sin\omega} = \frac{c}{\sin B}$$

$$\Rightarrow \quad OB = \frac{c\sin\omega}{\sin B} \qquad \text{...(i)}$$

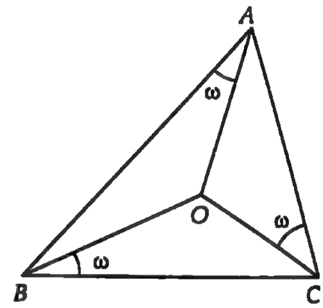


Fig. 10.7

Applying sine rule in ΔOBC, we get

$$\frac{OB}{\sin\angle BCO} = \frac{BC}{\sin\angle BOC}$$

$$\Rightarrow \quad \frac{OB}{\sin(C-\omega)} = \frac{BC}{\sin(180° - C)} \Rightarrow \frac{OB}{\sin(C-\omega)} = \frac{a}{\sin C} \Rightarrow OB = \frac{a\sin(C-\omega)}{\sin C} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{c\sin\omega}{\sin B} = \frac{a\sin(C-\omega)}{\sin C}$$

$$\Rightarrow \quad \frac{k\sin C\sin\omega}{\sin B} = \frac{k\sin A\sin(C-\omega)}{\sin C} \qquad \text{[Using sine rule]}$$

$$\Rightarrow \quad \sin^2 C\sin\omega = \sin A\sin B\sin(C-\omega)$$

$$\Rightarrow \quad \sin C\sin(A+B)\sin\omega = \sin A\sin B\sin(C-\omega) \; [\because \sin C = \sin(\pi - (A+B)) = \sin(A+B)]$$

$$\Rightarrow \quad \frac{\sin(A+B)}{\sin A\sin B} = \frac{\sin(C-\omega)}{\sin C\sin\omega}$$

$$\Rightarrow \quad \frac{\sin A\cos B + \cos A\sin B}{\sin A\sin B} = \frac{\sin C\cos\omega - \cos C\sin\omega}{\sin C\sin\omega}$$

$$\Rightarrow \quad \cot B + \cot A = \cot\omega - \cot C \Rightarrow \cot\omega = \cot A + \cot B + \cot C$$

(ii) From (i), we have

$$\cot \omega = \cot A + \cot B + \cot C$$

$$\Rightarrow \quad \cot^2 \omega = \cot^2 A + \cot^2 B + \cot^2 C + 2(\cot A \cot B + \cot B \cot C + \cot C \cot A)$$

$$\Rightarrow \quad \cot^2 \omega = \cot^2 A + \cot^2 B + \cot^2 C + 2 \quad [\because \cot A \cot B + \cot B \cot C + \cot C \cot A = 1]$$

$$\Rightarrow \quad \operatorname{cosec}^2 \omega - 1 = (\operatorname{cosec}^2 A - 1) + (\operatorname{cosec}^2 B - 1) + (\operatorname{cosec}^2 C - 1) + 2$$

$$\Rightarrow \quad \operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$$

EXAMPLE 17 *The angle of elevation of the top of a tower from a point A due South of the tower is α and from B due East of the tower is β. If AB = d, show that the height of the tower is* $\dfrac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta}}$.

SOLUTION Let OP be the tower and let A and B be two points due South and East respectively of the tower such that $\angle OAP = \alpha$ and $\angle OBP = \beta$. Then,

$$\angle OPA = \frac{\pi}{2} - \alpha \text{ and } \angle OPB = \frac{\pi}{2} - \beta.$$

Using sine rule in Δ's OAP and OBP, we have

$$\frac{OA}{\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{OP}{\sin \alpha} \text{ and, } \frac{OB}{\sin\left(\frac{\pi}{2} - \beta\right)} = \frac{OP}{\sin \beta}$$

$$\Rightarrow \quad \frac{OA}{\cos \alpha} = \frac{OP}{\sin \alpha} \text{ and } \frac{OB}{\cos \beta} = \frac{OP}{\sin \beta}$$

$$\Rightarrow \quad OA = OP \cot \alpha \text{ and } OB = OP \cot \beta$$

Using Pythagoras theorem in Δ AOB, we get

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow \quad d^2 = OP^2 \cot^2 \alpha + OP^2 \cot^2 \beta$$

$$\Rightarrow \quad OP = \frac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta}}$$

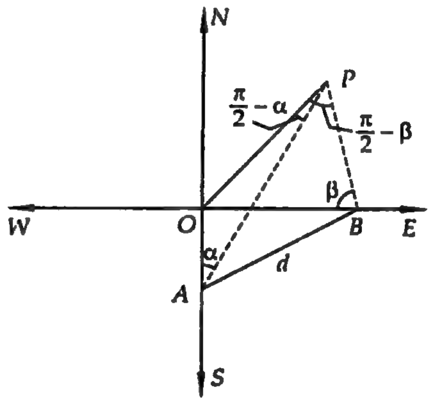


Fig. 10.8

EXAMPLE 18 *The elevation of a tower at a station A due North of it is α and at a station B due West of A is β. Prove that the height of the tower is* $\dfrac{AB \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}$.

SOLUTION Let OP be the tower and let A be a point due north of the tower OP and let B be the point due west of A such that $\angle OAP = \alpha$ and $\angle OBP = \beta$.

Clearly, triangles AOP and BOP are right triangles right angled at O.

$$\therefore \quad \angle OPA = \frac{\pi}{2} - \alpha \text{ and } \angle OPB = \frac{\pi}{2} - \beta$$

Using sine rule in triangles AOP and BOP, we get

$$\frac{OA}{\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{OP}{\sin \alpha} \text{ and } \frac{OB}{\sin\left(\frac{\pi}{2} - \beta\right)} = \frac{OP}{\sin \beta}$$

$$\Rightarrow \quad OA = OP \cot \alpha \text{ and } OB = OP \cot \beta$$

Applying Pythagoras theorem in ΔOAB, we get

$$OB^2 = OA^2 + AB^2$$

$$\Rightarrow \quad OB^2 - OA^2 = AB^2$$

$$\Rightarrow \quad OP^2 \cot^2 \beta - OP^2 \cot \alpha^2 = AB^2$$

$$\Rightarrow \quad OP^2 (\cot^2 \beta - \cot^2 \alpha) = AB^2$$

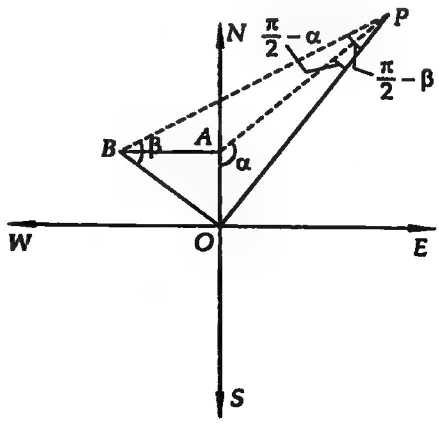


Fig. 10.9

$$\Rightarrow \quad OP^2 (\text{cosec}^2 \beta - \text{cosec}^2 \alpha) = AB^2$$

$$\Rightarrow \quad OP^2 \frac{(\sin^2 \alpha - \sin^2 \beta)}{\sin^2 \alpha \sin^2 \beta} = AB^2 \Rightarrow OP = \frac{AB \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}$$

EXAMPLE 19 *An object is observed from three points A, B, C in the same horizontal line passing through the base of the object. The angle of elevation at B is twice and at C thrice that at A. If $AB = a$, $BC = b$ prove that the height of the object is $\frac{a}{2b}\sqrt{(a+b)(3b-a)}$.*

SOLUTION Let the object be at P at a height h from OA. Let the object when observed from A, B and C the angles of elevation are θ, 2θ and 3θ respectively.

In ΔPAB, we have

$$2\theta = \theta + \angle APB \Rightarrow \angle APB = \theta$$

$$\therefore \quad \angle PAB = \angle APB = \theta \Rightarrow AB = BP = a$$

Similarly, in triangle BPC, $\angle BPC = \theta$.

In ΔOPB

$$\sin 2\theta = \frac{h}{a} \Rightarrow h = a \sin 2\theta$$

$$\Rightarrow \quad h = 2a \sin\theta \cos\theta \quad \text{...(i)}$$

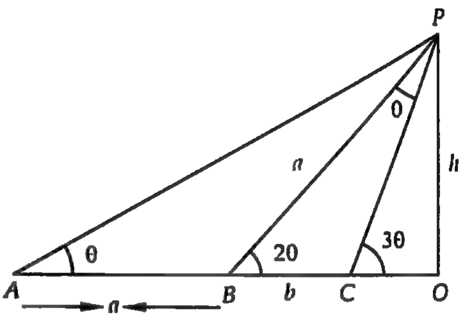


Fig. 10.10

In ΔPBC

$$\frac{PB}{\sin(\pi - 3\theta)} = \frac{BC}{\sin\theta} \quad \text{[Using sine rule]}$$

$$\Rightarrow \quad \frac{a}{\sin 3\theta} = \frac{b}{\sin\theta}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{\sin 3\theta}{\sin\theta}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta}$$

$$\Rightarrow \quad \frac{a}{b} = 3 - 4\sin^2\theta \Rightarrow 4\sin^2\theta = 3 - \frac{a}{b} \Rightarrow \sin^2\theta = \frac{3b-a}{4b} \Rightarrow \sin\theta = \sqrt{\frac{3b-a}{4b}}$$

$$\therefore \quad \cos^2\theta = 1 - \sin^2\theta \Rightarrow \cos^2\theta = 1 - \frac{3b-a}{4b} = \frac{a+b}{4b} \Rightarrow \cos\theta = \sqrt{\frac{a+b}{4b}}$$

Substituting the values of $\sin\theta$ and $\cos\theta$ in (i), we get

$$h = 2a\sqrt{\frac{3b-a}{4b}} \times \sqrt{\frac{a+b}{4b}} = \frac{a}{2b}\sqrt{(a+b)(3b-a)}$$

EXERCISE 10.1

LEVEL-1

1. If in a ΔABC, $\angle A = 45°$, $\angle B = 60°$, and $\angle C = 75°$; find the ratio of its sides.
2. If in any ΔABC, $\angle C = 105°$, $\angle B = 45°$, $a = 2$, then find b.
3. In ΔABC, if $a = 18$, $b = 24$ and $C = 30$ and $\angle C = 90°$, find $\sin A$, $\sin B$ and $\sin C$.

In any triangle ABC, prove the following: (4-24)

4. $$\frac{a-b}{a+b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$$

5. $(a-b)\cos\frac{C}{2} = c\sin\left(\frac{A-B}{2}\right)$ [NCERT]

6. $\frac{c}{a-b} = \frac{\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)}{\tan\left(\frac{A}{2}\right)-\tan\left(\frac{B}{2}\right)}$

7. $\frac{c}{a+b} = \frac{1-\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}{1+\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}$

8. $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$ [NCERT]

9. $\sin\left(\frac{B-C}{2}\right) = \frac{b-c}{a}\cos\frac{A}{2}$

10. $\frac{a^2-c^2}{b^2} = \frac{\sin(A-C)}{\sin(A+C)}$

11. $b\sin B - c\sin C = a\sin(B-C)$

12. $a^2\sin(B-C) = (b^2-c^2)\sin A$

13. $\frac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}+\sqrt{\sin B}} = \frac{a+b-2\sqrt{ab}}{a-b}$

14. $a(\sin B-\sin C)+b(\sin C-\sin A)+c(\sin A-\sin B) = 0$

15. $\frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C} = 0$

16. $a^2(\cos^2 B-\cos^2 C)+b^2(\cos^2 C-\cos^2 A)+c^2(\cos^2 A-\cos^2 B) = 0$

17. $b\cos B+c\cos C = a\cos(B-C)$

18. $\frac{\cos 2A}{a^2}-\frac{\cos 2B}{b^2} = \frac{1}{a^2}-\frac{1}{b^2}$

19. $\frac{\cos^2 B-\cos^2 C}{b+c}+\frac{\cos^2 C-\cos^2 A}{c+a}+\frac{\cos^2 A-\cos^2 B}{a+b} = 0$

20. $a\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right)+b\sin\frac{B}{2}\sin\left(\frac{C-A}{2}\right)+c\sin\frac{C}{2}\sin\left(\frac{A-B}{2}\right)=0.$

21. $\frac{b\sec B+c\sec C}{\tan B+\tan C} = \frac{c\sec C+a\sec A}{\tan C+\tan A} = \frac{a\sec A+b\sec B}{\tan A+\tan B}.$

22. $a\cos A+b\cos B+c\cos C = 2b\sin A\sin C = 2c\sin A\sin B$

23. $a(\cos B\cos C+\cos A) = b(\cos C\cos A+\cos B) = c(\cos A\cos B+\cos C).$

24. $a(\cos C-\cos B) = 2(b-c)\cos^2\frac{A}{2}.$ [NCERT]

25. In ΔABC prove that, if θ be any angle, then $b\cos\theta = c\cos(A-\theta)+a\cos(C+\theta)$.

26. In a ΔABC, if $\sin^2 A+\sin^2 B = \sin^2 C$, show that the triangle is right angled.

LEVEL-2

27. In any ΔABC, if a^2, b^2, c^2 are in A.P., prove that $\cot A$, $\cot B$ and $\cot C$ are also in A.P.
28. The upper part of a tree broken over by the wind makes an angle of 30° with the ground and the distance from the root to the point where the top of the tree touches the ground is 15 m. Using sine rule, find the height of the tree.
29. At the foot of a mountain the elevation of its summit is 45°; after ascending 1000 m towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. Find the height of the mountain.
30. A person observes the angle of elevation of the peak of a hill from a station to be α. He walks c metres along a slope inclined at the angle β and finds the angle of elevation of the peak of the hill to be γ. Show that the height of the peak above the ground is $\dfrac{c \sin \alpha \sin(\gamma-\beta)}{(\sin \gamma - \alpha)}$.
31. If the sides a, b, c of a ΔABC are in H.P., prove that $\sin^2 \dfrac{A}{2}$, $\sin^2 \dfrac{B}{2}$, $\sin^2 \dfrac{C}{2}$ are in H.P.

ANSWERS

1. $2 : \sqrt{6} : \sqrt{3}+1$ 2. $2\sqrt{2}$ 3. $\sin A = \dfrac{3}{5}$, $\sin B = \dfrac{4}{5}$

28. $15\sqrt{3}$ m 29. $500(\sqrt{3}+1)$ metres

10.3 THE LAW OF COSINES

THEOREM *In any ΔABC, we have:*

$$\text{(i)}\quad a^2 = b^2 + c^2 - 2bc \cos A \quad or, \quad \cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\text{(ii)}\quad b^2 = c^2 + a^2 - 2ac \cos B \quad or, \quad \cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\text{(iii)}\quad c^2 = a^2 + b^2 - 2ab \cos C \quad or, \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

PROOF The following cases may arise:

CASE I *When ΔABC is an acute angled triangle:*

Draw perpendicular AD from A on BC.

In ΔABD, we have

$$\cos B = \frac{BD}{c} \Rightarrow BD = c \cos B \qquad \text{...(i)}$$

In ΔACD, we have

$$\cos C = \frac{CD}{b} \Rightarrow CD = b \cos C$$

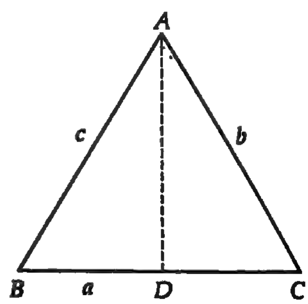


Fig. 10.11

In ΔACD, using Pythagoras theorem, we have

$$AC^2 = AD^2 + CD^2$$

$$\Rightarrow \quad AC^2 = AD^2 + (BC - BD)^2$$

$$\Rightarrow \quad AC^2 = AD^2 + BC^2 + BD^2 - 2\,BC\,.\,BD$$

$$\Rightarrow \quad AC^2 = BC^2 + (AD^2 + BD^2) - 2BC\,.\,BD$$

$$\Rightarrow \quad AC^2 = BC^2 + AB^2 - 2\,BC\,.\,BD \qquad [\because AB^2 = BD^2 + AD^2]$$

$$\Rightarrow \quad b^2 = a^2 + c^2 - 2ac \cos B \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad b^2 = c^2 + a^2 - 2ca \cos B$$

$$\Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

CASE II *When ΔABC is an obtuse angled triangle:*

Draw perpendicular AD from A on CB produced.

In ΔABD, we have

$$\cos(180 - B) = \frac{BD}{AB} \Rightarrow BD = -AB \cos B = -c \cos B \quad \text{...(i)}$$

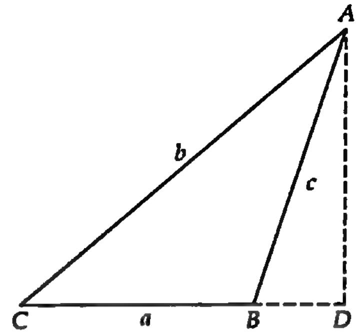


Fig. 10.12

Using Pythagoras theorem in ΔACD, we have

$$AC^2 = AD^2 + CD^2$$

$$\Rightarrow \quad AC^2 = AD^2 + (CB + BD)^2$$

$$\Rightarrow \quad AC^2 = AD^2 + CB^2 + BD^2 + 2CB \,.\, BD$$

$$\Rightarrow \quad AC^2 = BC^2 + (BD^2 + AD^2) + 2BC \,.\, BD$$

$$\Rightarrow \quad AC^2 = BC^2 + AB^2 + 2BC \,.\, BD \qquad [\text{In } \Delta ABD,\ AB^2 = AD^2 + BD^2]$$

$$\Rightarrow \quad b^2 = a^2 + c^2 + 2a(-c \cos B) \qquad [\text{Using (i)}]$$

$$\Rightarrow \quad b^2 = c^2 + a^2 - 2ac \cos B$$

$$\Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

CASE III When ΔABC is a right angled triangle:

Let ΔABC be a right angled triangle with right angle at B. Then, by Pythagoras theorem, we obtain

$$b^2 = a^2 + c^2$$

$$\Rightarrow \quad b^2 = a^2 + c^2 - 2ac \cos B \qquad \left[\because B = \frac{\pi}{2} \therefore \cos B = 0\right]$$

$$\Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

Hence, in all the cases, we have

$$b^2 = c^2 + a^2 - 2ac \cos B \Rightarrow \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

In a similar manner other results can be proved.

10.4 PROJECTION FORMULAE

THEOREM *In any ΔABC, we have*

(i) $a = b \cos C + c \cos B$ (ii) $b = c \cos A + a \cos C$ (iii) $c = a \cos B + b \cos A$

i.e. any side of a triangle is equal to the sum of the projections of other two sides on it.

PROOF The following cases arise:

CASE I When ΔABC is an acute angled triangle:

In Fig. 10.1, we have

$$\cos B = \frac{BD}{AB} \Rightarrow BD = AB \cos B \Rightarrow BD = c \cos B$$

and, $$\cos C = \frac{CD}{AC} \Rightarrow CD = AC \cos C \Rightarrow CD = b \cos C$$

Hence, $a = BC = BD + CD \Rightarrow a = c \cos B + b \cos C$

CASE II *When ΔABC is an obtuse angled triangle*

In Fig. 10.2, we have

$$\cos C = \frac{CD}{AC} \Rightarrow CD = AC \cos C \Rightarrow CD = b \cos C$$

and, $$\cos(180 - B) = \frac{BD}{AB} \Rightarrow BD = AB\cos(180 - B) \Rightarrow BD = -c\cos B$$

$\therefore$ $$a = BC = CD - BD \Rightarrow a = b\cos C + c\cos B$$

Hence, in each case, we obtain $a = b\cos C + c\cos B$

Similarly, it can be proved that

$$b = c\cos A + a\cos C \text{ and } c = a\cos B + b\cos A$$ Q.E.D.

REMARK *In Fig. 10.1, BD and CD are the projections of AB and AC respectively on BC.*

10.5 NAPIER'S ANALOGY (LAW OF TANGENTS)

THEOREM *In any ΔABC, we have*

(i) $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$ (ii) $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$

(iii) $\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$

PROOF Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k\sin A, b = k\sin B, c = k\sin C$ ...(i)

(i) $$\text{RHS} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C}\cot\frac{A}{2}$$ [Using (i)]

$$= \left(\frac{\sin B - \sin C}{\sin B + \sin C}\right)\cot\frac{A}{2} = \left\{\frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}\right\}$$

$$= \tan\left(\frac{B-C}{2}\right)\cot\left(\frac{B+C}{2}\right)\cot\frac{A}{2} = \tan\left(\frac{B-C}{2}\right)\cot\left(\frac{\pi}{2}-\frac{A}{2}\right)\cot\frac{A}{2}$$

$$= \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}\cot\frac{A}{2} = \tan\left(\frac{B-C}{2}\right) = \text{LHS}$$

Similarly, (ii) and (iii) can be proved.

10.6 AREA OF A TRIANGLE

THEOREM *Prove that the area of ΔABC is given by*

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

PROOF Let ABC be a triangle. Then the following cases arise :

CASE I *When ΔABC is an acute angled triangle:*

In Fig. 10.1, we have

$$\sin B = \frac{AD}{AB} \Rightarrow AD = AB\sin B = c\sin B$$

$\therefore \quad \Delta = \text{Area of } \Delta ABC = \frac{1}{2} BC \times AD = \frac{1}{2} a c \sin B$

CASE II *When ΔABC is an obtuse angled triangle:*

In Fig. 10.2, we have

$$\sin (180 - B) = \frac{AD}{AB} \Rightarrow AD = AB \sin B = c \sin B$$

$\therefore \quad \Delta = \text{Area of } \Delta ABC = \frac{1}{2} BC \times AD = \frac{1}{2} a c \sin B$

Thus, in each case, we have $\Delta = \frac{1}{2} ac \sin B$

Similarly, it can be proved that $\Delta = \frac{1}{2} ab \sin C$ and $\Delta = \frac{1}{2} bc \sin A$

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS ON APPLICATIONS OF COSINE FORMULA AND SINE RULE**

EXAMPLE 1 *In a ΔABC, if $a = 3, b = 5$ and $c = 7$, find $\cos A$, $\cos B$ and $\cos C$.*

SOLUTION We have, $a = 3, b = 5$ and $c = 7$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{25 + 49 - 9}{2 \times 5 \times 7} = \frac{65}{70} = \frac{13}{14}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2bc} = \frac{49 + 9 - 25}{2 \times 3 \times 7} = \frac{33}{42} = \frac{11}{14}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{9 + 25 - 49}{2 \times 3 \times 5} = -\frac{15}{30} = -\frac{1}{2}$$

EXAMPLE 2 *If the sides of a Δ ABC are $a = 4, b = 6$ and $c = 8$, show that $4 \cos B + 3 \cos C = 2$.*

SOLUTION We have, $a = 4, b = 6$ and, $c = 8$

$$\therefore \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{16 + 64 - 36}{2 \times 4 \times 8} = \frac{44}{64} = \frac{11}{16}$$

$$\text{and,} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{16 + 36 - 64}{2 \times 4 \times 6} = -\frac{12}{48} = -\frac{1}{4}$$

$$\therefore \quad 4 \cos B + 3 \cos C = 4 \times \frac{11}{16} - \frac{3}{4} = \frac{11}{4} - \frac{3}{4} = 2$$

EXAMPLE 3 *In any ΔABC, prove that:*

(i) $a (b \cos C - c \cos B) = b^2 - c^2$ [NCERT]

(ii) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$ [NCERT]

(iii) $2 (bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$

SOLUTION (i) LHS $= a (b \cos C - c \cos B) = ab \cos C - ac \cos B$

$$= ab \left(\frac{a^2 + b^2 - c^2}{2ab} \right) - ac \left(\frac{a^2 + c^2 - b^2}{2ac} \right)$$

$$= \frac{1}{2} \left\{ (a^2 + b^2 - c^2) - (a^2 + c^2 - b^2) \right\} = b^2 - c^2 = \text{RHS}$$

(ii) $\text{LHS} = \dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c}$

$$= \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a^2-b^2}{2acb} + \frac{a^2+b^2-c^2}{2abc} = \frac{a^2+b^2+c^2}{2abc} = \text{RHS}$$

(iii) $\text{LHS} = 2bc\cos A + 2ca\cos B + 2ab\cos C$

$$= 2bc\left(\frac{b^2+c^2-a^2}{2bc}\right) + 2ca\left(\frac{c^2+a^2-b^2}{2ac}\right) + 2ab\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$= (b^2+c^2-a^2)+(c^2+a^2-b^2)+(a^2+b^2-c^2) = a^2+b^2+c^2 = \text{RHS}$$

EXAMPLE 4 *In any ΔABC, prove that:* $(a-b)^2\cos^2\dfrac{C}{2} + (a+b)^2\sin^2\dfrac{C}{2} = c^2$

PROOF We have,

$$\text{LHS} = (a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2}$$

$$= a^2\left(\cos^2\frac{C}{2}+\sin^2\frac{C}{2}\right) + b^2\left(\cos^2\frac{C}{2}+\sin^2\frac{C}{2}\right) - 2ab\left(\cos^2\frac{C}{2}-\sin^2\frac{C}{2}\right)$$

$$= a^2+b^2-2ab\cos C = c^2 = \text{RHS}$$

EXAMPLE 5 *In a ΔABC, prove that:*

(i) $(b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C = 0$ [NCERT]

(ii) $\left(\dfrac{b^2-c^2}{a^2}\right)\sin 2A + \left(\dfrac{c^2-a^2}{b^2}\right)\sin 2B + \left(\dfrac{a^2-b^2}{c^2}\right)\sin 2C = 0$ [NCERT]

SOLUTION Let $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k$. Then, $\sin A = ak$, $\sin B = bk$ and $\sin C = ck$

(i) $\text{LHS} = (b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C$

$$= (b^2-c^2)\frac{\cos A}{\sin A} + (c^2-a^2)\frac{\cos B}{\sin B} + (a^2-b^2)\frac{\cos C}{\sin C}$$

$$= \frac{(b^2-c^2)}{ka}\left(\frac{b^2+c^2-a^2}{2bc}\right) + \frac{(c^2-a^2)}{kb}\left(\frac{a^2+c^2-b^2}{2ac}\right) + \frac{(a^2-b^2)}{kc}\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$= \frac{1}{2kabc}\left\{(b^2-c^2)(b^2+c^2-a^2) + (c^2-a^2)(a^2+c^2-b^2) + (a^2-b^2)(a^2+b^2-c^2)\right\}$$

$$= \frac{1}{2kabc}\left\{(b^2-c^2)(b^2+c^2) - a^2(b^2-c^2) + (c^2-a^2)(c^2+a^2) - b^2(c^2-a^2) + (a^2-b^2)(a^2+b^2) - c^2(a^2-b^2)\right\}$$

$$= \frac{1}{2kabc}\left\{(b^2-c^2)(b^2+c^2) + (c^2-a^2)(c^2+a^2) + (a^2-b^2)(a^2+b^2) - a^2(b^2-c^2) - b^2(c^2-a^2) - c^2(a^2-b^2)\right\}$$

$$= \frac{1}{2kabc}\left\{(b^4-c^4)+(c^4-a^4)+(a^4-b^4)-(a^2b^2-a^2c^2)-(b^2c^2-b^2a^2)-(c^2a^2-c^2b^2)\right\}$$

$$= \frac{1}{2kabc}\times 0 = 0 = \text{RHS}$$

(ii) LHS $= \left(\dfrac{b^2-c^2}{a^2}\right)\sin 2A + \left(\dfrac{c^2-a^2}{b^2}\right)\sin 2B + \left(\dfrac{a^2-b^2}{c^2}\right)\sin 2C$

$$= \left(\frac{b^2-c^2}{a^2}\right) 2\sin A\cos A + \left(\frac{c^2-a^2}{b^2}\right) 2\sin B\cos B + \left(\frac{a^2-b^2}{c^2}\right) 2\sin C\cos C$$

$$= \left(\frac{b^2-c^2}{a^2}\right) 2ka\left(\frac{b^2+c^2-a^2}{2bc}\right) + \left(\frac{c^2-a^2}{b^2}\right) 2kb\left(\frac{a^2+c^2-b^2}{2ac}\right) + \left(\frac{a^2-b^2}{c^2}\right) 2kc\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$= \frac{k}{abc}\left\{(b^2-c^2)(b^2+c^2-a^2)+(c^2-a^2)(c^2+a^2-b^2)+(a^2-b^2)(a^2+b^2-c^2)\right\}$$

$$= \frac{k}{abc}\times 0 = 0 = \text{RHS}$$

Type II **PROBLEMS BASED ON COSINE, SINE AND PROJECTION FORMULAE**

EXAMPLE 6 *In any ΔABC, prove that:*

(i) $\dfrac{\sin B}{\sin C} = \dfrac{c-a\cos B}{b-a\cos C}$

(ii) $2\left\{a\sin^2\dfrac{C}{2}+c\sin^2\dfrac{A}{2}\right\} = a+c-b$

(iii) $2\left\{b\cos^2\dfrac{C}{2}+c\cos^2\dfrac{B}{2}\right\} = a+b+c$

(iv) $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C = a+b+c$

SOLUTION Let $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k$. Then, $a = k\sin A, b = k\sin B, c = k\sin C$

(i) RHS $= \dfrac{c-a\cos B}{b-a\cos C} = \dfrac{(a\cos B+b\cos A)-a\cos B}{(a\cos C+c\cos A)-a\cos C} = \dfrac{b\cos A}{c\cos A} = \dfrac{b}{c} = \dfrac{k\sin B}{k\sin C} = \dfrac{\sin B}{\sin C} =$ LHS

(ii) LHS $= 2\left(a\sin^2\dfrac{C}{2}+c\sin^2\dfrac{A}{2}\right) = \left\{a(1-\cos C)+c(1-\cos A)\right\}$

$= a+c-(a\cos C+c\cos A) = a+c-b =$ RHS

(iii) LHS $= 2\left(b\cos^2\dfrac{C}{2}+c\cos^2\dfrac{B}{2}\right) = \left\{b(1+\cos C)+c(1+\cos B)\right\}$

$= (b+c+b\cos C+c\cos B) = b+c+a = a+b+c =$ RHS

(iv) LHS $= (b+c)\cos A+(c+a)\cos B+(a+b)\cos C$

$= (b\cos A+a\cos B)+(c\cos A+a\cos C)+(b\cos C+c\cos B)$

$= c+b+a = a+b+c =$ RHS

EXAMPLE 7 *In any ΔABC, prove that:*

$$\frac{\cos A}{b\cos C+c\cos B}+\frac{\cos B}{c\cos A+a\cos C}+\frac{\cos C}{a\cos B+b\cos A} = \frac{a^2+b^2+c^2}{2abc}$$

SOLUTION We have,

$$\text{LHS} = \frac{\cos A}{b\cos C+c\cos B}+\frac{\cos B}{c\cos A+a\cos C}+\frac{\cos C}{a\cos B+b\cos A}$$

$$= \frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$$

$$= \frac{b^2+c^2-a^2}{2abc}+\frac{c^2+a-b^2}{2abc}+\frac{a^2+b^2-c^2}{2abc} = \frac{a^2+b^2+c^2}{2abc} = \text{RHS}$$

EXAMPLE 8 *In a triangle ABC, if* $\cos A = \dfrac{\sin B}{2 \sin C}$, *show that the triangle is isosceles.*

SOLUTION Let $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k$. Then, $\sin A = ka$, $\sin B = kb$, $\sin C = kc$.

Now, $\cos A = \dfrac{\sin B}{2 \sin C}$

$\Rightarrow \quad 2 \cos A \sin C = \sin B$

$\Rightarrow \quad 2\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right) kc = kb$

$\Rightarrow \quad b^2 + c^2 - a^2 = b^2$

$\Rightarrow \quad c^2 = a^2 \Rightarrow c = a$

$\Rightarrow \quad \Delta ABC$ is isosceles.

EXAMPLE 9 *If in a triangle ABC,* $\dfrac{2 \cos A}{a} + \dfrac{\cos B}{b} + \dfrac{2 \cos C}{c} = \dfrac{a}{bc} + \dfrac{b}{ca}$, *then prove that the triangle is right angled.*

SOLUTION We have,

$$\frac{2 \cos A}{a} + \frac{\cos B}{b} + \frac{2 \cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$$

$\Rightarrow \quad 2\left(\dfrac{b^2 + c^2 - a^2}{2abc}\right) + \left(\dfrac{c^2 + a^2 - b^2}{2abc}\right) + 2\left(\dfrac{a^2 + b^2 - c^2}{2abc}\right) = \dfrac{a}{bc} + \dfrac{b}{ca}$

$\Rightarrow \quad 2(b^2 + c^2 - a^2) + (c^2 + a^2 - b^2) + 2(a^2 + b^2 - c^2) = 2a^2 + 2b^2$

$\Rightarrow \quad b^2 + c^2 = a^2$

$\Rightarrow \quad \Delta ABC$ is a right angled triangle

Type III **ON FINDING THE AREA OF A TRIANGLE WHEN ITS PARTS ARE GIVEN**

EXAMPLE 10 *Find the area of a triangle ABC in which* $\angle A = 60°$, $b = 4$ *cm and* $c = \sqrt{3}$ *cm.*

SOLUTION The area Δ of triangle ABC is given by

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} \times 4\sqrt{3} \times \sin 60° = 2\sqrt{3} \times \frac{\sqrt{3}}{2} = 3 \text{ sq. cm.}$$

EXAMPLE 11 *In any triangle ABC, prove that:* $\Delta = \dfrac{b^2 + c^2 - a^2}{4 \cot A}$.

SOLUTION We have,

$$\text{RHS} = \frac{b^2 + c^2 - a^2}{4 \cot A} = \frac{b^2 + c^2 - a^2}{4 \cos A} \sin A = \frac{b^2 + c^2 - a^2}{4(b^2 + c^2 - a^2)} \times 2bc \sin A$$

$$= \frac{1}{2} bc \sin A = \Delta = \text{LHS}$$

EXAMPLE 12 *In any* ΔABC, *prove that:* $\Delta = \dfrac{a^2 - b^2}{2} \cdot \dfrac{\sin A \sin B}{\sin (A - B)}$

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then, $a = k \sin A$, $b = k \sin B$, $c = k \sin C$

$$\text{RHS} = \frac{a^2 - b^2}{2} \times \frac{\sin A \sin B}{\sin (A - B)} = \frac{k^2 \sin^2 A - k^2 \sin^2 B}{2} \times \frac{\sin A \sin B}{\sin (A - B)}$$

$$= \frac{k^2}{2}(\sin^2 A - \sin^2 B) \times \frac{\sin A \sin B}{\sin (A-B)} = \frac{k^2}{2} \sin (A+B) \sin (A-B) \frac{\sin A \sin B}{\sin (A-B)}$$

$$= \frac{1}{2} k^2 \sin (A+B) \sin A \sin B = \frac{1}{2}(k \sin A)(k \sin B) \sin (\pi - C)$$

$$= \frac{1}{2} ab \sin C = \Delta = \text{LHS.}$$

EXAMPLE 13 *In any ΔABC, prove that:* $a \cos A + b \cos B + c \cos C = \dfrac{8\Delta^2}{abc}$. **[NCERT]**

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then, $a = k \sin A,\ b = k \sin B,\ c = k \sin C$

Now, $a \cos A + b \cos B + c \cos C$

$$= k \sin A \cos A + k \sin B \cos B + k \sin C \cos C$$

$$= \frac{k}{2}(\sin 2A + \sin 2B + \sin 2C)$$

$$= \frac{k}{2}(4 \sin A \sin B \sin C) = 2k \sin A \sin B \sin C = 2a \sin B \sin C$$

$$= 2a \times \frac{2\Delta}{ac} \times \frac{2\Delta}{ab} \qquad \left[\because \Delta = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B \therefore \sin B = \frac{2\Delta}{ac}, \sin C = \frac{2\Delta}{ab}\right]$$

$$= \frac{8\Delta^2}{abc} = \text{RHS}$$

EXAMPLE 14 *Two ships leave a port at the same time. One goes 24 km per hour in the direction N 45° E and other travels 32 km per hour in the direction S 75° E. Find the distance between the ships at the end of 3 hours.* **[NCERT]**

SOLUTION Let P and Q be the positions of two ships at the end of 3 hours. Then,

$$OP = 3 \times 24 = 72 \text{ km and } OQ = 3 \times 32 = 96 \text{ km}$$

Using cosine formula in ΔOPQ, we get

$$PQ^2 = OP^2 + OQ^2 - 2\,OP \times OQ \cos 60°$$

$$\Rightarrow \quad PQ^2 = 72^2 + 96^2 - 2 \times 72 \times 96 \times \frac{1}{2}$$

$$\Rightarrow \quad PQ^2 = 5184 + 9216 - 6912 = 7488$$

$$\Rightarrow \quad PQ = \sqrt{7488} \text{ km} = 86{\cdot}533 \text{ km}$$

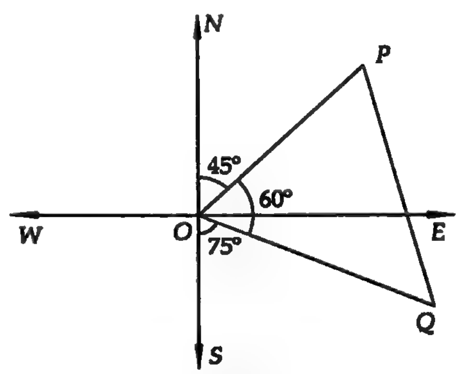


Fig. 10.13

EXAMPLE 15 *Two boats leave a place at the same time. One travels 56 km in the direction N 50° E, while other travels 48 km in the direction S 80° E. What is the distance between the two positions of the boats?*

SOLUTION Let A and B be the position of the boats such that $AB = x$.

Clearly, $\angle AOB = 180° - (50° + 80°) = 50°$

Using cosine formula, we have

$$AB^2 = OA^2 + OB^2 - 2\,OA\,OB \cos \angle AOB$$

$$\Rightarrow \quad x^2 = (56)^2 + (48)^2 - 2 \times 56 \times 48 \cos 50°$$

$$\Rightarrow \quad x^2 = 3136 + 2304 - 2 \times 56 \times 48 \times 0.6428$$

$$\Rightarrow \quad x^2 = 5440 - 3455.69 = 1984.31$$

$$\Rightarrow \quad x = \sqrt{1984.31} = 44.54 \text{ m}$$

Hence, the distance between the boats is 44.54 km.

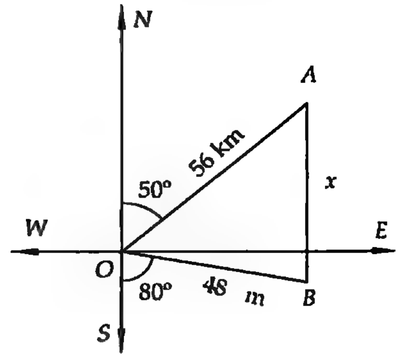


Fig. 10.14

EXAMPLE 16 *A lamp-post is situated at the middle point M of the side AC of a triangular plot ABC with* $BC = 7$ m, $CA = 8$ m *and* $AB = 9$ m. *Lamp-post subtends an angle of* $15°$ *at the point B. Determine the height of the lamp-post.* [NCERT]

SOLUTION Using cosine formula in ΔABC, we get

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49 + 64 - 81}{2 \times 7 \times 8} = \frac{32}{112} = \frac{2}{7}$$

Using cosine formula in ΔBMC, we get

$$BM^2 = BC^2 + CM^2 - 2BC\;CM \cos C$$

$$\Rightarrow \quad BM^2 = 49 + 16 - 2 \times 7 \times 4 \times \frac{2}{7} \qquad \left[\because CM = \frac{1}{2} AC = 4\right]$$

$$\Rightarrow \quad BM^2 = 49 \Rightarrow BM = 7$$

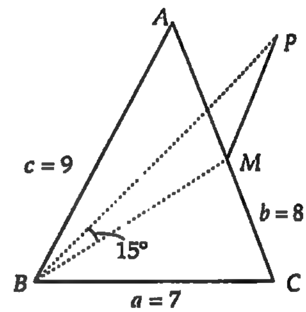


Fig. 10.15

In right triangle BMP, we have

$$\tan 15° = \frac{PM}{BM} \Rightarrow \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{PM}{7} \Rightarrow PM = 7\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) = 7\,(2-\sqrt{3}) \text{ m}$$

Hence, the height of the Lamp-post is $7\,(2-\sqrt{3})$ m.

LEVEL-2

EXAMPLE 17 *In any triangle ABC, prove that:*

$$a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B) = 3abc$$

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k \sin A, b = k \sin B, c = k \sin C$

$$\text{LHS} = a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B)$$

$$= a^2 k \sin A \cos (B - C) + b^2 k \sin B \cos (C - A) + c^2 k \sin C \cos (A - B)$$

$$= \frac{k}{2}\left[a^2\left\{2 \sin A \cos (B - C)\right\} + b^2\left\{2 \sin B \cos (C - A)\right\} + c^2\left\{2 \sin C \cos (A - B)\right\}\right]$$

$$= \frac{k}{2}\left[a^2\left\{2 \sin (B + C) \cos (B - C)\right\} + b^2\left\{2 \sin (C + A) \cos (C - A)\right\} + c^2\left\{2 \sin (A + B) \cos (A - B)\right\}\right]$$

$$= \frac{k}{2}\left[a^2 (\sin 2B + \sin 2C) + b^2 (\sin 2C + \sin 2A) + c^2 (\sin 2A + \sin 2B)\right]$$

$$= \frac{k}{2}\Big[2a^2(\sin B\cos B+\sin C\cos C)+2b^2(\sin C\cos C+\sin A\cos A)$$
$$+2c^2(\sin A\cos A+\sin B\cos B)\Big]$$
$$= \Big[a^2(k\sin B\cos B+k\sin C\cos C)+b^2(k\sin C\cos C+k\sin A\cos A)$$
$$+c^2(k\sin A\cos A+k\sin B\cos B)\Big]$$
$$= \Big[a^2(b\cos B+c\cos C)+b^2(c\cos C+a\cos A)+c^2(a\cos A+b\cos B)\Big]$$
$$[\because k\sin A=a,\ k\sin B=b,\ k\sin C=c]$$
$$= ab(a\cos B+b\cos A)+bc(b\cos C+c\cos B)+ca(a\cos C+c\cos A)$$
$$= abc+bca+cab = 3abc$$

EXAMPLE 18 *With usual notations, if in a triangle ABC* $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$, *then prove that:*
$$\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$$

SOLUTION Let $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}=\lambda$ (say) Then, $b+c=11\lambda,\ c+a=12\lambda,\ a+b=13\lambda$

$\therefore\quad (b+c+c+a+a+b)=11\lambda+12\lambda+13\lambda \Rightarrow 2(a+b+c)=36\lambda \Rightarrow a+b+c=18\lambda$

Now, $b+c=11\lambda$ and $a+b+c=18\lambda \Rightarrow a=7\lambda$

$c+a=12\lambda$ and $a+b+c=18\lambda \Rightarrow b=6\lambda$

$a+b=13\lambda$ and $a+b+c=18\lambda \Rightarrow c=5\lambda$

$$\therefore\quad \cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{36\lambda^2+25\lambda^2-49\lambda^2}{60\lambda^2}=\frac{12}{60}=\frac{1}{5}$$
$$\cos B=\frac{c^2+a^2-b^2}{2ac}=\frac{25\lambda^2+49\lambda^2-36\lambda^2}{70\lambda^2}=\frac{38}{70}=\frac{19}{35}$$
and,
$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{49\lambda^2+36\lambda^2-25\lambda^2}{84\lambda^2}=\frac{60}{84}=\frac{5}{7}$$
$$\therefore\quad \cos A:\cos B:\cos C=\frac{1}{5}:\frac{19}{35}:\frac{5}{7}=7:19:25$$

EXAMPLE 19 *If a^2, b^2, c^2 are in A.P., prove that* $\cot A$, $\cot B$, $\cot C$ *are in A.P.*

SOLUTION Let $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k$. Then, $\sin A=ka,\ \sin B=kb,\ \sin C=kc$.

Now, $\cot A, \cot B, \cot C$ will be in A.P.

$\Leftrightarrow\quad 2\cot B=\cot A+\cot C$

$$\Leftrightarrow\quad \frac{2\cos B}{\sin B}=\frac{\cos A}{\sin A}+\frac{\cos C}{\sin C}$$
$$\Leftrightarrow\quad \frac{2\cos B}{kb}=\frac{\cos A}{ka}+\frac{\cos C}{kc}$$
$$\Leftrightarrow\quad 2\left(\frac{a^2+c^2-b^2}{2abc}\right)=\left(\frac{b^2+c^2-a^2}{2abc}\right)+\left(\frac{a^2+b^2-c^2}{2abc}\right)$$
$$\Leftrightarrow\quad 2(a^2+c^2-b^2)=(b^2+c^2-a^2)+(a^2+b^2-c^2) \Leftrightarrow a^2+c^2=2b^2 \Leftrightarrow a^2, b^2, c^2 \text{ are in A.P.}$$

ALITER It is given that a^2, b^2, c^2 are in A.P.

$\therefore$ $-2a^2, -2b^2, -2c^2$ are in A.P.

$\Rightarrow$ $(a^2+b^2+c^2)-2a^2, (a^2+b^2+c^2)-2b^2, (a^2+b^2+c^2)-2c^2$ are in A.P.

$\Rightarrow$ $b^2+c^2-a^2,\ c^2+a^2-b^2,\ a^2+b^2-c^2$ are in A.P.

$\Rightarrow$ $\dfrac{b^2+c^2-a^2}{2abc}, \dfrac{c^2+a^2-b^2}{2abc}, \dfrac{a^2+b^2-c^2}{2abc}$ are in A.P.

$\Rightarrow$ $\dfrac{1}{a}\left(\dfrac{b^2+c^2-a^2}{2bc}\right), \dfrac{1}{b}\left(\dfrac{c^2+a^2-b^2}{2ac}\right), \dfrac{1}{c}\left(\dfrac{a^2+b^2+c^2}{2ab}\right)$ are in A.P.

$\Rightarrow$ $\dfrac{\cos A}{k \sin A}, \dfrac{\cos B}{k \sin B}, \dfrac{\cos C}{k \sin C}$ are in A.P. $\quad [\because a = k \sin A, b = k \sin B, c = k \sin C]$

$\Rightarrow$ $\cot A, \cot B, \cot C$ are in A.P.

EXAMPLE 20 *If in a triangle ABC, $\cos A + 2\cos B + \cos C = 2$ prove that the sides of the triangle are in A.P.*

SOLUTION We have,

$$\cos A + 2\cos B + \cos C = 2$$

$\Rightarrow$ $\cos A + \cos C = 2 - 2\cos B$

$\Rightarrow$ $\cos A + \cos C = 2(1-\cos B)$

$\Rightarrow$ $2\cos\left(\dfrac{A+C}{2}\right)\cos\left(\dfrac{A-C}{2}\right) = 2\left(2\sin^2\dfrac{B}{2}\right)$

$\Rightarrow$ $2\sin\dfrac{B}{2}\cos\left(\dfrac{A-C}{2}\right) = 4\sin^2\dfrac{B}{2}$ $\quad \left[\because \cos\dfrac{A+C}{2} = \cos\left(\dfrac{\pi}{2}-\dfrac{B}{2}\right) = \sin\dfrac{B}{2}\right]$

$\Rightarrow$ $\cos\left(\dfrac{A-C}{2}\right) = 2\sin\dfrac{B}{2}$ $\quad \left[\because 2\sin\dfrac{B}{2} \neq 0\right]$

$\Rightarrow$ $2\cos\dfrac{B}{2}\cos\left(\dfrac{A-C}{2}\right) = 4\sin\dfrac{B}{2}\cos\dfrac{B}{2}$ $\quad$ [Multiplying both sides by $2\cos\dfrac{B}{2}$]

$\Rightarrow$ $2\sin\dfrac{A+C}{2}\cos\dfrac{A-C}{2} = 2\left(2\sin\dfrac{B}{2}\cos\dfrac{B}{2}\right)$ $\quad \left[\because \cos\dfrac{B}{2} = \sin\dfrac{A+C}{2}\right]$

$\Rightarrow$ $\sin A + \sin C = 2\sin B$

$\Rightarrow$ $ka + kc = 2kb$ $\quad \left[\because \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k\right]$

$\Rightarrow$ $a + c = 2b \Rightarrow a, b, c$ are in A.P.

EXAMPLE 21 *In a triangle ABC, $\angle C = 60°$, then prove that :* $\dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$.

SOLUTION We have, $\angle C = 60°$

$\Rightarrow$ $\cos C = \dfrac{1}{2} \Rightarrow \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{1}{2} \Rightarrow a^2+b^2-c^2 = ab \Rightarrow a^2+b^2-ab = c^2$ $\quad$ ...(i)

Now, $$\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$$

if $$\frac{a+b+2c}{(a+c)(b+c)} = \frac{3}{a+b+c}$$

i.e. if $(a+b+2c)(a+b+c) = 3(a+c)(b+c)$

i.e. if $(a+b)^2 + 2c^2 + 3c(a+b) = 3(ab+ac+bc+c^2)$

i.e. if $a^2+b^2+2ab+2c^2+3ac+3bc = 3ab+3ac+3bc+3c^2$

i.e. if $a^2+b^2-ab = c^2$, which is given $\quad$ [see (i)]

EXAMPLE 22 *Two trees, A and B are on the same side of a river. From a point C in the river the distance of trees A and B are 250 m and 300 m respectively. If the angle C is 45°, find the distance between the trees (Use $\sqrt{2}=1.44$).*

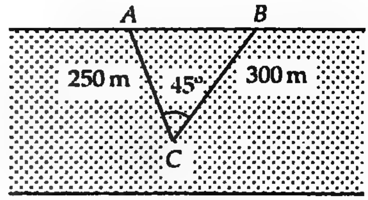


Fig. 10.16

SOLUTION Using cosine formula in ΔABC, we have

$$AB^2 = AC^2 + BC^2 - 2AC\cdot BC\cos\frac{\pi}{4}$$

$$\Rightarrow \quad AB = \sqrt{(250)^2+(300)^2-2\times 250\times 300\times\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \quad AB = \sqrt{62500+90000-75000\sqrt{2}}$$

$$\Rightarrow \quad AB = \sqrt{152500-75000\times 1.44}$$

$$\Rightarrow \quad AB = \sqrt{152500-108000} = \sqrt{44500} = 210.95 \text{ m}$$

EXERCISE 10.2

LEVEL-1

In any ΔABC, prove the following : (1-13)

1. In a ΔABC, if $a=5$, $b=6$ and $C=60°$, show that its area is $\frac{15\sqrt{3}}{2}$ sq. units.
2. In a ΔABC, if $a=\sqrt{2}$, $b=\sqrt{3}$ and $c=\sqrt{5}$, show that its area is $\frac{1}{2}\sqrt{6}$ sq. units.
3. The sides of a triangle are $a=5$, $b=6$ and $c=8$, show that: $8\cos A+16\cos B+4\cos C=17$.
4. In a ΔABC, if $a=18$, $b=24$, $c=30$, find $\cos A$, $\cos B$ and $\cos C$.
5. $b(c\cos A-a\cos C) = c^2-a^2$
6. $c(a\cos B-b\cos A) = a^2-b^2$
7. $2(bc\cos A+ca\cos B+ab\cos C)=a^2+b^2+c^2$
8. $(c^2-a^2+b^2)\tan A = (a^2-b^2+c^2)\tan B = (b^2-c^2+a^2)\tan C$
9. $\dfrac{c-b\cos A}{b-c\cos A} = \dfrac{\cos B}{\cos C}$
10. $a(\cos B+\cos C-1)+b(\cos C+\cos A-1)+c(\cos A+\cos B-1)=0$
11. $a\cos A+b\cos B+c\cos C=2b\sin A\sin C$
12. $a^2=(b+c)^2-4bc\cos^2\frac{A}{2}$
13. $4\left(bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}\right) = (a+b+c)^2$

LEVEL-2

14. In a ΔABC, prove that
$\sin^3 A\cos(B-C)+\sin^3 B\cos(C-A)+\sin^3 C\cos(A-B)=3\sin A\sin B\sin C$
15. In in any ΔABC, $\frac{b+c}{12}=\frac{c+a}{13}=\frac{a+b}{15}$, then prove that $\frac{\cos A}{2}=\frac{\cos B}{7}=\frac{\cos C}{11}$.
16. In a ΔABC, if $\angle B=60°$, prove that $(a+b+c)(a-b+c) = 3ca$
17. If in a ΔABC, $\cos^2 A+\cos^2 B+\cos^2 C = 1$, prove that the triangle is right angled.

18. In a $\Delta\ ABC$, if $\cos C = \dfrac{\sin A}{2 \sin B}$, prove that the triangle is isosceles.

19. Two ships leave a port at the same time. One goes 24 km/hr in the direction $N\ 38°\ E$ and other travels 32 km/hr in the direction $S\ 52°\ E$. Find the distance between the ships at the end of 3 hrs.

ANSWERS

4. $\cos A = \dfrac{4}{5}, \cos B = \dfrac{3}{5}, \cos C = 0$ 19. 120 km

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Find the area of the triangle ΔABC in which $a = 1, b = 2$ and $\angle c = 60°$.
2. In a ΔABC, if $b = \sqrt{3}$, $c = 1$ and $\angle A = 30°$, find a.
3. In a ΔABC, if $\cos A = \dfrac{\sin B}{2 \sin C}$, then show that $c = a$.
4. In a ΔABC, if $b = 20$, $c = 21$ and $\sin A = \dfrac{3}{5}$, find a.
5. In a ΔABC, if $\sin A$ and $\sin B$ are the roots of the equation $c^2x^2 - c(a+b)x + ab = 0$, then find $\angle C$.
6. In ΔABC, if $a = 8, b = 10, c = 12$ and $C = \lambda A$, find the value of λ.
7. If the sides of a triangle are proportional to $2, \sqrt{6}$ and $\sqrt{3} - 1$, find the measure of its greatest angle.
8. If in a ΔABC, $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} = \dfrac{\cos C}{c}$, then find the measures of angles A, B, C.
9. In any triangle ABC, find the value of $a \sin(B - C) + b \sin(C - A) + c \sin(A - B)$.
10. In any ΔABC, find the value of $\Sigma\, a(\sin B - \sin C)$

ANSWERS

1. $\sqrt{3}$ sq. units 2. 1 4. 13 5. 90° 6. 2 7. 120° 8. $A = B = C = 60°$ 9. 0 10. 0

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. In any ΔABC, $\sum a^2 (\sin B - \sin C) =$

   (a) $a^2 + b^2 + c^2$ (b) a^2 (c) b^2 (d) 0

2. In a ΔABC, if $a = 2$, $\angle B = 60°$ and $\angle C = 75°$, then $b =$

   (a) $\sqrt{3}$ (b) $\sqrt{6}$ (c) $\sqrt{9}$ (d) $1 + \sqrt{2}$

3. In the sides of a triangle are in the ratio $1 : \sqrt{3} : 2$, then the measure of its greatest angle is

   (a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{2}$ (d) $\dfrac{2\pi}{3}$

4. In any $\Delta\ ABC$, $2(bc \cos A + ca \cos B + ab \cos C) =$

   (a) abc (b) $a + b + c$ (c) $a^2 + b^2 + c^2$ (d) $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$

5. In a triangle ABC, $a = 4, b = 3, \angle A = 60°$ then c is a root of the equation
   (a) $c^2 - 3c - 7 = 0$ (b) $c^2 + 3c + 7 = 0$ (c) $c^2 - 3c + 7 = 0$ (d) $c^2 + 3c - 7 = 0$
6. In a ΔABC, if $(c + a + b)(a + b - c) = ab$, then the measure of angle C is
   (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{2}$
7. In any ΔABC, the value of $2ac \sin\left(\frac{A - B + C}{2}\right)$ is
   (a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$ (c) $b^2 - c^2 - a^2$ (d) $c^2 - a^2 - b^2$
8. In any ΔABC, $a(b \cos C - c \cos B) =$
   (a) a^2 (b) $b^2 - c^2$ (c) 0 (d) $b^2 + c^2$

ANSWERS

1. (d) 2. (b) 3. (c) 4. (c) 5. (a) 6. (a) 7. (c) 8. (c)

CHAPTER 11

TRIGONOMETRIC EQUATIONS

11.1 SOME DEFINITIONS

TRIGONOMETRIC EQUATIONS *The equations containing trigonometric functions of unknown angles are known as trigonometric equations.*

$\cos\theta = \frac{1}{2}$, $\sin\theta = 0$, $\tan\theta = \sqrt{3}$ etc. are trigonometric equations.

SOLUTION OF A TRIGONOMETRIC EQUATION *A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.*

Consider the equation $\sin\theta = \frac{1}{2}$. This equation is clearly satisfied by $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ etc. So, these are its solutions.

Solving an equation means to find the set of all values of the unknown angle which satisfy the given equation.

Consider the equation $2\cos\theta + 1 = 0$ or $\cos\theta = -1/2$. This equation is clearly satisfied by $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ etc.

Since the trigonometric functions are periodic. Therefore, if a trigonometric equation has a solution, it will have infinitely many solutions. For example, $\theta = \frac{2\pi}{3}$, $2\pi \pm \frac{2\pi}{3}$, $4\pi \pm \frac{2\pi}{3}$, ... are solutions of $2\cos\theta + 1 = 0$. These solutions can be put together in compact form as $2n\pi \pm \frac{2\pi}{3}$, where n is an integer. This solution is known as the general solution.

Thus, a solution generalised by means of periodicity is known as the general solution.

It also follows from the above discussion that solving an equation means to find its general solution.

11.2 GENERAL SOLUTIONS OF TRIGONOMETRIC EQUATIONS

In this section, we shall obtain the general solutions of the trigonometric equations $\sin\theta = 0$, $\cos\theta = 0$, $\tan\theta = 0$ and $\cot\theta = 0$.

THEOREM 1 *Prove that the general solution of* $\sin\theta = 0$ *is given by* $\theta = n\pi, n \in Z$.

PROOF In ΔOMP, we obtain

$$\sin\theta = \frac{PM}{OP}$$

$\therefore \quad \sin\theta = 0$

$\Rightarrow \quad \frac{PM}{OP} = 0$

$\Rightarrow \quad PM = 0$

$\Rightarrow \quad OP$ coincides with OX or, OX'

$\Rightarrow \quad \theta = 0, \pi, 2\pi, \ldots, -\pi, -2\pi, -3\pi, \ldots$

$\Rightarrow \quad \theta = n\pi, n \in Z.$

Hence, $\theta = n\pi, n \in Z$ is the general solution of $\sin\theta = 0$.

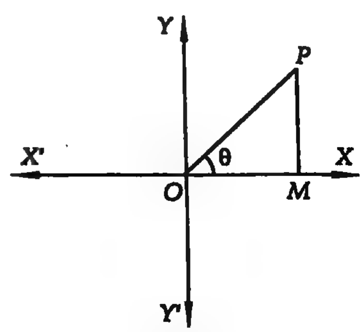


Fig. 11.1

Q.E.D.

THEOREM 2 *Prove that the general solution of* $\tan\theta = 0$ *is* $\theta = n\pi$, $n \in Z$.

PROOF By definition,

$$\tan\theta = \frac{PM}{OM} \quad \text{[See Fig. 11.1]}$$

$\therefore$ $\tan\theta = 0$

$\Rightarrow$ $\frac{PM}{OM} = 0$

X or, OX'

π, -2π, ...

ral solution of $\sin\theta = 0$. Q.E.D.

ral solution of $\cos\theta = 0$ is $\theta = (2n+1)\frac{\pi}{2}$, $n \in Z$.

[See Fig. 11.1]

OY'

.....

2

2, $n \in Z$.

Hence, the general solution of $\cos\theta = 0$ is $\theta = (2n+1)\frac{\pi}{2}$, $n \in Z$. Q.E.D.

THEOREM 4 *Prove that the general solution of* $\cot\theta = 0$ *is* $\theta = (2n+1)\frac{\pi}{2}$, $n \in Z$.

PROOF By definition,

$$\cot\theta = \frac{OM}{PM} \quad \text{[See Fig. 11.1]}$$

$\therefore$ $\cot\theta = 0$

$\Rightarrow$ $\frac{OM}{PM} = 0$

$\Rightarrow$ $OM = 0$

$\Rightarrow$ OP coincides with OY or, OY'

$\Rightarrow$ $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\Rightarrow$ $\theta = (2n+1)\frac{\pi}{2}$, $n \in Z$.

Hence, $\theta = (2n+1)\frac{\pi}{2}, n \in Z$ is the general solution of $\cot\theta = 0$. Q.E.D.

NOTE *Since* $\sec\theta \geq 1$, or $\sec\theta \leq -1$, *therefore* $\sec\theta = 0$ *does not have any solution. Similarly,* $\operatorname{cosec}\theta = 0$ *has no solution.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the general solutions of the following equations:*

(i) $\sin 2\theta = 0$ (ii) $\sin \frac{3\theta}{2} = 0$ (iii) $\sin^2 2\theta = 0$

SOLUTION (i) We have,

$$\sin 2\theta = 0$$
$$\Rightarrow 2\theta = n\pi, \text{ where } n \in Z \qquad [\because \sin\theta = 0 \Rightarrow \theta = n\pi]$$
$$\Rightarrow \theta = \frac{n\pi}{2}, \ n \in Z.$$

(ii)
$$\sin\frac{3\theta}{2} = 0$$
$$\Rightarrow \frac{3\theta}{2} = n\pi, \ n \in Z \qquad [\sin\theta = 0 \Rightarrow \theta = n\pi]$$
$$\Rightarrow \theta = \frac{2n\pi}{3}, n \in Z.$$

(iii) $\sin^2 2\theta = 0 \Rightarrow \sin 2\theta = 0 \Rightarrow 2\theta = n\pi, \ n \in Z \Rightarrow \theta = \frac{n\pi}{2}, \ n \in Z.$

EXAMPLE 2 *Find the general solutions of the following equations:*

(i) $\cos 3\theta = 0$ (ii) $\cos\frac{3\theta}{2} = 0$ (iii) $\cos^2 3\theta = 0$

SOLUTION We know that the general solution of the equation $\cos\theta = 0$ is $\theta = (2n+1)\frac{\pi}{2}, n \in Z$.

Therefore,

(i) $\cos 3\theta = 0 \Rightarrow 3\theta = (2n+1)\frac{\pi}{2}, n \in Z \Rightarrow \theta = (2n+1)\frac{\pi}{6}, \ n \in Z$

(ii) $\cos\frac{3\theta}{2} = 0 \Rightarrow \frac{3\theta}{2} = (2n+1)\frac{\pi}{2}, n \in Z \Rightarrow \theta = (2n+1)\frac{\pi}{3}, \ n \in Z.$

(iii) $\cos^2 3\theta = 0 \Rightarrow \cos 3\theta = 0 \Rightarrow 3\theta = (2n+1)\frac{\pi}{2}, \ n \in Z \Rightarrow \theta = (2n+1)\frac{\pi}{6}, \ n \in Z.$

EXAMPLE 3 *Find the general solutions of the following equations:*

(i) $\tan 2\theta = 0$ (ii) $\tan\frac{\theta}{2} = 0$ (iii) $\tan\frac{3\theta}{4} = 0$

SOLUTION We know that the general solution of the equation $\tan\theta = 0$ is $\theta = n\pi, n \in Z$.
Therefore,

(i) $\tan 2\theta = 0 \Rightarrow 2\theta = n\pi, n \in Z \Rightarrow \theta = \frac{n\pi}{2}, \ n \in Z$

(ii) $\tan\frac{\theta}{2} = 0 \Rightarrow \frac{\theta}{2} = n\pi, n \in Z \Rightarrow \theta = 2n\pi, \ n \in Z$

(iii) $\tan\frac{3\theta}{4} = 0 \Rightarrow \frac{3\theta}{4} = n\pi, n \in Z \Rightarrow \theta = \frac{4n\pi}{3}, n \in Z$

THEOREM 5 *Prove that the general solution of* $\sin\theta = \sin\alpha$ *is given by:* $\theta = n\pi + (-1)^n \alpha, \ n \in Z.$

<u>PROOF</u> We have,

$$\sin\theta = \sin\alpha$$
$$\Leftrightarrow \sin\theta - \sin\alpha = 0$$
$$\Leftrightarrow 2\sin\left(\frac{\theta-\alpha}{2}\right)\cos\left(\frac{\theta+\alpha}{2}\right) = 0$$

$$\Leftrightarrow \quad \sin\left(\frac{\theta-\alpha}{2}\right)=0 \text{ or, } \cos\left(\frac{\theta+\alpha}{2}\right)=0$$

$$\Leftrightarrow \quad \frac{\theta-\alpha}{2} = m\pi, \text{ or, } \frac{\theta+\alpha}{2} = (2m+1)\frac{\pi}{2}, m \in Z$$

$$\Leftrightarrow \quad \theta = 2m\pi+\alpha, \in Z \text{ or, } \theta = (2m+1)\pi-\alpha, m \in Z$$

$$\Leftrightarrow \quad \theta = (\text{Any even multiple of } \pi)+\alpha \text{ or, } \theta = (\text{Any odd multiple of } \pi)-\alpha$$

$$\Leftrightarrow \quad \theta = n\pi+(-1)^n \alpha, \text{ where } n \in Z.$$

Q.E.D.

REMARK 1 *The equation* $\operatorname{cosec}\theta=\operatorname{cosec}\alpha$ *is equivalent to* $\sin\theta=\sin\alpha$. *Thus,* $\operatorname{cosec}\theta=\operatorname{cosec}\alpha$ *and* $\sin\theta=\sin\alpha$ *have the same general solution.*

THEOREM 6 *Prove that the general solution of* $\cos\theta=\cos\alpha$ *is given by:* $\theta=2n\pi\pm\alpha$, where $n \in Z$.

PROOF We have,

$$\cos\theta = \cos\alpha$$

$$\Leftrightarrow \quad \cos\theta-\cos\alpha = 0$$

$$\Leftrightarrow \quad -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Leftrightarrow \quad \sin\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or, } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Leftrightarrow \quad \frac{\theta+\alpha}{2} = n\pi, \text{ or } \frac{\theta-\alpha}{2} = n\pi, n \in Z$$

$$\Leftrightarrow \quad \theta = 2n\pi-\alpha \text{ or, } \theta = 2n\pi+\alpha, n \in Z$$

$$\Leftrightarrow \quad \theta = 2n\pi\pm\alpha, n \in Z.$$

Q.E.D.

REMARK 2 *Since* $\sec\theta=\sec\alpha \Leftrightarrow \cos\theta=\cos\alpha$. *So, the general solutions of* $\cos\theta=\cos\alpha$ *and* $\sec\theta=\sec\alpha$ *are same.*

THEOREM 7 *Prove that the general solution of* $\tan\theta=\tan\alpha$ *is given by:* $\theta = n\pi+\alpha, n \in Z$.

PROOF We have,

$$\tan\theta = \tan\alpha$$

$$\Leftrightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$$

$$\Leftrightarrow \quad \sin\theta\cos\alpha-\cos\theta\sin\alpha = 0$$

$$\Leftrightarrow \quad \sin(\theta-\alpha) = 0$$

$$\Leftrightarrow \quad \theta-\alpha = n\pi, n \in Z$$

$$\Leftrightarrow \quad \theta=n\pi+\alpha, n \in Z$$

Q.E.D.

REMARK 3 *Since* $\tan\theta=\tan\alpha \Leftrightarrow \cot\theta=\cot\alpha$. *So, general solutions of* $\cot\theta=\cot\alpha$ *and* $\tan\theta=\tan\alpha$ *are same.*

In order to find the general solutions of trigonometrical equations of the formsin $\theta=\sin\alpha$, $\cos\theta=\cos\alpha$ and $\tan\theta=\tan\alpha$, we may use the following algorithm.

ALGORITHM

STEP I *Find a value of* θ, *preferably between 0 and* 2π or between $-\pi$ and π, *satisfying the given equation and call it* α.

STEP II *If the equation is* $\sin\theta=\sin\alpha$, write $\theta=n\pi+(-1)^n\alpha, n \in Z$ *as the general solution.*

For the equation $\cos\theta = \cos\alpha$, write $\theta=2n\pi\pm\alpha, n \in Z$ *as the general solution.*

For the equation $\tan\theta = \tan\alpha$, write $\theta = n\pi+\alpha, n \in Z$ *as the general solution.*

Following examples illustrate the algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE GENERAL SOLUTIONS OF THE EQUATIONS OF THE FORM** $\sin\theta = \sin\alpha, \cos\theta = \cos\alpha, \tan\theta = \tan\alpha$

EXAMPLE 1 *Find the general solutions of the following equations:*

(i) $\sin\theta = \frac{\sqrt{3}}{2}$ (ii) $2\sin\theta + 1 = 0$ (iii) $\text{cosec}\,\theta = 2$

SOLUTION (i) A value of θ satisfying $\sin\theta = \frac{\sqrt{3}}{2}$ is $\frac{\pi}{3}$.

$\therefore$ $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{3} \Rightarrow \theta = n\pi + (-1)^n \frac{\pi}{3}, n \in Z$

(ii) We have,

$2\sin\theta + 1 = 0 \Rightarrow \sin\theta = -\frac{1}{2}$

A value of θ satisfying this equation is $-\pi/6$.

$\therefore$ $\sin\theta = -\frac{1}{2}$

$\Rightarrow$ $\sin\theta = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow$ $\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right), n \in Z$

$\Rightarrow$ $\theta = n\pi + (-1)^{n+1}\frac{\pi}{6}, n \in Z.$

(iii) We have,

$\text{cosec}\,\theta = 2 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{6} \Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{6}, n \in Z.$

EXAMPLE 2 *Find the general solutions of the following equations:*

(i) $\cos\theta = \frac{1}{2}$ (ii) $\cos 3\theta = -\frac{1}{2}$ (iii) $\sqrt{3}\sec 2\theta = 2$

SOLUTION (i) $\cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$

(ii) $\cos 3\theta = -\frac{1}{2}$

$\Rightarrow$ $\cos 3\theta = \cos\frac{2\pi}{3}$

$\Rightarrow$ $3\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

$\Rightarrow$ $\theta = \frac{2n\pi}{3} \pm \frac{2\pi}{9}, n \in Z$

(iii) $\sqrt{3}\sec 2\theta = 2$

$\Rightarrow$ $\cos 2\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $\cos 2\theta = \cos\frac{\pi}{6}$

$\Rightarrow$ $2\theta = 2n\pi \pm \frac{\pi}{6}, n \in Z$

$\Rightarrow$ $\theta = n\pi \pm \frac{\pi}{12}, n \in Z$

EXAMPLE 3 *Solve the following trigonometric equations:*

(i) $\tan\theta = \frac{1}{\sqrt{3}}$ (ii) $\tan 2\theta = \sqrt{3}$ (iii) $\tan 3\theta = -1$

SOLUTION (i) $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \tan\theta = \tan\frac{\pi}{6} \Rightarrow \theta = n\pi + \frac{\pi}{6}, n \in Z$

(ii) $\tan 2\theta = \sqrt{3}$

$\Rightarrow \tan 2\theta = \tan\frac{\pi}{3}$

$\Rightarrow 2\theta = n\pi + \frac{\pi}{3}, n \in Z$

$\Rightarrow \theta = \frac{n\pi}{2} + \frac{\pi}{6}, n \in Z$

(iii) $\tan 3\theta = -1$

$\Rightarrow \tan 3\theta = \tan\left(-\frac{\pi}{4}\right)$

$\Rightarrow 3\theta = n\pi + \left(-\frac{\pi}{4}\right), n \in Z$

$\Rightarrow \theta = \frac{n\pi}{3} - \frac{\pi}{12}, n \in Z.$

EXAMPLE 4 *Solve the following trigonometric equations:*

(i) $\sin\frac{\theta}{2} = -1$ (ii) $\cos\frac{3\theta}{2} = \frac{1}{2}$ (iii) $\tan\left(\frac{2}{3}\theta\right) = \sqrt{3}$

SOLUTION (i) $\sin\frac{\theta}{2} = -1$

$\Rightarrow \sin\frac{\theta}{2} = \sin\left(-\frac{\pi}{2}\right)$

$\Rightarrow \frac{\theta}{2} = n\pi + (-1)^n\left(-\frac{\pi}{2}\right), n \in Z$

$\Rightarrow \theta = 2n\pi + (-1)^{n+1}\pi, n \in Z$

(ii) $\cos\frac{3\theta}{2} = \frac{1}{2}$

$\Rightarrow \cos\frac{3\theta}{2} = \cos\frac{\pi}{3}$

$\Rightarrow \frac{3\theta}{2} = 2n\pi \pm \frac{\pi}{3}, n \in Z$

$\Rightarrow \theta = \frac{4n\pi}{3} \pm \frac{2\pi}{9}, n \in Z$

(iii) $\tan\left(\frac{2\theta}{3}\right) = \sqrt{3}$

$\Rightarrow \tan\left(\frac{2\theta}{3}\right) = \tan\frac{\pi}{3}$

$\Rightarrow \frac{2\theta}{3} = n\pi + \frac{\pi}{3}, n \in Z$

$\Rightarrow \theta = \frac{3n\pi}{2} + \frac{\pi}{2}, n \in Z.$

Type II **ON FINDING THE GENERAL SOLUTION OF THE EQUATIONS REDUCIBLE TO THE FORMS** $\sin\theta = \sin\alpha$, $\cos\theta = \cos\alpha$, $\tan\theta = \tan\alpha$.

EXAMPLE 5 *Solve the equation:* $\sin\theta + \sin 3\theta + \sin 5\theta = 0$. **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\sin\theta + \sin 3\theta + \sin 5\theta = 0$$
$$\Rightarrow \quad (\sin 5\theta + \sin\theta) + \sin 3\theta = 0$$
$$\Rightarrow \quad 2\sin 3\theta\cos 2\theta + \sin 3\theta = 0$$
$$\Rightarrow \quad \sin 3\theta(2\cos 2\theta + 1) = 0$$
$$\Rightarrow \quad \sin 3\theta = 0 \text{ or } 2\cos 2\theta + 1 = 0$$
$$\Rightarrow \quad \sin 3\theta = 0 \text{ or, } \cos 2\theta = -\frac{1}{2}$$

Now, $\sin 3\theta = 0 \Rightarrow 3\theta = n\pi, n \in Z \Rightarrow \theta = \frac{n\pi}{3}, n \in Z$

And, $\cos 2\theta = -\frac{1}{2}$

$$\Rightarrow \quad \cos 2\theta = \cos\frac{2\pi}{3}$$
$$\Rightarrow \quad 2\theta = 2m\pi \pm \frac{2\pi}{3}, m \in Z$$
$$\Rightarrow \quad \theta = m\pi \pm \frac{\pi}{3}, m \in Z.$$

Hence, the general solution of the given equation is: $\theta = \frac{n\pi}{3}$ or, $\theta = m\pi \pm \frac{\pi}{3}$, where $m, n \in Z$.

EXAMPLE 6 *Solve the equation:* $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$

SOLUTION We have,

$$\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$$
$$\Leftrightarrow \quad 2\cos 2\theta\cos\theta - 2\cos 2\theta = 0$$
$$\Leftrightarrow \quad 2\cos 2\theta(\cos\theta - 1) = 0$$
$$\Rightarrow \quad \cos 2\theta = 0 \text{ or, } \cos\theta - 1 = 0$$

Now, $\cos 2\theta = 0 \Rightarrow 2\theta = (2n+1)\frac{\pi}{2}, n \in Z \quad \Rightarrow \theta = (2n+1)\frac{\pi}{4}, n \in Z$

And, $\cos\theta - 1 = 0$

$$\Rightarrow \quad \cos\theta = 1$$
$$\Rightarrow \quad \cos\theta = \cos 0$$
$$\Rightarrow \quad \theta = 2m\pi \pm 0, m \in Z$$
$$\Rightarrow \quad \theta = 2m\pi, m \in Z$$

Hence, $\theta = (2n+1)\frac{\pi}{4}$ or, $\theta = 2m\pi$, where $m, n \in Z$.

EXAMPLE 7 *Solve the equation:* $\sin m\theta + \sin n\theta = 0$.

SOLUTION We have,

$$\sin m\theta + \sin n\theta = 0$$
$$\Rightarrow \quad 2\sin\left(\frac{m+n}{2}\right)\theta\cos\left(\frac{m-n}{2}\right)\theta = 0$$
$$\Rightarrow \quad \sin\left(\frac{m+n}{2}\right)\theta = 0 \text{ or, } \cos\left(\frac{m-n}{2}\right)\theta = 0$$

Now, $\sin\left(\frac{m+n}{2}\right)\theta = 0$

$\Rightarrow \left(\frac{m+n}{2}\right)\theta = r\pi, r \in Z$

$\Rightarrow \theta = \frac{2r\pi}{m+n}, \quad r \in Z$

And, $\cos\left(\frac{m-n}{2}\right)\theta = 0$

$\Rightarrow \left(\frac{m-n}{2}\right)\theta = (2s+1)\frac{\pi}{2}, s \in Z$

$\Rightarrow \theta = \frac{(2s+1)\pi}{m-n}, s \in Z$

Hence, $\theta = \frac{2r\pi}{m+n}$ or, $\theta = \frac{(2s+1)\pi}{m-n}$, where $r, s \in Z$.

EXAMPLE 8 *Solve the following equations:*

(i) $\sin 2\theta + \cos\theta = 0$ [NCERT] (ii) $\sin 3\theta + \cos 2\theta = 0$

(iii) $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$

SOLUTION (i) $\sin 2\theta + \cos\theta = 0$

$\Rightarrow \cos\theta = -\sin 2\theta$

$\Rightarrow \cos\theta = \cos\left(\frac{\pi}{2} + 2\theta\right)$

$\Rightarrow \theta = 2n\pi \pm \left(\frac{\pi}{2} + 2\theta\right), \ n \in Z$

Taking positive sign, we have

$\theta = 2n\pi + \frac{\pi}{2} + 2\theta$

$\Rightarrow -\theta = 2n\pi + \frac{\pi}{2}, n \in Z$

$\Rightarrow \theta = -2n\pi - \frac{\pi}{2}, \ n \in Z$

$\Rightarrow \theta = 2m\pi - \frac{\pi}{2}, \quad$ where $m = -n \in Z$.

Taking negative sign, we have

$\theta = 2n\pi - \left(\frac{\pi}{2} + 2\theta\right)$

$\Rightarrow 3\theta = 2n\pi - \frac{\pi}{2} \Rightarrow \theta = \frac{2n\pi}{3} - \frac{\pi}{6}, n \in Z.$

Hence, $\theta = 2m\pi - \frac{\pi}{2}$, or, $\theta = \frac{2n\pi}{3} - \frac{\pi}{6}$, where $m, n \in Z$.

(ii) $\sin 3\theta + \cos 2\theta = 0$

$\Rightarrow \cos 2\theta = -\sin 3\theta$

$\Rightarrow \cos 2\theta = \cos\left(\frac{\pi}{2} + 3\theta\right)$

$\Rightarrow 2\theta = 2n\pi \pm \left(\frac{\pi}{2} + 3\theta\right), \ n \in Z$

Taking positive sign, we have

$$2\theta = 2n\pi + \frac{\pi}{2} + 3\theta$$

$$\Rightarrow \quad -\theta = 2n\pi + \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = -2n\pi - \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = 2m\pi - \frac{\pi}{2}, \text{ where } -n = m.$$

Taking negative sign, we have

$$2\theta = 2n\pi - \frac{\pi}{2} - 3\theta \Rightarrow 5\theta = 2n\pi - \frac{\pi}{2} \Rightarrow \theta = \frac{2n\pi}{5} - \frac{\pi}{10}, \ n \in Z$$

Hence, $\theta = \dfrac{2n\pi}{5} - \dfrac{\pi}{10}$ or, $\theta = 2m\pi - \dfrac{\pi}{2}$, where $m, n \in Z$.

(iii) We have,

$$\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$$

$$\Rightarrow \quad \sin 4\theta + (\sin 2\theta + \sin 6\theta) = 0$$

$$\Rightarrow \quad \sin 4\theta + 2\sin 4\theta \cos 2\theta = 0$$

$$\Rightarrow \quad \sin 4\theta (1 + 2\cos 2\theta) = 0$$

$$\Rightarrow \quad \sin 4\theta = 0 \text{ or, } 1 + 2\cos 2\theta = 0 \Rightarrow \sin 4\theta = 0 \text{ or, } \cos 2\theta = -\frac{1}{2}$$

Now, $\sin 4\theta = 0 \Rightarrow 4\theta = n\pi, n \in Z \Rightarrow \theta = \dfrac{n\pi}{4}, \ n \in Z$

And, $\cos 2\theta = -\dfrac{1}{2}$

$$\Rightarrow \quad \cos 2\theta = \cos \frac{2\pi}{3}$$

$$\Rightarrow \quad 2\theta = 2m\pi \pm \frac{2\pi}{3}, \ m \in Z$$

$$\Rightarrow \quad \theta = m\pi \pm \frac{\pi}{3}, \ m \in Z$$

Hence, $\theta = \dfrac{n\pi}{4}$ or, $\theta = m\pi \pm \dfrac{\pi}{3}$, where $m, n \in Z$.

EXAMPLE 9 *Solve the following equations:*

(i) $2\cos^2\theta + 3\sin\theta = 0$ **[NCERT]** (ii) $\cot^2\theta + \dfrac{3}{\sin\theta} + 3 = 0$

(iii) $2\tan\theta - \cot\theta = -1$ (iv) $4\cos\theta - 3\sec\theta = \tan\theta$

(v) $\tan^2\theta + (1 - \sqrt{3})\tan\theta - \sqrt{3} = 0$ (vi) $\sec^2 2x = 1 - \tan 2x$ **[NCERT]**

SOLUTION (i) $2\cos^2\theta + 3\sin\theta = 0$

$$\Rightarrow \quad 2(1 - \sin^2\theta) + 3\sin\theta = 0$$

$$\Rightarrow \quad 2\sin^2\theta - 3\sin\theta - 2 = 0$$

$$\Rightarrow \quad 2\sin^2\theta - 4\sin\theta + \sin\theta - 2 = 0$$

$$\Rightarrow \quad 2\sin\theta(\sin\theta - 2) + 1(\sin\theta - 2) = 0$$

$$\Rightarrow \quad (\sin\theta - 2)(2\sin\theta + 1) = 0$$

$$\Rightarrow \quad 2\sin\theta + 1 = 0 \qquad [\because \sin\theta \neq 2 \ \therefore \sin\theta - 2 \neq 0]$$

$\Rightarrow \quad \sin\theta = -\frac{1}{2}$

$\Rightarrow \quad \sin\theta = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow \quad \theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right),\ n \in Z$

$\Rightarrow \quad \theta = n\pi + (-1)^{n+1}\ \frac{\pi}{6},\ n \in Z.$

(ii) $\cot^2\theta + \frac{3}{\sin\theta} + 3$

$\Rightarrow$ cosec^2

$\Rightarrow$ cr

$\Rightarrow$

$\Rightarrow$

Now,

$\Rightarrow$

$\Rightarrow$ sin

$\Rightarrow \quad \sin\theta = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow \quad \theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right),\ n \in Z$

$\Rightarrow \quad \theta = n\pi + (-1)^{n+1}\ \frac{\pi}{6},\ n \in Z$

And, $\quad \text{cosec}\,\theta + 1 = 0$

$\Rightarrow \quad \frac{1}{\sin\theta} + 1 = 0$

$\Rightarrow \quad \sin\theta = -1$

$\Rightarrow \quad \sin\theta = \sin\left(-\frac{\pi}{2}\right)$

$\Rightarrow \quad \theta = m\pi + (-1)^m\left(-\frac{\pi}{2}\right),\ m \in Z$

$\Rightarrow \quad \theta = m\pi + (-1)^{m+1}\ \frac{\pi}{2},\ m \in Z$

Hence, $\theta = n\pi + (-1)^{n+1}\frac{\pi}{6}$ or, $\theta = m\pi + (-1)^{m+1}\frac{\pi}{2}, m, n \in Z$

(iii) $2\tan\theta - \cot\theta = -1$

$\Rightarrow \quad 2\tan\theta - \frac{1}{\tan\theta} = -1$

$\Rightarrow \quad 2\tan^2\theta + \tan\theta - 1 = 0$

$\Rightarrow \quad 2\tan^2\theta + 2\tan\theta - \tan\theta - 1 = 0$

$\Rightarrow \quad 2\tan\theta(\tan\theta + 1) - (\tan\theta + 1) = 0$

$\Rightarrow \quad (\tan\theta + 1)(2\tan\theta - 1) = 0$

$\Rightarrow \quad \tan\theta = -1$ or, $\tan\theta = \dfrac{1}{2}$

Now, $\tan\theta = -1$

$\Rightarrow \quad \tan\theta = \tan\left(-\dfrac{\pi}{4}\right)$

$\Rightarrow \quad \theta = n\pi + \left(-\dfrac{\pi}{4}\right),\ n \in Z$

$\Rightarrow \quad \theta = n\pi - \dfrac{\pi}{4},\ n \in Z$

And, $\tan\theta = \dfrac{1}{2}$

$\Rightarrow \quad \tan\theta = \tan\alpha$, where $\tan\alpha = \dfrac{1}{2}$

$\Rightarrow \quad \theta = m\pi + \alpha$, where $\tan\alpha = \dfrac{1}{2}$ and $m \in Z$

Hence, $\theta = n\pi - \dfrac{\pi}{4}$ or, $\theta = m\pi + \alpha$, where $m,\ n \in Z$ and $\tan\alpha = \dfrac{1}{2}$

(iv) $4\cos\theta - 3\sec\theta = \tan\theta$

$\Rightarrow \quad 4\cos\theta - \dfrac{3}{\cos\theta} = \dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow \quad 4\cos^2\theta - 3 = \sin\theta$

$\Rightarrow \quad 4(1 - \sin^2\theta) - 3 = \sin\theta$

$\Rightarrow \quad 4\sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \quad \sin\theta = \dfrac{-1 \pm \sqrt{1+16}}{8}$

$\Rightarrow \quad \sin\theta = \dfrac{-1 \pm \sqrt{17}}{8}$

$\Rightarrow \quad \sin\theta = \dfrac{-1 + \sqrt{17}}{8}$ or, $\sin\theta = \dfrac{-1 - \sqrt{17}}{8}$

Now, $\sin\theta = \dfrac{-1 + \sqrt{17}}{8}$

$\Rightarrow \quad \sin\theta = \sin\alpha$, where $\sin\alpha = \dfrac{-1 + \sqrt{17}}{8}$

$\Rightarrow \quad \theta = n\pi + (-1)^n\alpha$, where $\sin\alpha = \dfrac{-1 + \sqrt{17}}{8}$ and $n \in Z$

And, $\sin\theta = \dfrac{-1 - \sqrt{17}}{8}$

$\Rightarrow \quad \sin\theta = \sin\beta$, where $\sin\beta = \dfrac{-1 - \sqrt{17}}{8}$

$\Rightarrow \quad \theta = n\pi + (-1)^n\beta$, where $\sin\beta = \dfrac{-1 - \sqrt{17}}{8}$

(v) $\tan^2\theta + (1 - \sqrt{3})\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan^2\theta + \tan\theta - \sqrt{3}\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan(\tan\theta + 1) - \sqrt{3}(\tan\theta + 1) = 0$

$\Rightarrow \quad (\tan\theta + 1)(\tan\theta - \sqrt{3}) = 0$

$\Rightarrow \quad \tan\theta + 1 = 0$ or $\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan\theta = -1$ or, $\tan\theta = \sqrt{3}$

Now, $\tan\theta = -1 \Rightarrow \tan\theta = \tan\left(-\frac{\pi}{4}\right) \Rightarrow \theta = n\pi - \frac{\pi}{4},\ n \in Z$

And, $\tan\theta = \sqrt{3} \Rightarrow \tan\theta = \tan\frac{\pi}{3} \Rightarrow \theta = m\pi + \frac{\pi}{3},\ m \in Z$

Hence, $\theta = n\pi - \frac{\pi}{4}$ or, $\theta = m\pi + \frac{\pi}{3}$, where $m, n \in Z$.

(vi) $\sec^2 2x = 1 - \tan 2x$

$\Rightarrow \quad 1 + \tan^2 2x = 1 - \tan 2x$

$\Rightarrow \quad \tan^2 2x + \tan 2x = 0$

$\Rightarrow \quad \tan 2x(\tan 2x + 1) = 0$

$\Rightarrow \quad \tan 2x = 0$ or, $\tan 2x + 1 = 0$

$\Rightarrow \quad \tan 2x = 0$ or, $\tan 2x = -1$

$\Rightarrow \quad \tan 2x = 0$ or, $\tan 2x = \tan\frac{3\pi}{4}$

$\Rightarrow \quad 2x = n\pi$ or, $2x = n\pi + \frac{3\pi}{4},\ n \in Z$

$\Rightarrow \quad x = \frac{n\pi}{2}$ or, $x = \frac{n\pi}{2} + \frac{3\pi}{8},\ n \in Z$

EXAMPLE 10 *Solve the following equations:*

(i) $\tan\theta + \tan 2\theta + \tan\theta\tan 2\theta = 1$ (ii) $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta\tan 2\theta\tan 3\theta$

(iii) $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$ (iv) $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 3$

SOLUTION (i) $\tan\theta + \tan 2\theta + \tan\theta\tan 2\theta = 1$

$\Rightarrow \quad \tan\theta + \tan 2\theta = 1 - \tan\theta\tan 2\theta$

$\Rightarrow \quad \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = 1$

$\Rightarrow \quad \tan 3\theta = 1$

$\Rightarrow \quad \tan 3\theta = \tan\frac{\pi}{4}$

$\Rightarrow \quad 3\theta = n\pi + \frac{\pi}{4},\ n \in Z$

$\Rightarrow \quad \theta = \frac{n\pi}{3} + \frac{\pi}{12},\ n \in Z$

(ii) $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta\tan 2\theta\tan 3\theta$

$\Rightarrow \quad \tan\theta + \tan 2\theta = -\tan 3\theta + \tan\theta\tan 2\theta\tan 3\theta$

$\Rightarrow \quad \tan\theta + \tan 2\theta = -\tan 3\theta(1 - \tan\theta\tan 2\theta)$

$\Rightarrow \quad \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = -\tan 3\theta$

$\Rightarrow \quad \tan(\theta + 2\theta) = -\tan 3\theta$

$\Rightarrow \quad \tan 3\theta = -\tan 3\theta$

$\Rightarrow \quad 2\tan 3\theta = 0$

$\Rightarrow \quad \tan 3\theta = 0$

$\Rightarrow \quad 3\theta = n\pi,\ n \in Z \Rightarrow \theta = \frac{n\pi}{3},\ n \in Z.$

(iii) $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$

$\Rightarrow \tan\theta + \tan 2\theta = \sqrt{3}(1 - \tan\theta\tan 2\theta)$

$\Rightarrow \dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \sqrt{3}$

$\Rightarrow \tan 3\theta = \sqrt{3}$

$\Rightarrow \tan 3\theta = \tan\dfrac{\pi}{3}$

$\Rightarrow 3\theta = n\pi + \dfrac{\pi}{3}, n \in Z$

$\Rightarrow \theta = \dfrac{n\pi}{3} + \dfrac{\pi}{9}, n \in Z$

(iv) $\tan\theta + \tan\left(\theta + \dfrac{\pi}{3}\right) + \tan\left(\theta + \dfrac{2\pi}{3}\right) = 3$

$\Rightarrow \tan\theta + \dfrac{\tan\theta + \tan\frac{\pi}{3}}{1 - \tan\theta\tan\frac{\pi}{3}} + \dfrac{\tan\theta + \tan\frac{2\pi}{3}}{1 - \tan\theta\tan\frac{2\pi}{3}} = 3$

$\Rightarrow \tan\theta + \dfrac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} + \dfrac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} = 3$

$\Rightarrow \tan\theta + \dfrac{8\tan\theta}{1 - 3\tan^2\theta} = 3$

$\Rightarrow \dfrac{\tan\theta - 3\tan^3\theta + 8\tan\theta}{1 - 3\tan^2\theta} = 3$

$\Rightarrow \dfrac{3(3\tan\theta - \tan^3\theta)}{1 - 3\tan^2\theta} = 3$

$\Rightarrow 3\tan 3\theta = 3$

$\Rightarrow \tan 3\theta = 1$

$\Rightarrow \tan 3\theta = \tan\dfrac{\pi}{4}$

$\Rightarrow 3\theta = n\pi + \dfrac{\pi}{4}, n \in Z$

$\Rightarrow \theta = \dfrac{n\pi}{3} + \dfrac{\pi}{12}, n \in Z.$

11.3 GENERAL SOLUTIONS OF TRIGONOMETRICAL EQUATIONS OF THE FORM $\sin^2\theta = \sin^2\alpha, \cos^2\theta = \cos^2\alpha, \tan^2\theta = \tan^2\alpha$

THEOREM *Prove that:*

(i) $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in Z$ (ii) $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in Z$

(iii) $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in Z$

PROOF (i) $\sin^2\theta = \sin^2\alpha$

$\Rightarrow 2\sin^2\theta = 2\sin^2\alpha$

$\Rightarrow 1 - \cos 2\theta = 1 - \cos 2\alpha$

$\Rightarrow \cos 2\theta = \cos 2\alpha$

$\Rightarrow 2\theta = 2n\pi \pm 2\alpha, n \in Z$

$\Rightarrow \theta = n\pi \pm \alpha, n \in Z$

(ii) $\cos^2\theta = \cos^2\alpha$

$\Rightarrow \quad 2\cos^2\theta = 2\cos^2\alpha$

$\Rightarrow \quad 1+\cos 2\theta = 1+\cos 2\alpha$

$\Rightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Rightarrow \quad 2\theta = 2n\pi \pm 2\alpha, n \in Z$

$\Rightarrow \quad \theta = n\pi \pm \alpha, n \in Z$

(iii) $\tan^2\theta = \tan^2\alpha$

$$\Rightarrow \quad \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

$\Rightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Rightarrow \quad 2\theta = 2n\pi \pm 2\alpha, n \in Z$

$\Rightarrow \quad \theta = n\pi \pm \alpha, \; n \in Z$ Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve:* $7\cos^2\theta + 3\sin^2\theta = 4$

SOLUTION We have,

$$7\cos^2\theta + 3\sin^2\theta = 4$$

$\Rightarrow \quad 7(1-\sin^2\theta) + 3\sin^2\theta = 4$

$\Rightarrow \quad 4\sin^2\theta = 3$

$\Rightarrow \quad 4\sin^2\theta = 3$

$$\Rightarrow \quad \sin^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \quad \sin^2\theta = \sin^2\frac{\pi}{3} \Rightarrow \theta = n\pi \pm \frac{\pi}{3}, \; n \in Z$$

EXAMPLE 2 *Solve:* $2\sin^2 x + \sin^2 2x = 2$

SOLUTION We have,

$$2\sin^2 x + \sin^2 2x = 2$$

$\Rightarrow \quad 2\sin^2 x + (2\sin x\cos x)^2 = 2$

$\Rightarrow \quad 4\sin^2 x\cos^2 x + 2\sin^2 x = 2$

$\Rightarrow \quad 2\sin^2 x\cos^2 x + \sin^2 x = 1$

$\Rightarrow \quad 2\sin^2 x\cos^2 x - (1-\sin^2 x) = 0$

$\Rightarrow \quad 2\sin^2 x\cos^2 x - \cos^2 x = 0$

$\Rightarrow \quad \cos^2 x(2\sin^2 x - 1) = 0 \Rightarrow \cos^2 x = 0$ or $2\sin^2 x - 1 = 0$

$$\Rightarrow \quad \cos^2 x = 0 \text{ or, } \sin^2 x = \frac{1}{2}$$

Now, $\cos^2 x = 0 \Rightarrow \cos^2 x = \cos^2\frac{\pi}{2} \Rightarrow x = n\pi \pm \frac{\pi}{2}, \; n \in Z$

And, $\sin^2 x = \frac{1}{2} \Rightarrow \sin^2 x = \sin^2\frac{\pi}{4} \Rightarrow x = m\pi \pm \frac{\pi}{4}, \; m \in Z$

Hence, $x = n\pi \pm \frac{\pi}{2}$ or $x = m\pi \pm \frac{\pi}{4}$, where $m, n \in Z$

LEVEL-2

EXAMPLE 3 *Solve:* $\sin 3\alpha = 4\sin\alpha \sin(x+\alpha)\sin(x-\alpha)$, where $\alpha \neq n\pi, n \in Z$

SOLUTION We have,

$$\sin 3\alpha = 4\sin\alpha \sin(x+\alpha)\sin(x-\alpha)$$

$$\Rightarrow \quad \sin 3\alpha = 4\sin\alpha(\sin^2 x - \sin^2\alpha)$$

$$\Rightarrow \quad 3\sin\alpha - 4\sin^3\alpha = 4\sin^2 x \sin\alpha - 4\sin^3\alpha$$

$$\Rightarrow \quad 3\sin\alpha = 4\sin^2 x\sin\alpha$$

$$\Rightarrow \quad \sin^2 x = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \quad \sin^2 x = \sin^2\frac{\pi}{3} \Rightarrow x = n\pi \pm \frac{\pi}{3}, n \in Z$$

EXAMPLE 4 *Solve:* $4\sin x \sin 2x \sin 4x = \sin 3x$

SOLUTION We have,

$$4\sin x\sin 2x\sin 4x = \sin 3x$$

$$\Rightarrow \quad 4\sin x\sin(3x-x)\,.\sin(3x+x) = \sin 3x$$

$$\Rightarrow \quad 4[\sin x(\sin^2 3x - \sin^2 x)] = 3\sin x - 4\sin^3 x$$

$$\Rightarrow \quad 4\sin x\sin^2 3x - 4\sin^3 x = 3\sin x - 4\sin^3 x$$

$$\Rightarrow \quad 4\sin x\sin^2 3x = 3\sin x$$

$$\Rightarrow \quad \sin x(4\sin^2 3x - 3) = 0$$

$$\Rightarrow \quad \sin x = 0 \text{ or, } 4\sin^2 3x - 3 = 0$$

$$\Rightarrow \quad \sin x = 0 \text{ or, } \sin^2 3x = \frac{3}{4}$$

Now, $\sin x = 0 \Rightarrow x = n\pi, n \in Z$

And, $\sin^2 3x = \frac{3}{4}$

$$\Rightarrow \quad \sin^2 3x = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \quad \sin^2 3x = \sin^2\frac{\pi}{3}$$

$$\Rightarrow \quad 3x = m\pi \pm \frac{\pi}{3}, m \in Z$$

$$\Rightarrow \quad x = \frac{m\pi}{3} \pm \frac{\pi}{9}$$

Hence, $x = n\pi$ or, $x = m\pi \pm \frac{\pi}{3}$, where $m, n \in Z$

11.4 TRIGONOMETRIC EQUATIONS OF THE FORM

$a\cos\theta + b\sin\theta = c$, where $a, b, c \in R$ such that $|c| \leq \sqrt{a^2+b^2}$

To solve this type of equations, we first reduce them in the form $\cos\theta = \cos\alpha$, or $\sin\theta = \sin\alpha$.

The following algorithm provides the method of solution.

ALGORITHM

STEP I *Obtain the equation* $a\cos\theta + b\sin\theta = c$.

STEP II *Put* $a = r\cos\alpha$ *and* $b = r\sin\alpha$, *where* $r = \sqrt{a^2+b^2}$ *and* $\tan\alpha = b/a$ i.e. $\alpha = \tan^{-1}(b/a)$.

STEP III *Using the substitution in step II, the equation reduces to*

$$r\cos(\theta-\alpha) = c \Rightarrow \cos(\theta-\alpha) = \frac{c}{r} = \cos\beta \text{ (say)}.$$

STEP IV *Solve the equation obtained in step III by using the formulas discussed earlier.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve:* $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$ [NCERT EXEMPLAR]

SOLUTION We have,

$$\sqrt{3}\cos\theta + \sin\theta = \sqrt{2} \quad ...(i)$$

This is of the form $a\cos\theta + b\sin\theta = c$, where $a = \sqrt{3}, b = 1$ and $c = \sqrt{2}$.

Let $a = r\cos\alpha$ and $b = r\sin\alpha$. Then,

$\sqrt{3} = r\cos\alpha$ and $1 = r\sin\alpha$.

$$\Rightarrow \quad r = \sqrt{a^2+b^2} = \sqrt{(\sqrt{3})^2+1^2} = 2 \text{ and } \tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$$

Substituting $a = \sqrt{3} = r\cos\alpha$ and $b = 1 = r\sin\alpha$ in the equation (i) it reduces to

$$r\cos\alpha\cos\theta + r\sin\alpha\sin\theta = \sqrt{2}$$
$$\Rightarrow \quad r\cos(\theta-\alpha) = \sqrt{2}$$
$$\Rightarrow \quad 2\cos\left(\theta-\frac{\pi}{6}\right) = \sqrt{2}$$
$$\Rightarrow \quad \cos\left(\theta-\frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$$
$$\Rightarrow \quad \cos\left(\theta-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}$$
$$\Rightarrow \quad \theta-\frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}, n \in Z$$
$$\Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}, \ n \in Z$$
$$\Rightarrow \quad \theta = 2n\pi + \frac{\pi}{4} + \frac{\pi}{6} \text{ or, } \theta = 2n\pi - \frac{\pi}{4} + \frac{\pi}{6}$$
$$\Rightarrow \quad \theta = 2n\pi + \frac{5\pi}{12} \text{ or, } \theta = 2n\pi - \frac{\pi}{12}$$

Hence, $\theta = 2n\pi + \frac{5\pi}{12}$ or, $\theta = 2n\pi - \frac{\pi}{12}$, where $n \in Z$

EXAMPLE 2 *Solve:* $\sqrt{2}\sec\theta + \tan\theta = 1$

SOLUTION We have,

$$\sqrt{2}\sec\theta + \tan\theta = 1$$
$$\Rightarrow \quad \frac{\sqrt{2}}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = 1$$
$$\Rightarrow \quad \sqrt{2} + \sin\theta = \cos\theta$$
$$\Rightarrow \quad \cos\theta - \sin\theta = \sqrt{2} \quad ...(i)$$

This is of the form, $a\cos\theta - b\sin\theta = c$, where $a = 1$, $b = 1$ and $c = \sqrt{2}$

Let $a = r\cos\alpha$, and $b = r\sin\alpha$.

$\Rightarrow \quad 1 = r\cos\alpha$ and $1 = r\sin\alpha$.

$\Rightarrow \quad r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$ and, $\tan\alpha = \dfrac{r\sin\alpha}{r\cos\alpha} = 1$

$\Rightarrow \quad r = \sqrt{2}$ and, $\alpha = \dfrac{\pi}{4}$

Substituting $a = 1 = r\cos\alpha$ and $b = 1 = r\sin\alpha$ in (i), we get

$$r\cos 0\cos\alpha - r\sin\theta\sin\alpha = \sqrt{2}$$

$$\Rightarrow \quad r\cos(0+\alpha) = \sqrt{2}$$

$$\Rightarrow \quad \cos\left(\theta+\frac{\pi}{4}\right) = 1$$

$$\Rightarrow \quad \cos\left(\theta+\frac{\pi}{4}\right) = \cos 0°$$

$$\Rightarrow \quad \theta+\frac{\pi}{4} = 2n\pi \pm 0,\ n\in Z \quad \Rightarrow \theta = 2n\pi - \frac{\pi}{4},\ n\in Z$$

EXAMPLE 3 *Solve:* $\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$

SOLUTION We have,

$$\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$$

$$\Rightarrow \quad \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = \sqrt{3}$$

$$\Rightarrow \quad \cos\theta + 1 = \sqrt{3}\sin\theta$$

$$\Rightarrow \quad \sqrt{3}\sin\theta - \cos\theta = 1 \qquad \text{...(i)}$$

This is of the form $a\sin\theta + b\cos\theta = c$, where $a = \sqrt{3}, b = -1$ and $c = 1$.

$\therefore \quad \sqrt{3} = r\sin\alpha$ and $1 = r\cos\alpha$

$\Rightarrow \quad r = \sqrt{a^2+b^2} = \sqrt{3+1} = 2$ and, $\tan\alpha = \dfrac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow r = 2$ and $\alpha = \pi/3$

Substituting $a = \sqrt{3} = r\sin\alpha$ and $b = 1 = r\cos\alpha$ in (i), we get

$$r\sin\alpha\sin\theta - r\cos\alpha\cos\theta = 1$$

$$\Rightarrow \quad -r\cos(\theta+\alpha) = 1$$

$$\Rightarrow \quad -2\cos\left(\theta+\frac{\pi}{3}\right) = 1$$

$$\Rightarrow \quad \cos\left(\theta+\frac{\pi}{3}\right) = -\frac{1}{2}$$

$$\Rightarrow \quad \cos\left(\theta+\frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\Rightarrow \quad \theta+\frac{\pi}{3} = 2n\pi \pm \frac{2\pi}{3},\ n\in Z$$

$$\Rightarrow \quad \theta = 2n\pi \pm \frac{2\pi}{3} - \frac{\pi}{3}, n\in Z$$

$$\Rightarrow \quad \theta = 2n\pi + \frac{\pi}{3},\ n\in Z \quad \text{or,} \quad \theta = 2n\pi - \pi = (2n-1)\pi,\ n\in Z$$

But, θ cannot be equal to $(2n-1)\pi$ as it makes $\sin\theta = 0$.

Hence, $\theta = 2n\pi + \dfrac{\pi}{3}, n\in Z$

EXERCISE 11.1

LEVEL-1

1. Find the general solutions of the following equations:

(i) $\sin\theta = \frac{1}{2}$ (ii) $\cos\theta = -\frac{\sqrt{3}}{2}$ (iii) $\operatorname{cosec}\theta = -\sqrt{2}$

(iv) $\sec\theta = \sqrt{2}$ (v) $\tan\theta = -\frac{1}{\sqrt{3}}$ (vi) $\sqrt{3}\sec\theta = 2$

2. Find the general solutions of the following equations:

(i) $\sin 2\theta = \frac{\sqrt{3}}{2}$ (ii) $\cos 3\theta = \frac{1}{2}$ (iii) $\sin 9\theta = \sin\theta$

(iv) $\sin 2\theta = \cos 3\theta$ (v) $\tan\theta + \cot 2\theta = 0$ (vi) $\tan 3\theta = \cot\theta$

(vii) $\tan 2\theta \tan\theta = 1$ (viii) $\tan m\theta + \cot n\theta = 0$ (ix) $\tan p\theta = \cot q\theta$

(x) $\sin 2\theta + \cos\theta = 0$ (xi) $\sin\theta = \tan\theta$ (xii) $\sin 3\theta + \cos 2\theta = 0$

3. Solve the following equations:

(i) $\sin^2\theta - \cos\theta = \frac{1}{4}$ (ii) $2\cos^2\theta - 5\cos\theta + 2 = 0$

(iii) $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$ (iv) $4\sin^2\theta - 8\cos\theta + 1 = 0$

(v) $\tan^2 x + (1-\sqrt{3})\tan x - \sqrt{3} = 0$

(vi) $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$ (vii) $\cos 4\theta = \cos 2\theta$

4. Solve the following equations:

(i) $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ (ii) $\cos\theta + \cos 3\theta - \cos 2\theta = 0$ [NCERT]

(iii) $\sin\theta + \sin 5\theta = \sin 3\theta$ (iv) $\cos\theta\cos 2\theta\cos 3\theta = \frac{1}{4}$

(v) $\cos\theta + \sin\theta = \cos 2\theta + \sin 2\theta$ (vi) $\sin\theta + \sin 2\theta + \sin 3\theta = 0$

(vii) $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

(viii) $\sin 3\theta - \sin\theta = 4\cos^2\theta - 2$ (ix) $\sin 2\theta - \sin 4\theta + \sin 6\theta = 0$ [NCERT]

5. Solve the following equations:

(i) $\tan\theta + \tan 2\theta + \tan 3\theta = 0$ (ii) $\tan\theta + \tan 2\theta = \tan 3\theta$

(iii) $\tan 3\theta + \tan\theta = 2\tan 2\theta$

6. Solve the following equations:

(i) $\sin\theta + \cos\theta = \sqrt{2}$ (ii) $\sqrt{3}\cos\theta + \sin\theta = 1$

(iii) $\sin\theta + \cos\theta = 1$ (iv) $\operatorname{cosec}\theta = 1 + \cot\theta$

(v) $(\sqrt{3}-1)\cos\theta + (\sqrt{3}+1)\sin\theta = 2$ [NCERT EXEMPLAR]

7. Solve the following equations:

(i) $\cot\theta + \tan\theta = 2$ [NCERT EXEMPLAR]

(ii) $2\sin^2\theta = 3\cos\theta, 0 \le \theta \le 2\pi$ [NCERT EXEMPLAR]

(iii) $\sec\theta\cos 5\theta + 1 = 0, 0 < \theta < \frac{\pi}{2}$ [NCERT EXEMPLAR]

(iv) $5\cos^2\theta + 7\sin^2\theta - 6 = 0$ [NCERT EXEMPLAR]

(v) $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ [NCERT EXEMPLAR]

ANSWERS

1. (i) $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in Z$ (ii) $\theta = 2n\pi \pm \frac{7\pi}{6}, n \in Z$

(iii) $\theta = n\pi + (-1)^{n+1}\frac{\pi}{4}, n \in Z$ (iv) $\theta = 2n\pi \pm \frac{\pi}{4}, n \in Z$

(v) $\theta = n\pi - \frac{\pi}{6}$, $n \in Z$ (vi) $\theta = 2n\pi \pm \frac{\pi}{6}$, $n \in Z$

2. (i) $\theta = \frac{n\pi}{2} + (-1)^n \frac{\pi}{6}$, $n \in Z$ (ii) $\theta = \frac{2n\pi}{3} \pm \frac{\pi}{9}$, $n \in Z$

(iii) $\theta = \frac{r\pi}{4}$ or $\theta = (2r+1)\frac{\pi}{10}$, where $r \in Z$

(iv) $\theta = (4n+1)\frac{\pi}{10}$ or $\theta = (4n-1)\frac{\pi}{2}$, where $n \in Z$

(v) $\theta = n\pi - \frac{\pi}{2}$, $n \in Z$ (vi) $\theta = \frac{n\pi}{4} + \frac{\pi}{8}$, $n \in Z$

(vii) $\theta = \frac{n\pi}{3} + \frac{\pi}{6}$, $n \in Z$ (viii) $\theta = \frac{(2r+1)\pi}{m-n}$, $r \in Z$

(ix) $\theta = \left(\frac{2n+1}{p+q}\right)\frac{\pi}{2}$, $n \in Z$

(x) $\theta = (4n-1)\frac{\pi}{2}$ or $\theta = (4m-1)\frac{\pi}{6}$, where $m, n \in Z$

(xi) $\theta = m\pi$ or $\theta = 2n\pi$, where $m, n \in Z$

(xii) $\theta = (4n-1)\frac{\pi}{10}$ or $\theta = (4m-1)\frac{\pi}{2}$, $m, n \in Z$

3. (i) $\theta = 2n\pi \pm \frac{\pi}{3}$, $n \in Z$ (ii) $\theta = 2n\pi \pm \frac{\pi}{3}$, $n \in Z$

(iii) $\theta = 2n\pi \pm \frac{5\pi}{6}$, $n \in Z$ (iv) $\theta = 2n\pi \pm \frac{\pi}{3}$, $n \in Z$

(v) $\theta = n\pi - \frac{\pi}{4}$ or $\theta = m\pi + \frac{\pi}{3}$, where $m, n \in Z$

(vi) $\theta = n\pi - \frac{\pi}{3}$ or $\theta = m\pi + \frac{\pi}{6}$, where $m, n \in Z$

(vii) $x = n\pi, x = \frac{n\pi}{3}, n \in Z$

4. (i) $\theta = (2n+1)\frac{\pi}{4}$ or $\theta = 2m\pi \pm \frac{2\pi}{3}$, where $m, n \in Z$

(ii) $\theta = (2n+1)\frac{\pi}{4}$ or $\theta = 2m\pi \pm \frac{\pi}{3}$ where $m, n \in Z$

(iii) $\theta = \frac{n\pi}{3}$ or $\theta = m\pi \pm \frac{\pi}{6}$, where $m, n \in Z$

(iv) $\theta = 2n + 1\frac{\pi}{8}$ or $\theta = m\pi \pm \frac{\pi}{3}$, where $m, n \in Z$

(v) $\theta = \frac{(2n\pi)}{3} + \frac{\pi}{6}$ or $\theta = 2m\pi$, where $m, n \in Z$

(vi) $\theta = \frac{n\pi}{2}$ or $\theta = 2n\pi \pm \frac{2\pi}{3}$, where $m, n \in Z$

(vii) $\theta = n\pi + \frac{\pi}{2}$, $\theta = (2m+1)\pi$, $\theta = \frac{2r\pi}{5}$, where $m, n\, r \in Z$

(viii) $\theta = n\pi + (-1)^n \frac{\pi}{2}$ or $\theta = (2m+1)\frac{\pi}{4}$, where $m, n \in Z$

(ix) $\theta = \frac{n\pi}{4}$, $\theta = n\pi \pm \frac{\pi}{6}$, $n \in Z$

5. (i) $\theta = \frac{m\pi}{3}$ or $\theta = n\pi \pm \alpha$, where $\alpha = \tan^{-1}\frac{1}{\sqrt{2}}$ and $m, n \in Z$

(ii) $\theta = m\pi$ or $\theta = \frac{n\pi}{3}$, where $m, n \in Z$

(iii) $\theta = n\pi$, where $n \in Z$

6. (i) $\theta = (8n+)\frac{\pi}{4}$, $n \in Z$

(ii) $\theta = (4n+1)\frac{\pi}{2}$ or $\theta = (12m-1)\frac{\pi}{6}$, where $m, n \in Z$

(iii) $\theta = 2n\pi$ or $\theta = 2m\pi + \frac{\pi}{2}$, where $m, n \in Z$

(iv) $\theta = 2m\pi + \frac{\pi}{2}$, where $m, n \in Z$

(v) $\theta = 2n\pi + \frac{\pi}{3}$ or, $\theta = 2n\pi - \frac{\pi}{6}, n \in Z$

7. (i) $\theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$

(ii) $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

(iii) $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}$

(iv) $\theta = n\pi \pm \frac{\pi}{4}, n \in Z$

(v) $x = \frac{n\pi}{2} \pm \frac{\pi}{8}, n \in Z$

HINTS TO NCERT & SELECTED PROBLEMS

4. (ii)

$2n+1)\frac{\pi}{2}, n \in Z \Rightarrow \theta = (2n+1)\frac{\pi}{4}, n \in Z$

$= \cos\frac{\pi}{3} \Rightarrow \theta = 2m\pi \pm \frac{\pi}{3}, m \in Z$

(iv) $3\theta = \frac{1}{4}$

$\Rightarrow$ $2(2 \ldots \theta)\cos 3\theta = 1$

$\Rightarrow$ $2(\cos 3\theta + \cos\theta)\cos 3\theta = 1$

$\Rightarrow$ $2\cos^2 3\theta + 2\cos 3\theta\cos\theta = 1$

$\Rightarrow$ $2\cos^2 3\theta + \cos 4\theta + \cos 2\theta = 1$

$\Rightarrow$ $(2\cos^2 3\theta - 1) + \cos 4\theta + \cos 2\theta = 0$

$\Rightarrow$ $\cos 6\theta + \cos 2\theta + \cos 4\theta = 0$

$\Rightarrow$ $2\cos 4\theta\cos 2\theta + \cos 4\theta = 0$

$\Rightarrow$ $\cos 4\theta(2\cos 2\theta + 1) = 0$

$\Rightarrow$ $\cos 4\theta = 0, 2\cos 2\theta + 1 = 0$

$\Rightarrow$ $\cos 4\theta = 0, \cos 2\theta = \cos\frac{2\pi}{3}$

$\Rightarrow$ $4\theta = (2n+1)\frac{\pi}{2}, 2\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

$\Rightarrow$ $\theta = (2n+1)\frac{\pi}{8}, \theta = n\pi \pm \frac{\pi}{3}, n \in Z$

(ix) We have,

$\sin 2\theta - \sin 4\theta + \sin 6\theta = 0$

$\Rightarrow$ $\sin 6\theta + \sin 2\theta - \sin 4\theta = 0$

$\Rightarrow$ $2\sin 4\theta\cos 2\theta - \sin 4\theta = 0$

$\Rightarrow \quad \sin 4\theta(2\cos 2\theta - 1) = 0$

$\Rightarrow \quad \sin 4\theta = 0$ or, $2\cos 2\theta - 1 = 0$

Now, $\sin 4\theta = 0 \Rightarrow 4\theta = n\pi, n \in Z \Rightarrow \theta = \frac{n\pi}{4}, n \in Z$

and, $2\cos 2\theta - 1 = 0$

$\Rightarrow \quad \cos 2\theta = \frac{1}{2} \Rightarrow \cos 2\theta = \cos\frac{\pi}{3} \Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{3}, n \in Z \Rightarrow \theta = n\pi \pm \frac{\pi}{6}, n \in Z$

5. (i) $\tan\theta + \tan 2\theta + \tan 3\theta = 0$

$\Rightarrow \quad \tan\theta + \tan 2\theta + \tan(\theta + 2\theta) = 0$

$\Rightarrow \quad \tan\theta + \tan 2\theta + \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = 0$

$\Rightarrow \quad (\tan\theta + \tan 2\theta)\ (2 - \tan\theta\tan 2\theta) = 0$

$\Rightarrow \quad \tan\theta + \tan 2\theta = 0$ or $\tan\theta\tan 2\theta = 2$

Now, $\tan\theta + \tan 2\theta = 0$

$\Rightarrow \quad \tan\theta + \frac{2\tan\theta}{1 - \tan^2\theta} = 0$

$\Rightarrow \quad \tan\theta(3 - \tan^2\theta) = 0$

$\Rightarrow \quad \tan\theta = 0$ or, $\tan^2\theta = 3$

$\Rightarrow \quad \tan\theta = 0$ or $\tan^2\theta = \tan^2\frac{\pi}{3}$

$\Rightarrow \quad \theta = n\pi,\ \theta = n\pi \pm \frac{\pi}{3}, n \in Z$

and, $\tan\theta\tan 2\theta = 2$

$\Rightarrow \quad \tan\theta \times \frac{2\tan\theta}{1 - \tan^2\theta} = 2$

$\Rightarrow \quad \tan^2\theta = 1 - \tan^2\theta$

$\Rightarrow \quad 2\tan^2\theta = 1$

$\Rightarrow \quad \tan^2\theta = \left(\frac{1}{\sqrt{2}}\right)^2 = \tan^2\alpha$ (say), where $\tan\alpha = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = n\pi \pm \alpha$, where $\tan\alpha = \frac{1}{\sqrt{2}}$

(iii) $\tan 3\theta - \tan 2\theta = \tan 2\theta - \tan\theta$

$\Rightarrow \quad \frac{\sin(3\theta - 2\theta)}{\cos 3\theta\cos 2\theta} = \frac{\sin(2\theta - \theta)}{\cos 2\theta\cos\theta}$

$\Rightarrow \quad \frac{\sin\theta}{\cos 3\theta\cos 2\theta} = \frac{\sin\theta}{\cos 2\theta\cos\theta}$

$\Rightarrow \quad \sin\theta\cos 2\theta(\cos 3\theta - \cos\theta) = 0$

$\Rightarrow \quad -2\sin\theta\cos 2\theta\sin 2\theta\sin\theta = 0$

$\Rightarrow \quad 2\sin^2\theta\sin 2\theta = 0$ $\quad [\because \cos 2\theta \neq 0]$

$\Rightarrow \quad \sin\theta = 0$ or, $\sin 2\theta = 0 \Rightarrow \theta = n\pi$ or, $2\theta = m\pi \Rightarrow \theta = n\pi$ or, $\theta = \frac{m\pi}{2}$, where $n, m \in Z$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the number of solutions of the equation $\tan x + \sec x = 2\cos x$ in the interval $[0, 2\pi]$.
2. Write the number of solutions of the equation $4\sin x - 3\cos x = 7$.
3. Write the general solution of $\tan^2 2x = 1$.
4. Write the set of values of a for which the equation $\sqrt{3}\sin x - \cos x = a$ has no solution.
5. If $\cos x = k$ has exactly one solution in $[0, 2\pi]$, then write the value(s) of k.
6. Write the number of points of intersection of the curves $2y = 1$ and $y = \cos x$, $0 \le x \le 2\pi$.
7. Write the values of x in $[0, \pi]$ for which $\sin 2x$, $\frac{1}{2}$ and $\cos 2x$ are in A.P.
8. Write the number of points of intersection of the curves $2y = -1$ and $y = \operatorname{cosec} x$.
9. Write the solution set of the equation $(2\cos\theta + 1)(4\cos\theta + 5) = 0$ in the interval $[0, 2\pi]$.
10. Write the number of values of θ in $[0, 2\pi]$ that satisfy the equation $\sin^2\theta - \cos\theta = \frac{1}{4}$.
11. If $3\tan(\theta - 15°) = \tan(\theta + 15°)$, $0 < \theta < 90°$, find θ.
12. If $2\sin^2\theta = 3\cos\theta$, where $0 \le \theta \le 2\pi$, then find the value of θ.
13. If $\sec x \cos 5x + 1 = 0$, where $0 < x \le \frac{\pi}{2}$, find the value of x.

ANSWERS

1. 2	2. 0	3. $\frac{n\pi}{2} + \frac{\pi}{8}, n \in Z$	4. $a \in (-\infty, -2) \cup (2, \infty)$
5. -1	6. 2	7. $0, \frac{\pi}{4}, \pi$	8. 0
9. $\frac{2\pi}{3}, \frac{4\pi}{3}$	10. 2	11. $\frac{\pi}{4}$	12. $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$
13. $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}$			

MULTIPLE CHOICES QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The smallest value of θ satisfying the equation $\sqrt{3}(\cot\theta + \tan\theta) = 4$ is
   (a) $2\pi/3$ (b) $\pi/3$ (c) $\pi/6$ (d) $\pi/12$
2. If $\cos\theta + \sqrt{3}\sin\theta = 2$, then $\theta =$
   (a) $\pi/3$ (b) $2\pi/3$ (c) $4\pi/3$ (d) $5\pi/3$
3. If $\tan p\theta - \tan q\theta = 0$, then the values of θ form a series in
   (a) AP (b) GP (c) HP (d) none of these
4. If a is any real number, the number of roots of $\cot x - \tan x = a$ in the first quadrant is (are).
   (a) 2 (b) 0 (c) 1 (d) none of these
5. The general solution of the equation $7\cos^2\theta + 3\sin^2\theta = 4$ is
   (a) $\theta = 2n\pi \pm \frac{\pi}{6}, n \in Z$ (b) $\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

(c) $\theta = n\pi \pm \frac{\pi}{3}, n \in Z$ (d) none of these

6. A solution of the equation $\cos^2\theta + \sin\theta + 1 = 0$, lies in the interval

(a) $(-\pi/4, \pi/4)$ (b) $(\pi/4, 3\pi/4)$
(c) $(3\pi/4, 5\pi/4)$ (d) $(5\pi/4, 7\pi/4)$

7. The number of solution in $[0, \pi/2]$ of the equation $\cos 3x \tan 5x = \sin 7x$ is

(a) 5 (b) 7 (c) 6 (d) none of these

8. The general value of x satisfying the equation $\sqrt{3}\sin x + \cos x = \sqrt{3}$ is given by

(a) $x = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{3}, n \in Z$ (b) $x = n\pi + (-1)^n \frac{\pi}{3} - \frac{\pi}{6}, n \in Z$

(c) $x = n\pi \pm \frac{\pi}{6}, n \in Z$ (d) $x = n\pi \pm \frac{\pi}{3}, n \in Z$

9. The smallest positive angle which satisfies the equation
$2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$ is

(a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

10. If $4\sin^2\theta = 1$, then the values of θ are

(a) $2n\pi \pm \frac{\pi}{3}, n \in Z$ (b) $n\pi \pm \frac{\pi}{3}, n \in Z$ (c) $n\pi \pm \frac{\pi}{6}, n \in Z$ (d) $2n\pi \pm \frac{\pi}{6}, n \in Z$

11. If $\cot\theta - \tan\theta = \sec\theta$, then, θ is equal to

(a) $2n\pi + \frac{3\pi}{2}, n \in Z$ (b) $n\pi + (-1)^n \frac{\pi}{6}, n \in Z$

(c) $n\pi + \frac{\pi}{2}, n \in Z$ (d) none of these.

12. A value of θ satisfying $\cos\theta + \sqrt{3}\sin\theta = 2$ is

(a) $\frac{5\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{3}$

13. In $(0, \pi)$, the number of solutions of the equation
$\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta \tan 2\theta \tan 3\theta$ is

(a) 7 . (b) 5 (c) 4 (d) 2.

14. The number of values of θ in $[0, 2\pi]$ that satisfy the equation $\sin^2\theta - \cos\theta = \frac{1}{4}$

(a) 1 (b) 2 (c) 3 (d) 4

15. If $e^{\sin x} - e^{-\sin x} - 4 = 0$, then $x =$

(a) 0 (b) $\sin^{-1}\{\log_e(2-\sqrt{5})\}$

(c) 1 (d) none of these

16. The equation $3\cos x + 4\sin x = 6$ has .... solution.

(a) finite (b) infinite (c) one (d) no

17. If $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$, then general value of θ is

(a) $n\pi + (-1)^n \frac{\pi}{4}, n \in Z$ (b) $(-1)^n \frac{\pi}{4} - \frac{\pi}{3}, n \in Z$

(c) $n\pi + \frac{\pi}{4} - \frac{\pi}{3}, n \in Z$ (d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}, n \in Z$

18. General solution of $\tan 5\theta = \cot 2\theta$ is

(a) $\frac{n\pi}{7} + \frac{\pi}{2}, n \in Z$ (b) $\theta = \frac{n\pi}{7} + \frac{\pi}{3}, n \in Z$

(c) $\theta = \frac{n\pi}{7} + \frac{\pi}{14}, n \in Z$ (d) $\theta = \frac{n\pi}{7} - \frac{\pi}{14}, n \in Z$

19. The solution of the equation $\cos^2\theta + \sin\theta + 1 = 0$ lies in the interval

(a) $(-\pi/4, \pi/4)$ (b) $(\pi/4, 3\pi/4)$ (c) $(3\pi/4, 5\pi/4)$ (d) $(5\pi/4, 7\pi/4)$

20. If $\cos\theta = -\frac{1}{2}$ and $0 < \theta < 360°$, then the solutions are

(a) $\theta = 60°, 240°$ (b) $\theta = 120°, 240°$

(c) $\theta = 120°, 210°$ (d) $\theta = 120°, 300°$

21. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is

(a) 0 (b) 5 (c) 6 (d) 10

ANSWERS

1. (c)	2. (a)	3. (a)	4. (c)	5. (a)	6. (d)	7. (c)	8. (b)
9. (a)	10. (c)	11. (b)	12. (d)	13. (d)	14. (b)	15. (d)	16. (d)
17. (d)	18. (c)	19. (d)	20. (b)	21. (c)			

SUMMARY

1. An equation containing trigonometric functions of unknown angles is known as a trigonometric equation.
2. A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.
3. Following are the general solutions of trigonometric equations in standard forms:

Trigonometric equation	*General solution*
(i) $\sin\theta = 0$	$\theta = n\pi, n \in Z$
(ii) $\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}, n \in Z$
(iii) $\tan\theta = 0$	$\theta = n\pi, n \in Z$
(iv) $\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n \alpha, n \in Z$
(v) $\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha, n \in Z$
(vi) $\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha, n \in Z$
(vii) $\left.\begin{aligned}\sin^2\theta &= \sin^2\alpha\\ \cos^2\theta &= \cos^2\alpha\\ \tan^2\theta &= \tan^2\alpha\end{aligned}\right\}$	$\theta = n\pi \pm \alpha, n \in Z$

4. The equation $a\cos\theta + b\sin\theta = c$ is solvable for $|c| \le \sqrt{a^2+b^2}$.

CHAPTER 12

MATHEMATICAL INDUCTION

12.1 STATEMENTS

A sentence or description which can be judged to be true or false is called a statement.

Following are some examples of statements:

EXAMPLE 1 *2 divides 6.*

EXAMPLE 2 *Jaipur is the capital of Rajasthan.*

EXAMPLE 3 *There are 5 days in a week.*

EXAMPLE 4 $(x+1)$ *is a factor of* $x^2 - 3x + 2$.

EXAMPLE 5 $A \cup B = B \cup A$.

Clearly, statements in Examples 1, 2 and 5 are true statements whereas statements in Examples 3 and 4 are false.

MATHEMATICAL STATEMENTS *Statements involving mathematical relations are known as the mathematical statements.*

Clearly, statements in examples 1, 4 and 5 are mathematical statements. In this chapter, we shall be mainly discussing mathematical statements concerning natural numbers. We shall be using notations $P(n)$ or $P_1(n)$ or $P_2(n)$ etc. to denote such statements.

EXAMPLE 1 *Let* $P(n)$ *be the statement "*$10n + 3$ *is prime". Then,*

$P(2)$ is the statement "$10 \times 2 + 3$ is prime" i.e. "23 is prime".

Clearly, $P(2)$ is true.

$P(3)$ is the statement "$10 \times 3 + 3$ is prime" i.e. "33 is prime".

Clearly $P(3)$ is not true.

EXAMPLE 2 *If* $P(n)$ *is the statement "*$n^3 + n$ *is divisible by 3", is the statement* $P(3)$ *true ? Is the statement* $P(4)$ *true ?*

SOLUTION $P(3)$ is the statement "$3^3 + 3 = 30$ is divisible by 3".

Clearly, it is true.

$P(4)$ is the statement "$4^3 + 4 = 68$ is divisible by 3".

Clearly, it is not true.

EXAMPLE 3 *If* $P(n)$ *is the statement "*$n(n+1)(n+2)$ *is divisible by 12", prove that the statements* $P(3)$ *and* $P(4)$ *are true, but that* $P(5)$ *is not true.*

SOLUTION $P(3)$ is the statement "$3(3+1)(3+2) = 60$ is divisible by 12".

It is true.

$P(4)$ is the statement "$4(4+1)(4+2) = 120$ is divisible by 12".

It is also true.

$P(5)$ is the statement "$5(5+1)(5+2) = 210$ is divisible by 12".

Clearly it is not true.

EXAMPLE 4 *Let $P(n)$ be the statement "7 divides $(2^{3n}-1)$". What is $P(n+1)$?*

SOLUTION $P(n+1)$ is the statement "7 divides $(2^{3(n+1)}-1)$".

Clearly, $P(n+1)$ is obtained by replacing n by $(n+1)$ in $P(n)$.

EXAMPLE 5 *If $P(n)$ is the statement "$n^2>100$", prove that whenever $P(r)$ is true, $P(r+1)$ is also true.*

SOLUTION The statement $P(n)$ is "$n^2>100$". Let $P(r)$ be true. Then $r^2>100$.

We wish to prove that the statement $P(r+1)$ is true i.e. "$(r+1)^2>100$".

Now,

$P(r)$ is true

$\Rightarrow \quad r^2>100$

$\Rightarrow \quad r^2+2r+1>100+2r+1$ [Adding $(2r+1)$ on both sides]

$\Rightarrow \quad (r+1)^2>100+2r+1$

$\Rightarrow \quad (r+1)^2>100$

$\Rightarrow \quad P(r+1)$ is true [$\because$ $100+2r+1>100$ for every natural number r]

Thus, whenever $P(r)$ is true, $P(r+1)$ is also true.

EXAMPLE 6 *Let $P(n)$ be the statement "$3^n>n$". If $P(n)$ is true, prove that $P(n+1)$ is true.*

SOLUTION We are given that $P(n)$ is true i.e. $3^n>n$, and we wish to prove that $P(n+1)$ is true i.e. $3^{(n+1)}>(n+1)$.

Now,

$P(n)$ is true

$\Rightarrow \quad 3^n>n$

$\Rightarrow \quad 3.3^n>3n$ [Multiplying both sides by 3]

$\Rightarrow \quad 3^{n+1}>n+2n$

$\Rightarrow \quad 3^{n+1}>n+1$ [$\because$ $2n>1$ for every $n\in N \Rightarrow 2n+n>n+1$ for every $n\in N$]

$\Rightarrow \quad P(n+1)$ is true

EXAMPLE 7 *If $P(n)$ is the statement "$2^{3n}-1$ is an integral multiple of 7", and if $P(r)$ is true, prove that $P(r+1)$ is true.*

SOLUTION Let $P(r)$ be true. Then, $2^{3r}-1$ is an integral multiple of 7.

We wish to prove that $P(r+1)$ is true i.e. $2^{3(r+1)}-1$ is an integral multiple of 7.

Now,

$P(r)$ is true

$\Rightarrow \quad 2^{3r}-1$ is an integral multiple of 7

$\Rightarrow \quad 2^{3r}-1=7\lambda$, for some $\lambda\in N$.

$\Rightarrow \quad 2^{3r}=7\lambda+1$ ...(i)

Now, $\quad 2^{3(r+1)}-1=2^{3r}\times 2^3-1=(7\lambda+1)\times 8-1$ [Using (i)]

$\Rightarrow \quad 2^{3(r+1)}-1=56\lambda+8-1=56\lambda+7=7(8\lambda+1)$

$\Rightarrow \quad 2^{3(r+1)}-1=7\mu$, where $\mu=8\lambda+1\in N$

$\Rightarrow \quad 2^{3(r+1)}-1$ is an integral multiple of 7

$\Rightarrow \quad P(r+1)$ is true

EXERCISE 12.1

1. If $P(n)$ is the statement "$n(n+1)$ is even", then what is $P(3)$?
2. If $P(n)$ is the statement "$n^3 + n$ is divisible by 3", prove that $P(3)$ is true but $P(4)$ is not true.
3. If $P(n)$ is the statement "$2^n \geq 3n$", and if $P(r)$ is true, prove that $P(r+1)$ is true.
4. If $P(n)$ is the statement "$n^2 + n$ is even", and if $P(r)$ is true, then $P(r+1)$ is true.
5. Given an example of a statement $P(n)$ such that it is true for all $n \in N$.
6. If $P(n)$ is the statement "$n^2 - n + 41$ is prime", prove that $P(1)$, $P(2)$ and $P(3)$ are true. Prove also that $P(41)$ is not true.
7. Give an example of a statement $P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true. Justify your answer.

ANSWERS

1. $P(3): 3(3+1)$ is even 5. $P(n): 1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$ 7. $P(n): 2n < n!$

HINTS TO SELECTED PROBLEMS

3. Let $P(r)$ be true. Then, $P(r)$ is true

$\Rightarrow 2^r \geq 3r$

$\Rightarrow 2 \cdot 2^r \geq 6r$

$\Rightarrow 2^{r+1} \geq 3r + 3r$

$\Rightarrow 2^{r+1} \geq 3r + 3$ $[\because 3r \geq 3 \Rightarrow 3r + 3r \geq 3r + 3]$

$\Rightarrow 2^{r+1} \geq 3(r+1) \Rightarrow P(r+1)$ is true

5. See the statement in Q. No. 4

12.2 THE PRINCIPLES OF MATHEMATICAL INDUCTION

FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n such that

(I) $P(1)$ is true i.e. $P(n)$ is true for $n = 1$.

and, (II) $P(m+1)$ is true, whenever $P(m)$ is true.

i.e. $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Then, $P(n)$ is true for all natural numbers n.

SECOND PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n such that

(I) $P(1)$ is true i.e. $P(n)$ is true for $n = 1$.

and, (II) $P(m+1)$ is true, whenever $P(n)$ is true for all n, where $1 \leq n \leq m$.

Then, $P(n)$ is true for all natural numbers.

ILLUSTRATIVE EXAMPLES

Type I PROBLEMS BASED UPON FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

Recall that the first principle of mathematical induction consists of two parts. First we must show that the given statement $P(n)$ is true for $n = 1$. The second part has two steps. The first step is to assume that the statement $P(n)$ is true for some $m \in N$. The second step is to use this assumption to prove that the statement $P(n)$ is true for $n = m + 1$.

In order to prove that a statement is true for all natural numbers using first principle of mathematical induction, we may use the following algorithm:

ALGORITHM

STEP I *Obtain P (n) and understand its meaning.*

STEP II *Prove that the statement P (1) is true i.e. P (n) is true for n = 1.*

STEP III *Assume that the statement P (n) is true for n = m (say) i.e. P (m) is true.*

STEP IV *Using assumption in step III prove that P (m + 1) is true.*

STEP V *Combining the results of step II and step IV, conclude by the first principle of mathematical induction that P (n) is true for all $n \in N$.*

The following examples illustrate the above algorithm.

LEVEL-1

EXAMPLE 1 *Prove by the principle of mathematical induction that for all $n \in N$, $n^2 + n$ is even natural number*

SOLUTION Let $P(n)$ be the statement "$n^2 + n$ is even".

STEP I We have, $P(n): n^2 + n$ is even

$\because$ $1^2 + 1 = 2$, which is even

$\therefore$ $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$P(m)$ is ture $\Rightarrow m^2 + m$ is even $\Rightarrow m^2 + m = 2\lambda$ for some $\lambda \in N$ ...(i)

Now, we shall show that $P(m+1)$ is true. For this we have to show that $(m+1)^2 + (m+1)$ is an even natural number.

Now,

$$(m+1)^2 + (m+1) = (m^2 + 2m + 1) + (m+1) = (m^2 + m) + (2m + 2)$$

$$\Rightarrow \quad (m+1)^2 + (m+1) = m^2 + m + 2(m+1) = 2\lambda + 2(m+1) \quad \text{[Using (i)]}$$

$$\Rightarrow \quad (m+1)^2 + (m+1) = 2(\lambda + m + 1) = 2\mu, \text{ where } \mu = \lambda + m + 1 \in N$$

$\Rightarrow$ $(m+1)^2 + (m+1)$ is an even natural number

$\Rightarrow$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$ i.e. $n^2 + n$ is even for all $n \in N$.

EXAMPLE 2 *Prove by the principle of mathematical induction that: $n(n+1)(2n+1)$ is divisible by 6 for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement "$n(n+1)(2n+1)$ is divisible by 6".

i.e. $P(n): n(n+1)(2n+1)$ is divisible by 6

STEP I We have, $P(1): 1(1+1)(2+1)$ is divisible by 6.

$\because$ $1(1+1)(2+1) = 6$ which is divisible by 6

$\therefore$ $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$m(m+1)(2m+1)$ is divisible by 6

$\Rightarrow$ $m(m+1)(2m+1) = 6\lambda$, for some $\lambda \in N$ ...(i)

Now, we shall show that $P(m+1)$ is true. For this we have to show that

$(m+1)(m+1+1)\{2(m+1)+1\}$ is divisible by 6.

Now,

$$\begin{aligned}(m+1)(m+1+1)\{2(m+1)+1\} &= (m+1)(m+2)\{(2m+1)+2\}\\ &= (m+1)(m+2)(2m+1)+2(m+1)(m+2)\\ &= m(m+1)(2m+1)+2(m+1)(2m+1)+2(m+1)(m+2)\\ &= m(m+1)(2m+1)+2(m+1)(2m+1+m+2)\\ &= m(m+1)(2m+1)+2(m+1)(3m+3)\\ &= m(m+1)(2m+1)+6(m+1)^2 = 6\lambda+6(m+1)^2 \quad \text{[Using (i)]}\\ &= 6\{\lambda+(m+1)^2\}, \text{ which is divisible by 6}\end{aligned}$$

$\Rightarrow$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction, the given statement is true for all $n \in N$.

EXAMPLE 3 *Prove by the principle of mathematical induction that for all $n \in N$:*

$$1+4+7+\ldots+(3n-2) = \frac{1}{2}n(3n-1)$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1+4+7+\ldots+(3n-2) = \frac{1}{2}n(3n-1)$$

STEP I We have,

$$P(1): 1 = \frac{1}{2}\times(1)\times(3\times1-1).$$

$$\because \quad 1 = \frac{1}{2}\times(1)\times(3\times1-1)$$

So, $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$$1+4+7+\ldots+(3m-2) = \frac{1}{2}m(3m-1) \qquad \text{...(i)}$$

We wish to show that $P(m+1)$ is true. For this we have to show that

$$1+4+7+\ldots+(3m-2)+\{3(m+1)-2\} = \frac{1}{2}(m+1)\{3(m+1)-1\}$$

Now, $1+4+7+\ldots+(3m-2)+\{3(m+1)-2\}$

$$\begin{aligned}&= \frac{1}{2}m(3m-1)+\{3(m+1)-2\} \qquad \text{[Using (i)]}\\ &= \frac{1}{2}m(3m-1)+(3m+1) = \frac{1}{2}\{3m^2-m+6m+2\}\\ &= \frac{1}{2}\{3m^2+5m+2\} = \frac{1}{2}(m+1)(3m+2) = \frac{1}{2}(m+1)\{3(m+1)-1\}\end{aligned}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 4 *Prove by the principle of mathematical induction that for all $n \in N$:*

$$1^2+2^2+3^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1)$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1^2+2^2+3^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1)$$

STEP I We have,

$$P(1): 1^2 = \frac{1}{6}(1)(1+1)(2\times1+1)$$

$\because \quad 1^2 = 1 = \frac{1}{6}(1)(1+1)(2\times 1+1)$

$\therefore \quad P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$$1^2+2^2+3^2+\ldots+m^2 = \frac{1}{6}m(m+1)(2m+1) \quad \ldots(i)$$

We wish to show that $P(m+1)$ is true. For this we have to show that

$$1^2+2^2+3^2+\ldots+m^2+(m+1)^2 = \frac{1}{6}(m+1)\{(m+1)+1\}\{2(m+1)+1\}$$

Now, $1^2+2^2+3^2+\ldots+m^2+(m+1)^2$

$$= \{1^2+2^2+3^2+\ldots+m^2\}+(m+1)^2$$

$$= \frac{1}{6}m(m+1)(2m+1)+(m+1)^2 \quad \text{[Using (i)]}$$

$$= \frac{1}{6}(m+1)\{m(2m+1)+6(m+1)\} = \frac{1}{6}(m+1)\{2m^2+7m+6\}$$

$$= \frac{1}{6}(m+1)(m+2)(2m+3) = \frac{1}{6}(m+1)\{(m+1)+1\}\{2(m+1)+1\}$$

$\therefore \quad P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 5 *Using the principle of mathematical induction prove that:*

$$1^3+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 \text{ for all } n \in N$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1^3+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

STEP I We have,

$$P(1): 1^3 = \left\{\frac{1(1+1)}{2}\right\}^2$$

Clearly, $1^3 = 1 = \left\{\frac{1(1+1)}{2}\right\}^2$

$\therefore \quad P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$1^3+2^3+3^3+\ldots+m^3 = \left\{\frac{m(m+1)}{2}\right\}^2 \quad \ldots(i)$$

We shall now prove that $P(m+1)$ is true. For this we have to prove that

$$1^3+2^3+3^3+\ldots+m^3+(m+1)^3 = \left\{\frac{(m+1)\{(m+1)+1\}}{2}\right\}^2$$

Now,

$$1^3+2^3+3^3+\ldots+m^3+(m+1)^3$$

$$= \left\{1^3+2^3+\ldots+m^3\right\}+(m+1)^3$$

$$= \left\{\frac{m(m+1)}{2}\right\}^2+(m+1)^3 \quad \text{[Using (i)]}$$

$$= (m+1)^2 \left\{\frac{m^2}{4} + (m+1)\right\}$$

$$= (m+1)^2 \left\{\frac{m^2+4m+4}{4}\right\} = \frac{(m+1)^2 (m+2)^2}{4} = \left\{\frac{(m+1)\{(m+1)+1\}}{2}\right\}^2$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 6 *Using the principle of mathematical induction, prove that*

$$1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4} \text{ for all } n \in N.$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

STEP I We have,

$$P(1): 1.2.3 = \frac{1(1+1)(1+2)(1+3)}{4}$$

$\because$ $1.2.3 = 6$ and $\frac{1(1+1)(1+2)(1+3)}{4} = \frac{2\times 3\times 4}{4} = 6$

$\therefore$ $1.2.3 = \frac{1(1+1)(1+2)(1+3)}{4}$

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$1.2.3 + 2.3.4 + \ldots + m(m+1)(m+2) = \frac{m(m+1)(m+2)(m+3)}{4} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true. For this we will prove that

$$1.2.3 + 2.3.4 + \ldots + m(m+1)(m+2) + (m+1)(m+2)(m+3) = \frac{(m+1)(m+2)(m+3)(m+4)}{4}$$

Now, $1.2.3 + 2.3.4 + \ldots + m(m+1)(m+2) + (m+1)(m+2)(m+2)$

$$= \frac{m(m+1)(m+2)(m+3)}{4} + (m+1)(m+2)(m+3) \quad \text{[Using (i)]}$$

$$= (m+1)(m+2)(m+3)\left(\frac{m}{4}+1\right) = \frac{(m+1)(m+2)(m+3)(m+4)}{4}$$

$\therefore$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 7 *Using the principle of mathematical induction prove that*

$$1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)\,3^{n+1}+3}{4} \text{ for all } n \in N$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)\,3^{n+1}+3}{4}$$

STEP I $P(1): 1.3 = \frac{(2\times 1-1)\times 3^{1+1}+3}{4}$

$\because$ $1.3 = 3$ and $\frac{(2\times 1-1)\times 3^{1+1}+3}{4} = \frac{9+3}{4} = 3$

$\therefore \quad 1.3 = \dfrac{(2 \times 1 - 1) \times 3^{1+1} + 3}{4}$

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$1.3 + 2.3^2 + 3.3^3 + \ldots + m.3^m = \frac{(2m-1)\, 3^{m+1} + 3}{4} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true.

i.e. $$1.3 + 2.3^2 + 3.3^3 + \ldots + m.3^m + (m+1).3^{m+1} = \frac{\{2(m+1)-1\}\, 3^{(m+1)+1} + 3}{4}$$

Now,

$$1.3 + 2.3^2 + 3.3^3 + \ldots + m.3^m + (m+1).3^{m+1}$$

$$= \frac{(2m-1)\, 3^{m+1} + 3}{4} + (m+1)\, 3^{m+1}$$

$$= \frac{(2m-1)\, 3^{m+1} + 3 + (4m+4)\, 3^{m+1}}{4}$$

$$= \frac{(2m-1) \times 3^{m+1} + (4m+4) \times 3^{m+1} + 3}{4}$$

$$= \frac{(2m-1+4m+4)\, 3^{m+1} + 3}{4}$$

$$= \frac{(6m+3)\, 3^{m+1} + 3}{4} = \frac{(2m+1)\, 3^{m+2} + 3}{4} = \frac{\{2(m+1)-1\}\, 3^{(m+1)+1} + 3}{4}$$

$\therefore \quad P(m+1)$ is true.

Hence, by the principal of mathematical induction $P(n)$ is true for all $n \in N$ i.e., the given result is true for all $n \in N$.

EXAMPLE 8 *Prove by the principle of mathematical induction that for all $n \in N$:*

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n) : \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

STEP I We have, $P(1) : \dfrac{1}{1.2} = \dfrac{1}{1+1}$

$$\because \quad \frac{1}{1.2} = \frac{1}{1+1} = \frac{1}{2}$$

So, $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{m(m+1)} = \frac{m}{m+1} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true. For this we have to show that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+1+1)} = \frac{(m+1)}{(m+1)+1}$$

Now, $$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{m(m+1)} + \frac{1}{(m+1)((m+1)+1)}$$

$$= \left\{\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{m(m+1)}\right\} + \frac{1}{(m+1)((m+1)+1)}$$

$$= \frac{m}{m+1} + \frac{1}{(m+1)((m+1)+1)} = \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} \qquad \text{[Using (i)]}$$

$$= \frac{1}{(m+1)}\left\{\frac{m}{1} + \frac{1}{m+2}\right\} = \frac{1}{(m+1)} \times \frac{(m^2+2m+1)}{(m+2)} = \frac{(m+1)^2}{(m+1)(m+2)}$$

$$= \frac{m+1}{m+2} = \frac{(m+1)}{(m+1)+1}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true

Hence, by the principle of mathematical induction, the given statement is true for all $n \in N$.

EXAMPLE 9 *Using the principle of mathematical induction prove that*

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1} \textit{ for all } n \in N.$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}$$

<u>STEP I</u> We have, $P(1): 1 = \frac{2 \times 1}{1+1}$

Clearly, $\frac{2\times 1}{1+1} = \frac{2}{2} = 1$

$\therefore$ $1 = \frac{2 \times 1}{1+1}$

So, $P(1)$ is true.

<u>STEP II</u> Let $P(m)$ be true. Then,

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+m} = \frac{2m}{m+1} \qquad \ldots\text{(i)}$$

We shall now show that $P(m+1)$ is true. For this we will prove that

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+m} + \frac{1}{1+2+3+\ldots+(m+1)} = \frac{2(m+1)}{(m+1)+1}$$

Now, $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3\ldots+m} + \frac{1}{1+2+3+\ldots+(m+1)}$

$$= \frac{2m}{m+1} + \frac{1}{1+2+3+\ldots+(m+1)} \qquad \text{[Using (i)]}$$

$$= \frac{2m}{m+1} + \frac{1}{\frac{(m+1)(m+2)}{2}} \qquad \left[\because 1+2+\ldots+m+(m+1) = \frac{(m+1)(m+2)}{2}\right]$$

$$= \frac{2m}{m+1} + \frac{2}{(m+1)(m+2)} \qquad \text{[Using (i)]}$$

$$= \frac{2}{m+1}\left\{m + \frac{1}{(m+2)}\right\} = \frac{2}{m+1}\left\{\frac{m^2+2m+1}{(m+2)}\right\} = \frac{2}{m+1} \times \frac{(m+1)^2}{m+2} = \frac{2(m+1)}{(m+1)+1}$$

$\therefore$ $P(m+1)$ is true.

Hence, by the principle of of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 10 *Prove by induction that the sum $S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3 for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by

$P(n): S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3

STEP I We have, $P(1): S_1 = 1^3 + 3(1)^2 + 5(1) + 3$ is divisible by 3.

Since, $1^3 + 3(1)^2 + 5(1) + 3 = 12$, which is divisible by 3

$\therefore$ $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$S_m = m^3 + 3m^2 + 5m + 3$ is divisible by 3

$\Rightarrow$ $S_m = m^3 + 3m^2 + 5m + 3 = 3\lambda$, for some $\lambda \in N$ ...(i)

We now wish to show that $P(m+1)$ is true. For this we have to show that $(m+1)^3 + 3(m+1)^2 + 5(m+1) + 3$ is divisible by 3.

Now, $(m+1)^3 + 3(m+1)^2 + 5(m+1) + 3$

$= (m^3 + 3m^2 + 5m + 3) + 3m^2 + 9m + 9$

$= 3\lambda + 3(m^2 + 3m + 3)$

$= 3(\lambda + m^2 + 3m + 3)$ [Using (i)]

$= 3\mu$, where $\mu = \lambda + m^2 + 3m + 3 \in N$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction the statement is true for all $n \in N$.

EXAMPLE 11 *Prove by the principle of mathematical induction that for all $n \in N$:*

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n) : \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

STEP I We have, $P(1): \frac{1}{1.3} = \frac{1}{(2 \times 1 + 1)}$.

Clearly, $\frac{1}{1.3} = \frac{1}{(2 \times 1 + 1)}$.

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2m-1)(2m+1)} = \frac{m}{2m+1} \quad \text{...(i)}$$

We shall now show that $P(m+1)$ is true. For this we shall show that

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2m-1)(2m+1)} + \frac{1}{(2m+1)(2m+3)} = \frac{m+1}{2m+3}$$

Now,

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2m-1)(2m+1)} + \frac{1}{(2m+1)(2m+3)}$$

$$= \frac{m}{2m+1} + \frac{1}{(2m+1)(2m+3)} \quad \text{[Using (i)]}$$

$$= \frac{2m^2 + 3m + 1}{(2m+1)(2m+3)} = \frac{(2m+1)(m+1)}{(2m+1)(2m+3)} = \frac{m+1}{2m+3}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 12 *Using the principle of mathematical induction, prove that*

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)} \text{ for all } n \in N.$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

STEP I We have,

$$P(1) = \frac{1}{1.2.3} = \frac{1(1+3)}{4(1+1)(1+2)}$$

$$\because \quad \frac{1}{1.2.3} = \frac{1}{6} \text{ and } \frac{1(1+3)}{4(1+1)(1+2)} = \frac{4}{4 \times 2 \times 3} = \frac{1}{6}$$

$$\therefore \quad \frac{1}{1.2.3} = \frac{1(1+3)}{4(1+1)(1+2)}$$

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{m(m+1)(m+2)} = \frac{m(m+3)}{4(m+1)(m+2)} \qquad \ldots(i)$$

We shall now show that $P(m+1)$ is true.

i.e., $$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \ldots + \frac{1}{m(m+1)(m+2)} + \frac{1}{(m+1)(m+2)(m+3)} = \frac{(m+1)(m+4)}{4(m+2)(m+3)}$$

Now,

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{m(m+1)(m+2)} + \frac{1}{(m+1)(m+2)(m+3)}$$

$$= \frac{m(m+3)}{4(m+1)(m+2)} + \frac{1}{(m+1)(m+2)(m+3)} \qquad \text{[Using (i)]}$$

$$= \frac{m(m+3)^2 + 4}{4(m+1)(m+2)(m+3)}$$

$$= \frac{m^3 + 6m^2 + 9m + 4}{4(m+1)(m+2)(m+3)} = \frac{(m+1)^2(m+4)}{4(m+1)(m+2)(m+3)} = \frac{(m+1)(m+4)}{4(m+2)(m+3)}$$

$\therefore$ $P(m+1)$ is true.

Hence, $P(n)$ is true for all $n \in N$.

EXAMPLE 13 *If x and y are any two distinct integers, then prove by mathematical induction that $(x^n - y^n)$ is divisible by $(x - y)$ for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by

$P(n)$: $(x^n - y^n)$ is divisible by $(x - y)$

STEP I $P(1): (x^1 - y^1)$ is divisible by $(x - y)$.

$\because$ $x^1 - y^1 = (x - y)$ is divisible by $(x - y)$.

$\therefore$ $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$(x^m - y^m)$ is divisible by $(x - y)$

$\Rightarrow$ $(x^m - y^m) = \lambda(x - y)$, for some $\lambda \in Z$ ...(i)

We shall now show that $P(m + 1)$ is true. For this it is sufficient to show that $(x^{m+1} - y^{m+1})$ is divisible by $(x - y)$.

Now,

$$x^{m+1} - y^{m+1}$$
$$= x^{m+1} - x^m y + x^m y - y^{m+1}$$
$$= x^m (x - y) + y(x^m - y^m)$$
$$= x^m (x - y) + y\lambda(x - y) \quad \text{[Using (i)]}$$
$$= (x - y)(x^m + y\lambda), \text{ which is divisible by } (x - y)$$

So, $P(m + 1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m + 1)$ is true

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

i.e. $(x^n - y^n)$ is divisible by $(x - y)$ for all $n \in N$.

EXAMPLE 14 *Using principle of mathematical induction, prove that $x^{2n} - y^{2n}$ is divisible by $x + y$ for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by

$P(n): (x^{2n} - y^{2n})$ is divisisble by $(x + y)$.

STEP I $P(1): (x^2 - y^2)$ is divisible by $(x + y)$.

$\because$ $(x^2 - y^2) = (x - y)(x + y)$, which is divisble by $(x + y)$

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$x^{2m} - y^{2m}$ is divisible by $(x + y)$

$\Rightarrow$ $x^{2m} - y^{2m} = \lambda(x + y)$ ...(i)

We shall now show that $P(m + 1)$ is true i.e., $x^{2m+2} - y^{2m+2}$ is divisible by $(x + y)$.

Now,

$$x^{2m+2} - y^{2m+2} = x^{2m+2} - x^{2m} y^2 + x^{2m} y^2 - y^{2m+2}$$
$$\Rightarrow x^{2m+2} - y^{2m+2} = x^{2m}(x^2 - y^2) + y^2(x^{2m} - y^{2m})$$
$$\Rightarrow x^{2m+2} - y^{2m+2} = x^{2m}(x^2 - y^2) + y^2\lambda(x + y) \quad \text{[Using (i)]}$$
$$\Rightarrow x^{2m+2} - y^{2m+2} = (x + y)\left\{x^{2m}(x - y) + \lambda y^2\right\}$$

Clearly, it is divisible by $(x + y)$.

$\therefore$ $P(m + 1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m + 1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e., $x^{2n} - y^{2n}$ is divisible by $(x + y)$ for all $n \in N$.

EXAMPLE 15 *Using principle of mathematical induction, prove that*

(i) $41^n - 14^n$ *is a multiple of* 27 (ii) $7^n - 3^n$ *is divisible by* 4.

SOLUTION (i) Let $P(n)$ be the statement given by

$P(n): 41^n - 14^n$ is a multiple of 27.

STEP I $P(1): 41^1 - 14^1$ is a multiple of 27.

$\because$ $41^1 - 14^1 = 41 - 14 = 27$, which is a multiple of 27.

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$41^m - 14^m$ is a multiple of 27

$\Rightarrow$ $41^m - 14^m = 27\lambda$ for some $\lambda \in N$ ...(i)

Now, $41^{m+1} - 14^{m+1} = 41^{m+1} - 41 \times 14^m + 41 \times 14^m - 14^{m+1}$

$\Rightarrow$ $41^{m+1} - 14^{m+1} = 41(41^m - 14^m) + (41 - 14)\,14^m$

$\Rightarrow$ $41^{m+1} - 14^{m+1} = 41 \times 27\lambda + 27 \times 14^m$ [Using (i)]

$\Rightarrow$ $41^{m+1} - 14^{m+1} = 27(41\lambda + 14^m)$, which is a multiple of 27.

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, $P(n)$ is true for all $n \in N$.

(ii) Proceed as in (i).

EXAMPLE 16 *Using the principle of mathematical induction, prove that* $(2^{3n} - 1)$ *is divisible by 7 for all* $n \in N$. **[NCERT EXEMPLAR]**

SOLUTION Let $P(n)$ be the statement given by

$P(n)$: $2^{3n} - 1$ is divisible by 7

STEP I $P(1): 2^{3\times 1} - 1$ is divisible by 7.

Clearly, $2^{3\times 1} - 1 = 8 - 1 = 7$, which is divisible by 7.

So, $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$2^{3m} - 1$ is divisible by 7.

$\Rightarrow$ $2^{3m} - 1 = 7\lambda$, for some $\lambda \in N$ ...(i)

We shall now show that $P(m+1)$ is true. For this we have to show that $2^{3(m+1)} - 1$ is divisible by 7.

Now,

$$2^{3(m+1)} - 1 = 2^{3m} \times 2^3 - 1 = (7\lambda + 1)\,2^3 - 1 \quad \text{[Using (i)]}$$

$$= 56\lambda + 8 - 1 = 7(8\lambda + 1), \text{ which is divisible by 7}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$ i.e. $2^{3n} - 1$ is divisible by 7 for all $n \in N$.

EXAMPLE 17 *Prove by the principle of induction that for all* $n \in N$, $(10^{2n-1} + 1)$ *is divisible by* 11.

SOLUTION Let $P(n)$ be the statement given by

$P(n)$: $10^{2n-1} + 1$ is divisible by 11

STEP I We have

$P(1): 10^{2\times 1-1}+1$ is divisible by 11.

Since $10^{2\times 1-1}+1=11$, which is divisible by 11.

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$10^{2m-1}+1$ is divisible by 11

$\Rightarrow$ $10^{2m-1}+1 = 11\lambda$, for some $\lambda \in N$ ...(i)

We shall now show that $P(m+1)$ is true. For this we have to show that $10^{2(m+1)}+1$ is divisible by 11.

Now, $10^{2(m+1)-1}+1 = 10^{2m+1}+1 = 10^{2m-1}\times 10^2+1$

$\Rightarrow$ $10^{2(m+1)-1}+1 = (11\lambda-1)\,100+1$ [Using (i)]

$\Rightarrow$ $10^{2(m+1)-1}+1 = 1100\lambda-99 = 11(100\lambda-9) = 11\mu$, where $\mu = 100\lambda-9 \in N$

$\Rightarrow$ $10^{2(m+1)}-1$ is divisible by 11

$\Rightarrow$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction $P(m)$ is true for all $n \in N$ i.e. $10^{2n-1}+1$ is divisible by 11 for all $n \in N$.

EXAMPLE 18 *Prove that for $n \in N$, $10^n + 3.4^{n+2}+5$ is divisible by 9.*

SOLUTION Let $P(n)$ be the statement given by

$P(n)$: $10^n + 3.4^{n+2}+5$ is divisible by 9

STEP I $P(1): 10^1 + 3(4^{1+2})+5$ is divisible by 9.

$\because$ $10^1+3(4^{1+2})+5 = 10+192+5 = 207$, which is divisible by 9

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$10^m + 3(4^{m+2})+5$ is divisible by 9

$\Rightarrow$ $10^m + 3(4^{m+2})+5 = 9\lambda,\ \lambda \in N$ ...(i)

We shall now show that $P(m+1)$ is true for which we have to show that $10^{(m+1)}+3(4^{m+3})+5$ is divisible by 9.

Now,

$$\begin{aligned}
10^{m+1}+3(4^{m+3})+5 &= 10^m\,(10)+3(4^{m+3})+5\\
&= \{9\lambda-3(4^{m+2})-5\}\times 10+3\times 4^{m+3}+5 \qquad \text{[Using (i)]}\\
&= 90\lambda-30\times 4^{m+2}-50+3\times 4\times 4^{m+2}+5\\
&= 90\lambda-30\times 4^{m+2}+12\times 4^{m+2}-45\\
&= 90\lambda-18\times 4^{m+2}-45\\
&= 9(10\lambda-2\times 4^{m+2}-5) = 9\mu, \text{ where } \mu = 10\lambda-2\times 4^{m+2}-5
\end{aligned}$$

$\Rightarrow$ $10^{m+1}+3.4^{m+3}+5$ is divisible by 9

$\Rightarrow$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 19 *Prove by induction that the sum of the cubes of three consecutive natural numbers is divisible by 9.*

SOLUTION Let $P(n)$ be the statement given by

$P(n)$: Sum of the cubes of three consecutive natural numbers starting from n is divisible by 9.

STEP I $P(1)$: Sum of the cubes of first three consecutive natural numbers is divisible by 9.

Since $1^3 + 2^3 + 3^3 = 36$, which is divisible by 9.

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then, sum of the cubes of three consecutive natural numbers starting with m is divisible by 9.

i.e. $m^3 + (m+1)^3 + (m+2)^3$ is divisible by 9

$\Rightarrow$ $m^3 + (m+1)^3 + (m+2)^3 = 9\lambda, \lambda \in N$

We shall now show that $P(m+1)$ is true for which we have to show that

$(m+1)^3 + (m+2)^3 + (m+3)^3$ is divisible by 9.

Now, $(m+1)^3 + (m+2)^3 + (m+3)^3$

$= (m+1)^3 + (m+2)^3 + m^3 + 9m^2 + 27m + 27$

$= m^3 + (m+1)^3 + (m+2)^3 + 9(m^2 + 3m + 3)$

$= 9\lambda + 9(m^2 + 3m + 3)$ [Using (i)]

$= 9(\lambda + m^2 + 3m + 3)$, which is divisible by 9.

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 20 *Using principle of mathematical induction prove that $4^n + 15n - 1$ is divisible by 9 for all natural numbers n.*

SOLUTION Let $P(n)$ be the statement given by

$P(n) : 4^n + 15n - 1$ is divisible by 9

STEP I $P(1) : 4^1 + 15 \times 1 - 1$ is divisible by 9.

$\because$ $4^1 + 15 \times 1 - 1 = 18$, which is divisible by 9

$\therefore$ $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$4^m + 15m - 1$ is divisible by 9

$\Rightarrow$ $4^m + 15m - 1 = 9\lambda$, for some $\lambda \in N$

We shall now show that $P(m+1)$ is true, for this we have to show that $4^{m+1} + 15(m+1) - 1$ is divisible by 9.

Now,

$$4^{m+1} + 15(m+1) - 1 = 4^m \cdot 4 + 15(m+1) - 1$$
$$= (9\lambda - 15m + 1) \times 4 + 15(m+1) - 1$$
$$= 36\lambda - 45m + 18 = 9(4\lambda - 5m + 2), \text{ which is divisible by 9.}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e., $4^n + 15n - 1$ is divisible by 9.

EXAMPLE 21 *Prove that:* $2.7^n + 3.5^n - 5$ *is divisible by 24, for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by

$P(n): 2.7^n + 3.5^n - 5$ is divisible by 24.

STEP I We have,

$P(1): 2 \times 7^1 + 3 \times 5^1 - 5$ is divisible by 24

$\because \quad 2 \times 7^1 + 3 \times 5^1 - 5 = 14 + 15 - 5 = 24$, which is divisible by 24.

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$2 \times 7^m + 3 \times 5^m - 5$ is divisible by 24

$\Rightarrow \quad 2 \times 7^m + 3 \times 5^m - 5 = 24\lambda$ for some $\lambda \in N$

$$\Rightarrow \quad 3 \times 5^m = 24\lambda + 5 - 2 \times 7^m \qquad ...(i)$$

Now, $2 \times 7^{m+1} + 3 \times 5^{m+1} - 5$

$= 2 \times 7^{m+1} + (3 \times 5^m) 5 - 5$

$= 2 \times 7^{m+1} + (24\lambda + 5 - 2 \times 7^m) 5 - 5$ [Using (i)]

$= 2 \times 7^{m+1} + 120\lambda + 25 - 10 \times 7^m - 5$

$= (2 \times 7^{m+1} - 10 \times 7^m) + 120\lambda + 20$

$= (2 \times 7 \times 7^m - 10 \times 7^m) + 120\lambda + 24 - 4$

$= (14 - 10) 7^m - 4 + 24(5\lambda + 1)$

$= 4(7^m - 1) + 24(5\lambda + 1)$

$= 4 \times 6\mu + 24(5\lambda + 1)$ [$\because$ $7^m - 1$ is a multiple of 6 for all $m \in N$ $\therefore 7^m - 1 = 6\mu, \mu \in N$]

$= 24(\mu + 5\lambda + 1)$, which is divisible by 24.

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

EXAMPLE 22 *Prove that :*

(i) $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)...\left(1 + \frac{1}{n}\right) = (n+1)$ *for all* $n \in N$.

(ii) $\left(1 + \frac{1}{3}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right)...\left(1 + \frac{2n+1}{n^2}\right\} = (n+1)^2$ *for all* $n \in N$.

SOLUTION (i) Let $P(n)$ be the statement given by

$$P(n): \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)...\left(1 + \frac{1}{n}\right) = n + 1$$

STEP I We have,

$$P(1): \left(1 + \frac{1}{1}\right) = (1 + 1)$$

$$\because \quad \left(1 + \frac{1}{1}\right) = 2 = (1 + 1)$$

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\dots\left(1+\frac{1}{m}\right)=m+1 \quad \text{....(i)}$$

Now,

$P(m)$ is true

$$\Rightarrow \quad \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\dots\left(1+\frac{1}{m}\right)=(m+1) \quad \text{[From (i)]}$$

$$\Rightarrow \quad \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\dots\left(1+\frac{1}{m}\right)\left(1+\frac{1}{m+1}\right)$$

$$=(m+1)\left(1+\frac{1}{m+1}\right) \quad \left[\text{Multiplying both sides by}\left(1+\frac{1}{m+1}\right)\right]$$

$$=\frac{(m+1)(m+2)}{m+1}=m+2$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

(ii) Let $P(n)$ be the statement given by

$$P(n):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\dots\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$$

STEP I We have,

$$P(1):\left(1+\frac{3}{1}\right)=(1+1)^2$$

$$\because \quad 1+\frac{3}{1}=1+3=4=(1+1)^2$$

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\dots\left(1+\frac{2m+1}{m^2}\right)=(m+1)^2 \quad \text{...(i)}$$

We shall now prove that $P(m+1)$ is true.

i.e. $$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\dots\left(1+\frac{2m+1}{m^2}\right)\left(1+\frac{2(m+1)+1}{(m+1)^2}\right)=\left\{(m+1)+1\right\}^2$$

Now,

$P(m)$ is true

$$\Rightarrow \quad \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\dots\left(1+\frac{2m+1}{m^2}\right)=(m+1)^2 \quad \text{[From (i)]}$$

$$\Rightarrow \quad \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\dots\left(1+\frac{2m+1}{m^2}\right)\left(1+\frac{2m+3}{(m+1)^2}\right)$$

$$=(m+1)^2\left(1+\frac{2m+3}{(m+1)^2}\right) \quad \left[\text{Multiplying both sides by } 1+\frac{2m+3}{(m+1)^2}\right]$$

$$=(m+1)^2\left\{\frac{(m+1)^2+2m+3)}{(m+1)^2}\right\}=(m^2+4m+4)=(m+2)^2=\{(m+1)+1\}^2$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

EXAMPLE 23 *Prove by induction that* $4 + 8 + 12 + \ldots + 4n = 2n(n+1)$ *for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 4 + 8 + 12 + \ldots + 4n = 2n(n+1)$$

STEP I $P(1): 4 = 2 \times 1 \times (1+1)$, which is true.

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$4 + 8 + 12 + \ldots + 4m = 2m(m+1) \qquad \ldots(i)$$

We shall now show that $P(m+1)$ is true

i.e. $4 + 8 + \ldots + 4m + 4(m+1) = 2(m+1)\{(m+1)+1\}$.

Now,

$$4 + 8 + \ldots + 4m + 4(m+1)$$
$$= 2m(m+1) + 4(m+1) \qquad \text{[Using (i)]}$$
$$= (m+1)(2m+4) = 2(m+1)(m+2) = 2(m+1)\{(m+1)+1\}$$

$\therefore$ $P(m+1)$ is true.

Thus $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true

Hence, by induction $P(n)$ is true for all $n \in N$.

LEVEL-2

EXAMPLE 24 *For all positive integer n, prove that* $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2}{3}n^3 - \frac{n}{105}$ *is an integer*

SOLUTION Let $P(n)$ be the statement given by

$$P(1): \frac{n^7}{7} + \frac{n^5}{5} + 2\frac{n^3}{3} - \frac{n}{105} \text{ is an integer}$$

STEP I $P(1): \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105}$ is an integer.

Since $\frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105} = \frac{15 + 21 + 70 - 1}{105} = 1$, which is an integer.

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then, $\frac{m^7}{7} + \frac{m^5}{5} + \frac{2m^3}{3} - \frac{m}{105}$ is an integer

$$\text{Let } \frac{m^7}{7} + \frac{m^5}{5} + \frac{2m^3}{3} - \frac{m}{105} = \lambda, \lambda \in Z \qquad \ldots(i)$$

We shall now show that $P(m+1)$ is true for which we have to show that

$$\frac{(m+1)^7}{7} + \frac{(m+1)^5}{5} + \frac{2(m+1)^3}{3} - \frac{(m+1)}{105} \text{ is an integer.}$$

Now, $\frac{(m+1)^7}{7} + \frac{(m+1)^5}{5} + \frac{2(m+1)^3}{3} - \frac{(m+1)}{105}$

$$= \frac{1}{7}(m^7 + 7m^6 + 21m^5 + 35m^4 + 35m^3 + 21m^2 + 7m + 1)$$
$$+ \frac{1}{5}(m^5 + 5m^4 + 10m^3 + 10m^2 + 5m + 1) + \frac{2}{3}(m^3 + 3m^2 + 3m + 1) - \frac{m}{105} - \frac{1}{105}$$

$$= \left\{\frac{m^7}{7} + \frac{m^5}{5} + 2\frac{m^3}{3} - \frac{m}{105}\right\} + m^6 + 3m^5 + 6m^4 + 7m^3 + 7m^2 + 4m + 1$$

$$= \lambda + m^6 + 3m^5 + 6m^4 + 7m^3 + 7m^2 + 4m + 1 \qquad \text{[Using (i)]}$$

$= $ an integer

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

i.e. $\frac{n^7}{7} + \frac{n^5}{5} + 2\frac{n^3}{3} - \frac{n}{105}$ is an integer.

EXAMPLE 25 *Prove by the principle of mathematical induction that* $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ *is a natural number for all* $n \in N$. **[NCERT EXEMPLAR]**

SOLUTION Let $P(n)$ be the statement given by

$$P(n): \frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15} \text{ is a natural number}$$

STEP I $P(1): \frac{1}{5} + \frac{1}{3} + \frac{7}{15}$ is a natural number.

$\therefore$ $\frac{1}{5} + \frac{1}{3} + \frac{7}{15} = \frac{15}{15} = 1$, which is a natural number.

So, $P(1)$ is true.

STEP II *Let $P(m)$ be true.*

Then, $\frac{m^5}{5} + \frac{m^3}{3} + \frac{7m}{15}$ is a natural number. Let $\frac{m^5}{5} + \frac{m^3}{3} + \frac{7m}{15} = \lambda$ ...(i)

We shall now show that $P(m+1)$ is true, for which it is sufficient to prove that $\frac{(m+1)^5}{5} + \frac{(m+1)^3}{3} + \frac{7(m+1)}{15}$ is a natural number.

Now, $\frac{(m+1)^5}{5} + \frac{(m+1)^3}{3} + \frac{7(m+1)}{15}$

$$= \frac{1}{5}(m^5 + 5m^4 + 10m^3 + 10m^2 + 5m + 1) + \frac{1}{3}(m^3 + 3m^2 + 3m + 1) + \frac{7}{15}m + \frac{7}{15}$$

$$= \left(\frac{m^5}{5} + \frac{m^3}{3} + \frac{7}{15}m\right) + (m^4 + 2m^3 + 3m^2 + 2m) + \frac{1}{5} + \frac{1}{3} + \frac{7}{15}$$

$$= \lambda + m^4 + 2m^3 + 3m^2 + 2m + 1 \qquad \text{[Using (i)]}$$

$= $ an integer

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

i.e. $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7}{15}n$ is a natural number for all $n \in N$.

EXAMPLE 26 ***Prove by the principle of mathematical induction that for all $n \in N$, 3^{2n} when divided by 8, the remainder is always 1.***

SOLUTION Let $P(n)$ be the statement given by

$P(n)$: 3^{2n} when divided by 8, the remainder is 1

or, $P(n)$: $3^{2n} = 8\lambda + 1$ for some $\lambda \in N$

STEP I $P(1): 3^2 = 8\lambda + 1$ for some $\lambda \in N$.

$\because$ $3^2 = 8 \times 1 + 1 = 8\lambda + 1$, where $\lambda = 1$

$\therefore$ $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$$3^{2m} = 8\lambda + 1 \text{ for some } \lambda \in N \quad \text{...(i)}$$

We shall now show that $P(m+1)$ is true for which we have to show that $3^{2(m+1)}$ when divided by 8, the remainder is 1 i.e. $3^{2(m+1)} = 8\mu + 1$ for some $\mu \in N$.

Now, $3^{2(m+1)} = 3^{2m} \times 3^2 = (8\lambda + 1) \times 9$ [Using (i)]

$= 72\lambda + 9 = 72\lambda + 8 + 1 = 8(9\lambda + 1) + 1 = 8\mu + 1$, where $\mu = 9\lambda + 1 \in N$

$\Rightarrow$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e. 3^{2n} when divided by 8 the remainder is always 1.

EXAMPLE 27 *Prove by the principle of mathematical induction that $n < 2^n$ for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by $P(n)$: $n < 2^n$.

STEP I $P(1)$: $1 < 2^1$

$\because$ $1 < 2^1$

$\therefore$ $P(1)$ is true

STEP II Let $P(m)$ be true. Then, $m < 2^m$

We shall now show that $P(m+1)$ is true for which we will have to prove that $(m+1) < 2^{m+1}$.

Now,

$P(m)$ is true

$\Rightarrow$ $m < 2^m$

$\Rightarrow$ $2m < 2.2^m$

$\Rightarrow$ $2m < 2^{m+1}$

$\Rightarrow$ $(m+m) < 2^{m+1}$

$\Rightarrow$ $m + 1 \leq m + m < 2^{m+1}$ [$\because 1 \leq m \ \therefore m + 1 \leq m + m$]

$\Rightarrow$ $(m+1) < 2^{m+1}$

$\Rightarrow$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e. $n < 2^n$ for all $n \in N$.

EXAMPLE 28 *Prove by induction the inequality $(1+x)^n \geq 1 + nx$ whenever x is positive and n is a positive integer.*

SOLUTION Let $P(n)$ be the statement given by $P(n)$: $(1+x)^n \geq 1 + nx$

STEP I $P(1): (1+x)^1 \geq 1 + 1(x)$

$\because \quad (1+x)^1 \geq 1 + 1\,(x)$

$\therefore \quad P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$(1+x)^m \geq 1 + m\,x$ ...(i)

We shall now prove that $P(m+1)$ is true whenever $P(m)$ is true. For this we have to show that $(1+x)^{m+1} \geq 1 + (m+1)\,x$.

Now, $P(m)$ is true

$\Rightarrow \quad (1+x)^m \geq 1 + mx$

$\Rightarrow \quad (1+x)(1+x)^m \geq (1+x)(1+m\,x)$ [Multiplying both sides by $(1+x)$]

$\Rightarrow \quad (1+x)^{m+1} \geq 1 + (m+1)\,x + mx^2$

$\Rightarrow \quad (1+x)^{m+1} \geq 1 + (m+1)\,x + mx^2 \geq 1 + (m+1)\,x$ $[\because m\,x^2 \geq 0]$

$\Rightarrow \quad (1+x)^{m+1} \geq 1 + (m+1)\,x$

$\Rightarrow \quad P(m+1)$ is true

Hence, by the principle of induction, $P(n)$ is true for all $n \in N$ i.e. $(1+x)^n \geq 1 + nx$ for all $n \in N$.

EXAMPLE 29 *Prove by induction that* $(2n+7) < (n+3)^2$ *for all natural numbers n. Using this, prove by induction that* $(n+3)^2 \leq 2^{n+3}$ *for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by

$P(n)\ :\ (2n+7) < (n+3)^2$

STEP I $P(1) : (2 \times 1 + 7) < (1+3)^2$

$\because \quad (2 \times 1 + 7) = 9 < (1+3)^2$

$\therefore \quad P(1)$ is true

STEP II Let $P(m)$ be true. Then, $2m + 7 < (m+3)^2$. ... (i)

We shall now show that $P(m+1)$ is true whenever $P(m)$ is true. For this we have to show that $2(m+1) + 7 < (m+1+3)^2$.

Now,

$P(m)$ is true

$\Rightarrow \quad 2m + 7 < (m+3)^2$

$\Rightarrow \quad 2m + 7 + 2 < (m+3)^2 + 2$

$\Rightarrow \quad 2(m+1) + 7 < m^2 + 6m + 11$

$\Rightarrow \quad 2(m+1) + 7 < m^2 + 6m + 11 < m^2 + 8m + 16$

$\Rightarrow \quad 2(m+1) + 7 < (m+4)^2$

$\Rightarrow \quad \{2(m+1) + 7\} < \{(m+1) + 3\}^2$

$\Rightarrow \quad P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Now, let $P'(n)$ be the statement given by $P'(n)\ :\ (n+3)^2 \leq 2^{n+3}$

STEP I $P'(1)\ :\ (1+3)^2 \leq 2^{1+3}$

$\because \quad (1+3)^2 = 16 \leq 2^{1+3}$

$\therefore \quad P'(1)$ is true

STEP II Let $P'(m)$ be true. Then, $(m+3)^2 \leq 2^{m+3}$.

We shall now show that $P'(m+1)$ is true whenever $P'(m)$ is true. For this we have to show that $\{(m+1)+3\}^2 \leq 2^{(m+1)+3}$.

Now, $P'(m)$ is true

$\Rightarrow \quad (m+3)^2 \leq 2^{m+3}$

$\Rightarrow \quad (m+3)^2 + (2m+7) \leq 2^{m+3} + (2m+7)$

$\Rightarrow \quad (m+4)^2 \leq 2^{m+3} + (m+3)^2 \quad [\because\ 2m+7 < (m+3)^2 \therefore 2^{m+3} + (2m+7) < 2^{m+3} + (m+3)^2]$

$\Rightarrow \quad (m+4)^2 \leq 2^{m+3} + 2^{m+3} \quad [\because\ (m+3)^2 \leq 2^{m+3} \Rightarrow (m+3)^2 + 2^{m+3} \leq 2^{m+3} + 2^{m+3}]$

$\Rightarrow \quad (m+4)^2 \leq 2.2^{m+3}$

$\Rightarrow \quad (m+4)^2 \leq 2^{m+4}$

$\Rightarrow \quad \{(m+1)+3\}^2 \leq 2^{(m+1)+3}$

$\Rightarrow \quad P'(m+1)$ is true

Hence, by the principle of mathematical induction, $P'(n)$ is true for all $n \in N$ i.e. $(n+3)^2 \leq 2^{n+3}$ for all $n \in N$.

EXAMPLE 30 *Prove that:* $1^2 + 2^2 + 3^2 + \ldots + n^2 > \frac{n^3}{3}$ *for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 > \frac{n^3}{3}$$

STEP I $P(1): 1^2 > \frac{1^3}{3}$

$\because \quad 1^2 = 1 > \frac{1}{3} = \frac{1^3}{3}$

$\therefore \quad P(1)$ is true.

STEP II Let $P(n)$ be true for $n = m$. Then,

$$1^2 + 2^2 + 3^2 + \ldots + m^2 > \frac{m^3}{3} \quad \ldots(i)$$

We shall now prove that $P(m+1)$ is true.

i.e. $\quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{(m+1)^3}{3}$

Now, $P(m)$ is true

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 > \frac{m^3}{3}$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{m^3}{3} + (m+1)^2$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{1}{3}(m^3 + 3m^2 + 6m + 3)$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{1}{3}\left\{(m^3 + 3m^2 + 3m + 1) + (3m + 2)\right\}$

$$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{1}{3}\left\{(m+1)^3 + (3m+2)\right\} > \frac{(m+1)^3}{3}$$

$\Rightarrow$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

EXAMPLE 31 *Prove that:* $1 + 2 + 3 + \ldots + n < \frac{(2n+1)^2}{8}$ *for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1 + 2 + 3 + \ldots + n < \frac{(2n+1)^2}{8}$$

STEP I We have,

$$P(1): 1 < \frac{(2 \times 1 + 1)^2}{8}$$

$$\because \quad 1 < \frac{(2 \times 1 + 1)^2}{8} = \frac{9}{8}$$

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$1 + 2 + 3 + \ldots + m < \frac{(2m+1)^2}{8} \quad \ldots \text{(i)}$$

We shall now show that $P(m+1)$ is true.

i.e., $1 + 2 + 3 + \ldots + m + (m+1) < \frac{\{2(m+1)+1\}^2}{8}$

Now,

$P(m)$ is true

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + m < \frac{(2m+1)^2}{8}$$

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(2m+1)^2}{8} + (m+1)$$

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(2m+1)^2 + 8(m+1)}{8}$$

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(4m^2 + 12m + 9)}{8}$$

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(2m+3)^2}{8} = \frac{\{2(m+1)+1\}^2}{8}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 32 *Prove by the principle of mathematical induction that for all* $n \in N$,

$$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\left(\frac{n+1}{2}\right)\theta \, \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$ **[NCERT EXEMPLAR]**

SOLUTION Let $P(n)$ be the statement given by

$$P(n): \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\left(\frac{n+1}{2}\right)\theta \, \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

STEP I We have, $P(1): \sin\theta = \dfrac{\sin\left(\frac{1+1}{2}\right)\theta \sin\left(\frac{1\times\theta}{2}\right)}{\sin\frac{\theta}{2}}$

$$\because \quad \sin\theta = \frac{\sin\left(\frac{1+1}{2}\right)\theta . \sin\left(\frac{1\times\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\sin\theta + \sin 2\theta + \ldots + \sin m\theta = \frac{\sin\left(\frac{m+1}{2}\right)\theta \, \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} \qquad \ldots(i)$$

We shall now show that $P(m+1)$ is true.

i.e. $$\sin\theta + \sin 2\theta + \ldots + \sin m\theta + \sin(m+1)\theta = \frac{\sin\left\{\frac{(m+1)+1}{2}\right\}\theta \, \sin\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

Now,

$$\sin\theta + \sin 2\theta + \ldots + \sin m\theta + \sin(m+1)\theta$$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \, \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} + \sin(m+1)\theta \qquad \text{[Using (i)]}$$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \, \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} + 2\sin\left(\frac{m+1}{2}\right)\theta \, \cos\left(\frac{m+1}{2}\right)\theta$$

$$= \sin\left(\frac{m+1}{2}\right)\theta \left\{\frac{\sin\left(\frac{m\theta}{2}\right)}{\sin\frac{\theta}{2}} + 2\cos\left(\frac{m+1}{2}\right)\theta\right\}$$

$$= \sin\left(\frac{m+1}{2}\right)\theta \left\{\frac{\sin\left(\frac{m\theta}{2}\right) + 2\sin\frac{\theta}{2}\cos\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}\right\}$$

$$= \sin\left(\frac{m+1}{2}\right)\theta \left\{\frac{\sin\left(\frac{m\theta}{2}\right) + \sin\left(\frac{m+2}{2}\right)\theta - \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}}\right\}$$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \sin\left(\frac{m+2}{2}\right)\theta}{\sin\frac{\theta}{2}} = \frac{\sin\left\{\frac{(m+1)+1}{2}\right\}\theta \sin\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 33 *Using principle of mathematical induction, prove that*

$$\cos\alpha \cos 2\alpha \cos 4\alpha \ldots \cos(2^{n-1}\alpha) = \frac{\sin 2^n \alpha}{2^n \sin\alpha} \text{ for all } n \in N.$$ **[NCERT EXEMPLAR]**

SOLUTION Let $P(n)$ be the statement given by

$$P(n): \cos\alpha \cos 2\alpha \cos 4\alpha \ldots \cos(2^{n-1}\alpha) = \frac{\sin(2^n\alpha)}{2^n \sin\alpha}$$

STEP I $P(1): \cos\alpha = \dfrac{\sin(2^1\alpha)}{2^1 \sin\alpha}$

$$\because \frac{\sin(2^1\alpha)}{2^1\sin\alpha} = \frac{\sin 2\alpha}{2\sin\alpha} = \frac{2\sin\alpha\cos\alpha}{2\sin\alpha} = \cos\alpha$$

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\cos\alpha \cos 2\alpha \cos 4\alpha \ldots \cos(2^{m-1}\alpha) = \frac{\sin(2^m\alpha)}{2^m\sin\alpha} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true. For this we have to show that

$$\cos\alpha \cos 2\alpha \cos 2^2\alpha \ldots \cos(2^{m-1}\alpha)\cos(2^m\alpha) = \frac{\sin(2^{(m+1)}\alpha)}{2^{m+1}\sin\alpha}$$

We have,

$$\cos\alpha \cos 2\alpha \cos 2^2\alpha \ldots \cos(2^{m-1}\alpha)\cos(2^m\alpha)$$

$$= \{\cos\alpha \cos 2\alpha \cos 2^2\alpha \ldots \cos(2^{m-1}\alpha)\}\cos(2^m\alpha)$$

$$= \frac{\sin(2^m\alpha)}{2^m\sin\alpha} \times \cos(2^m\alpha) \quad \text{[Using (i)]}$$

$$= \frac{2\sin(2^m\alpha)\cos(2^m\alpha)}{2^{m+1}\sin\alpha} = \frac{\sin(2 \cdot 2^m\alpha)}{2^{m+1}\sin\alpha} = \frac{\sin(2^{m+1}\alpha)}{2^{m+1}\sin\alpha}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Type II **PROBLEMS BASED UPON SECOND PRINCIPLE OF MATHEMATICAL INDUCTION**

EXAMPLE 34 *Let $U_1 = 1$, $U_2 = 1$ and $U_{n+2} = U_{n+1} + U_n$ for $n \geq 1$. Use mathematical induction to show that:*

$$U_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \text{ for all } n \geq 1.$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): U_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

We have,

$$U_1 = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right\} = 1$$

and,

$$U_2 = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right\} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+5+2\sqrt{5}}{4}\right) - \left(\frac{1+5-2\sqrt{5}}{4}\right)\right\} = 1$$

$\therefore$ $P(1)$ and $P(2)$ are true.

Let $P(n)$ be true for all $n \leq m$.

i.e. $$U_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \text{ for all } n \leq m \qquad \text{...(i)}$$

We shall now show that $P(n)$ is true for $n = m + 1$.

i.e. $$U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m+1}\right\}$$

We have,

$$U_{n+2} = U_{n+1} + U_n \quad \text{for } n \geq 1$$

$$\Rightarrow \quad U_{m+1} = U_m + U_{m-1} \quad \text{for } m \geq 2 \qquad \text{[On replacing } n \text{ by } (m-1)]$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right\} + \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\right\}$$

[Using (i)]

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left[\left\{\left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\right\} - \left\{\left(\frac{1-\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\right\}\right]$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{1-\sqrt{5}}{2}+1\right)\right\}$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{3-\sqrt{5}}{2}\right)\right\}$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{6+2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{6-2\sqrt{5}}{4}\right)\right\}$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right\}$$

$$\Rightarrow \qquad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m+1}\right\}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(n)$ is true for all $n \le m \Rightarrow P(n)$ is true for all $n \le m+1$.

Hence, $P(n)$ is true for all $n \in N$.

EXERCISE 12.2

LEVEL-1

Prove the following by the principle of mathematical induction: (1-38)

1. $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ i.e, the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.
2. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $1 + 3 + 3^2 + \ldots + 3^{n-1} = \frac{3^n - 1}{2}$
4. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
5. $1 + 3 + 5 + \ldots + (2n-1) = n^2$ i.e., the sum of first n odd natural numbers is n^2.
6. $\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \ldots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$
7. $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$
8. $\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$
9. $\frac{1}{3.7} + \frac{1}{7.11} + \frac{1}{11.15} + \ldots + \frac{1}{(4n-1)(4n+3)} = \frac{n}{3(4n+3)}$
10. $1.2 + 2.2^2 + 3.2^3 + \ldots + n.2^n = (n-1)\,2^{n+1} + 2$
11. $2 + 5 + 8 + 11 + \ldots + (3n-1) = \frac{1}{2}n(3n+1)$
12. $1.3 + 2.4 + 3.5 + \ldots + n.(n+2) = \frac{1}{6}n(n+1)(2n+7)$
13. $1.3 + 3.5 + 5.7 + \ldots + (2n-1)(2n+1) = \frac{n(4n^2 + 6n - 1)}{3}$
14. $1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$
15. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$
16. $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$
17. $a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{r^n - 1}{r - 1}\right), r \ne 1$

18. $a+(a+d)+(a+2d)+\ldots+(a+(n-1)\,d) = \frac{n}{2}[2a+(n-1)\,d]$

19. $5^{2n}-1$ is divisible by 24 for all $n \in N$

20. $3^{2n}+7$ is divisible by 8 for all $n \in N$

21. $5^{2n+2}-24n-25$ is divisible by 576 for all $n \in N$

22. $3^{2n+2}-8n-9$ is divisible by 8 for all $n \in N$

23. $(ab)^n = a^n b^n$ for all $n \in N$

24. $n(n+1)(n+5)$ is a multiple of 3 for all $n \in N$

25. $7^{2n}+2^{3n-3}\,.\,3^{n-1}$ is divisible by 25 for all $n \in N$

26. $2.7^n + 3.5^n - 5$ is divisible by 24 for all $n \in N$

27. $11^{n+2}+12^{2n+1}$ is divisible by 133 for all $n \in N$

28. $1\times 1!+2\times 2!+3\times 3!+\ldots+n\times n! = (n+1)!-1$ for all $N \in N$. [NCERT EXEMPLAR]

29. n^3-7n+3 is divisible by 3 for all $n \in N$.

30. $1+2+2^2+\ldots+2^n = 2^{n+1}-1$ for all $n \in N$.

LEVEL-2

31. Prove that $7+77+777+\ldots+\underbrace{777\ldots\ldots\ldots 7}_{n\text{-digits}} = \frac{7}{81}(10^{n+1}-9n-10)$ for all $n \in N$

32. Prove that $\frac{n^7}{7}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{n^2}{2}-\frac{37}{210}n$ is a positive integer for all $n \in N$

33. Prove that $\frac{n^{11}}{11}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{62}{165}n$ is a positive integer for all $n \in N$.

34. Prove that $\frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\ldots+\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right)=\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)-\cot x$ for all $n \in N$ and $0<x<\frac{\pi}{2}$

35. Prove that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for all natural numbers, $n \geq 2$.

36. Prove that $\frac{(2n)!}{2^{2n}(n!)^2} \leq \frac{1}{\sqrt{3n+1}}$ for all $n \in N$.

37. Prove that $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{n^2}<2-\frac{1}{n}$ for all $n>2, n \in N$.

38. Prove that $x^{2n-1}+y^{2n-1}$ is divisible by $x+y$ for all $n \in N$.

39. Prove that $\sin x+\sin 3x+\ldots+\sin(2n-1)\,x = \frac{\sin^2 nx}{\sin x}$ for all $n \in N$.

40. Prove that $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

for all $n \in N$. **[NCERT EXEMPLAR]**

41. Prove that $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$, for all natural numbers $n > 1$.

[NCERT EXEMPLAR]

42. Given $a_1 = \frac{1}{2}\left(a_0 + \frac{A}{a_0}\right), a_2 = \frac{1}{2}\left(a_1 + \frac{A}{a_1}\right)$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{A}{a_n}\right)$ for $n \geq 2$,

where $a > 0, A > 0$.

Prove that $\frac{a_n - \sqrt{A}}{a_n + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)2^{n-1}$.

43. Let $P(n)$ be the statement: $2^n \geq 3n$. If $P(r)$ is true, show that $P(r+1)$ is true. Do you conclude that $P(n)$ is true for all $n \in N$?

44. Show by the Principle of Mathematical induction that the sum S_n of the n terms of the series $1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + 7^2 + \ldots$ is given by

$$S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

[NCERT EXEMPLAR]

45. Prove that the number of subsets of a set containing n distinct elements is 2^n for all $n \in N$.

[NCERT EXEMPLAR]

46. A sequence $a_1, a_2, a_3, \ldots\ldots$ is defined by letting $a_1 = 3$ and $a_k = 7\,a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3.7^{n-1}$ for all $n \in N$.

[NCERT EXEMPLAR]

47. A sequence $x_1, x_2, x_3, \ldots$ is defined by letting $x_1 = 2$ and $x_k = \frac{x_{k-1}}{n}$ for all natural numbers $k, k \geq 2$. Show that $x_n = \frac{2}{n!}$ for all $n \in N$. **[NCERT EXEMPLAR]**

48. A sequence $x_0, x_1, x_2, x_3, \ldots$ is defined by letting $x_0 = 5$ and $x_k = 4 + x_{k-1}$ for all natural number k. Show that $x_n = 5 + 4n$ for all $n \in N$ using mathematical induction.

[NCERT EXEMPLAR]

49. Using principle of mathematical induction prove that

$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}$ for all natural numbers $n \geq 2$. **[NCERT EXEMPLAR]**

50. The distributive law from algebra states that for all real numbers c, a_1 and a_2, we have

$$c(a_1 + a_2) = ca_1 + ca_2$$

Use this law and mathematical induction to prove that, for al natural numbers, $n \geq 2$, if $c, a_1, a_2, \ldots, a_n$ are any real numbers, then

$$c(a_1 + a_2 + \ldots + a_n) = ca_1 + ca_2 + \ldots + ca_n.$$

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per the requirement of the question.

1. State the first principle of mathematical induction.
2. Write the set of value of n for which the statement $P(n): 2n < n!$ is true.
3. State the second principle of mathematical induction.
4. If $P(n): 2 \times 4^{2n+1} + 3^{3n+1}$ is divisible by λ for all $n \in N$ is true, then find the value of λ.

ANSWERS

2. $\{n \in N : n \geq 4\}$ 4. 11

MULTIPLE CHOICES QUESTIONS (MCQS)

Make the correct alternative in each of the following.

1. If $x^n - 1$ is divisible by $x - \lambda$, then the least positive integral value of λ is
   (a) 1 (b) 2 (c) 3 (d) 4
2. For all $n \in N$, $3 \times 5^{2n+1} + 2^{3n+1}$ is divisible by
   (a) 19 (b) 17 (c) 23 (d) 25
3. If $10^n + 3 \times 4^{n+2} + \lambda$ is divisible by 9 for all $n \in N$, then the least positive integral value of λ is
   (a) 5 (b) 3 (c) 7 (d) 1
4. Let $P(n): 2^n < (1 \times 2 \times 3 \times \ldots \times n)$. Then the smallest positive integer for which $P(n)$ is true is
   (a) 1 (b) 2 (c) 3 (d) 4
5. A student was asked to prove a statement $P(n)$ by induction. He proved $P(k+1)$ is true whenever $P(k)$ is true for all $k > 5 \in N$ and also $P(5)$ is true. On the basis of this he could conclude that $P(n)$ is true.
   (a) for all $n \in N$ (b) for all $n > 5$ (c) for all $n \geq 5$ (d) for all $n < 5$
6. If $P(n): 49^n + 16^n + \lambda$ is divisible by 64 for $n \in N$ is true, then the least negative integral value of λ is
   (a) -3 (b) -2 (c) -1 (d) -4

ANSWERS

1. (a) 2. (b) 3. (a) 4. (d) 5. (c) 6. (c)

SUMMARY

1. A sentence or description which can be judged to be true or false is called a statement. Statements involving mathematical relations are called mathematical statements.
2. Let $P(n)$ be a statement involving the natural number n such that
   (i) $P(1)$ is true.
   and, (ii) $P(m+1)$ is true, whenever $P(m)$ is true.
   Then, $P(n)$ is true for all $n \in N$.
   This is called first principle of mathematical induction.
3. Let $P(n)$ be a statement involving the natural number n such that
   (i) $P(1)$ is true
   and, (ii) $P(m+1)$ is true, whenever $P(n)$ is true for all $n \leq m$.
   Then, $P(n)$ is true for all $n \in N$.
   This is called second principle of mathematical induction.

CHAPTER 13

COMPLEX NUMBERS

13.1 NEED FOR COMPLEX NUMBERS

If a, b are natural numbers such that $a > b$, then the equation $x + a = b$ is not solvable in N, the set of natural numbers i.e. there is no natural number satisfying the equation $x + a = b$. So, the set of natural numbers is extended to form the set I of integers in which every equation of the form $x + a = b$; $a, b \in N$ is solvable. But, equations of the form $xa = b$, where $a, b \in I$, $a \neq 0$ are not solvable in I also. Therefore, the set I of integers is extended to obtain the set Q of all rational numbers in which every equation of the form $xa = b$, $a \neq 0$, $a, b \in I$ is uniquely solvable. The equations of the form $x^2 = 2$, $x^2 = 3$ etc. are not solvable in Q because there is no rational number whose square is 2. Such numbers are known as irrational numbers. The set Q of all rational numbers is extended to obtain the set R which includes both rational and irrational numbers. This set is known as the set of real numbers. The equations of the form $x^2 + 1 = 0$, $x^2 + 4 = 0$ etc. are not solvable in R i.e. there is no real number whose square is a negative real number. *Euler was the first mathematician to introduce the symbol i (iota) for the square root of -1 i.e. a solution of $x^2 + 1 = 0$ with the property $i^2 = -1$. He also called this symbol as the imaginary unit.*

13.2 INTEGRAL POWERS OF IOTA (i)

Positive integral powers of i: We have, $i = \sqrt{-1}$

$$\therefore \quad i^2 = -1,\ i^3 = i^2 \times i = -i,\ i^4 = (i^2)^2 = (-1)^2 = 1$$

In order to compute i^n for $n > 4$, we divide n by 4 and obtain the remainder r. Let m be the quotient when n is divided by 4. Then,

$$n = 4m + r, \text{ where } 0 \leq r < 4$$

$$\Rightarrow \quad i^n = i^{4m+r} = (i^4)^m\, i^r = i^r$$

Thus, the value of i^n for $n > 4$ is i^r, where r is the remainder when n is divided by 4.

Negative integral powers of i: By the law of indices, we have

$$i^{-1} = \frac{1}{i} = \frac{i^3}{i^4} = i^3 = -i,\ i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

$$i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = i, \qquad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

If $n > 4$, then $i^{-n} = \dfrac{1}{i^n} = \dfrac{1}{i^r}$, where r is the remainder when n is divided by 4

<u>NOTE</u> *i^0 is defined as 1.*

The above discussion suggests the following algorithm to find integral exponents of i.

ALGORITHM

To find the value of i^n for $n \in Z$, we may follow the following steps.

STEP I *If $n = 0$, then write $i^n = 1$.*

STEP II *If $n > 0$, then*

$$i^n = \begin{cases} i, & \text{if } n = 1 \\ -1, & \text{if } n = 2 \\ -i, & \text{if } n = 3 \\ 1, & \text{if } n = 4 \\ i^r, & \text{if } n > 4, \text{ where } r \text{ is the remainder when } n \text{ is divided by } 4 \end{cases}$$

STEP III *If $n < 0$, then $n = -m$, where $m > 0$.*

$$\therefore \quad i^n = \begin{cases} i^{-1} = \frac{1}{i} = -i, & \text{if } n = -1 \\ i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1, & \text{if } n = -2 \\ i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = i, & \text{if } n = -3 \\ i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1, & \text{if } n = -4 \\ i^{-m} = \frac{1}{i^m} = \frac{1}{i^r}, & \text{where } r \text{ is the remainder when } m \text{ is divided by } 4, \text{ if } n < -4. \end{cases}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate the following:*

(i) i^{135} (ii) i^{19} (iii) i^{-999} (iv) $(-\sqrt{-1})^{4n+3}, n \in N$

SOLUTION (i) 135 leaves remainder as 3 when it is divided by 4.

$\therefore \quad i^{135} = i^3 = -i$

(ii) The remainder is 3 when 19 is divided by 4.

$\therefore \quad i^{19} = i^3 = -i$

(iii) We have, $i^{-999} = 1/i^{999}$

On dividing 999 by 4, we obtain 3 as the remainder. Therefore, $i^{999} = i^3$.

Hence, $i^{-999} = \frac{1}{i^{999}} = \frac{1}{i^3} = \frac{i}{i^4} = \frac{i}{1} = i$

(iv) We have, $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3} = (-i)^{4n}(-i)^3 = \{(-i)^4\}^n(-i^3) = 1 \times -i^3 = i$

EXAMPLE 2 *Show that:*

(i) $\left\{i^{19} + \left(\frac{1}{i}\right)^{25}\right\}^2 = -4$ (ii) $\left\{i^{17} - \left(\frac{1}{i}\right)^{34}\right\}^2 = 2i$

(iii) $\left\{i^{18} + \left(\frac{1}{i}\right)^{24}\right\}^3 = 0$ (iv) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$, *for all $n \in N$.*

SOLUTION (i) $\left\{i^{19} + \left(\frac{1}{i}\right)^{25}\right\}^2 = \left\{i^{19} + \frac{1}{i^{25}}\right\}^2 = \left\{i^3 + \frac{1}{i}\right\}^2 = \left\{-i + \frac{i^3}{i^4}\right\}^2$

$$= [-i + i^3]^2 = (-i - i)^2 = 4i^2 = -4.$$

(ii) $\left\{i^{17}-\left(\frac{1}{i}\right)^{34}\right\}^2=\left\{i^{17}-\frac{1}{i^{34}}\right\}^2=\left\{i-\frac{1}{i^2}\right\}^2=\left\{i-\frac{1}{(-1)}\right\}^2=(i+1)^2$

$= i^2+2i+1=-1+2i+1=2i$

(iii) $\left\{i^{18}+\left(\frac{1}{i}\right)^{24}\right\}^3=\left\{i^{18}+\frac{1}{i^{24}}\right\}^3=\left(i^2+\frac{1}{1}\right)^3=(-1+1)^3=0$

(iv) $i^n+i^{n+1}+i^{n+2}+i^{n+3}=i^n+i^n\times i+i^n\times i^2+i^n\times i^3$

$= i^n(1+i+i^2+i^3)$

$= i^n(1+i-1-i)=i^n(0)=0$

LEVEL-2

EXAMPLE 3 *Evaluate* $\sum_{n=1}^{13}(i^n+i^{n+1})$, *where* $n\in N$.

SOLUTION $\sum_{n=1}^{13}(i^n+i^{n+1})$

$=\sum_{n=1}^{13}(i+1)\,i^n$

$=(i+1)\sum_{n=1}^{13}i^n$

$=(i+1)(i+i^2+i^3+\ldots+i^{13})$

$=(i+1)\times i\left(\frac{i^{13}-1}{i-1}\right)$

$=(i^2+i)\left(\frac{i-1}{i-1}\right)$ $\quad [\because\ i^{13}=i]$

$=(-1+i)$

EXAMPLE 4 *Evaluate* $1+i^2+i^4+i^6+\ldots+i^{2n}$.

SOLUTION Let $S=1+i^2+i^4+i^6+\ldots+i^{2n}$. Then

$S=1+i^2+(i^2)^2+(i^2)^3+\ldots+(i^2)^n$

$\Rightarrow\quad S=1\left\{\frac{1-(i^2)^{n+1}}{1-i^2}\right\}=\frac{1-(i^2)^{n+1}}{1+1}=\frac{1}{2}\left\{1-(-1)^{n+1}\right\}$

$\Rightarrow\quad S=\begin{cases}\frac{1}{2}(1-1)=0, \text{ if } n \text{ is odd}\\ \frac{1}{2}(1+1)=1, \text{ if } n \text{ is even}\end{cases}$

EXERCISE 13.1

LEVEL-1

1. Evaluate the following:

(i) i^{457} (ii) i^{528} (iii) $\frac{1}{i^{58}}$ (iv) $i^{37}+\frac{1}{i^{67}}$

(v) $\left(i^{41}+\frac{1}{i^{257}}\right)^9$ (vi) $(i^{77}+i^{70}+i^{87}+i^{414})^3$ (vii) $i^{30}+i^{40}+i^{60}$

(viii) $i^{49}+i^{68}+i^{89}+i^{110}$

2. Show that $1 + i^{10} + i^{20} + i^{30}$ is a real number.

3. Find the values of the following expressions:

(i) $i^{49} + i^{68} + i^{89} + i^{110}$ (ii) $i^{30} + i^{80} + i^{120}$

(iii) $i + i^2 + i^3 + i^4$ (iv) $i^5 + i^{10} + i^{15}$

(v) $\dfrac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$ (vi) $1 + i^2 + i^4 + i^6 + i^8 + \ldots + i^{20}$

(vii) $(1+i)^6 + (1-i)^3$

ANSWERS

1. (i) i	(ii) 1	(iii) -1	(iv) $2i$	(v) 0	(vi) -8	(vii) 1	(viii) $2i$
3. (i) $2i$	(ii) 1	(iii) 0	(iv) -1	(v) -1	(vi) 1	(vii) $-2-10i$	

13.3 IMAGINARY QUANTITIES

The square root of a negative real number is called an imaginary quantity or an imaginary number.

For example, $\sqrt{-3}, \sqrt{-4}, \sqrt{-9/4}$ etc. are imaginary quantities.

THEOREM *If a, b are positive real numbers, then* $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$.

PROOF We have,

$$\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1} \times \sqrt{a} = i\sqrt{a} \text{ and, } \sqrt{-b} = \sqrt{-1 \times b} = \sqrt{-1} \times \sqrt{b} = i\sqrt{b}$$

$$\therefore \quad \sqrt{-a} \times \sqrt{-b} = (i\sqrt{a})(i\sqrt{b}) = i^2 (\sqrt{a} \times \sqrt{b}) = -1(\sqrt{ab}) = -\sqrt{ab}$$

NOTE 1 *For any two real numbers* $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ *is true only when at least one of a and b is either positive or zero. In other words,* $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ *is not valid if a and b both are negative.*

NOTE 2 *For any positive real number a, we have* $\sqrt{-a} = \sqrt{-1 \times a}\ \sqrt{-1} \times \sqrt{a} = i\sqrt{a}$.

ILLUSTRATION 1 *Compute the following:*

(i) $\sqrt{-144}$ (ii) $\sqrt{-4} \times \sqrt{\dfrac{-9}{4}}$

(iii) $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9}$

SOLUTION

(i) $\sqrt{-144} = \sqrt{-1 \times 144} = \sqrt{-1} \times \sqrt{144} = 12i$

(ii) $\sqrt{-4} \times \sqrt{\dfrac{-9}{4}} = (2i)\left(\dfrac{3i}{2}\right) = 3i^2 = -3$

(iii) $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9} = 5i + 6i + 6i = 17i$

ILLUSTRATION 2 *A student writes the formula* $\sqrt{ab} = \sqrt{a}\sqrt{b}$. *Then he substitutes* $a = -1$ *and* $b = -1$ *and finds* $1 = -1$. *Explain where is he wrong?*

SOLUTION Since a and b both are negative. Therefore, $\sqrt{ab}$ cannot be written as $\sqrt{a}\sqrt{b}$. In fact, for a and b both negative, we have $\sqrt{a}\sqrt{b} = -\sqrt{ab}$.

ILLUSTRATION 3 *Is the following computation correct? If not give the correct computation:*

$$\text{“}[\sqrt{(-2)} \cdot \sqrt{(-3)}] = \sqrt{(-2) \cdot (-3)} = \sqrt{6}\text{”}$$

SOLUTION The said computation is not correct, because -2 and -3 both are negative and $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is true when at least one of a and b is positive or zero. The correct computation is

$$(\sqrt{-2})(\sqrt{-3}) = (i\sqrt{2})(i\sqrt{3}) = i^2\sqrt{6} = -\sqrt{6}$$

13.4 COMPLEX NUMBERS

COMPLEX NUMBER *If a, b are two real numbers, then a number of the form $a + ib$ is called a complex number.*

For example, $7 + 2i, -1 + i, 3 - 2i, 0 + 2i, 1 + 0i$ etc. are complex numbers.

Real and imaginary parts of a complex number: If $z = a + ib$ is a complex number, then 'a' is called the real part of z and 'b' is known as the imaginary part of z. The real part of z is denoted by Re (z) and the imaginary part by Im (z).

If $z = 3 - 4i$, then $\text{Re}(z) = 3$ and $\text{Im}(z) = -4$.

Purely real and purely imaginary complex numbers: A complex number z is purely real if its imaginary part is zero i.e. $\text{Im}(z) = 0$ *and purely imaginary if its real part is zero i.e.* $\text{Re}(z) = 0$.

Set of complex numbers: The set of all complex numbers is denoted by C i.e. $C = \{a + ib : a, b \in R\}$.

Since a real number 'a' can be written as $a + 0i$. Therefore, every real number is a complex number. Hence, $R \subset C$, where R is the set of all real numbers.

13.5 EQUALITY OF COMPLEX NUMBERS

DEFINITION *Two complex numbers $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$.*

i.e. $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

Thus, $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

ILLUSTRATION 1 *If $z_1 = 2 - iy$ and $z_2 = x + 3i$ are equal, find x and y.*

SOLUTION We have,

$$z_1 = z_2 \Rightarrow 2 - iy = x + 3i \Rightarrow 2 = x \text{ and } -y = 3 \Rightarrow x = 2 \text{ and } y = -3.$$

ILLUSTRATION 2 *If $(a + b) - i(3a + 2b) = 5 + 2i$, find a and b.*

SOLUTION We have,

$$(a + b) - i(3a + 2b) = 5 + 2i \Rightarrow a + b = 5 \text{ and } -(3a + 2b) = 2 \Rightarrow a = -12, b = 17$$

13.6 ADDITION OF COMPLEX NUMBERS

DEFINITION *Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ be two complex numbers. Then their sum $z_1 + z_2$ is defined as the complex number $(a_1 + a_2) + i(b_1 + b_2)$.*

It follows from this definition that the sum $z_1 + z_2$ is a complex number such that

$$\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2) \text{ and, } \text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$$

For example, If $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$, then $z_1 + z_2 = (2 + 3) + (3 - 2)i = 5 + i$

13.6.1 PROPERTIES OF ADDITION OF COMPLEX NUMBERS

(i) Addition is Commutative: For any two complex numbers z_1 and z_2

$$z_1 + z_2 = z_2 + z_1$$

PROOF Let $z_1 = a_1 + i\,b_1, z_2 = a_2 + ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$$\begin{aligned} z_1 + z_2 &= (a_1 + a_2) + i(b_1 + b_2) && \text{[By definition of addition]} \\ &= (a_2 + a_1) + i(b_2 + b_1) && \text{[By commutativity of addition of real numbers]} \\ &= z_2 + z_1 && \text{[By definition of addition]} \end{aligned}$$

Thus, $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in C$.

Hence, addition of complex number is commutative.

(ii) Addition is Associative: For any three complex numbers z_1, z_2, z_3

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

PROOF Let $z_1 = a_1 + i\,b_1, z_2 = a_2 + ib_2$ and $z_3 = a_3 + i\,b_3$, where a_1, a_2, a_3 and b_1, b_2, b_3 are real numbers. Then,

$$\begin{aligned}(z_1+z_2)+z_3 &= [(a_1+a_2)+i(b_1+b_2)]+(a_3+ib_3) && \text{[By definition of addition]}\\ &= [(a_1+a_2)+a_3]+i[(b_1+b_2)+b_3] && \text{[By definition of addition]}\\ &= [(a_1+(a_2+a_3)]+i[b_1+(b_2+b_3)] && \text{[By associativity of addition on } R]\\ &= (a_1+ib_1)+[(a_2+a_3)+i(b_2+b_3)] && \text{[By definition of addition]}\\ &= z_1+(z_2+z_3) && \text{[By definition of addition]}\end{aligned}$$

Thus, $(z_1+z_2)+z_3=z_1+(z_2+z_3)$ for all $z_1, z_2, z_3 \in C$.

Hence, addition of complex numbers is associative.

(iii) Existence of Additive Identity: The complex number $0=0+i\,0$ is the identity element for addition i.e. $z+0=z=0+z$ for all $z \in C$.

PROOF Let $z=a+ib$ be an arbitrary complex number. Then,

$$z+0 = (a+ib)+(0+i0) = (a+0)+i(b+0) = a+ib = z$$

and, $$0+z = (0+i0)+(a+ib) = (0+a)+i(0+b) = a+ib = z$$

Thus, $z+0 = z = 0+z$ for all $z \in C$

Hence, the complex number $0=0+i\,0$ is the identity element for addition.

(iv) Existence of Additive Inverse: For any complex number $z=a+ib$, there exists $-z=(-a)+i(-b)$ such that $z+(-z)=0=(-z)+z$.

PROOF Let $z=a+ib$ be an arbitrary complex number. Then, $-z=(-a)+i(-b)$ is also a complex number such that

$$z+(-z) = (a+ib)+\{(-a)+i(-b)\} = \{a+(-a)\}+i\{b+(-b)\} = 0+i0 = 0$$

and $$(-z)+z = \{(-a)+i(-b)\}+(a+ib) = \{(-a)+a\}+i\{(-b)+b\} = 0+i0 = 0.$$

Thus, for each complex number $z=a+ib$, there exists a complex number $-z=(-a)+i(-b)$ such that $z+(-z)=0=(-z)+z$.

The complex number $-z$ is called the *additive inverse of z*.

13.7 SUBTRACTION OF COMPLEX NUMBERS

DEFINITION *Let $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$ be two complex numbers. Then the subtraction of z_2 from z_1 is denoted by z_1-z_2 and is defined as the addition of z_1 and $-z_2$.*

Thus, $z_1-z_2=z_1+(-z_2)=(a_1+ib_1)+(-a_2-ib_2)=(a_1-a_2)+i(b_1-b_2)$

For example, If $z_1=-2+3i$ and $z_2=4+5i$, then

$$z_1-z_2 = (-2+3i)+(-4-5i) = (-2-4)+i(3-5) = -6-2i$$

13.8 MULTIPLICATION OF COMPLEX NUMBERS

Let $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$ be two complex numbers. Then the multiplication of z_1 with z_2 is denoted by z_1z_2 and is defined as the complex number $(a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1)$.

Thus, $z_1z_2 = (a_1+ib_1)(a_2+ib_2)$

$\Rightarrow$ $z_1z_2 = (a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1)$

$\Rightarrow$ $z_1z_2 = [\mathrm{Re}(z_1)\,\mathrm{Re}(z_2)-\mathrm{Im}(z_1)\,\mathrm{Im}(z_2)]+i[\mathrm{Re}(z_1)\,\mathrm{Im}(z_2)+\mathrm{Re}(z_2)\,\mathrm{Im}(z_1)]$

For example, If $z_1=3+2i$ and $z_2=2-3i$, then

$$z_1z_2 = (3+2i)(2-3i) = (3\times2-2\times(-3))+i(3\times-3+2\times2) = 12-5i$$

NOTE *The product z_1z_2 can also be obtained if we actually carry out the multiplication $(a_1+ib_1)(a_2+ib_2)$ as given below:*

$$\begin{aligned}(a_1+ib_1)(a_2+ib_2) &= a_1a_2+ia_1b_2+ib_1a_2+i^2b_1b_2\\ &= (a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1) && [\because i^2=-1]\end{aligned}$$

13.8.1 PROPERTIES OF MULTIPLICATION

(i) Multiplication is commutative: For any two complex numbers z_1 and z_2

$$z_1z_2=z_2z_1$$

PROOF Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$, where a_1, a_2, and b_1, b_2 are real numbers. Then,

$$z_1 z_2 = (a_1 + i b_1)(a_2 + i b_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

and, $z_2 z_1 = (a_2 + i b_2)(a_1 + i b_1) = (a_2 a_1 - b_2 b_1) + i(b_2 a_1 + b_1 a_2)$

$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$ [By commutativity of mult. of real numbers]

$\therefore \quad z_1 z_2 = z_2 z_1$

Thus, $z_1 z_2 = z_2 z_1$ for all $z_1, z_2 \in C$.

Hence, the multiplication of complex numbers is commutative on C.

(ii) Multiplication is associative: For any three complex numbers z_1, z_2, z_3

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

PROOF Let $z_1 = a_1 + i b_1, z_2 = a_2 + i b_2$ and $z_3 = a_3 + i b_3$ be any three complex numbers. Then,

$$\begin{aligned}
(z_1 z_2) z_3 &= \{(a_1 + i b_1)(a_2 + i b_2)\}(a_3 + i b_3) \\
&= \{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)\}(a_3 + i b_3) \\
&= \{(a_1 a_2 - b_1 b_2) a_3 - (a_1 b_2 + a_2 b_1) b_3\} + i\{(a_1 a_2 - b_1 b_2) b_3 + (a_1 b_2 + a_2 b_1) a_3\} \\
&= \{a_1 (a_2 a_3 - b_2 b_3) - b_1 (a_2 b_3 + a_3 b_2)\} + i\{b_1 (a_2 a_3 - b_2 b_3) + a_1 (a_3 b_2 + a_2 b_3)\} \\
&= (a_1 + i b_1)\{(a_2 a_3 - b_2 b_3) + i(a_2 b_3 + a_3 b_2)\} \\
&= z_1 (z_2 z_3)
\end{aligned}$$

Thus, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ for all $z_1, z_2, z_3 \in C$.

Hence, multiplication is associative on C.

(iii) Existence of identity element for multiplication: The complex number $1 = 1 + i0$ is the identity element for multiplication i.e. for every complex number z, $z \cdot 1 = z = 1 \cdot z$.

PROOF Let $z = a + ib$. Then,

$$z \cdot 1 = (a + ib)(1 + i0) = (a \times 1 - b \times 0) + i(a \times 0 + 1 \times b) = a + ib.$$

Similarly, we obtain $1 \cdot z = z$

Thus, $z \cdot 1 = z = 1 \cdot z$, for all $z \in C$.

Hence, $1 = 1 + 0i$ is the multiplicative identity in C.

(iv) Existence of multiplicative inverse: Corresponding to every non-zero complex number $z = a + ib$ there exists a complex number $z_1 = x + iy$ such that $z \cdot z_1 = 1 = z_1 \cdot z$.

PROOF Clearly,

$$\begin{aligned}
& z \cdot z_1 = 1 \\
\Rightarrow \quad & (a + ib)(x + iy) = 1 + i0 \\
\Rightarrow \quad & (ax - by) + i(ay + bx) = 1 + i0 \\
\Rightarrow \quad & ax - by = 1 \text{ and } ay + bx = 0.
\end{aligned}$$

Solving these two equations, we get

$$x = \frac{a}{a^2 + b^2}, \quad y = -\frac{b}{a^2 + b^2} \qquad [\because a \neq 0, b \neq 0]$$

Thus, every non-zero complex number $z = a + ib$ possesses multiplicative inverse given by

$$\left\{\frac{a}{a^2 + b^2}\right\} + i\left\{\frac{-b}{a^2 + b^2}\right\}$$

NOTE *The multiplicative inverse of z is denoted by z^{-1} or, $\frac{1}{z}$*

ILLUSTRATION *Find the multiplicative inverse of $z = 3 - 2i$.*

SOLUTION Using the above formula, we have

$$z^{-1} = \frac{3}{3^2 + (-2)^2} + \frac{i(-(-2))}{3^2 + (-2)^2} = \frac{3}{13} + \frac{2}{13} i$$

(v) Multiplication of complex numbers is distributive over addition of complex numbers : For any three complex numbers z_1, z_2, z_3

(i) $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$ *(Left distributivity)*

(ii) $(z_2+z_3) z_1 = z_2 z_1 + z_3 z_1$ *(Right distributivity)*

PROOF Let $z_1 = a_1 + i b_1, z_2 = a_2 + i b_2$ and $z_3 = a_3 + i b_3$. Then,

$$\begin{aligned} z_1(z_2+z_3) &= (a_1+ib_1)\{(a_2+a_3)+i(b_2+b_3)\} \\ &= \{a_1(a_2+a_3)-b_1(b_2+b_3)\}+i\{a_1(b_2+b_3))+b_1(a_2+a_3)\} \\ &= [(a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1)]+[(a_1a_3-b_1b_3)+i(a_1b_3+a_3b_1)] \\ &= z_1z_2 + z_1z_3 \end{aligned}$$

Similarly, it can be established that $(z_2+z_3) z_1 = z_2 z_1 + z_3 z_1$.

13.9 DIVISION OF COMPLEX NUMBERS

The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\frac{z_1}{z_2}$.

Thus, $\frac{z_1}{z_2} = z_1 z_2^{-1} = z_1 \left\{\frac{1}{z_2}\right\}$

Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$. Then,

$$\frac{z_1}{z_2} = (a_1+ib_1)\left\{\frac{a_2}{a_2^2+b_2^2}+i\frac{(-b_2)}{a_2^2+b_2^2}\right\} \qquad \left[\because z=a+ib \Rightarrow \frac{1}{z}=\frac{a}{a^2+b^2}+\frac{i(-b)}{a^2+b^2}\right]$$

$$\Rightarrow \frac{z_1}{z_2} = \left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right)+i\left(\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right) \qquad \text{[By definition of multiplication]}$$

For example, If $z_1 = 2+3i$ and $z_2 = 1+2i$, then

$$\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2} = (2+3i)\times\frac{1}{1+2i} = (2+3i)\frac{1}{5}-\frac{2}{5}i = \left(\frac{2}{5}+\frac{6}{5}\right)+i\left(-\frac{4}{5}+\frac{3}{5}\right) = \frac{8}{5}-\frac{1}{5}i$$

13.10 CONJUGATE OF A COMPLEX NUMBER

DEFINITION *Let $z = a + ib$ be a complex number. Then the conjugate of z is denoted by $\bar{z}$ and is equal to $a - ib$.*

Thus, $z = a+ib \Rightarrow \bar{z} = a - ib$

It follows from this definition that the conjugate of a complex number is obtained by replacing i by $-i$.

For example, if $z = 3+4i$, then $\bar{z} = 3-4i$.

13.10.1 PROPERTIES OF CONJUGATE

THEOREM *If z, z_1, z_2 are complex numbers, then*

(i) $(\bar{\bar{z}}) = z$ *(ii)* $z+\bar{z} = 2\,\mathrm{Re}(z)$

(iii) $z-\bar{z} = 2i\,\mathrm{Im}(z)$ *(iv)* $z=\bar{z} \Leftrightarrow z$ *is purely real*

(v) $z+\bar{z}=0 \Rightarrow z$ *is purely imaginary* *(vi)* $z\bar{z} = \{\mathrm{Re}(z)\}^2+\{\mathrm{Im}(z)\}^2$

(vii) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$ *(viii)* $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$

(ix) $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$ *(x)* $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$.

PROOF Let $z = a + ib, z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$.

(i) $z = a + ib \Rightarrow \bar{z} = a - ib \Rightarrow (\bar{\bar{z}}) = \overline{(a - ib)} = a + ib \Rightarrow (\bar{\bar{z}}) = z.$

(ii) $z + \bar{z} = (a + ib) + (a - ib) = 2a = 2\,\mathrm{Re}(z)$

(iii) $z - \bar{z} = (a + ib) - (a - ib) = 2ib = 2i\,\mathrm{Im}(z)$

(iv) $z = \bar{z} \Leftrightarrow a + ib = a - ib \Leftrightarrow 2ib = 0 \Leftrightarrow b = 0 \Leftrightarrow \mathrm{Im}(z) = 0 \Rightarrow z$ is purely real

(v) $z + \bar{z} = 0 \Leftrightarrow (a + ib) + (a - ib) = 0 \Leftrightarrow 2a = 0 \Leftrightarrow a = 0 \Leftrightarrow \mathrm{Re}(z) = 0 \Leftrightarrow z$ is purely imaginary

(vi) $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = \{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2$

(vii) We have,

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$\therefore \quad \overline{z_1 + z_2} = (a_1 + a_2) - i(b_1 + b_2) = (a_1 - ib_1) + (a_2 - ib_2) = \overline{(a_1 + ib_1)} + \overline{(a_2 + ib_2)} = \bar{z}_1 + \bar{z}_2.$$

(viii) We have, $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$

$$\therefore \quad \overline{z_1 - z_2} = (a_1 - a_2) - i(b_1 - b_2) = (a_1 - ib_1) - (a_2 - ib_2) = \overline{(a_1 + ib_1)} - \overline{(a_2 + ib_2)} = \bar{z}_1 - \bar{z}_2$$

(ix) We have, $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

$$\therefore \quad \overline{z_1 z_2} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1) = (a_1 - ib_1)(a_2 - ib_2) = \bar{z}_1 \bar{z}_2$$

(x) We have, $\dfrac{z_1}{z_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2} = \left(\dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + i\left(\dfrac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)$

$$\therefore \quad \overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) - i\left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right) \quad \text{...(i)}$$

Now, $\dfrac{\bar{z}_1}{\bar{z}_2} = \bar{z}_1 \times \dfrac{1}{\bar{z}_2} = (a_1 - ib_1)\left(\dfrac{a_2}{a_2^2 + b_2^2} + i\dfrac{b_2}{a_2^2 + b_2^2}\right) = \left(\dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) - i\left(\dfrac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)$ ... (ii)

From (i) and (ii), we get $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}.$

13.11 MODULUS OF A COMPLEX NUMBER

DEFINITION *The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as*

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2}$$

Clearly, $|z| \geq 0$ for all $z \in C$.

For example, If $z_1 = 3 - 4i, z_2 = -5 + 2i$ and $z_3 = 1 + \sqrt{-3}$, then

$$|z_1| = \sqrt{3^2 + (-4)^2} = 5, \; |z_2| = \sqrt{(-5)^2 + 2^2} = \sqrt{29} \text{ and, } |z_3| = |1 + i\sqrt{3}| = \sqrt{1^2 + (\sqrt{3})^2} = 2.$$

REMARK *In the set C of all complex numbers, the order relation is not defined. As such $z_1 > z_2$ or, $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or, $|z_1| < |z_2|$ has got its meaning as $|z_1|$ and $|z_2|$ are real numbers.*

13.11.1 PROPERTIES OF MODULUS

THEOREM *If $z, z_1, z_2 \in C$, then*

(i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\mathrm{Re}(z) = \mathrm{Im}(z) = 0$

(ii) $|z| = |\bar{z}| = |-z|$

(iii) $-|z| \leq \mathrm{Re}(z) \leq |z|$; $-|z| \leq \mathrm{Im}(z) \leq |z|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 z_2| = |z_1||z_2|$

(vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$; $z_2 \neq 0$

(vii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1 \overline{z_2})$

(viii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1 \overline{z_2})$

(ix) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

(x) $|a z_1 - b z_2|^2 + |b z_1 + a z_2|^2 = (a^2 + b^2)\left(|z_1|^2 + |z_2|^2\right)$, *where* $a, b \in R$.

[NCERT EXEMPLAR]

<u>PROOF</u> Let $z = a + ib$. Then,

(i) $|z| = 0 \Leftrightarrow \sqrt{a^2 + b^2} = 0 \Leftrightarrow a^2 + b^2 = 0 \Leftrightarrow a = 0$ and $b = 0 \Leftrightarrow \text{Re}(z) = \text{Im}(z) = 0$

(ii) Let $z = a + ib$. Then, $\bar{z} = a - ib$ and $-z = -a - ib$.

$\therefore$ $|z| = \sqrt{a^2 + b^2}, |\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$ and, $|-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2}$

Clearly, $|z| = |\bar{z}| = |-z|$

(iii) Let $z = a + ib$. Then, $|z| = \sqrt{a^2 + b^2}$.

Clearly, $-\sqrt{a^2 + b^2} \leq a \leq \sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2} \leq b \leq \sqrt{a^2 + b^2}$

$\Rightarrow$ $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(iv) Let $z = a + ib$. Then, $\bar{z} = a - ib$.

$\therefore$ $z\bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 = \left\{\sqrt{a^2 + b^2}\right\}^2 = |z|^2$

(v) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

$\Rightarrow$ $|z_1 z_2| = \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2}$

$\Rightarrow$ $|z_1 z_2| = \sqrt{a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2}$

$\Rightarrow$ $|z_1 z_2| = \sqrt{a_1^2 (a_2^2 + b_2^2) + b_1^2 (a_2^2 + b_2^2)} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}$

$\Rightarrow$ $|z_1 z_2| = |z_1||z_2|$

(vi) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$$\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2} = (a_1 + ib_1)\left(\frac{a_2}{a_2^2 + b_2^2} + i\frac{(-b_2)}{a_2^2 + b_2^2}\right) = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + i\left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right)^2 + \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)^2}$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(a_1 a_2 + b_1 b_2)^2 + (a_2 b_1 - a_1 b_2)^2}{(a_2^2 + b_2^2)^2}}$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{a_1^2 a_2^2 + b_1^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_2^2}{(a_2^2 + b_2^2)^2}}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(a_1^2+b_1^2)(a_2^2+b_2^2)}{(a_2^2+b_2^2)^2}} = \sqrt{\frac{a_1^2+b_1^2}{a_2^2+b_2^2}} = \frac{\sqrt{a_1^2+b_1^2}}{\sqrt{a_2^2+b_2^2}} = \frac{|z_1|}{|z_2|}$$

(vii) Clearly,

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) \qquad [\because z\bar{z}=|z|^2]$$

$$\Rightarrow \quad |z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2) \qquad [\because \overline{z_1+z_2}=\bar{z}_1+\bar{z}_2]$$

$$\Rightarrow \quad |z_1+z_2|^2 = z_1\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_2+z_2\bar{z}_1 \qquad \text{[By distributivity of multiplication]}$$

$$\Rightarrow \quad |z_1+z_2|^2 = |z_1|^2+|z_2|^2+z_1\bar{z}_2+(\overline{z_1\bar{z}_2}) \qquad [\because (\overline{z_1\bar{z}_2})=\bar{z}_1(\bar{\bar{z}}_2)=\bar{z}_1z_2=z_2\bar{z}_1]$$

$$\Rightarrow \quad |z_1+z_2|^2 = |z_1|^2+|z_2|^2+2\,\mathrm{Re}(z_1\bar{z}_2) \qquad [\because z+\bar{z}=2\,\mathrm{Re}(z)]$$

(viii) Clearly,

$$|z_1-z_2|^2 = (z_1-z_2)(\overline{z_1-z_2}) \qquad [\because z\bar{z}=|z|^2]$$

$$\Rightarrow \quad |z_1-z_2|^2 = (z_1-z_2)(\bar{z}_1-\bar{z}_2) \qquad [\because \overline{z_1+z_2}=\bar{z}_1-\bar{z}_2]$$

$$\Rightarrow \quad |z_1-z_2|^2 = z_1\bar{z}_1+z_2\bar{z}_2-z_1\bar{z}_2-z_2\bar{z}_1 \qquad \text{[By distributivity of multiplication]}$$

$$\Rightarrow \quad |z_1-z_2|^2 = |z_1|^2+|z_2|^2-z_1\bar{z}_2-(\overline{z_1\bar{z}_2}) \qquad [\because (\overline{z_1\bar{z}_2})=\bar{z}_1(\bar{\bar{z}}_2)=\bar{z}_1z_2]$$

$$\Rightarrow \quad |z_1-z_2|^2 = |z_1|^2+|z_2|^2-2\,\mathrm{Re}(z_1\bar{z}_2)$$

(ix) Using (vii) and (viii), we get

$$|z_1+z_2|^2+|z_1-z_2|^2 = |z_1|^2+|z_2|^2+2\,\mathrm{Re}(z_1\bar{z}_2)+|z_1|^2+|z_2|^2-2\,\mathrm{Re}(z_1\bar{z}_2)$$

$$= 2(|z_1|^2+|z_2|^2)$$

(x) We have,

$$\begin{aligned}
|az_1-bz_2|^2 &= (az_1-bz_2)(\overline{az_1-bz_2})\\
&= (az_1-bz_2)(a\bar{z}_1-b\bar{z}_2)\\
&= a^2z_1\bar{z}_1-(az_1)(b\bar{z}_2)-(bz_2)(a\bar{z}_1)+b^2z_2\bar{z}_2\\
&= a^2|z_1|^2-ab(z_1\bar{z}_2+\bar{z}_1z_2)+b^2|z_2|^2\\
&= a^2|z_1|^2-ab(z_1\bar{z}_2+(\overline{z_1\bar{z}_2}))+b^2|z_2|^2\\
&= a^2|z_1|^2-ab\{2\,\mathrm{Re}(z_1\bar{z}_2)\}+b^2|z_2|^2 \qquad [\because z_1\bar{z}_2+(\overline{z_1\bar{z}_2})=2\,\mathrm{Re}(z_1\bar{z}_2)]\\
&= a^2|z_1|^2-2ab\,\mathrm{Re}(z_1\bar{z}_2)+b^2|z_2|^2
\end{aligned}$$

Similarly, we obtain

$$|bz_1+az_2|^2 = b^2|z_1|^2+a^2|z_2|^2+2ab\,\mathrm{Re}(z_1\bar{z}_2)$$

$$\therefore \quad |az_1-bz_2|^2+|bz_1+az_2|^2$$

$$\begin{aligned}
&= a^2|z_1|^2-2ab\,\mathrm{Re}(z_1\bar{z}_2)+b^2|z_2|^2+b^2|z_1|^2+a^2|z_2|^2+2ab\,\mathrm{Re}(z_1\bar{z}_2)\\
&= |z_1|^2(a^2+b^2)+|z_2|^2(b^2+a^2)\\
&= (a^2+b^2)\left(|z_1|^2+|z_2|^2\right)
\end{aligned}$$

13.12 RECIPROCAL OF A COMPLEX NUMBER

Let $z = a + ib$ be a non-zero complex number. Then,

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} \quad \left[\begin{array}{r}\text{Multiplying numerator and denominator}\\ \text{by conjugate of denominator}\end{array}\right]$$

$$\Rightarrow \quad \frac{1}{z} = \frac{a-ib}{a^2 - i^2 b^2} = \frac{a-ib}{a^2+b^2}$$

$$\Rightarrow \quad \frac{1}{z} = \frac{a}{a^2+b^2} + \frac{i(-b)}{a^2+b^2}$$

Clearly, $\frac{1}{z}$ is equal to the multiplicative inverse of z.

Also, $\frac{1}{z} = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$

Thus, the multiplicative inverse of a non-zero complex number z is same as its reciprocal and is given by

$$\frac{\text{Re}(z)}{|z|^2} + i\frac{(-\text{Im}(z))}{|z|^2} = \frac{\bar{z}}{|z|^2}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **EXPRESSING A COMPLEX NUMBER IN THE STANDARD FORM a + ib**

In order to express a complex number in the standard form, we may follow the following algorithm.

ALGORITHM

STEP I *Write the complex number in the form* $\frac{a+ib}{c+id}$ *by using fundamental operations of addition, subtraction and multiplication.*

STEP II *Multiply the numerator and denominator by the conjugate of the denominator.*

EXAMPLE 1 *Express the following in the form $a + ib$:*

(i) $(-5i)\left(\frac{1}{8}i\right)$ [NCERT] (ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3$ [NCERT]

(iii) $(5i)\left(-\frac{3}{5}i\right)$ [NCERT] (iv) $i^9 + i^{19}$

(v) i^{-39} [NCERT] (vi) $(1-i)^4$ [NCERT]

SOLUTION (i) $(-5i)\left(\frac{1}{8}i\right) = -\frac{5}{8}i^2 = -\frac{5}{8} \times -1 = \frac{5}{8} = \frac{5}{8} + 0\,i$

(ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3 = -2i^2 \times -\frac{1}{512}i^3 = \frac{1}{256} \times i^2 \times i^3 = \frac{1}{256}i^5 = \frac{i}{256} = 0 + \frac{1}{256}i$

(iii) $(5i)\left(-\frac{3}{5}i\right) = -3i^2 = -3 \times -1 = 3 = 3 + 0\,i$

(iv) $i^9 + i^{19} = (i^4)^2\, i + (i^4)^4\, i^3 = i + i^3 = i - i = 0 = 0 + 0\,i$

(v) $i^{-39} = (i^4)^{-10}\, i = i = 0 + 1.i$

(vi) $(1-i)^4 = \left\{(1-i)^2\right\}^2 = (1 - 2i + i^2)^2 = (1 - 2i - 1)^2 = (-2i)^2 = 4i^2 = -4 = -4 + 0i$

EXAMPLE 2 *Express each of the following in the form a + ib:*

(i) $3(7+7i)+i(7+7i)$ [NCERT] (ii) $(1-i)-(-1+6i)$ [NCERT]

(iii) $\left(\frac{1}{5}+\frac{2}{5}i\right)-\left(4+\frac{5}{2}i\right)$ [NCERT] (iv) $\left\{\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)\right\}-\left(-\frac{4}{3}+i\right)$ [NCERT]

SOLUTION (i) $3(7+7i)+i(7+7i)=21+21i+7i+7i^2=21+21i+7i-7=14+28i$

(ii) $(1-i)-(-1+6i)=1-i+1-6i=2-7i$

(iii) $\left(\frac{1}{5}+\frac{2}{5}\right)i-\left(4+\frac{5}{2}i\right)=\left(\frac{1}{5}-4\right)+\frac{2i}{5}-\frac{5i}{2}=-\frac{19}{5}-\frac{21}{10}i$

(iv) $$\left\{\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)\right\}-\left(-\frac{4}{3}+i\right)=\left\{\left(\frac{1}{3}+4\right)+i\left(\frac{7}{3}+\frac{1}{3}\right)\right\}-\left(-\frac{4}{3}+i\right)$$
$$=\left(\frac{13}{3}+\frac{8}{3}i\right)+\frac{4}{3}-i$$
$$=\left(\frac{13}{3}+\frac{4}{3}\right)+\left(\frac{8}{3}-1\right)i=\frac{17}{3}+\frac{5}{3}i$$

EXAMPLE 3 *Express each of the following in the form a + ib:*

(i) $\left(\frac{1}{3}+3i\right)^3$ [NCERT] (ii) $\left(-2-\frac{1}{3}i\right)^3$ [NCERT]

(iii) $(5-3i)^3$ [NCERT] (iv) $(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)$

SOLUTION (i) $$\left(\frac{1}{3}+3i\right)^3=\left(\frac{1}{3}\right)^3+(3i)^3+3\times\frac{1}{3}\times 3i\left(\frac{1}{3}+3i\right)=\frac{1}{27}+27i^3+3i\left(\frac{1}{3}+3i\right)$$
$$=\frac{1}{27}+27i^3+i+9i^2=\frac{1}{27}-27i+i-9=-\frac{242}{27}-26i$$

(ii) $$\left(-2-\frac{1}{3}i\right)^3=(-2)^3+\left(-\frac{1}{3}i\right)^3+3\times-2\times-\frac{1}{3}i\left(-2-\frac{1}{3}i\right)=-8-\frac{1}{27}i^3+2i\left(-2-\frac{1}{3}i\right)$$
$$=-8+\frac{1}{27}i-4i-\frac{2}{3}i^2=-8+\frac{1}{27}i-4i+\frac{2}{3}=-\frac{22}{3}-\frac{107}{27}i$$

(iii) $(5-3i)^3=5^3+(-3i)^3+3\times 25\times-3i+3\times 5\times(-3i)^2=125+27i-225i-135=-10-198i$

(v) $$(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)=(-\sqrt{3}+i\sqrt{2})(2\sqrt{3}-i)=-6+\sqrt{3}i+2\sqrt{6}i-\sqrt{2}i^2$$
$$=-6+(\sqrt{3}+2\sqrt{6})i+\sqrt{2}=(\sqrt{2}-6)+(\sqrt{3}+2\sqrt{6})i$$

EXAMPLE 4 *Express each one of the following in the standard form a + ib.*

(i) $\frac{1}{3-4i}$ (ii) $\frac{5+4i}{4+5i}$ (iii) $\frac{(1+i)^2}{3-i}$

(iv) $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ (v) $\frac{1}{-2+\sqrt{-3}}$ (vi) $\left(\frac{1}{1-2i}+\frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$

(vii) $\frac{1}{1-\cos\theta+2i\sin\theta}$ (viii) $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$ [NCERT]

SOLUTION (i) $\frac{1}{3-4i}=\frac{1}{3-4i}\times\frac{3+4i}{3+4i}=\frac{3+4i}{9-16i^2}=\frac{3+4i}{9+16}=\frac{3}{25}+\frac{4}{25}i$

(ii) $\dfrac{5+4i}{4+5i} = \dfrac{5+4i}{4+5i} \times \dfrac{4-5i}{4-5i} = \dfrac{(20+20)+i(16-25)}{16-25i^2} = \dfrac{40-9i}{41} = \dfrac{40}{41} - \dfrac{9}{41}i$

(iii) $\dfrac{(1+i)^2}{3-i} = \dfrac{1+2i+i^2}{3-i} = \dfrac{2i}{3-i} = \dfrac{2i}{3-i} \cdot \dfrac{3+i}{3+i} = \dfrac{6i+2i^2}{9-i^2} = \dfrac{-2+6i}{10} = -\dfrac{1}{5} + \dfrac{3}{5}i$

(iv) $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \dfrac{(6+6)+i(-4+9)}{(2+2)+i(4-1)} = \dfrac{12+5i}{4+3i} = \dfrac{12+5i}{4+3i} \times \dfrac{4-3i}{4-3i}$

$$= \frac{(48+15)+i(-36+20)}{16-9i^2} = \frac{63}{25} - \frac{16}{25}i$$

(v) $\dfrac{1}{-2+\sqrt{-3}} = \dfrac{1}{-2+i\sqrt{3}} = \dfrac{1}{-2+i\sqrt{3}} \times \dfrac{-2-i\sqrt{3}}{-2-i\sqrt{3}} = \dfrac{-2-i\sqrt{3}}{4-3i^2} = -\dfrac{2}{7} - \dfrac{\sqrt{3}}{7}i$

(vi) $\left(\dfrac{1}{1-2i} + \dfrac{3}{1+i}\right)\left(\dfrac{3+4i}{2-4i}\right) = \dfrac{1+i+3-6i}{(1+2)+i(-2+1)} \times \dfrac{3+4i}{2-4i} = \dfrac{4-5i}{3-i} \times \dfrac{3+4i}{2-4i} = \dfrac{(12+20)+i(16-15)}{(6-4)+i(-2-12)}$

$$= \frac{32+i}{2-14i} = \frac{32+i}{2-14i} \times \frac{2+14i}{2+14i} = \frac{(64-14)+i(2+448)}{4-196i^2} = \frac{50+450i}{200} = \frac{1}{4} + \frac{9}{4}i$$

(vii) $\dfrac{1}{1-\cos\theta+2i\sin\theta} = \dfrac{1}{1-\cos\theta+2i\sin\theta} \times \dfrac{1-\cos\theta-2i\sin\theta}{1-\cos\theta-2i\sin\theta}$

$$= \frac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta)^2-4i^2\sin^2\theta} = \frac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta)^2+4\sin^2\theta}$$

$$= \frac{1-\cos\theta-2i\sin\theta}{1-2\cos\theta+\cos^2\theta+4\sin^2\theta} = \frac{1-\cos\theta-2i\sin\theta}{2-2\cos\theta+3\sin^2\theta}$$

$$= \left(\frac{1-\cos\theta}{2-2\cos\theta+3\sin^2\theta}\right) + i\left(\frac{-2\sin\theta}{2-2\cos\theta+3\sin^2\theta}\right)$$

(viii) $\dfrac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})} = \dfrac{(9-\sqrt{5}\times-\sqrt{5})+i(3\times-\sqrt{5}+3\sqrt{5})}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i}$

$$= \frac{(9+5)+i\times 0}{2\sqrt{2}i} = \frac{14}{2\sqrt{2}i} = \frac{7}{\sqrt{2}i} = \frac{-7}{\sqrt{2}}i = 0 - \frac{7}{\sqrt{2}}i$$

EXAMPLE 5 *Prove that the following complex numbers are purely real:*

(i) $\left(\dfrac{2+3i}{3+4i}\right)\left(\dfrac{2-3i}{3-4i}\right)$ (ii) $\left(\dfrac{3+2i}{2-3i}\right) + \left(\dfrac{3-2i}{2+3i}\right)$

SOLUTION (i) $\left(\dfrac{2+3i}{3+4i}\right)\left(\dfrac{2-3i}{3-4i}\right) = \dfrac{(2+3i)(2-3i)}{(3+4i)(3-4i)} = \dfrac{4-9i^2}{9-16i^2} = \dfrac{13}{25}$, which is purely real.

(ii) $\left(\dfrac{3+2i}{2-3i}\right) + \left(\dfrac{3-2i}{2+3i}\right) = \dfrac{3+2i}{2-3i} \times \dfrac{2+3i}{2+3i} + \dfrac{3-2i}{2+3i} \times \dfrac{2-3i}{2-3i}$

$$= \frac{(3+2i)(2+3i)}{4-9i^2} + \frac{(3-2i)(2-3i)}{4-9i^2}$$

$= \dfrac{13i}{13} - \dfrac{13i}{13} = 0$, which is purely real.

EXAMPLE 6 *Express* $(1-2i)^{-3}$ *in the standard form* $a+ib$.

SOLUTION We have,

$$(1-2i)^{-3} = \frac{1}{(1-2i)^3} = \frac{1}{1-8i^3-6i+12i^2} = \frac{1}{1+8i-6i-12} = \frac{1}{-11+2i}$$

$$= \frac{1}{-11+2i} \times \frac{-11-2i}{-11-2i} = \frac{-11-2i}{(-11)^2-(2i)^2} = \frac{-11-2i}{125} = \frac{-11}{125} - \frac{2}{125}i$$

EXAMPLE 7 *Perform the indicated operation and find the result in the form a + ib.*

(i) $\dfrac{2-\sqrt{-25}}{1-\sqrt{-16}}$ (ii) $\dfrac{3-\sqrt{-16}}{1-\sqrt{-9}}$

SOLUTION We have,

(i) $$\frac{2-\sqrt{-25}}{1-\sqrt{-16}} = \frac{2-5i}{1-4i} = \frac{2-5i}{1-4i} \times \frac{1+4i}{1+4i} = \frac{(2+20)+i(8-5)}{1-16i^2} = \frac{22+3i}{17} = \frac{22}{17} + \frac{3}{17}i$$

(ii) $$\frac{3-\sqrt{-16}}{1-\sqrt{-9}} = \frac{3-4i}{1-3i} = \frac{3-4i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{(3+12)+i(-4+9)}{1-9i^2} = \frac{15}{10} + \frac{5}{10}i = \frac{3}{2} + \frac{1}{2}i$$

EXAMPLE 8 *If* z_1, z_2 *are* $1-i, -2+4i$, *respectively, find* $\text{Im}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$.

SOLUTION Clearly,

$$\frac{z_1 z_2}{\bar{z}_1} = \frac{(1-i)(-2+4i)}{(\overline{1-i})} = \frac{(-2+4)+i(2+4)}{1+i} = \frac{2+6i}{1+i}$$

$$= \frac{2+6i}{1+i} \times \frac{1-i}{1-i} = \frac{(2+6)+i(6-2)}{1+1} = 4+2i$$

$$\therefore \quad \text{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = 2$$

Type II ON EQUALITY OF COMPLEX NUMBERS

Recall that two complex numbers z_1 *and* z_2 *are equal iff* $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

EXAMPLE 9 *Find the real values of x and y, if*

(i) $(3x-7)+2iy = -5y+(5+x)i$ (ii) $(1-i)x+(1+i)y = 1-3i$

(iii) $(x+iy)(2-3i) = 4+i$ (iv) $\dfrac{x-1}{3+i} + \dfrac{y-1}{3-i} = i$

SOLUTION (i) We have

$$(3x-7)+2iy = -5y+(5+x)i$$
$$\Rightarrow \quad 3x-7 = -5y \text{ and } 2y = 5+x$$
$$\Rightarrow \quad 3x+5y = 7 \text{ and } x-2y = -5$$
$$\Rightarrow \quad x = -1, y = 2.$$

(ii) We have,

$$(1-i)x+(1+i)y = 1-3i$$
$$\Rightarrow \quad (x+y)+i(-x+y) = 1-3i$$
$$\Rightarrow \quad x+y = 1 \text{ and } -x+y = -3 \qquad \text{[On equating real and imaginary parts]}$$
$$\Rightarrow \quad x = 2, y = -1$$

(iii) We have,

$$(x+iy)(2-3i) = 4+i$$
$$\Rightarrow \quad (2x+3y)+i(-3x+2y) = 4+i$$
$$\Rightarrow \quad 2x+3y = 4 \text{ and } -3x+2y = 1 \qquad \text{[On equating real and imaginary parts]}$$

$$\Rightarrow \quad x = \frac{5}{13}, y = \frac{14}{13}$$

(iv) We have,

$$\frac{x-1}{3+i} + \frac{y-1}{3-i} = i$$

$$\Rightarrow \quad \frac{(x-1)(3-i) + (y-1)(3+i)}{(3+i)(3-i)} = i$$

$$\Rightarrow \quad \frac{(3x+3y-6) + i(y-x)}{9-i^2} = i$$

$$\Rightarrow \quad \left(\frac{3x+3y-6}{10}\right) + i\left(\frac{y-x}{10}\right) = 0 + i$$

$$\Rightarrow \quad \frac{3x+3y-6}{10} = 0 \text{ and } \frac{y-x}{10} = 1 \qquad \text{[On equating real and imaginary parts]}$$

$$\Rightarrow \quad x + y - 2 = 0 \text{ and } y - x = 10$$

$$\Rightarrow \quad x = -4, \; y = 6.$$

EXAMPLE 10 *Find real values of x and y for which the following equalities hold:*

(i) $(1+i)\,y^2 + (6+i) = (2+i)\,x$ (ii) $(x^4 + 2xi) - (3x^2 + iy) = (3-5i) + (1+2iy)$

SOLUTION (i) We have,

$$(1+i)\,y^2 + (6+i) = (2+i)\,x$$

$$\Rightarrow \quad (y^2+6) + i(y^2+1) = 2x + ix$$

$$\Rightarrow \quad y^2 + 6 = 2x \quad \text{...(i)} \qquad \text{and,} \quad y^2 + 1 = x \quad \text{...(ii)}$$

From (i) and (ii), we get

$$y^2 + 6 = 2(y^2+1) \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

Substituting $y = \pm 2$ in (ii), we get $x = 5$.

Thus, $x = 5$ and $y = 2$ or, $x = 5$ and $y = -2$

(ii) We have,

$$(x^4 + 2xi) - (3x^2 + iy) = (3-5i) + (1+2iy)$$

$$\Rightarrow \quad (x^4 - 3x^2) + i(2x - y) = 4 + i(2y-5)$$

$$\Rightarrow \quad x^4 - 3x^2 = 4 \text{ and, } 2x - y = 2y - 5 \qquad \text{[On equating real and imaginary parts]}$$

$$\Rightarrow \quad x^4 - 3x^2 - 4 = 0, \; 2x - 3y + 5 = 0$$

Now, $x^4 - 3x^2 - 4 = 0 \Rightarrow (x^2-4)(x^2+1) = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$

Putting $x = \pm 2$ in $2x - 3y + 5 = 0$, we get

$y = 3$ when $x = 2$ and $y = 1/3$ when $x = -2$

Thus, $x = 2$ and $y = 3$ or, $x = -2$ and $y = 1/3$.

EXAMPLE 11 *If* $a + ib = \dfrac{c+i}{c-i}$, *where c is real, prove that:* $a^2 + b^2 = 1$ *and* $\dfrac{b}{a} = \dfrac{2c}{c^2-1}$.

SOLUTION We have,

$$a + ib = \frac{c+i}{c-i}$$

$$\Rightarrow \quad a + ib = \frac{(c+i)(c+i)}{(c-i)(c+i)}$$

$$\Rightarrow \quad a + ib = \frac{(c+i)^2}{c^2 - i^2}$$

$$\Rightarrow \quad a+ib = \frac{c^2+2ic+i^2}{c^2-i^2}$$

$$\Rightarrow \quad a+ib = \frac{c^2-1}{c^2+1}+\frac{i\,2c}{c^2+1}$$

$$\Rightarrow \quad a = \frac{c^2-1}{c^2+1} \text{ and } b = \frac{2c}{c^2+1}$$

$$\Rightarrow \quad a^2+b^2 = \left(\frac{c^2-1}{c^2+1}\right)^2+\frac{4c^2}{(c^2+1)^2} \text{ and, } \frac{b}{a} = \left(\frac{2c}{c^2+1}\right)\div\left(\frac{c^2-1}{c^2+1}\right)$$

$$\Rightarrow \quad a^2+b^2 = \frac{(c^2+1)^2}{(c^2+1)^2} = 1 \text{ and, } \frac{b}{a} = \frac{2c}{c^2-1}$$

EXAMPLE 12 *If* $(x+iy)^{1/3} = a+ib$, $x, y, a\, b \in R$. *Show that*

(i) $\frac{x}{a}+\frac{y}{b} = 4(a^2-b^2)$ (ii) $\frac{x}{a}-\frac{y}{b} = -2(a^2+b^2)$ [NCERT EXEMPLAR]

SOLUTION We have,

$$(x+iy)^{1/3} = a+ib$$

$$\Rightarrow \quad (x+iy) = (a+ib)^3 \qquad \text{[On cubing both sides]}$$

$$\Rightarrow \quad x+iy = a^3+3a^2\,i\,b+3a\,i^2\,b^2+i^3\,b^3$$

$$\Rightarrow \quad x+iy = (a^3-3ab^2)+i(3a^2\,b-b^3)$$

$$\Rightarrow \quad x = a^3-3ab^2 \text{ and } y = 3a^2\,b-b^3$$

$$\Rightarrow \quad \frac{x}{a} = a^2-3b^2 \text{ and } \frac{y}{b} = 3a^2-b^2$$

$$\Rightarrow \quad \frac{x}{a}+\frac{y}{b} = a^2-3b^2+3a^2-b^2 = 4(a^2-b^2) \text{ and } \frac{x}{a}-\frac{y}{b} = (a^2-3b^2)-(3a^2-b^2) = -2(a^2+b^2)$$

Type III **ON CONJUGATE OF A COMPLEX NUMBER**

EXAMPLE 13 *Multiply* $3-2i$ *by its conjugate.*

SOLUTION The conjugate of $3-2i$ is $3+2i$.

$\therefore$ Required product $= (3-2i)(3+2i) = 9-4i^2 = 9+4 = 13$

ALITER Let $z = 3-2i$. Then, $\bar{z} = 3+2i$

$\therefore \quad z\bar{z} = |z|^2 \Rightarrow z\bar{z} = 3^2+(-2)^2 = 13$

EXAMPLE 14 *Find the conjugate of* $\frac{1}{3+4i}$.

SOLUTION Let $z = \frac{1}{3+4i}$. Then,

$$z = \frac{1}{3+4i} = \frac{1}{3+4i}\times\frac{3-4i}{3-4i} = \frac{3-4i}{9+16} = \frac{3}{25}-\frac{4}{25}i$$

$$\therefore \quad \bar{z} = \frac{3}{25}+\frac{4}{25}i.$$

EXAMPLE 15 *Express the following complex numbers in the standard form. Also, find their conjugate:*

(i) $\dfrac{1-i}{1+i}$ (ii) $\dfrac{(1+i)^2}{3-i}$

(iii) $\dfrac{(2+3i)^2}{2-i}$ (iv) $\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5-12i}-\sqrt{5-12i}}$ [NCERT EXEMPLAR]

SOLUTION (i) We have,

$$z = \frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{(1-i)^2}{1^2-i^2} = \frac{1-2i+i^2}{1-i^2} = \frac{1-2i-1}{1+1} = 0-i$$

$\therefore \quad \bar{z} = 0+i$

(ii) We have,

$$z = \frac{(1+i)^2}{3-i} = \frac{1+2i+i^2}{3-i} \times \frac{3+i}{3+i} = \frac{2i}{3-i} \times \frac{3+i}{3+i} = \frac{6i+2i^2}{9-i^2} = \frac{6i-2}{10} = -\frac{1}{5}+\frac{3}{5}i$$

$\therefore \quad \bar{z} = -\frac{1}{5} - \frac{3}{5}i$

(iii) We have,

$$z = \frac{(2+3i)^2}{2-i} = \frac{4+12i+9i^2}{2-i} = \frac{4+12i-9}{2-i} \times \frac{2+i}{2+i} = \frac{-5+12i}{2-i} \times \frac{2+i}{2+i} = \frac{-22+19i}{4-i^2} = -\frac{22}{5}+\frac{19}{5}i$$

$\therefore \quad \bar{z} = -\frac{22}{5} - \frac{19}{5}i$

(iv) Let $\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5-12i}-\sqrt{5-12i}}$. Then,

$$z = \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}} \times \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}+\sqrt{5-12i}} = \frac{\left(\sqrt{5+12i}+\sqrt{5-12i}\right)^2}{(5+12i)-(5-12i)}$$

$$\Rightarrow \quad z = \frac{5+12i+5-12i+2\sqrt{5+12i}\sqrt{5-12i}}{5+12i-5+12i} = \frac{10+2\sqrt{25+144}}{24i} = \frac{3}{2i} = -\frac{3}{2}i = 0-\frac{3}{2}i$$

$\therefore \quad \bar{z} = 0+\frac{3}{2}i$

EXAMPLE 16 *Find real values of x and y for which the complex numbers* $-3+ix^2y$ *and* x^2+y+4i *are conjugate of each other.*

SOLUTION Since $-3+ix^2y$ and x^2+y+4i are complex conjugates.

$\therefore \quad -3+ix^2y = \overline{x^2+y+4i}$

$\Rightarrow \quad -3+ix^2y = x^2+y-4i$

$\Rightarrow \quad -3 = x^2+y \quad$ ...(i) $\quad$ and, $\quad x^2y = -4 \quad$ ...(ii)

$\Rightarrow \quad -3 = x^2-\dfrac{4}{x^2} \quad$ [Putting $y=-4/x^2$ from (ii) in (i)]

$\Rightarrow \quad x^4+3x^2-4 = 0$

$\Rightarrow \quad (x^2+4)(x^2-1) = 0$

$\Rightarrow \quad x^2-1 = 0 \quad$ [$\because x^2+4 \neq 0$ for any real x]

$\Rightarrow \quad x = \pm 1$

From (ii), $y=-4$, when $x=\pm 1$.

Hence, $x=1, y=-4$ or, $x=-1, y=-4$

EXAMPLE 17 *Find the real numbers x and y, if $(x-iy)(3+5i)$ is the conjugate of $-6-24i$.* [NCERT]

SOLUTION We have,

$$(x-iy)(3+5i)=(3x+5y)+i(5x-3y)$$

It is given that $(x-iy)(3+5i)$ is the conjugate of $-6-24i$.

$\therefore \quad (x-iy)(3+5i)=\overline{-6-24i}$

$\Rightarrow \quad (3x+5y)+i(5x-3y)=-6+24i$

$\Rightarrow \quad 3x+5y=-6$ and $5x-3y=24$ [On equating real and imaginary parts]

Solving these equations, we get $x=3$, $y=-3$.

Type IV **ON FINDING THE MULTIPLICATIVE INVERSE OR RECIPROCAL OF A NON-ZERO COMPLEX NUMBER**

EXAMPLE 18 *Find the multiplicative inverse of the following complex numbers:*

(i) $3+2i$ [NCERT] (ii) $(2+\sqrt{3}\,i)^2$

SOLUTION (i) Let $z=3+2i$. Then,

$$\frac{1}{z}=\frac{1}{3+2i}=\frac{3-2i}{(3+2i)(3-2i)}=\frac{3-2i}{9-4i^2}=\frac{3}{13}-\frac{2}{13}i$$

ALITER Let $z=3+2i$. Then,

$$\frac{1}{z}=\frac{\bar{z}}{|z|^2}=\frac{3-2i}{9+4}=\frac{3}{13}-\frac{2i}{13}$$

(ii) Let $z=(2+\sqrt{3}i)^2$. Then,

$$z=4+3i^2+4\sqrt{3}i=4-3+4\sqrt{3}i=1+4\sqrt{3}i$$

$$\therefore \quad \frac{1}{z}=\frac{1}{4+\sqrt{3}i}=\frac{1-4\sqrt{3}i}{(1+4\sqrt{3}i)(1-4\sqrt{3}i)}=\frac{1-4\sqrt{3}i}{1+48}=\frac{1}{49}-\frac{4\sqrt{3}i}{49}$$

Type V **PROBLEMS BASED UPON CONJUGATE AND MODULUS OF A COMPLEX NUMBER**

EXAMPLE 19 *If $\dfrac{a+ib}{c+id}=x+iy$, prove that $\dfrac{a-ib}{c-id}=x-iy$ and $\dfrac{a^2+b^2}{c^2+d^2}=x^2+y^2$.*

SOLUTION We have,

$$\frac{a+ib}{c+id}=x+iy$$

$$\Rightarrow \quad \overline{\left(\frac{a+ib}{c+id}\right)}=\overline{x+iy} \qquad \text{[Taking Conjugate of both sides]}$$

$$\Rightarrow \quad \frac{\overline{a+ib}}{\overline{c+id}}=\overline{x+iy} \qquad \left[\because \overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}\right]$$

$$\Rightarrow \quad \frac{a-ib}{c-id}=x-iy$$

Thus, we have $\dfrac{a+ib}{c+id}=x+iy$ and $\dfrac{a-ib}{c-id}=x-iy$

$$\Rightarrow \quad \frac{a+ib}{c+id}\times\frac{a-ib}{c-id}=(x+iy)(x-iy)$$

$$\Rightarrow \quad \frac{(a+ib)(a-ib)}{(c+id)(c-id)}=(x+iy)(x-iy)$$

$$\Rightarrow \quad \frac{a^2+b^2}{c^2+d^2} = x^2+y^2 \qquad [\text{Using: } z\bar{z} = |z|^2]$$

EXAMPLE 20 *If* $\dfrac{(a+i)^2}{(2a-i)} = p+iq$, *show that:* $p^2+q^2 = \dfrac{(a^2+1)^2}{(4a^2+1)}$.

SOLUTION We have,

$$\frac{(a+i)^2}{(2a-i)} = (p+iq) \qquad \text{...(i)}$$

$$\Rightarrow \quad \overline{\left\{\frac{(a+i)^2}{(2a-i)}\right\}} = \overline{(p+iq)} \qquad [\text{Taking conjugate of both sides}]$$

$$\Rightarrow \quad \frac{(\overline{a+i})^2}{(\overline{2a-i})} = \overline{(p+iq)}$$

$$\Rightarrow \quad \frac{(a-i)^2}{(2a+i)} = p-iq \qquad \text{...(ii)}$$

Multiplying (i) and (ii), we get

$$\frac{(a+i)^2}{(2a-i)} \times \frac{(a-i)^2}{(2a+i)} = (p+iq)(p-iq)$$

$$\Rightarrow \quad \frac{\{(a+i)(a-i)\}^2}{(2a-i)(2a+i)} = (p+iq)(p-iq)$$

$$\Rightarrow \quad \frac{(a^2+1)^2}{4a^2+1} = p^2+q^2 \qquad [\text{Using : } z\bar{z} = |z|^2]$$

EXAMPLE 21 *If* $a+ib = \dfrac{(x+i)^2}{2x^2+1}$, *prove that* $a^2+b^2 = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$. [NCERT]

SOLUTION We have,

$$a+ib = \frac{(x+i)^2}{2x^2+1} \qquad \text{...(i)}$$

$$\Rightarrow \quad \overline{a+ib} = \overline{\left\{\frac{(x+i)^2}{2x^2+1}\right\}} \qquad [\text{Taking conjugate of both sides}]$$

$$\Rightarrow \quad \overline{a+ib} = \frac{(\overline{x+i})^2}{(2x^2+1)}$$

$$\Rightarrow \quad a-ib = \frac{(x-i)^2}{2x^2+1} \qquad \text{...(ii)}$$

Multiplying (i) and (ii), we get

$$(a+ib)(a-ib) = \frac{(x+i)^2(x-i)^2}{(2x^2+1)(2x^2+1)}$$

$$\Rightarrow \quad a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2} \qquad \left[\because (x+i)(x-i) = x^2-i^2 = x^2+1\right]$$

EXAMPLE 22 *If* $x+iy = \sqrt{\dfrac{a+ib}{c+id}}$, *prove that:* $(x^2+y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$. [NCERT]

SOLUTION We have,

$$x+iy = \sqrt{\frac{a+ib}{c+id}}$$

$$\Rightarrow \quad x-iy = \sqrt{\frac{a-ib}{c-id}} \qquad \text{[Taking conjugate of both sides]}$$

$$\therefore \quad (x+iy)(x-iy) = \sqrt{\frac{a+ib}{c+id}} \times \sqrt{\frac{a-ib}{c-id}} = \sqrt{\frac{a+ib}{c+id} \times \frac{a-ib}{c-id}}$$

$$\Rightarrow \quad x^2+y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

$$\Rightarrow \quad (x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$$

EXAMPLE 23 *Find the least positive value of n, if* $\left(\frac{1+i}{1-i}\right)^n = 1.$ **[NCERT]**

SOLUTION We have,

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{1+2i+i^2}{1-i^2} = \frac{1+2i-1}{1+1} = i$$

$\therefore \quad \left(\frac{1+i}{1-i}\right)^n = 1 \Rightarrow i^n = 1 \Rightarrow n$ is a multiple of 4 $\Rightarrow$ The smallest positive value of n is 4.

EXAMPLE 24 *Find real* θ *such that* $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ *is purely real.* **[NCERT]**

SOLUTION Clearly,

$$\frac{3+2i\sin\theta}{1-2i\sin\theta} = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$$

$$= \frac{(3-4\sin\theta)+i(6\sin\theta+2\sin\theta)}{1+4\sin^2\theta}$$

$$= \frac{(3-4\sin\theta)+i(6\sin\theta+2\sin\theta)}{1+4\sin^2\theta} = \frac{3-4\sin\theta}{1+4\sin^2\theta} + \frac{i\,8\sin\theta}{1+4\sin^2\theta}$$

It is given that $\frac{3+2\sin\theta}{1-2i\sin\theta}$ is purely real. Therefore, its imaginary part is zero.

i.e. $\quad \frac{8\sin\theta}{1+4\sin^2\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = n\pi, n \in Z$

LEVEL-2

EXAMPLE 25 *The sum and product of two complex numbers are real if and only if they are conjugate of each other.*

SOLUTION First, let the two complex numbers be conjugate of each other. Let complex numbers be $z_1 = a+ib$ and $z_2 = a-ib$. Then,

$z_1 + z_2 = (a+ib)+(a-ib) = 2a,$ which is real.

And, $z_1 z_2 = (a+ib)(a-ib) = a^2 - i^2 b^2 = a^2 + b^2$, which is also real.

Thus, if z_1 and z_2 are conjugate of each other. Then, Their sum $z_1 + z_2$ and product $z_1 z_2$ both are real.

Conversely, let z_1 and z_2 be two complex numbers such that their sum $z_1 + z_2$ and product $z_1 z_2$ both are real. Then, we have to prove that z_1 and z_2 are conjugate of each other.

Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$. Then,

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2) \text{ and } z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Now, $z_1 + z_2$ and $z_1 z_2$ are real

$\Rightarrow$ $(a_1 + a_2) + i(b_1 + b_2)$ and, $(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$ are real

$\Rightarrow$ $b_1 + b_2 = 0$ and $a_1 b_2 + a_2 b_1 = 0$ [$\because$ z is real $\Leftrightarrow$ Im (z) = 0]

$\Rightarrow$ $b_2 = -b_1$ and $a_1 b_2 + a_2 b_1 = 0$

$\Rightarrow$ $b_2 = -b_1$ and $-a_1 b_1 + a_2 b_1 = 0$

$\Rightarrow$ $b_2 = -b_1$ and $(a_2 - a_1) b_1 = 0$

$\Rightarrow$ $b_2 = -b_1$ and $a_2 - a_1 = 0$

$\Rightarrow$ $b_2 = -b_1$ and $a_2 = a_1$

$\Rightarrow$ $z_2 = a_2 + i\,b_2 = a_1 - i\,b_1 \Rightarrow z_2 = \bar{z}_1$

$\Rightarrow$ z_1 and z_2 are conjugate of each other

EXAMPLE 26 *If* $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = (x + iy)$, *show that:* $2.5.10 \ldots (1 + n^2) = x^2 + y^2$.

SOLUTION We have,

$(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = x + iy$

$\Rightarrow$ $|(1 + i)(1 + 2i) \ldots (1 + ni)| = |x + iy|$ [Taking modulus of both sides]

$\Rightarrow$ $|1 + i|\,|1 + 2i| \ldots |1 + ni| = |x + iy|$ [$\because |z_1 z_2 \ldots z_n| = |z_1|\,|z_2| \ldots |z_n|$]

$\Rightarrow$ $\sqrt{1+1}\,\sqrt{1+4} \ldots \sqrt{1+n^2} = \sqrt{x^2 + y^2}$

$\Rightarrow$ $2.5.10 \ldots (1 + n^2) = (x^2 + y^2)$ [On squaring both sides]

EXAMPLE 27 *If* $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, *prove that*

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$$

SOLUTION We have,

$(a + ib)(c + id)(e + if)(g + ih) = A + iB$

$\Rightarrow$ $|(a + ib)(c + id)(e + if)(g + ih)| = |A + iB|$ [Taking modulus of both sides]

$\Rightarrow$ $|a + ib|\,|c + id|\,|e + if|\,|g + ih| = |A + iB|$ [Using: $|z_1 z_2 \ldots z_n| = |z_1|\,|z_2| \ldots |z_n|$]

$\Rightarrow$ $\sqrt{a^2 + b^2}\,\sqrt{c^2 + d^2}\,\sqrt{c^2 + f^2}\,\sqrt{g^2 + h^2} = \sqrt{A^2 + B^2}$

$\Rightarrow$ $(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$ [On squaring both sides]

EXAMPLE 28 *If* z_1, z_2 *are complex numbers such that* $\frac{2z_1}{3z_2}$ *is purely imaginary number, then find* $\left|\frac{z_1 - z_2}{z_1 + z_2}\right|$.

SOLUTION It is given that $\frac{2z_1}{3z_2}$ is purely imaginary. Therefore,

$\frac{2z_1}{3z_2} = \lambda i$ for some $\lambda \in R$.

$\Rightarrow$ $\frac{z_1}{z_2} = \frac{3\lambda}{2} i$

Now, $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = \left|\frac{\frac{z_1}{z_2} - 1}{\frac{z_1}{z_2} + 1}\right| = \left|\frac{\frac{3}{2}\lambda i - 1}{\frac{3}{2}\lambda i + 1}\right|$ $\left[\because \frac{z_1}{z_2} = \frac{3\lambda}{2} i\right]$

$$\Rightarrow \quad \left|\frac{z_1 - z_2}{z_1 + z_2}\right| = \left|\frac{3\lambda i - 2}{3\lambda i + 2}\right| = \left|\frac{-2 + 3\lambda i}{2 + 3\lambda i}\right| = \frac{|-2 + 3\lambda i|}{|2 + 3\lambda i|} = \frac{\sqrt{4 + 9\lambda^2}}{\sqrt{4 + 9\lambda^2}} = 1$$

Type VI **ON FINDING THE VALUE OF A POLYNOMIAL FOR A GIVEN VALUE OF THE VARIABLE**

EXAMPLE 29 *If* $x = -5 + 2\sqrt{-4}$, *find the value of* $x^4 + 9x^3 + 35x^2 - x + 4$.

SOLUTION We have,

$$x = -5 + 2\sqrt{-4}$$
$$\Rightarrow \quad x + 5 = 4i$$
$$\Rightarrow \quad (x+5)^2 = 16i^2 \Rightarrow x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$$

Now, $x^4 + 9x^3 + 35x^2 - x + 4$

$$= x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$$
$$= x^2(0) - x(0) + 4(0) - 160 = -160 \qquad [\because x^2 + 10x + 41 = 0]$$

Thus, the value of the given polynomial for $x = -5 + 2\sqrt{-4}$ is -160.

EXAMPLE 30 *Find the value of* $x^3 + 7x^2 - x + 16$, *when* $x = 1 + 2i$.

SOLUTION We have,

$$x = 1 + 2i \Rightarrow x - 1 = 2i \Rightarrow (x-1)^2 = 4i^2 \Rightarrow x^2 - 2x + 1 = -4 \Rightarrow x^2 - 2x + 5 = 0$$

$$\therefore \quad x^3 + 7x^2 - x + 16 = x(x^2 - 2x + 5) + 9(x^2 - 2x + 5) + (12x - 29)$$
$$= x(0) + 9(0) + 12x - 29 \qquad [\because x^2 - 2x + 5 = 0]$$
$$= 12(1 + 2i) - 29 \qquad [\because x = 1 + 2i]$$
$$= -17 + 24i$$

Hence, the value of the given polynomial when $x = 1 + 2i$ is $-17 + 24i$.

Type VII **MISCELLANEOUS PROBLEMS**

EXAMPLE 31 *Prove that:* $x^4 + 4 = (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$.

SOLUTION We have,

$$(x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$$
$$= \{(x+1)^2 - i^2\}\{(x-1)^2 - i^2\}$$
$$= \{(x+1)^2 + 1\}\{(x-1)^2 + 1\}$$
$$= \{x^2 + 2x + 2\}\{x^2 - 2x + 2\}$$
$$= \{x^2 + 2 + 2x\}\{x^2 + 2 - 2x\} = (x^2 + 2)^2 - (2x)^2 = x^4 + 4x^2 + 4 - 4x^2 = x^4 + 4$$

EXAMPLE 32 *If z is a complex number such that* $|z| = 1$, *prove that* $\frac{z-1}{z+1}$ *is purely imaginary. What will be your conclusion if* $z = 1$?

SOLUTION Let $z = x + iy$. Then,

$$|z| = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$$

Now,

$$\frac{z-1}{z+1} = \frac{x + iy - 1}{x + iy + 1} = \frac{(x-1) + iy}{(x+1) + iy} = \frac{(x-1) + iy}{(x+1) + iy} \times \frac{(x+1) - iy}{(x+1) - iy}$$

$$\Rightarrow \quad \frac{z-1}{z+1} = \frac{(x^2 - 1 + y^2) + i(xy + y - xy + y)}{(x+1)^2 + y^2} = \frac{(x^2 + y^2 - 1) + 2iy}{(x+1)^2 + y^2}$$

$$\Rightarrow \quad \frac{z-1}{z+1} = \frac{2iy}{(x+1)^2+y^2} \qquad [\because x^2+y^2=1]$$

$$\therefore \quad \frac{z-1}{z+1} = \frac{2iy}{(x+1)^2+y^2}\text{, which is purely imaginary.}$$

Again, when $z=1$, then $x+iy=1+i.0 \Rightarrow x=1$ and $y=0$.

$$\therefore \quad \frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{1+i0-1}{1+i0+1} = 0\text{, which is purely real}$$

EXAMPLE 33 *If $z=x+iy$ and $w=\dfrac{1-iz}{z-i}$, show that $|w|=1 \Rightarrow z$ is purely real.*

SOLUTION We have,

$$|w| = 1$$

$$\Rightarrow \quad \left|\frac{1-iz}{z-i}\right| = 1$$

$$\Rightarrow \quad \frac{|1-iz|}{|z-i|} = 1$$

$$\Rightarrow \quad |1-iz|=|z-i|$$

$$\Rightarrow \quad |1-i(x+iy)| = |x+iy-i)|\text{, where } z = x+iy$$

$$\Rightarrow \quad |1+y-ix| = |x+i(y-1)|$$

$$\Rightarrow \quad \sqrt{(1+y)^2+(-x)^2} = \sqrt{x^2+(y-1)^2}$$

$$\Rightarrow \quad (1+y)^2+x^2 = x^2+(y-1)^2$$

$$\Rightarrow \quad y = 0$$

$$\Rightarrow \quad z = x+i0 = x,\text{ which is purely real}$$

EXAMPLE 34 *If $z=2-3i$, show that $z^2-4z+13=0$ and hence find the value of $4z^3-3z^2+169$.*

SOLUTION We have,

$$z=2-3i \Rightarrow z-2=-3i \Rightarrow (z-2)^2=(-3i)^2 \Rightarrow z^2-4z+4=9i^2 \Rightarrow z^2-4z+13=0$$

$$\therefore \quad 4z^3-3z^2+169 = 4z(z^2-4z+13)+13(z^2-4z+13) = 4z(0)+13(0)=0 \qquad [\because z^2-4z+13=0]$$

EXAMPLE 35 *Show that a real value of x will satisfy the equation $\dfrac{1-ix}{1+ix}=a-ib$ if $a^2+b^2=1$, where a, b are real.*

SOLUTION We have,

$$\frac{1-ix}{1+ix} = \frac{a-ib}{1}$$

$$\Rightarrow \quad \frac{(1-ix)+(1+ix)}{(1-ix)-(1+ix)} = \frac{a-ib+1}{a-ib-1} \qquad \text{[Applying componendo and dividendo]}$$

$$\Rightarrow \quad \frac{2}{-2ix} = \frac{1+a-ib}{-(1-a+ib)}$$

$$\Rightarrow \quad ix = \frac{1-a+ib}{1+a-ib}$$

$$\Rightarrow \quad ix = \frac{(1-a+ib)}{(1+a-ib)} \times \frac{(1+a+ib)}{(1+a+ib)}$$

$$\Rightarrow \quad ix = \frac{1-a^2-b^2+2ib}{(1+a)^2-i^2b^2}$$

$$\Rightarrow \quad ix = \frac{1-a^2-b^2+2ib}{(1+a)^2+b^2}$$

$$\Rightarrow \quad ix = \frac{2ib}{(1+a)^2+b^2}, \text{ if } a^2+b^2 = 1$$

$$\Rightarrow \quad x = \frac{2b}{(1+a)^2+b^2}, \text{ which is real}$$

EXAMPLE 36 *If* α and β *are different complex numbers with* $|\beta| = 1$, *find* $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$. **[NCERT]**

SOLUTION Clearly,

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\overline{\left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)} = \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha\bar{\beta}}\right) = \frac{(\beta-\alpha)(\bar{\beta}-\bar{\alpha})}{(1-\bar{\alpha}\beta)(1-\alpha\bar{\beta})}$$

$$\Rightarrow \quad \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = \frac{\beta\bar{\beta}-\beta\bar{\alpha}-\alpha\bar{\beta}+\alpha\bar{\alpha}}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+\bar{\alpha}\beta\alpha\bar{\beta}} = \frac{|\beta|^2-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+(\alpha\bar{\alpha})(\beta\bar{\beta})}$$

$$\Rightarrow \quad \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = \frac{|\alpha|^2-\alpha\bar{\beta}-\bar{\alpha}\beta+|\beta|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2|\beta|^2} = \frac{|\alpha|^2-\alpha\bar{\beta}-\bar{\alpha}\beta+1}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2} = 1 \qquad [\because |\beta| = 1]$$

$$\therefore \quad \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1.$$

EXAMPLE 37 *If* $|z_1| = |z_2| = \ldots = |z_n| = 1$, *prove that* $|z_1+z_2+z_3+\ldots+z_n| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right|$.

SOLUTION Clearly, **[NCERT EXEMPLAR]**

$$|z_1+z_2+z_3+\ldots+z_n| = \left|\frac{z_1\bar{z}_1}{\bar{z}_1}+\frac{z_2\bar{z}_2}{\bar{z}_2}+\frac{z_3\bar{z}_3}{\bar{z}_3}+\ldots+\frac{z_n\bar{z}_n}{\bar{z}_n}\right|$$

$$= \left|\frac{|z_1|^2}{\bar{z}_1}+\frac{|z_2|^2}{\bar{z}_2}+\frac{|z_3|^2}{\bar{z}_3}+\ldots+\frac{|z_n|^2}{\bar{z}_n}\right|$$

$$= \left|\frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}+\ldots+\frac{1}{\bar{z}_n}\right| \qquad [\because |z_1| = |z_2| = \ldots = |z_n| = 1]$$

$$= \left|\overline{\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right)}\right| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right| \qquad [\because |z| = |\bar{z}|]$$

EXAMPLE 38 *Find non-zero integral solutions of* $|1-i|^x = 2^x$. **[NCERT]**

SOLUTION We have,

$$|1-i|^x = 2^x$$

$$\Rightarrow \quad (\sqrt{2})^x = 2^x \Rightarrow 2^{x/2} = 2^x \Rightarrow 2^{x/2} = 1 \Rightarrow 2^{x/2} = 2^0 \Rightarrow \frac{x}{2} = 0 \Rightarrow x = 0.$$

Hence, the given equation has no non-zero integral solution.

EXAMPLE 39 *Find all non-zero complex numbers z satisfying $\bar{z} = i z^2$.*

SOLUTION Let $z = x + iy$. Then,

$$\bar{z} = iz^2$$

$$\Rightarrow \quad x - iy = i(x^2 - y^2 + 2ixy)$$

$$\Rightarrow \quad x - iy = i(x^2 - y^2) - 2xy$$

$$\Rightarrow \quad (x + 2xy) - i(x^2 - y^2 + y) = 0$$

$$\Rightarrow \quad x + 2xy = 0 \quad \text{...(i)} \qquad \text{and,} \quad x^2 - y^2 + y = 0 \quad \text{...(ii)}$$

Now,

$$x + 2xy = 0 \Rightarrow x(1 + 2y) = 0 \Rightarrow x = 0 \text{ or, } 1 + 2y = 0 \Rightarrow x = 0 \text{ or, } y = -\frac{1}{2}$$

CASE I When $x = 0$:

Putting $x = 0$ in (ii), we have

$$\Rightarrow \quad -y^2 + y = 0 \Rightarrow y(y-1) = 0 \Rightarrow y = 0, y = 1$$

Thus, we have the following pairs of values of x and y:

$$x = 0, y = 0; x = 0, y = 1$$

$$\therefore \quad z = 0 + i0 = 0, \ z = 0 + 1i = i$$

CASE II When $y = -\frac{1}{2}$:

Putting $y = -\frac{1}{2}$ in (ii), we get

$$x^2 - y^2 + y = 0 \Rightarrow x^2 - \frac{1}{4} - \frac{1}{2} = 0 \Rightarrow x^2 - \frac{3}{4} = 0 \Rightarrow x = \pm\frac{\sqrt{3}}{2}$$

Thus, we have the following pairs of values of x and y:

$$x = \frac{\sqrt{3}}{2}, y = \frac{-1}{2} \text{ and, } x = \frac{-\sqrt{3}}{2}, y = \frac{-1}{2}$$

$$\therefore \quad z = \frac{\sqrt{3}}{2} - \frac{1}{2}i, \ z = \frac{-\sqrt{3}}{2} - \frac{1}{2}i$$

Hence, $z = 0, i, \frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

EXAMPLE 40 *If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$.*

SOLUTION We have,

$$iz^3 + z^2 - z + i = 0$$

$$\Rightarrow \quad z^3 - iz^2 + iz + 1 = 0 \qquad \text{[Dividing both sides by } i\text{]}$$

$$\Rightarrow \quad z^2(z - i) + i(z - i) = 0 \Rightarrow (z - i)(z^2 + i) = 0 \Rightarrow z = i \text{ or, } z^2 = -i$$

Now,

$$z = i \Rightarrow |z| = |i| = 1$$

and, $z^2 = -i$

$$\Rightarrow \quad |z^2| = |-i| = 1 \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1.$$

Hence, in either case, we have $|z| = 1$.

EXAMPLE 41 *Solve the equation $z^2 + |z| = 0$, where z is a complex number.*

SOLUTION Let $z = x + iy$. Then,

$$z^2 + |z| = 0$$

$$\Rightarrow (x + iy)^2 + \sqrt{x^2 + y^2} = 0$$

$$\Rightarrow (x^2 - y^2) + \sqrt{x^2 + y^2} + 2i\,xy = 0$$

$$\Rightarrow x^2 - y^2 + \sqrt{x^2 + y^2} = 0 \quad \text{...(i)} \qquad \text{and,} \quad 2xy = 0 \quad \text{...(ii)}$$

Now,

$$2xy = 0 \Rightarrow xy = 0 \Rightarrow x = 0 \text{ or, } y = 0$$

CASE I *When* $y = 0$

Putting $y = 0$ in (i), we get

$$x^2 + \sqrt{x^2} = 0 \Rightarrow x^2 + |x| = 0$$

Clearly, $x^2 + |x| > 0$ for all $x > 0$. So, let $x < 0$.

In this case, we have

$$x^2 + |x| = 0$$

$$\Rightarrow x^2 - x = 0 \qquad [\because x < 0 \therefore |x| = -x]$$

$$\Rightarrow x(x - 1) = 0$$

$$\Rightarrow x = 0, x = 1$$

But, $x < 0$. So, the equation $x^2 + |x| = 0$ has no solution for $x < 0$.

Clearly, $x = 0$ satisfies the equation $x^2 + |x| = 0$.

Thus, we have $x = 0,\ y = 0$.

$$\therefore z = 0$$

CASE II *When* $x = 0$

Putting $x = 0$ in (i), we get

$$-y^2 + \sqrt{y^2} = 0 \Rightarrow -y^2 + |y| = 0$$

If $y > 0$, then $|y| = y$.

$$\therefore -y^2 + |y| = 0$$

$$\Rightarrow -y^2 + y = 0$$

$$\Rightarrow y = 0, y = 1$$

$$\Rightarrow y = 1 \qquad [\because y > 0]$$

If $y < 0$, then $|y| = -y$.

$$-y^2 + |y| = 0$$

$$\Rightarrow -y^2 - y = 0$$

$$\Rightarrow y = 0, -1$$

$$\Rightarrow y = -1 \qquad [\because y < 0]$$

Thus, we obtain $x = 0, y = 1$ or, $x = 0, y = -1$.

$$\therefore z = 0 + i \text{ or, } z = 0 - i$$

Hence, $z = 0, i$ and $-i$ are solutions of $z^2 + |z| = 0$.

EXAMPLE 42 *Solve the equation* $z^2 = \bar{z}$. [NCERT EXEMPLAR]

SOLUTION Let $z = x + i\,y$. Then,

$$z^2 = \bar{z}$$

$$\Rightarrow (x + iy)^2 = x - iy$$

$\Rightarrow \quad x^2+2ixy+(iy)^2=x-iy$

$\Rightarrow \quad (x^2-y^2)+2ixy=x-iy$

$\Rightarrow \quad x^2-y^2=x \quad ...(i) \quad \text{and,} \quad 2xy=-y \quad ...(ii)$

Now, $2xy=-y \Rightarrow (2x+1)y=0 \Rightarrow 2x+1=0$ or $y=0 \Rightarrow x=-\frac{1}{2}$ or $y=0$

Following cases arise :

CASE I *When* $y=0$

Putting $y=0$ in (i), we obtain

$$x^2=x \Rightarrow x(x-1)=0 \Rightarrow x=0 \text{ or, } x=1$$

Thus, we obtain ($x=0$ and $y=0$) or ($x=1$ and $y=0$)

$\therefore \quad z=0+i0=0$ or $z=1+i0$

CASE II *When* $x=-\frac{1}{2}$

Putting $x=-\frac{1}{2}$ in (i), we obtain

$$\frac{1}{4}-y^2=-\frac{1}{2} \Rightarrow y^2=\frac{3}{4} \Rightarrow y=\pm\frac{\sqrt{3}}{2}$$

Thus, we obtain $\left(x=-\frac{1}{2} \text{ and } y=\frac{\sqrt{3}}{2}\right)$ or $\left(x=-\frac{1}{2} \text{ and } y=-\frac{\sqrt{3}}{2}\right)$

$\therefore \quad z=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$ or $z=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$

Hence the values of z satisfying the given equation are

$$z=0+i0, z=1+i0, z=-\frac{1}{2}+i\frac{\sqrt{3}}{2} \text{ and } z=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$$

EXAMPLE 43 *Solve the equation* $|z+1|=z+2(1+i)$. [NCERT EXEMPLAR]

SOLUTION Let $z=x+iy$. Then, $z+1=(x+1)+iy$

$\therefore \quad |z+1|=\sqrt{(x+1)^2+y^2}$

Now, $|z+1|=z+2(1+i)$

$\Rightarrow \quad \sqrt{(x+1)^2+y^2}=(x+iy)+2(1+i)$

$\Rightarrow \quad \sqrt{(x+1)^2+y^2}+0i=(x+2)+(y+2)i$

$\Rightarrow \quad \sqrt{(x+1)^2+y^2}=x+2$ and $y+2=0$ [On equating real and imaginary parts]

$\Rightarrow \quad (x+1)^2+y^2=(x+2)^2$ and $y=-2$

$\Rightarrow \quad y^2=2x+3$ and $y=-2$

$\Rightarrow \quad 4=2x+3$ and $y=-2$

$\Rightarrow \quad x=\frac{1}{2}$ and $y=-2$

Hence, $z=\frac{1}{2}-2i$

EXAMPLE 44 *If* $|z^2-1|=|z|^2+1$, *then show that z lies on the imaginary axis.* [NCERT EXEMPLAR]

SOLUTION Let $z=x+iy$. Then, $z^2=x^2-y^2+2ixy$ and $|z|^2=x^2+y^2$.

$\therefore \quad |z^2-1|=|z|^2+1$

$\Rightarrow \quad \left|(x^2-y^2)+2i\,xy\right| = x^2+y^2+1$

$\Rightarrow \quad \sqrt{(x^2-y^2-1)^2+4x^2y^2} = x^2+y^2+1$

$\Rightarrow \quad (x^2-y^2-1)^2+4x^2y^2 = (x^2+y^2+1)^2$

$\Rightarrow \quad x^4+y^4+1-2x^2+2y^2-2x^2y^2+4x^2y^2 = x^4+y^4+1+2x^2y^2+2x^2+2y^2$

$\Rightarrow \quad 4x^2 = 0$

$\Rightarrow \quad x = 0$

$\therefore \quad z = x+iy = 0+iy$

Thus, z is purely imaginary and hence it lies on y-axis.

EXAMPLE 45 *If the imaginary part of $\frac{2z+1}{iz+1}$ is -2, then show that the locus of the point representing z in the argand plane is a straight line.* **[NCERT EXEMPLAR]**

SOLUTION Let $z = x+iy$. Then,

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+ix}$$

$$= \frac{(2x+1)+i2y}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix} = \frac{(2x+1-y)+i(2y-2y^2-2x^2-x)}{(1-y)^2+x^2}$$

$$= \left\{\frac{2x+1-y}{x^2+(1-y)^2}\right\} + i\left\{\frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2}\right\}$$

$$\therefore \quad \mathrm{Im}\left(\frac{2z+1}{iz+1}\right) = \frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2}$$

But, it is given that Im (z) = −2.

$$\therefore \quad \frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2} = -2$$

$\Rightarrow \quad 2y-2y^2-2x^2-x = -2x^2-2(1-y)^2$

$\Rightarrow \quad x+2y-2=0$, which is a straight line.

Hence, the locus of z is a starlight line.

EXAMPLE 46 *If the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.* **[NCERT EXEMPLAR]**

SOLUTION Let $z = x+iy$. Then, $\bar{z} = x-iy$

$$\therefore \quad \frac{\bar{z}+2}{\bar{z}-1} = \frac{x-iy+2}{x-iy-1} = \frac{(x+2)-iy}{(x-1)-iy}$$

$$= \frac{(x+2)-iy}{(x-1)-iy} \times \frac{(x-1)+iy}{(x-1)+iy}$$

$$= \frac{(x^2+y^2+x-2)+3iy}{(x-1)^2+y^2} = \left\{\frac{x^2+y^2-x-2}{(x-1)^2+y^2}\right\} + i\left\{\frac{3y}{(x-1)^2+y^2}\right\}$$

It is given that the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4.

$$\therefore \quad \frac{x^2+y^2-x-y}{(x-1)^2+y^2} = 4$$

$\Rightarrow \quad 3x^2+3y^2-7x+y+4=0$, which represents a circle.

EXAMPLE 47 *If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + a = 0$, where $a \in R$, represents a circle.*
[NCERT EXEMPLAR]

SOLUTION We have,

$$z = x + iy \Rightarrow \bar{z} = x - iy$$

$$\therefore \quad z\bar{z} + 2(z + \bar{z}) + a = 0$$

$$\Rightarrow \quad (x + iy)(x - iy) + 2(x + iy + x - iy) + a = 0$$

$$\Rightarrow \quad x^2 + y^2 + 4x + a = 0$$

$$\Rightarrow \quad (x + 2)^2 + (y - 0)^2 = (\sqrt{4 - a})^2,$$ which represents a circle for all $a \le 4$.

EXAMPLE 48 *Show that $\left|\frac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.*

SOLUTION Let $z = x + iy$. Then,

$$\left|\frac{z-2}{z-3}\right| = 2$$

$$\Rightarrow \quad \left|\frac{(x-2)+iy}{(x-3)+iy}\right| = 2$$

$$\Rightarrow \quad |(x-2)+iy| = 2\,|(x-3)+iy|$$

$$\Rightarrow \quad \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2}$$

$$\Rightarrow \quad (x-2)^2 + y^2 = 4\{(x-3)^2 + y^2\}$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

$$\Rightarrow \quad \left(x - \frac{10}{3}\right)^2 + (y - 0)^2 = \left(\frac{2}{3}\right)^2,$$ which represents a circle with centre at $(10/3, 0)$ and radius $2/3$

EXAMPLE 49 *Find a complex number z satisfying the equation $z + \sqrt{2}\,|z + 1| + i = 0$.*
[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then,

$$z + \sqrt{2}\,|z+1| + i = 0$$

$$\Rightarrow \quad x + iy + \sqrt{2}\,|(x+1) + iy| + i = 0$$

$$\Rightarrow \quad x + \sqrt{2}\sqrt{(x+1)^2 + y^2} + (y+1)\,i = 0$$

$$\Rightarrow \quad x + \sqrt{2(x+1)^2 + 2y^2} = 0 \text{ and } (y+1) = 0$$

$$\Rightarrow \quad x + \sqrt{2(x+1)^2 + 2y^2} = 0 \text{ and } y = -1$$

$$\Rightarrow \quad x + \sqrt{2(x+1)^2 + 2} = 0 \text{ and } y = -1$$

$$\Rightarrow \quad \sqrt{2(x+1)^2 + 2} = -x \text{ and } y = -1$$

$$\Rightarrow \quad 2(x+1)^2 + 2 = x^2 \text{ and } y = -1$$

$$\Rightarrow \quad x^2 + 4x + 4 = 0 \text{ and } y = -1$$

$$\Rightarrow \quad (x+2)^2 = 0 \text{ and } y = -1$$

$$\Rightarrow \quad x = -2 \text{ and } y = -1$$

Hence, $z = x + iy = -2 - i$.

EXAMPLE 50 *Let z_1 and z_2 be two complex numbers such that*

$$\left|1-\bar{z}_1 z_2\right|^2-\left|z_1-z_2\right|^2=k\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$$

Find the value of k.

SOLUTION We have,

$$|1-\bar{z}_1 z_2|^2-|z_1-z_2|^2$$

$$=(1-\bar{z}_1 z_2)\,\overline{(1-\bar{z}_1 z_2)}-(z_1-z_2)\,\overline{(z_1-z_2)}$$

$$=(1-\bar{z}_1 z_2)\,(1-z_1 \bar{z}_2)-(z_1-z_2)\,(\bar{z}_1-\bar{z}_2)$$

$$=(1-\bar{z}_1 z_2-z_1 \bar{z}_2+\bar{z}_1 z_2 z_1 \bar{z}_2)-(z_1 \bar{z}_1-z_1 \bar{z}_2-\bar{z}_1 z_2+z_2 \bar{z}_2)$$

$$=1-\bar{z}_1 z_2-z_1 \bar{z}_2+(z_1 \bar{z}_1)(z_2 \bar{z}_2)-\left(|z_1|^2-z_1 \bar{z}_2-\bar{z}_1 z_2+|z_2|^2\right)$$

$$=1-\bar{z}_1 z_2-z_1 \bar{z}_2+|z_1|^2|z_2|^2-|z_1|^2+z_1 \bar{z}_2+\bar{z}_1 z_2-|z_2|^2$$

$$=1-|z_1|^2-|z_2|^2+|z_1|^2|z_2|^2=(1-|z_1|^2)(1-|z_2|^2)$$

$\therefore$ $$|1-\bar{z}_1 z_2|^2-|z_1-z_2|^2=k(1-|z|^2)(1-|z_2|^2)$$

$\Rightarrow$ $$\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)=k\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$$

$\Rightarrow$ $k=1.$

EXERCISE 13.2

LEVEL-1

1. Express the following complex numbers in the standard form $a+ib$:

(i) $(1+i)(1+2i)$

(ii) $\dfrac{3+2i}{-2+i}$

(iii) $\dfrac{1}{(2+i)^2}$

(iv) $\dfrac{1-i}{1+i}$

(v) $\dfrac{(2+i)^3}{2+3i}$

(vi) $\dfrac{(1+i)(1+\sqrt{3}i)}{1-i}$

(vii) $\dfrac{2+3i}{4+5i}$

(viii) $\dfrac{(1-i)^3}{1-i^3}$

(ix) $(1+2i)^{-3}$

(x) $\dfrac{3-4i}{(4-2i)(1+i)}$

(xi) $\left(\dfrac{1}{1-4i}-\dfrac{2}{1+i}\right)\left(\dfrac{3-4i}{5+i}\right)$ [NCERT]

(xii) $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i}$ [NCERT]

2. Find the real values of x and y, if

(i) $(x+iy)(2-3i)=4+i$

(ii) $(3x-2iy)(2+i)^2=10(1+i)$

(iii) $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$

(iv) $(1+i)(x+iy)=2-5i$

3. Find the conjugates of the following complex numbers:

(i) $4-5i$

(ii) $\dfrac{1}{3+5i}$

(iii) $\dfrac{1}{1+i}$ (iv) $\dfrac{(3-i)^2}{2+i}$

(v) $\dfrac{(1+i)(2+i)}{3+i}$ (vi) $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ [NCERT]

4. Find the multiplicative inverse of the following complex numbers:
(i) $1-i$ (ii) $(1+i\sqrt{3})^2$
(iii) $4-3i$ (iv) $\sqrt{5}+3i$

5. If $z_1 = 2-i, z_2 = 1+i$, find $\left|\dfrac{z_1+z_2+1}{z_1-z_2+i}\right|$. [NCERT]

6. If $z_1 = 2-i, z_2 = -2+i$, find
(i) $\text{Re}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$ (ii) $\text{Im}\left(\dfrac{1}{z_1 \bar{z}_1}\right)$ [NCERT]

7. Find the modulus of $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}$ [NCERT]

8. If $x+iy = \dfrac{a+ib}{a-ib}$, prove that $x^2+y^2=1$ [NCERT]

9. Find the least positive integral value of n for which $\left(\dfrac{1+i}{1-i}\right)^n$ is real.

10. Find the real values of θ for which the complex number $\dfrac{1+i\cos\theta}{1-2i\cos\theta}$ is purely real.

11. Find the smallest positive integer value of n for which $\dfrac{(1+i)^n}{(1-i)^{n-2}}$ is a real number.

12. If $\left(\dfrac{1+i}{1-i}\right)^3 - \left(\dfrac{1-i}{1+i}\right)^3 = x+iy$, find (x, y) [NCERT EXEMPLAR]

13. If $\dfrac{(1+i)^2}{2-i} = x+iy$, find $x+y$. [NCERT EXEMPLAR]

14. If $\left(\dfrac{1-i}{1+i}\right)^{100} = a+ib$, find (a, b). [NCERT EXEMPLAR]

15. If $a = \cos\theta + i\sin\theta$, find the value of $\dfrac{1+a}{1-a}$. [NCERT EXEMPLAR]

LEVEL-2

16. Evaluate the following:
(i) $2x^3+2x^2-7x+72$, when $x = \dfrac{3-5i}{2}$
(ii) $x^4-4x^3+4x^2+8x+44$, when $x = 3+2i$
(iii) $x^4+4x^3+6x^2+4x+9$, when $x = -1+i\sqrt{2}$
(iv) $x^6+x^4+x^2+1$, when $x = \dfrac{1+i}{\sqrt{2}}$.
(v) $2x^4+5x^3+7x^2-x+41$, when $x = -2-\sqrt{3}i$ [NCERT EXEMPLAR]

17. For a positive integer n, find the value of $(1-i)^n\left(1-\dfrac{1}{i}\right)^n$. [NCERT EXEMPLAR]

18. If $(1+i)z=(1-i)\bar{z}$, then show that $z=-i\bar{z}$. [NCERT EXEMPLAR]
19. Solve the system of equations $\text{Re}(z^2)=0, |z|=2$. [NCERT EXEMPLAR]
20. If $\frac{z-1}{z+1}$ is purely imaginary number $(z \neq -1)$, find the value of $|z|$. [NCERT EXEMPLAR]
21. If z_1 is a complex number other than -1 such that $|z_1|=1$ and $z_2=\frac{z_1-1}{z_1+1}$, then show that the real parts of z_2 is zero.
22. If $|z+1|=z+2(1+i)$, find z.
23. Solve the equation $|z|=z+1+2i$. [NCERT EXEMPLAR]
24. What is the smallest positive integer n for which $(1+i)^{2n}=(1-i)^{2n}$?
25. If z_1, z_2, z_3 are complex numbers such that $|z_1|=|z_2|=|z_3|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$, then find the value of $|z_1+z_2+z_3|$. [NCERT EXEMPLAR]
26. Find the number of solutions of $z^2+|z|^2=0$. [NCERT EXEMPLAR]

ANSWERS

1. (i) $-1+3i$ (ii) $-\frac{4}{5}-\frac{7}{5}i$ (iii) $\frac{3}{25}-\frac{4}{25}i$ (iv) $-i$
(v) $\frac{37}{13}+\frac{16}{13}i$ (vi) $-\sqrt{3}+i$ (vii) $\frac{23}{41}+\frac{2}{41}i$ (viii) $-2+0i$
(ix) $\frac{-11}{125}+\frac{2i}{125}$ (x) $\frac{1}{4}-\frac{3}{4}i$ (xi) $\frac{307}{442}+i\frac{599}{442}$ (xii) $1+2\sqrt{2}i$
2. (i) $x=\frac{5}{13}, y=\frac{14}{13}$ (ii) $x=\frac{14}{15}, y=\frac{1}{5}$ (iii) $x=3, y=-1$ (iv) $x=-\frac{3}{2}, y=-\frac{7}{2}$
3. (i) $4+5i$ (ii) $\frac{1}{34}(3+5i)$ (iii) $\frac{1}{2}+\frac{1}{2}i$ (iv) $2+4i$
(v) $\frac{3}{5}-\frac{4}{5}i$ (vi) $\frac{63}{25}+\frac{16}{25}i$
4. (i) $\frac{1}{2}+\frac{1}{2}i$ (ii) $-\frac{1}{8}-i\frac{\sqrt{3}}{8}$ (iii) $\frac{4}{25}+\frac{3}{25}i$ (iv) $\frac{\sqrt{5}}{14}-\frac{3i}{14}$
5. $\frac{4}{\sqrt{2}}$ 6. (i) $-\frac{2}{5}$ (ii) 0 7. 2 9. $n=2$
10. $\theta=2n\pi\pm\frac{\pi}{2}, n\in Z$ 11. 1 12. $(0,-2)$ 13. $\frac{2}{5}$ 14. $(1,0)$
15. $i\cot\frac{\theta}{2}$ 16. (i) 4 (ii) 5 (iii) 12 (iv) 0 (v) 6
17. 2^n 19. $\sqrt{2}(1\pm i), \sqrt{2}(-1\pm i)$ 20. 1 22. $\frac{1}{2}-2i$
23. $\frac{3}{2}-2i$ 24. $n=2$ 25. 1
26. Infinitely many solutions of the form $z=0+iy,\ y\in R$.

HINTS TO NCERT & SELECTED PROBLEMS

1. (xi) $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)=\frac{1+i-2(1-4i)}{(1-4i)(1+i)}\times\frac{3-4i}{5+i}=\frac{-1+9i}{5-3i}\times\frac{3-4i}{5+i}$

$$= \frac{(-1+9i)(3-4i)}{(5-3i)(5+i)} = \frac{33+31i}{28-10i} = \frac{33+31i}{28-10i} \times \frac{28+10i}{28+10i} = \frac{614+1198i}{784+100} = \frac{307}{442} + i\frac{599}{442}$$

(xii) $\frac{5+\sqrt{2}\,i}{1-\sqrt{2}\,i} = \frac{5+\sqrt{2}\,i}{1-\sqrt{2}\,i} \times \frac{1+\sqrt{2}\,i}{1+\sqrt{2}\,i} = \frac{3+6\sqrt{2}\,i}{1+2} = 1+2\sqrt{2}\,i$

3. (vi) $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \frac{12+5i}{4+3i} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i} = \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i$

5. We have, $z_1 = 2-i$ and $z_2 = 1+i$

$\therefore \quad z_1 + z_2 = 3$ and $z_1 - z_2 = 1-2i$

So, $\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right| = \left|\frac{3+1}{1-2i+i}\right| = \frac{4}{|1-i|} = \frac{4}{\sqrt{2}}$

6. (i) We have, $z_1 = 2-i$ and $z_2 = -2+i$

$\therefore \quad z_1 z_2 = (2-i)(-2+i) = -3+4i$

$\Rightarrow \quad \frac{z_1 z_2}{\bar{z}_1} = \frac{-3+4i}{2+i} = \frac{-3+4i}{2+i} \times \frac{2-i}{2-i} = \frac{-2+11i}{4+1} = \frac{-2}{5} + \frac{11}{5}i$

$\therefore \quad \mathrm{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = \frac{-2}{5}$

(ii) $z_1 = 2-i \Rightarrow \bar{z}_1 = 2+i$

$\therefore \quad z_1 \bar{z}_1 = (2-i)(2+i) = 5$

So, $\mathrm{Im}\left(\frac{1}{z_1 \bar{z}_1}\right) = \mathrm{Im}\left(\frac{1}{5} + 0i\right) = 0$

7. $z = \frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)} = \frac{4i}{1-i^2} = \frac{4i}{1+1} = 2i$

$\therefore \quad |z| = 2$

8. We have, $x+iy = \frac{a+ib}{a-b}$

$\Rightarrow \quad \overline{x+iy} = \overline{\left(\frac{a+ib}{a-ib}\right)} \Rightarrow x-iy = \frac{a-ib}{a+ib}$

$\therefore \quad (x+iy)(x-iy) = \frac{a+ib}{a-ib} \times \frac{a-ib}{a+ib}$

$\Rightarrow \quad x^2+y^2 = \frac{a^2+b^2}{a^2+b^2} \Rightarrow x^2+y^2 = 1.$

11. We have,

$$\frac{(1+i)^n}{(1-i)^{n-2}} = \frac{(1+i)^n}{(1-i)^n}(1-i)^2 = \left(\frac{1+i}{1-i}\right)^n (1-2i+i^2) = -2i\left\{\frac{(1+i)^2}{(1+i)(1-i)}\right\}^n$$

$$= -2i\left(\frac{1+2i+i^2}{1-i^2}\right)^n = -2i\left(\frac{1+2i-1}{2}\right)^n = -2i\,(i)^n = -2i^{n+1}$$

$\therefore \quad \frac{(1+i)^n}{(1-i)^{n-2}}$ is real $\Rightarrow -2i^{n+1}$ is real $\Rightarrow n+1 = 2, 4, 6 \Rightarrow n = 1, 3, 5, \ldots$

Hence, the least value of n is 1.

12. We have,

$$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x+iy$$

$$\Rightarrow \; i^3 - (-i)^3 = x+iy \qquad \left[\because \frac{1+i}{1-i} = \frac{(1+i)^2}{1-i^2} = \frac{2i}{2} = i \therefore \frac{1-i}{1+i} = \frac{1}{i} = -i\right]$$

$$\Rightarrow \; 2i^3 = x+iy \Rightarrow 0-2i = x+iy \Rightarrow x=0, y=-2$$

13. We have,

$$\frac{(1+i)^2}{2-i} = x+iy$$

$$\Rightarrow \frac{1+2i+i^2}{2-i} = x+iy$$

$$\Rightarrow \frac{2i}{2-i} = x+iy$$

$$\Rightarrow \frac{2i\,(2+i)}{(2-i)\,(2+i)} = x+iy$$

$$\Rightarrow \frac{4i+2i^2}{4-i^2} = x+iy$$

$$\Rightarrow -\frac{2}{5}+\frac{4}{5}i = x+iy \Rightarrow x=-\frac{2}{5} \text{ and } y=\frac{4}{5} \Rightarrow x+y=\frac{2}{5}$$

14. We have,

$$\left(\frac{1-i}{1+i}\right)^{100} = a+ib$$

$$\Rightarrow (-i)^{100} = a+ib \qquad \left[\because \frac{1-i}{1+i} = \frac{(1-i)^2}{1-i^2} = \frac{1-2i+i^2}{2} = -i\right]$$

$$\Rightarrow 1 = a+ib \Rightarrow 1+0i = a+ib \Rightarrow a=1, b=0$$

15. We have,

$$a = \cos\theta + i\sin\theta$$

$$\therefore \frac{1+a}{1-a} = \frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} = \left(\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}\right)\times\left(\frac{1-\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}\right)$$

$$\Rightarrow \frac{1+a}{1-a} = \frac{(1-\cos^2\theta-\sin^2\theta)+2i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta} = \frac{i\sin\theta}{1-\cos\theta} = \frac{2i\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = i\cot\frac{\theta}{2}$$

17. $(1-i)^n\left(1-\frac{1}{i}\right)^n = (1-i)^n(1+i)^n = \{(1-i)\,(1+i)^n\} = (1-i^2)^n = 2^n$

18. We have,

$$(1+i)\,z = (1-i)\,\bar{z}$$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{1-i}{1+i}\times\frac{1-i}{1-i} = \frac{1-2i+i^2}{1+i^2} = -i$$

$$\Rightarrow z = -i\bar{z}$$

19. Let $z = x+iy$. Then, $z^2 = x^2-y^2+2ixy$ and $|z| = \sqrt{x^2+y^2}$.

$$\therefore \operatorname{Re}(z^2) = 0 \text{ and } |z| = 2$$

$$\Rightarrow x^2-y^2 = 0 \text{ and } x^2+y^2 = 4$$

$$\Rightarrow x^2 = y^2 = 2$$

$$\Rightarrow x = \pm\sqrt{2}, y = \pm\sqrt{2}$$

$\therefore \quad z = \pm\sqrt{2} \pm \sqrt{2}i$

20. Let $z = x + iy$. Then,

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

If $\frac{z-1}{z+1}$ is purely imaginary, then

$$\text{Re}\left(\frac{z-1}{z+1}\right) = 0 \Rightarrow \frac{x^2+y-1}{(x+1)^2+y^2} = 0 \Rightarrow x^2+y^2 = 1 \Rightarrow |z| = 1$$

21. Let $z_1 = x + iy$. Then,

$$z_2 = \frac{z_1-1}{z_1+1} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

$$\Rightarrow \text{Re}(z_2) = \frac{x^2+y^2-1}{(x+1)^2+y^2} = 0 \qquad [\because |z_1| = 1 \Rightarrow x^2+y^2 = 1]$$

22. Let $z = x + iy$. Then, $z + 1 = (x+1) + iy$ and $z + 2(1+i) = (x+2) + i(y+2)$

$\therefore \quad |z+1| = z + 2(1+i)$

$\Rightarrow \sqrt{(x+1)^2+y^2} = (x+2) + i(y+2)$

$\Rightarrow \sqrt{(x+1)^2+y^2} = x+2$ and $y+2 = 0$

$\Rightarrow (x+1)^2+y^2 = (x+2)^2$ and $y = -2$

$\Rightarrow y^2 = 2x+3$ and $y = -2$

$\Rightarrow 4 = 2x+3$ and $y = -2$

$\Rightarrow x = \frac{1}{2}$ and $y = -2$

$\therefore \quad z = x + iy = \frac{1}{2} - 2i$

23. Let $z = x + iy$. Then, $|z| = \sqrt{x^2+y^2}$ and $z + 1 + 2i = (x+1) + i(y+2)$.

$\therefore \quad |z| = z + 1 + 2i$

$\Rightarrow \sqrt{x^2+y^2} = (x+1) + i(y+2)$

$\Rightarrow \sqrt{x^2+y^2} = x+1$ and $y+2 = 0$

$\Rightarrow x^2+y^2 = (x+1)^2$ and $y = -2$

$\Rightarrow x^2+4 = (x+1)^2$ and $y = -2$

$\Rightarrow x = \frac{3}{2}$ and $y = -2$

Hence, $z = x + iy = \frac{3}{2} - 2i$

24. $(1+i)^{2n} = (1-i)^{2n}$.

$\Rightarrow \{(1+i)^2\}^n = \{(1-i)^2\}^n$

$\Rightarrow (1+2i+i^2)^n = (1-2i+i^2)^n$

$\Rightarrow (2i)^n = (-2i)^n$

$\Rightarrow i^n = (-1)^n i^n$

$\Rightarrow (-1)^n = 1 \Rightarrow n$ is a multiple of 2.

25. Proceed as in Example No. 37.

26. Let $z = x + iy$. Then, $z^2 = x^2 - y^2 + 2ixy$ and $|z|^2 = x^2 + y^2$.

$\therefore \quad z^2 + |z|^2 = 0$

$\Rightarrow \quad x^2 - y^2 + 2i\,xy + x^2 + y^2 = 0$

$\Rightarrow \quad 2x^2 + 2ixy = 0 \Rightarrow 2x^2 = 0$ and $2xy = 0 \Rightarrow x = 0$ and $y \in R$

$\therefore \quad z = 0 + iy$, where $y \in R$.

13.13 SQUARE ROOTS OF A COMPLEX NUMBER

Let $a + ib$ be a complex number such that $\sqrt{a+ib} = x + iy$, where x and y are real numbers. Then,

$$\sqrt{a+ib} = x + iy$$

$$\Rightarrow \quad (a+ib) = (x+iy)^2$$

$$\Rightarrow \quad a + ib = (x^2 - y^2) + 2i\,xy$$

On equating real and imaginary parts, we get

$$x^2 - y^2 = a \qquad \text{...(i)}$$

and, $$2xy = b \qquad \text{...(ii)}$$

Now, $$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$$\Rightarrow \quad (x^2+y^2)^2 = a^2 + b^2$$

$$\Rightarrow \quad (x^2+y^2) = \sqrt{a^2+b^2} \qquad [\because x^2 + y^2 > 0] \qquad \text{...(iii)}$$

Solving the equations (i) and (ii), we get

$$x^2 = \frac{1}{2}\left\{\sqrt{a^2+b^2} + a\right\} \text{ and } y^2 = \frac{1}{2}\left\{\sqrt{a^2+b^2} - a\right\}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2} + a\right\}} \text{ and } y = \pm\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2} - a\right\}}$$

If b is positive, then by equation (ii), x and y are of the same sign.

$$\therefore \quad \sqrt{a+ib} = \pm\left[\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2} + a\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2} - a\right\}}\right]$$

If b is negative, then by equation (ii), x and y are of different signs.

$$\therefore \quad \sqrt{a+ib} = \pm\left[\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2} + a\right\}} - i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2} - a\right\}}\right]$$

<u>REMARK</u> *It is evident from the above discussion that for any complex number z, we have*

(i) $$\sqrt{z} = \pm\left\{\sqrt{\frac{|z| + \mathrm{Re}(z)}{2}} + i\sqrt{\frac{|z| - \mathrm{Re}(z)}{2}}\right\}, \text{ if } \mathrm{Im}(z) > 0$$

(ii) $$\sqrt{z} = \pm\left\{\sqrt{\frac{|z| + \mathrm{Re}(z)}{2}} - i\sqrt{\frac{|z| - \mathrm{Re}(z)}{2}}\right\}, \text{ if } \mathrm{Im}(z) < 0$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the square roots of the following:*

(i) $7 - 24i$ (ii) $5 + 12i$

SOLUTION (i) Let $\sqrt{7 - 24i} = x + iy$. Then,

$$\sqrt{7 - 24i} = x + iy$$

$\Rightarrow \quad 7-24i = (x+iy)^2$

$\Rightarrow \quad 7-24i = (x^2-y^2)+2ixy$

$\Rightarrow \quad x^2-y^2 = 7 \quad$ ...(i) $\quad$ and, $\quad 2xy = -24 \quad$ ...(ii)

Now, $\quad (x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$

$\Rightarrow \quad (x^2+y^2)^2 = 49+576 = 625$

$\Rightarrow \quad x^2+y^2 = 25 \quad [\because x^2+y^2>0] \quad$ ...(iii)

On solving (i) and (iii), we get

$$x^2 = 16 \text{ and } y^2 = 9 \Rightarrow x = \pm 4 \text{ and } y = \pm 3$$

From (ii) we observe that $2xy$ is negative. So, x and y are of opposite signs.

$\therefore \quad (x = 4 \text{ and } y = -3)$ or, $(x = -4 \text{ and } y = 3)$

Hence, $\sqrt{7-24i} = \pm(4-3i)$

ALITER Let $z = 7-24i$. Then, $\mathrm{Re}(z) = 7$ and $|z| = \sqrt{49+576} = 25$

$$\therefore \quad \sqrt{7-24i} = \pm\left\{\sqrt{\frac{|z|+\mathrm{Re}(z)}{2}} - i\sqrt{\frac{|z|-\mathrm{Re}(z)}{2}}\right\} \quad [\because \mathrm{Im}(z)<0]$$

$$\Rightarrow \quad \sqrt{7-24i} = \pm\left\{\sqrt{\frac{25+7}{2}} - i\sqrt{\frac{25-7}{2}}\right\} = \pm(4-3i)$$

(ii) Let $\sqrt{5+12i} = x+iy$. Then,

$\sqrt{5+12i} = x+iy$

$\Rightarrow \quad 5+12i = (x+iy)^2$

$\Rightarrow \quad 5+12i = (x^2-y^2)+2ixy$

$\Rightarrow \quad x^2-y^2 = 5 \quad$ ...(i) $\quad$ and, $\quad 2xy = 12 \quad$ ...(ii)

Now, $\quad (x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$

$\Rightarrow \quad (x^2+y^2)^2 = 5^2+12^2 = 169$

$\Rightarrow \quad x^2+y^2 = 13 \quad [\because x^2+y^2>0] \quad$ ...(iii)

On solving (i) and (iii), we get

$$x^2 = 9 \text{ and } y^2 = 4 \Rightarrow x = \pm 3 \text{ and } y = \pm 2$$

From (ii) we observe that $2xy$ is positive. So, x and y are of the same sign.

$\therefore \quad (x = 3 \text{ and } y = 2)$ or, $(x = -3 \text{ and } y = -2)$

Hence, $\sqrt{5+12i} = \pm(3+2i)$.

ALITER Let $z = 5+12i$. Then, $\mathrm{Re}(z) = 5$, and $|z| = \sqrt{25+144} = 13$

$$\therefore \quad \sqrt{5+12i} = \pm\left\{\sqrt{\frac{|z|+\mathrm{Re}(z)}{2}} + i\sqrt{\frac{|z|-\mathrm{Re}(z)}{2}}\right\} \quad [\because \mathrm{Im}(z)>0]$$

$$\Rightarrow \quad \sqrt{5+12i} = \pm\left\{\sqrt{\frac{13+5}{2}} + i\sqrt{\frac{13-5}{2}}\right\} = \pm(3+2i)$$

EXAMPLE 2 *Find the square roots of* $-15-8i$.

SOLUTION Let $\sqrt{-15-8i} = x+iy$. Then,

$\sqrt{-15-8i} = x+iy$

$\Rightarrow \quad -15-8i = (x+iy)^2$

$\Rightarrow \quad -15-8i = (x^2-y^2)+2i\,xy$

$\Rightarrow \quad -15 = x^2-y^2 \quad \ldots(i) \quad$ and, $\quad 2xy = -8 \quad \ldots(ii)$

Now, $\quad (x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$

$\Rightarrow \quad (x^2+y^2)^2 = (-15)^2+64 = 289$

$\Rightarrow \quad x^2+y^2 = 17 \quad \ldots(iii)$

On solving (i) and (iii), we get

$$x^2 = 1 \text{ and } y^2 = 16 \Rightarrow x = \pm 1 \text{ and } y = \pm 4$$

From (ii), we observe that $2xy$ is negative. So, x and y are of opposite signs.

$\therefore \quad (x = 1 \text{ and } y = -4)$ or, $(x = -1 \text{ and } y = 4)$

Hence, $\sqrt{-15-8i} = \pm(1-4i)$

EXAMPLE 3 *Find the square root of i.*

SOLUTION Let $\sqrt{i} = x+iy$. Then,

$\quad \sqrt{i} = x+iy$

$\Rightarrow \quad i = (x+iy)^2$

$\Rightarrow \quad (x^2-y^2)+2i\,xy = 0+i$

$\Rightarrow \quad x^2-y^2 = 0 \quad \ldots(i) \quad$ and, $\quad 2xy = 1 \quad \ldots(ii)$

Now, $\quad (x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$

$\Rightarrow \quad (x^2+y^2)^2 = 0+1 = 1$

$\Rightarrow \quad x^2+y^2 = 1 \quad [\because x^2+y^2>0] \quad \ldots(iii)$

Solving (i) and (iii), we get

$$x^2 = 1/2 \text{ and } y^2 = 1/2 \Rightarrow x = \pm 1/\sqrt{2} \text{ and } y = \pm 1/\sqrt{2}$$

From (ii) we observe that we find that $2xy$ is positive. So, x and y are of same sign.

$\therefore \quad \left(x = \frac{1}{\sqrt{2}} \text{ and } y = \frac{1}{\sqrt{2}}\right)$ or, $\left(x = -\frac{1}{\sqrt{2}} \text{ and } y = -\frac{1}{\sqrt{2}}\right)$

Hence, $\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = \pm\frac{1}{\sqrt{2}}(1+i)$

<u>ALITER</u> Let $z = i$. Then, Re (z) = 0 and $|z| = 1$.

$\therefore \quad \sqrt{i} = \pm\left\{\sqrt{\frac{|z|+\text{Re}(z)}{2}}+i\sqrt{\frac{|z|-\text{Re}(z)}{2}}\right\} \quad [\because \text{Im}(z)>0]$

$\Rightarrow \quad \sqrt{i} = \pm\left\{\sqrt{\frac{1+0}{2}}+i\sqrt{\frac{1-0}{2}}\right\} = \pm\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}(1+i)$

EXERCISE 13.3

LEVEL-1

1. Find the square root of the following complex numbers:
   (i) $-5+12i$ (ii) $-7-24i$ (iii) $1-i$
   (iv) $-8-6i$ (v) $8-15i$ (vi) $-11-60\sqrt{-1}$
   (vii) $1+4\sqrt{-3}$ (viii) $4i$ (ix) $-i$

ANSWERS

1. (i) $\pm(2+3i)$ (ii) $\pm(3-4i)$ (iii) $\pm\left\{\left(\sqrt{\frac{\sqrt{2}+1}{2}}\right)-\left(\sqrt{\frac{\sqrt{2}-1}{2}}\right)i\right\}$

(iv) $\pm(1-3i)$ (v) $\pm\frac{1}{\sqrt{2}}(5-3i)$ (vi) $\pm(5-6i)$ (vii) $\pm(2+\sqrt{3}i)$

(viii) $\pm\sqrt{2}(1+i)$ (ix) $\pm\frac{1}{\sqrt{2}}(1-i)$

13.14 REPRESENTATIONS OF A COMPLEX NUMBER

A complex number can be represented in the following forms:

(i) Geometrical form (ii) Vectorial form (iii) Trigonometrical form or, Polar form

In this section, we shall learn about these three representations of a complex number.

13.14.1 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z = x + iy$ can be represented by a point (x, y) on the plane which is known as the Argand plane. To represent $z = x + iy$ geometrically we take two mutually perpendicular straight lines $X'OX$ and $Y'OY$. Now plot a point whose x and y coordinates are respectively the real and imaginary parts of z. This point $P(x, y)$ represents the complex number $z = x + iy$.

If a complex number is purely real, then its imaginary part is zero. Therefore, a purely real number is represented by a point on x-axis. A purely imaginary complex number is represented by a point on y-axis. That is why x-axis is known as the *real axis* and y-axis, as the imaginary axis.

Conversely, *if* $P(x, y)$ is a point in the plane, then the point $P(x, y)$ represents a complex number $z = x + iy$. The complex number $z = x + iy$ is known as the *affix* of the point P.

Thus, there exists a one-one correspondence between the points of the plane and the members (elements) of the set C of all complex numbers, *i.e.*, for every complex number $z = x + iy$ there exists uniquely a point (x, y) on the plane and for every point (x, y) of the plane there exists uniquely a complex number $z = x + iy$.

The plane in which we represent a complex number geometrically is known as the **complex plane or Argand plane or the Gaussian plane.** The point P, plotted on the Argand plane, is called the Argand diagram.

The length of the line segment OP is called the *modulus* of z and is denoted by $|z|$.

From Fig. 13.1, we have

$$OP^2 = OM^2 + MP^2$$

$$\Rightarrow \quad OP^2 = x^2 + y^2$$

$$\Rightarrow \quad OP = \sqrt{x^2+y^2}$$

Thus, $$|z| = \sqrt{x^2+y^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$$

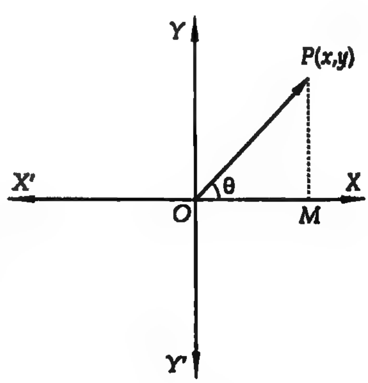


Fig. 13.1

The angle θ which OP makes with positive direction of x-axis in anticlockwise sense is called the *argument or amplitude of* z and is denoted by arg(z) or amp(z).

From Fig. 13.1, we have

$$\tan\theta = \frac{PM}{OM} = \frac{y}{x} = \frac{\text{Im}(z)}{\text{Re}(z)} \Rightarrow \theta = \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$$

This angle θ has infinitely many values differing by multiples of 2π. The unique value of θ such that $-\pi < \theta \le \pi$ is called the *principal value of the amplitude or principal argument*. This formula for determining the argument of $z = x + iy$ has severe drawback, because $z_1 = 1 + i\sqrt{3}$ and

$z_2 = -1 - i\sqrt{3}$ are two distinct complex numbers represented by two distinct points in the Argand plane but their arguments seem to be $\tan^{-1}\sqrt{3} = \pi/3$ or $4\pi/3$ which is not correct. In fact the argument is the common solution of the simultaneous trigonometric equations

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and, } \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

Since the above system of equations has infinitely many solutions. Therefore, there can be infinitely many arguments of $z = x + iy$. The argument θ which satisfies the inequality $-\pi < \theta \le \pi$ is usually known as the *principal argument* of z. The argument of z depends upon the quadrant in which the point P lies as discussed below.

13.14.2 ARGUMENT OR AMPLITUDE OF A COMPLEX NUMBER FOR DIFFERENT SIGNS OF REAL AND IMAGINARY PARTS

(i) *Argument of $z = x + iy$ when $x > 0$ and $y > 0$*: Since x and y both are positive, therefore the point $P(x, y)$ representing $z = x + iy$ in the Argand plane lies in the first quadrant. Let θ be the argument of z and let α be the acute angle satisfying $\tan\alpha = |y/x|$. Then it is evident from Fig. 13.2 that $\theta = \alpha$.

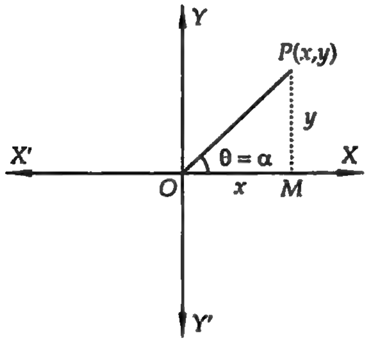


Fig. 13.2

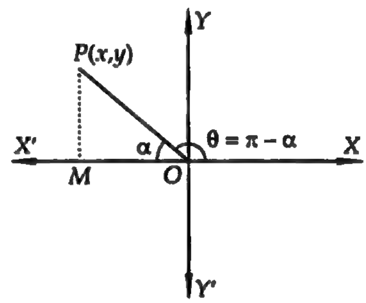


Fig. 13.3

Thus, if x and y both are positive, then the argument of $z = x + iy$ is the acute angle given by $\tan\alpha = \frac{y}{x}$.

(ii) *Argument of $z = x + iy$ when $x < 0$ and $y > 0$*: In this case, the point $P(x, y)$ representing $z = x + iy$ in the Argand plane lies in the second quadrant. Let θ be the argument of z and let α be the acute angle satisfying $\tan\alpha = |y/x|$. Then it is evident from Fig. 13.3 that $\theta = \pi - \alpha$.

Thus, if $x < 0$ and $y > 0$, then the argument of $z = x + iy$ is $\pi - \alpha$, where α is the acute angle given by $\tan\alpha = \left|\frac{y}{x}\right|$.

(iii) *Argument of $z = x + iy$ when $x < 0$ and $y < 0$*: In this case, the point $P(x, y)$ representing $z = x + iy$ lies in the third quadrant. Let θ be the argument of z and α be the acute angle given by $\tan\alpha = |y/x|$. Then from Fig. 13.4, we obtain $\theta = -(\pi - \alpha) = \alpha - \pi$.

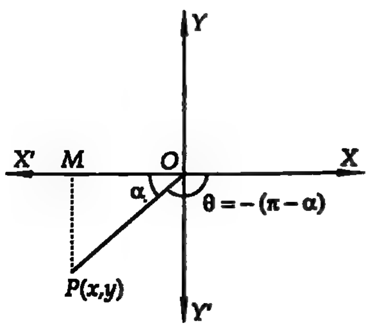


Fig. 13.4

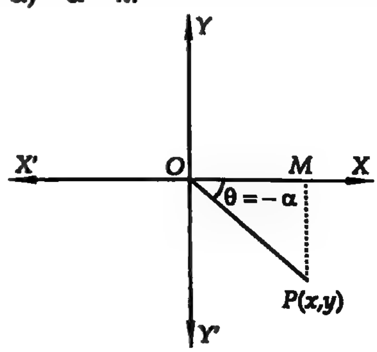


Fig. 13.5

Thus, if $x < 0$ and $y < 0$ then the argument of $z = x + iy$ is $\alpha - \pi$ where α is the acute angle given by $\tan \alpha = |y/x|$.

(iv) *Argument of $z = x + iy$ when $x > 0$ and $y < 0$:* In this case, the point $P(x, y)$ representing $z = x + iy$ lies in the fourth quadrant. Let θ be the argument of z and let α be the acute angle given by $\tan \alpha = |y/x|$. Then from Fig. 13.5, we obtain $\theta = -\alpha$.

Thus, if $x > 0$ and $y < 0$, then the argument of $z = x + iy$ is $-\alpha$ where α is the acute angle given by $\tan \alpha = |y/x|$.

The above discussion suggests us the following algorithm for finding the argument of a complex number $z = x + iy$.

ALGORITHM

STEP I *Find the acute angle α given by* $\tan \alpha = |y/x|$.

STEP II *Determine quadrant in which the point $P(x, y)$ lies.*
If $P(x, y)$ belongs to the first quadrant, then $\arg(z) = \alpha$.
If $P(x, y)$ belongs to the second quadrant, then $\arg(z) = \pi - \alpha$.
If $P(x, y)$ belongs to the third quadrant, the $\arg(z) = -(\pi - \alpha)$ or $\pi + \alpha$.
If $P(x, y)$ belongs to the fourth quadrant, then $\arg(z) = -\alpha$ or $2\pi - \alpha$.

ILLUSTRATION 1 *Find the modulus and argument of each of the following complex numbers:*

(i) $1 + i\sqrt{3}$ [NCERT] (ii) $-2 + 2i\sqrt{3}$
(iii) $-\sqrt{3} - i$ (iv) $2\sqrt{3} - 2i$

SOLUTION (i) Let $z = 1 + i\sqrt{3}$ and let α be the acute angle given by $\tan \alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan \alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

We observe that $\text{Re}(z) > 0$ and $\text{Im}(z) > 0$. So, the point representing z lies in the first quadrant.

$$\therefore \quad \arg(z) = \alpha = \frac{\pi}{3}.$$

Also, $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

(ii) Let $z = -2 + 2\sqrt{3}\,i$. Then, $|z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$.

Let α be the angle given by $\tan \alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$.

$$\tan \alpha = \left|\frac{2\sqrt{3}}{-2}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

Clearly, $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$. So, the point representing z lies in the second quadrant.

$$\therefore \quad \arg(z) = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

(iii) Let $z = -\sqrt{3} - i$. Then, $|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$.

Let α be the acute angle given by $\tan \alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$.

$$\therefore \quad \tan \alpha = \left|\frac{-1}{-\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$$

We find that $\text{Re}(z) < 0$ and $\text{Im}(z) < 0$. So, the point representing z lies in the third quadrant.

$$\therefore \quad \arg(z) = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}$$

(iv) Let $z = 2\sqrt{3} - 2i$. Then, $|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4$.

Let α be the acute angle given by $\tan\alpha \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-2}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$$

We observe that Re (z) > 0 and Im (z) < 0. So, the point representing z lies in the fourth quadrant.

$\therefore \quad \arg(z) = -\alpha = -\pi/6$

ILLUSTRATION 2 *Find the modulus and argument of the following complex numbers:*

(i) $\dfrac{1+i}{1-i}$ [NCERT] (ii) $\dfrac{1}{1+i}$ [NCERT]

SOLUTION (i) Let $z = \dfrac{1+i}{1-i}$. Then,

$$z = \frac{1+i}{1-i} = \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{1+2i+i^2}{1-i^2} = \frac{1+2i-1}{1+1} = i = 0 + i$$

$\therefore \quad |z| = \sqrt{0^2 + 1^2} = 1$

Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \frac{1}{0} = \infty$

$$\tan\alpha = \infty \Rightarrow \alpha = \frac{\pi}{2}$$

We find that $\text{Re}(z) = 0$ and $\text{Im}(z) = 1 > 0$. So the point representing z lies on y-axis. Consequently, $\arg(z) = \alpha = \frac{\pi}{2}$.

Hence, $|z| = 1$ and $\arg(z) = \frac{\pi}{2}$.

(ii) Let $z = \dfrac{1}{1+i}$. Then,

$$z = \frac{1}{1+i} = \frac{1}{1+i} \times \frac{1-i}{1-i} = \frac{1-i}{1-i^2} = \frac{1-i}{1+1} = \frac{1}{2} - \frac{1}{2}i$$

$$\therefore \quad |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-1/2}{1/2}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

We observe that $\text{Re}(z) = \frac{1}{2} > 0$ and $\text{Im}(z) = -\frac{1}{2} < 0$. So, the point representing z lies in the fourth quadrant.

$\therefore \quad \arg(z) = -\alpha = -\frac{\pi}{4}$

Hence, $|z| = \frac{1}{\sqrt{2}}$ and $\arg(z) = -\frac{\pi}{4}$.

13.14.3 VECTORIAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z = x + iy$ can be represented by the position vector OP of point $P(x, y)$ in a two dimensional plane because a complex number depends on two things viz. (i) its modulus and (ii) its argument which are also the requirements of a vector on a plane.

In Fig. 13.6, the complex number $z = x + i\,y$ is represented by the vector $\vec{OP}$ and in such a case $|z|$ is the length OP and arg (z) is the angle which the directed line OP makes with the positive direction of x-axis.

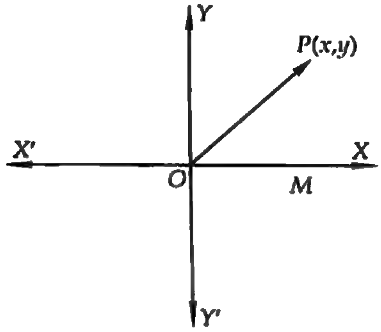


Fig. 13.6

13.14.4 POLAR OR TRIGONOMETRICAL FORM OF A COMPLEX NUMBER

Let $z = x + i\,y$ be a complex number represented by a point $P(x, y)$ in the Argand plane. Then by the geometrical representation of $z = x + i\,y$, we obtain

$$OP = |z| \text{ and, } \angle POX = \theta = \arg(z)$$

In $\Delta\, POM$, we obtain

$$\cos\theta = \frac{OM}{OP} = \frac{x}{|z|} \Rightarrow x = |z|\cos\theta$$

and, $$\sin\theta = \frac{PM}{OP} = \frac{y}{|z|} \Rightarrow y = |z|\sin\theta$$

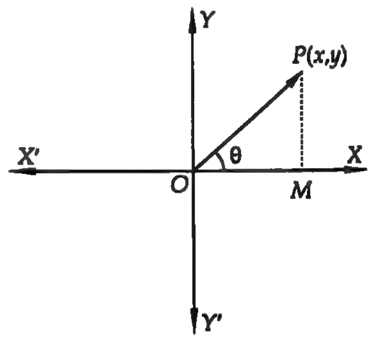


Fig. 13.7

$\therefore$ $z = x + i\,y$

$\Rightarrow$ $z = |z|\cos\theta + i|z|\sin\theta$

$\Rightarrow$ $z = |z|(\cos\theta + i\sin\theta)$

$\Rightarrow$ $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg(z)$

This form of z is called a polar form of z. *If* we use the general value of the argument of θ, then the polar form of z is given by

$$z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)], \text{ where } r = |z|,\ \theta = \arg(z) \text{ and } n \text{ is an integer.}$$

13.14.5 MULTIPLICATION OF A COMPLEX NUMBER BY IOTA

Let $z = x + i\,y$ be a complex number represented by a point $P(x, y)$ in the argand plane. Let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, $r = |z|$ and $\arg(z) = \theta$.

Now, $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $iz = ir(\cos\theta + i\sin\theta)$

$\Rightarrow$ $iz = r(-\sin\theta + i\cos\theta)$

$\Rightarrow$ $iz = r\{\cos(\pi/2 + \theta) + \sin(\pi/2 + \theta)\}$

This shows that iz is a complex number such that

$$|iz| = r = |z| \text{ and } \arg(iz) = \pi/2 + \theta = \pi/2 + \arg(z).$$

Thus, multiplication of a complex number by i results in rotating the vector joining the origin to point representing z through a right angle.

13.14.6 POLAR FORM OF A COMPLEX NUMBER FOR DIFFERENT SIGNS OF REAL AND IMAGINARY PARTS

Let $|z| = r$ and α be the acute angle given by $\tan\alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|$. Let θ be the argument of z.

CASE I *Polar form of $z = x + iy$ when $x > 0$ and $y > 0$:* In this case, we have $\theta = \alpha$.
So, the polar form of $z = x + iy$ is $z = r(\cos\alpha + i\sin\alpha)$

CASE II *Polar form of $z = x\ iy$ when $x < 0$ and $y > 0$:* In this case, we have $\theta = \pi - \alpha$.
So, the polar form of $z = x + iy$ is

$$z = r[\cos(\pi - \alpha) + i\sin(\pi - \alpha)] = r(-\cos\alpha + i\sin\alpha)$$

CASE III *Polar form of $z = x + iy$ when $x < 0$ and $y < 0$:* In this case, we have $\theta = -(\pi - \alpha)$.
So, the polar form of z is given by

$$z = r[\cos(\pi - \alpha) + i\sin(-(\pi - \alpha))] = r(-\cos\alpha - i\sin\alpha)$$

CASE IV *Polar form of $z = x + iy$ when $x > 0$ and $y < 0$:* In this case, we have $\theta = -\alpha$.
So, the polar form of z is

$$z = r[\cos(-\alpha) + i\sin(-\alpha)] = r(\cos\alpha - i\sin\alpha)$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Write the following complex numbers in the polar form:*

(i) $-3\sqrt{2} + 3\sqrt{2}\,i$ (ii) $1 + i$

(iii) $-1 - i$ [NCERT] (iv) $1 - i$ [NCERT EXEMPLAR]

SOLUTION (i) Let $z = -3\sqrt{2} + 3\sqrt{2}i$. Then, $r = |z| = \sqrt{(-3\sqrt{2})^2 + (3\sqrt{2})^2} = 6$.

Let α be the acute angle given by $\tan\alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = 1 \Rightarrow \alpha = \pi/4$$

The point representing z lies in the second quadrant. So, the argument θ of z is given by

$$\theta = \pi - \alpha = \pi - (\pi/4) = 3\pi/4.$$

Hence, the polar form of $z = -3\sqrt{2} + 3\sqrt{2}\,i$ is

$$z = r(\cos\theta + i\sin\theta) = 6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

(ii) Let $z = 1 + i$. Then, $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \frac{1}{1} = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

We find that the point (1, 1) representing z lies in first quadrant. Therefore, the argument of z is given by $\theta = \alpha = \dfrac{\pi}{4}$.

Hence, the polar form of $z = 1 + i$ is

$$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(iii) Let $z = -1 - i$. Then, $r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-1}{-1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Clearly, the point $(-1, -1)$ representing z lies in the third quadrant. Therefore, the argument of z is given by

$$\theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}.$$

Hence, the polar form of $z = -1 - i$ is

$$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left\{\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right\} = \sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

(iv) Let $z = 1 - i$. Then, $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}.$$

We find that the point $(1, -1)$ representing z lies in the fourth quadrant. Therefore, the argument of z is given by $\theta = -\alpha = -\frac{\pi}{4}$.

Hence, the polar form of $z = 1 - i$ is

$$r(\cos\theta + i\sin\theta) = \sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right\} = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

EXAMPLE 2 *Find the modulus and principal argument of* $(1 + i)$ *and hence express it in the polar form.* [NCERT]

SOLUTION Let $z = 1 + i$. Then, $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}.$$

Clearly, the point $(1, 1)$ representing $z = 1 + i$ lies in first quadrant. Therefore, $\theta = \arg(z) = \frac{\pi}{4}$.

Hence, the polar form of $z = 1 + i$ is $z = |z|(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

EXAMPLE 3 *Find the modulus and principal argument of* $-2i$.

SOLUTION Let $z = -2i = 0 + (-2)i$. Then, $|z| = \sqrt{0 + (-2)^2} = 2$.

Clearly, the point $(0, -2)$ representing $z = -2i$ lies on the negative side of imaginary axis. Therefore, principal argument of is $-\frac{\pi}{2}$.

EXAMPLE 4 *Find the modulus and principal argument of* -4. **[NCERT]**

SOLUTION Let $z = -4 + 0i$. Then, $|z| = \sqrt{(-4)^2 + 0} = 4$.

Clearly, the point $(-4, 0)$ representing $z = -4 + 0i$ lies on the negative side of real axis. Therefore, principal argument of z is π.

EXAMPLE 5 *Express the following complex numbers in the polar form:*

(i) $\dfrac{1+i}{1-i}$ (ii) $\dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$

SOLUTION (i) Let $z = \dfrac{1+i}{1-i}$, and, let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, $r = |z|$ and $\theta = \arg(z)$.

Now, $z = \dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{(1-i)(1+i)} = \dfrac{1-2i+i^2}{1-i^2} = \dfrac{1+2i-1}{1+1} = i = 0 + 1\,i$.

$\therefore \quad r = |z| = \sqrt{0+1} = 1$.

Clearly, the point (0, 1) representing $z = 0 + i$ lies on positive direction of imaginary axis. Therefore, $\arg(z) = \pi/2$.

Hence, the polar form of z is $z = 1\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right) = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$

(ii) Let $z = \dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$ and, let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, $r = |z|$ and $\theta = \arg(z)$.

Clearly, $z = \dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i} = \dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i} \cdot \dfrac{(5-\sqrt{3}\,i)}{(5-\sqrt{3}\,i)} = \dfrac{28+28\sqrt{3}\,i}{28} = 1 + i\sqrt{3}$

$\therefore \quad r = |z| = \sqrt{1+3} = 2$.

Let α be acute angle given by

$$\tan\alpha \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}. \text{ Then,}$$

$$\tan\alpha = \frac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

Clearly, the point $(1, \sqrt{3})$ representing z lies in first quadrant. Therefore, $\theta \arg(z) = \alpha = \pi/3$.

Hence, the polar form of z is $2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$.

EXAMPLE 6 *Put the complex number* $\dfrac{1+7i}{(2-i)^2}$ *in the form* $r(\cos\theta + i\sin\theta)$, *where r is a positive real number and* $-\pi < \theta \le \pi$. **[NCERT]**

SOLUTION Let $z = \dfrac{1+7i}{(2-i)^2}$. Then,

$$z = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{3-4i} = \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{-25+25i}{25} = -1 + i$$

$\therefore \quad r = |z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

Let α be the acute angle given by $\tan\alpha = \left|\dfrac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|-\frac{1}{1}\right| = 1 \Rightarrow \alpha = \pi/4$$

Clearly, the point (– 1, 1) representing z lies in the second quadrant. Therefore,

$\therefore \quad \theta = \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$

Hence, z in the polar form is given by $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

EXAMPLE 7 *Find the modulus and argument of the following complex numbers and convert them in polar form:*

(i) $\frac{1+2i}{1-3i}$ [NCERT] (ii) $\frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ [NCERT]

(iii) $\frac{1+3i}{1-2i}$ [NCERT]

SOLUTION (i) Let $z = \frac{1+2i}{1-3i}$. Then,

$$z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{(1-6)+i(2+3)}{1+9} = -\frac{1}{2} + \frac{1}{2}i$$

$$\therefore \quad r = |z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-1/2}{1/2}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

We find that $\text{Re}(z) = -\frac{1}{2} < 0$ and $\text{Im}(z) = \frac{1}{2} > 0$. So, the point representing z lies in the second quadrant.

$\therefore \quad \theta = \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence, the polar form of z is $r(\cos\theta + i\sin\theta) = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$.

(ii) Let $z = \frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$. Then,

$$z = \frac{i-1}{\frac{1}{2} + i\frac{\sqrt{3}}{2}} = \frac{2(-1+i)}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{2\{(-1+\sqrt{3}) + i(1+\sqrt{3})\}}{1+3} = \left(\frac{\sqrt{3}-1}{2}\right) + i\left(\frac{\sqrt{3}+1}{2}\right)$$

$$\therefore \quad |z| = \sqrt{\left(\frac{\sqrt{3}-1}{2}\right)^2 + \left(\frac{\sqrt{3}+1}{2}\right)^2} = \sqrt{\frac{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2}{4}} = \sqrt{\frac{2(3+1)}{4}} = \sqrt{2}$$

Let α be the acture angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{\frac{\sqrt{3}+1}{2}}{\frac{\sqrt{3}-1}{2}}\right| = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \tan\frac{5\pi}{12}$$

$\therefore \quad \alpha = \dfrac{5\pi}{12}$

Clearly, $\text{Re}(z) = \dfrac{\sqrt{3}-1}{2} > 0$ and, $\text{Im}(z) = \dfrac{\sqrt{3}+1}{2} > 0$. So, the point representing z lies in the first quadrant. Therefore, $\theta = \arg(z) = \dfrac{5\pi}{12}$.

Hence, the polar form of z is $r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right)$.

(iii) Let $z = \dfrac{1+3i}{1-2i}$. Then,

$$z = \frac{1+3i}{1-2i} = \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{(1-6)+i(3+2)}{1+4} = -1+i$$

$$\therefore \quad r = |z| = \sqrt{(-1^2 + 1^2} = \sqrt{2}$$

Let α be the acture angle given by $\tan\alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{1}{-1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

We find that $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$. So, the point representing z lies in the second quadrant.

$$\therefore \quad \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Hence, the polar form of z is $r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

LEVEL-2

EXAMPLE 8 *For any complex number z, prove that* $|\text{Re}(z)| + |\text{Im}(z) \le \sqrt{2}|z|$.

SOLUTION Let $z = r(\cos\theta + i\sin\theta)$. Then, $|z| = r$ and $\arg(z) = \theta$.

Now,

$$|\text{Re}(z)| + \text{Im}(z)| = |r\cos\theta| + |r\sin\theta|$$

$$\Rightarrow \quad |\text{Re}(z)| + \text{Im}(z)| = r\left\{|\cos\theta| + |\sin\theta|\right\} \qquad [\because r = |z| > 0]$$

$$\Rightarrow \quad \left\{|\text{Re}(z) + |\text{Im}(z)|\right\}^2 = r^2\left\{|\cos\theta| + |\sin\theta|\right\}^2$$

$$\Rightarrow \quad \left\{|\text{Re}(z) + |\text{Im}(z)|\right\}^2 = r^2\left\{1 + |2\sin\theta\cos\theta|\right\}$$

$$\Rightarrow \quad \left\{|\text{Re}(z) + |\text{Im}(z)|\right\}^2 = r^2\left\{1 + |\sin 2\theta|\right\}$$

$$\Rightarrow \quad \left\{|\text{Re}(z) + |\text{Im}(z)|\right\}^2 \le r^2(1+1) \qquad [\because |\sin 2\theta| \le 1]$$

$$\Rightarrow \quad |\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}\,r$$

$$\Rightarrow \quad |\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}|z|$$

EXAMPLE 9 *If z and w are two complex number such that $|zw| = 1$ and $\arg(z) - \arg(w) = \dfrac{\pi}{2}$, then show that $\bar{z}w = -i$.* **[NCERT EXEMPLAR]**

SOLUTION Let $|z| = r$ and $\arg(z) = \theta$. Then, $z = r(\cos\theta + \sin\theta)$.

Now, $|zw| = 1$ and $\arg(z) - \arg(w) = \dfrac{\pi}{2}$

$$\Rightarrow \quad |z||w| = 1 \text{ and } \arg(w) = \arg(z) - \frac{\pi}{2}$$

$$\Rightarrow \quad |w| = \frac{1}{r} \text{ and } \arg(w) = \theta - \frac{\pi}{2}$$

Thus, $\quad w = |w|\{\cos(\arg w) + i\sin(\arg w)\}$

$$\Rightarrow \quad w = \frac{1}{r}\left\{\cos\left(\theta - \frac{\pi}{2}\right) + i\sin\left(\theta - \frac{\pi}{2}\right)\right\}$$

$$\Rightarrow \quad w = \frac{1}{r}\{\sin\theta - i\cos\theta\} = -\frac{i}{r}(\cos\theta + i\sin\theta)$$

$$\therefore \quad \bar{z}w = r(\cos\theta - i\sin\theta) \times -\frac{i}{r}(\cos\theta + i\sin\theta)$$

$$\Rightarrow \quad \bar{z}w = -i(\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow \quad \bar{z}w = -i.$$

EXAMPLE 10 *What is the locus of z, if amplitude of $(z-2-3i)$ is $\frac{\pi}{4}$?*

SOLUTION Let $z = x + iy$. Then,

$$z - 2 - 3i = (x + iy) - 2 - 3i = (x+2) + i(y-3)$$

Let θ be the amplitude of $(x-2) + i(y-3)$. Then,

$$\tan\theta = \frac{y-3}{x-2}$$

$$\Rightarrow \quad \tan\frac{\pi}{4} = \frac{y-3}{x-2} \qquad \left[\because \theta = \frac{\pi}{4}\right]$$

$$\Rightarrow \quad 1 = \frac{y-3}{x-2}$$

$\Rightarrow \quad x - y + 1 = 0,$ which is a straight line.

Hence, the locus of z is a straight line.

EXAMPLE 11 *Show that the complex number z, satisfying $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ lies on a circle.*

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then,

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2-1+y^2)+2iy}{(x+1)^2+y^2}$$

$$\Rightarrow \quad \frac{z-1}{z+1} = \left\{\frac{x^2+y^2-1}{(x+1)^2+y^2}\right\} + i\left\{\frac{2y}{(x+1)^2+y^2}\right\}$$

Let θ be the argument of $\frac{z-1}{z+1}$. Then,

$$\tan\theta = \frac{2y/(x+1)^2+y^2}{(x^2+y^2-1)/(x+1)^2+y^2} = \frac{2y}{x^2+y^2-1}$$

But, it is given that $\arg\left(\frac{z-1}{z+1}\right)$ is $\frac{\pi}{4}$ i.e. $\theta = \frac{\pi}{4}$.

$$\therefore \quad \tan\frac{\pi}{4} = \frac{2y}{x^2+y^2-1}$$

$$\Rightarrow \quad x^2+y^2-1 = 2y$$

$$\Rightarrow \quad x^2+y^2-2y-1 = 0$$

$\Rightarrow \quad (x-0)^2+(y-1)^2=(\sqrt{2})^2$, which represents a circle.

EXAMPLE 12 *If arg (z − 1) = arg (z + 3i), then find (x − 1) : y, where z = x + iy.*

SOLUTION We have, $z=x+iy$. **[NCERT EXEMPLAR]**

$\therefore \quad z-1=(x-1)+iy$ and $z+3i=x+i(y+3)$

Let θ_1 and θ_2 be the arguments of $z-1$ and $z+3i$. Then,

$$\tan\theta_1=\frac{y}{x-1} \text{ and } \tan\theta_2=\frac{y+3}{x}$$

It is given that $\arg(z-1)=\arg(z+3i)$ i.e. $\theta_1=\theta_2$.

$\therefore \quad \tan\theta_1=\tan\theta_2$

$\Rightarrow \quad \dfrac{y}{x-1}=\dfrac{y+3}{x}$

$\Rightarrow \quad 3x-y-3=0$

$\Rightarrow \quad 3(x-1)=y$

$\Rightarrow \quad \dfrac{x-1}{y}=\dfrac{1}{3}$

$\Rightarrow \quad (x-1):y=1:3$

EXAMPLE 13 *If for complex numbers z_1 and z_2, $\arg(z_1)-\arg(z_2)=0$, then show that* $|z_1-z_2|=||z_1|-|z_2||$. **[NCERT EXEMPLAR]**

SOLUTION Let $|z|=r_1$ and $|z_2|=r_2$. It is given that $\arg(z_1)-\arg(z_2)=0$

i.e. $\quad \arg(z_1)=\arg(z_2)=\theta$ (say)

$\therefore \quad z_1=r_1(\cos\theta+i\sin\theta)$ and $z_2=r_2(\cos\theta+i\sin\theta)$

$\Rightarrow \quad z_1-z_2=(r_1-r_2)\cos\theta+i(r_1-r_2)\sin\theta$

$\Rightarrow \quad |z_1-z_2|^2=(r_1-r_2)^2\cos^2\theta+(r_1-r_2)^2\sin^2\theta$

$\Rightarrow \quad |z_1-z_2|^2=(r_1-r_2)^2(\cos^2\theta+\sin^2\theta)$

$\Rightarrow \quad |z_1-z_2|^2=(r_1-r_2)^2$

$\Rightarrow \quad |z_1-z_2|=|r_1-r_2|$

$\Rightarrow \quad |z_1-z_2|=||z_1|-|z_2||$

EXAMPLE 14 *If z, z_1 and z_2 are complex numbers, prove that:*

(i) $\arg(\bar{z})=-\arg(z)$. In general, $\arg(\bar{z})=2n\pi-\arg(z)$

(ii) $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$

(iii) $\arg(z_1\overline{z_2})=\arg(z_1)-\arg(z_2)$

(iv) $\arg\left(\dfrac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$

SOLUTION (i) Let $z=r(\cos\theta+i\sin\theta)$ be the polar form of z. Then, $|z|=r$ and $\arg(\bar{z})=\theta$.

Now, $\quad z=r(\cos\theta\, i+i\sin\theta)$

$\Rightarrow \quad \bar{z}=r(\cos\theta-i\sin\theta)$

$\Rightarrow \quad \bar{z}=r\{\cos(-\theta)+i\sin(-\theta)\}$

$\Rightarrow \quad |\bar{z}|=r$ and $\arg(\bar{z})=-\theta$

Since $\cos\theta$ and $\sin\theta$ are periodic functions with period 2π. Therefore, in general

$$\arg(\bar{z})=2n\pi-\arg(z)$$

(ii) Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$ be two complex numbers in their polar forms. Then,

$$|z_1|=r_1, |z_2|=r_2, \arg(z_1)=\theta_1 \text{ and } \arg(z_2)=\theta_2$$

$\therefore \quad z_1 z_2 = r_1 (\cos\theta_1 + i\sin\theta_1) \times r_2 (\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow \quad z_1 z_2 = r_1 r_2 \{(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)\}$

$\Rightarrow \quad z_1 z_2 = r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$

$\Rightarrow \quad |z_1 z_2| = r_1 r_2$ and, $\arg(z_1 z_2) = \theta_1 + \theta_2$

$\Rightarrow \quad |z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

REMARK *It follows from the above result that*

$$|z_1 z_2 \ldots z_n| = |z_1||z_2|\ldots|z_n|$$

and, $\quad \arg(z_1 z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$

Replacing $z_1, z_2, z_3, \ldots, z_n$ by z, we get

$$|z^n| = |z|^n \text{ and } \arg(z^n) = n \arg(z)$$

(iii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$\overline{z_2} = r_2(\cos\theta_2 - i\sin\theta_2) = r_2\{\cos(-\theta_2) + i\sin(-\theta_2)\}$

$\therefore \quad z_1 \overline{z_2} = r_1 r_2 [\cos\{\theta_1 + (-\theta_2)\} + i\sin\{\theta_1 + (-\theta_2)\}]$ [Using (ii)]

$\Rightarrow \quad z_1 \overline{z_2} = r_1 r_2 \{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$

$\Rightarrow \quad \arg(z_1 \overline{z_2}) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

(iv) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$|z_1| = r_1, |z_2| = r_2, \arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

$$\therefore \quad \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\left\{\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right\}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\left\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right\}$$

$$\Rightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

EXAMPLE 15 *Let* $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ *and* $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ *be two complex numbers. Then, prove that*

(i) $|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)$

or, $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(ii) $|z_1 - z_2|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$

or, $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

SOLUTION We have,

$z_1 = r_1 \cos\theta_1 + i\sin\theta_1$ and, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore \quad |z_1| = r_1, |z_2| = r_2, \arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

(i) We have,

$z_1 + z_2 = (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$

$\therefore \quad |z_1+z_2|^2=(r_1\cos\theta_1+r_2\cos\theta_2)^2+(r_1\sin\theta_1+r_2\sin\theta_2)^2$

$\Rightarrow \quad |z_1+z_2|^2=r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)$

$\Rightarrow \quad |z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$

(ii) We have,

$z_1-z_2=(r_1\cos\theta_1-r_2\cos\theta_2)+i(r_1\sin\theta_1-r_2\sin\theta_2)$

$\therefore \quad |z_1-z_2|^2=(r_1\cos\theta_1-r_2\cos\theta_2)^2+(r_1\sin\theta_1-r_2\sin\theta_2)^2$

$\Rightarrow \quad |z_1-z_2|^2=r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)$

$\Rightarrow \quad |z_1-z_2|^2=|z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)$

EXAMPLE 16 *For any two complex numbers* z_1 and z_2, *prove that :*

(i) $|z_1+z_2|=|z_1-z_2| \Leftrightarrow \arg(z_1)-\arg(z_2)=\frac{\pi}{2}\Leftrightarrow \frac{z_1}{z_2}$ *is purely imaginary.*

(ii) $|z_1+z_2|=|z_1||z_2| \Leftrightarrow \arg(z_1)=\arg(z_2) \Leftrightarrow \frac{z_1}{z_2}$ *is purely real.*

(iii) $|z_1+z_2|^2=|z_1|^2+|z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ *is purely imaginary*

SOLUTION Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$, $z_2=r_2(\cos\theta_2+i\sin\theta_2)$. Then, $|z_1|=r_1$, $|z_2|=r_2$, $\arg(z_1)=\theta_1$ and $\arg(z_2)=\theta_2$.

(i) We have,

$|z_1+z_2|=|z_1-z_2|$

$\Leftrightarrow \quad |z_1+z_2|^2=|z_1-z_2|^2$

$\Leftrightarrow \quad r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)=r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)$

$\Leftrightarrow \quad 4r_1r_2\cos(\theta_1-\theta_2)=0$

$\Leftrightarrow \quad \cos(\theta_1-\theta_2)=0$

$\Leftrightarrow \quad \theta_1-\theta_2=\frac{\pi}{2}$ i.e. $\arg(z_1)-\arg(z_2)=\frac{\pi}{2}$

$\Leftrightarrow \quad \arg\left(\frac{z_1}{z_2}\right)=\frac{\pi}{2} \qquad \left[\because \arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)\right]$

$\Leftrightarrow \quad \frac{z_1}{z_2}$ is purely imaginary.

(ii) We have,

$|z_1+z_2|=|z_1|+|z_2|$

$\Leftrightarrow \quad |z_1+z_2|^2=(r_1+r_2)^2 \qquad [\because |z_1|=r_1 \text{ and } |z_2|=r_2]$

$\Leftrightarrow \quad r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)=r_1^2+r_2^2+2r_1r_2$

$\Leftrightarrow \quad \cos(\theta_1-\theta_2)=1$

$\Leftrightarrow \quad \theta_1-\theta_2=0$ i.e. $\arg(z_1)-\arg(z_2)=0$ or, $\arg(z_1)=\arg(z_2)$

$\Leftrightarrow \quad \arg\left(\frac{z_1}{z_2}\right)=0 \qquad \left[\because \arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)\right]$

$\Leftrightarrow \quad \frac{z_1}{z_2}$ is purely real

(iii) We have,

$$|z_1+z_2|^2 = |z_1|^2+|z_2|^2$$

$\Leftrightarrow$ $r_1^2+r_2^2+2r_1 r_2\cos(\theta_1-\theta_2) = r_1^2+r_2^2$

$\Leftrightarrow$ $2r_1 r_2\cos(\theta_1-\theta_2) = 0$

$\Leftrightarrow$ $\cos(\theta_1-\theta_2) = 0$

$\Leftrightarrow$ $\theta_1-\theta_2 = \frac{\pi}{2}$

$\Leftrightarrow$ $\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$ $\left[\because \arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)=\theta_1-\theta_2\right]$

$\Leftrightarrow$ $\frac{z_1}{z_2}$ is purely imaginary.

EXAMPLE 17 *For any two complex numbers z_1 and z_2, prove that:*

(i) $|z_1+z_2|\le|z_1|+|z_2|$

ii) $|z_1-z_2|\le|z_1|+|z_2|$ *(Triangle inequalities)*

') $|z_1+z_2|\ge|z_1|-|z_2|$

) $|z_1-z_2|\ge|z_1|-|z_2|$

JTION (i) We have,

$$|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$$

$\cos(\theta_1-\theta_2)\le 1$

$\Rightarrow$ $2|z_1||z_2|\cos(\theta_1-\theta_2)\le 2|z_1||z_2|$ [Multiplying both sides by $2|z_1||z_2|$]

$\Rightarrow$ $|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)\le|z_1|^2+|z_2|^2+2|z_1||z_2|$

$\Rightarrow$ $|z_1+z_2|^2\le\left(|z_1|+|z_2|\right)^2$

$\Rightarrow$ $|z_1+z_2|\le|z_1|+|z_2|$

(ii) We have,

$$|z_1-z_2|^2 = |z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)$$

$\because$ $-1\le\cos(\theta_1-\theta_2)\le 1$

$\therefore$ $-1\le-\cos(\theta_1-\theta_2)\le 1$

$\Rightarrow$ $-\cos(\theta_1-\theta_2)\le 1$

$\Rightarrow$ $-2|z_1||z_2|\cos(\theta_1-\theta_2)\le 2|z_1||z_2|$

$\Rightarrow$ $|z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)\le|z_1|^2+|z_2|^2+2|z_1||z_2|$

$\Rightarrow$ $|z_1-z_2|^2\le\left(|z_1|+|z_2|\right)^2$

$\Rightarrow$ $|z_1-z_2|\le|z_1|+|z_2|$

(iii) We have,

$$|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$$

$\because$ $-1\le\cos(\theta_1-\theta_2)\le 1$

$\Rightarrow$ $\cos(\theta_1-\theta_2)\ge -1$

$\Rightarrow$ $2|z_1||z_2|\cos(\theta_1-\theta_2)\ge -2|z_1||z_2|$

$\Rightarrow$ $|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)\ge|z_1|^2+|z_2|^2-2|z_1||z_2|$

$\Rightarrow \quad |z_1 + z_2|^2 \geq \left(|z_1| - |z_2|\right)^2$

$\Rightarrow \quad |z_1 + z_2| > |z_1| - |z_2|$

(iv) We have,

$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

$\therefore \quad -1 \leq \cos(\theta_1 - \theta_2) \leq 1$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) \leq 1$

$\Rightarrow \quad -\cos(\theta_1 - \theta_2) \geq -1$

$\Rightarrow \quad -2|z_1||z_2|\cos(\theta_1 - \theta_2) \geq -2|z_1||z_2|$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2) \geq |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$

$\Rightarrow \quad |z_1 + z_2|^2 \geq \left(|z_1| - |z_2|\right)^2$

$\Rightarrow \quad |z_1 - z_2| \geq |z_1| - |z_2|$

EXAMPLE 18 *If* $z_r = \cos\left(\frac{\pi}{3^r}\right) + i \sin\left(\frac{\pi}{3^r}\right)$, $r = 1, 2, 3, \ldots$, *prove that* $z_1 z_2 z_3 \ldots z_\infty = i$.

SOLUTION We know that, if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, ... are complex numbers, then

$$z_1 z_2 z_3 \ldots z_n = r_1 r_2 r_3 \ldots r_n \{\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)\}$$

Here, $z_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$, $r = 1, 2, 3, \ldots$

$\therefore \quad |z_r| = \sqrt{\cos^2\frac{\pi}{3^r} + \sin^2\frac{\pi}{3^r}} = 1, r = 1, 2, 3, \ldots$ and, $\arg(z_r) = \frac{\pi}{3^r}, r = 1, 2, 3, \ldots$

$\therefore \quad z_1 z_2 z_3 \ldots z_n = \cos\left\{\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots \frac{\pi}{3^n}\right\} + i\sin\left\{\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots + \frac{\pi}{3^n}\right\}$

$\Rightarrow \quad z_1 z_2 z_3 \ldots z_n = \cos\left\{\dfrac{\frac{\pi}{3}\left(1 - \frac{1}{3^n}\right)}{\left(1 - \frac{1}{3}\right)}\right\} + i\sin\left\{\dfrac{\frac{\pi}{3}\left(1 - \frac{1}{3^n}\right)}{\left(1 - \frac{1}{3}\right)}\right\}$

$\Rightarrow \quad z_1 z_2 z_3 \ldots z_n = \cos\left\{\frac{\pi}{2}\left(1 - \frac{1}{3^n}\right)\right\} + i\sin\left\{\frac{\pi}{2}\left(1 - \frac{1}{3^n}\right)\right\}$

Hence, $z_1 z_2 z_3 \ldots z_\infty = \lim_{n\to\infty}(z_1 z_2 z_3 \ldots z_n)$

$$= \lim_{n\to\infty}\left[\cos\left\{\frac{\pi}{2}\left(1 - \frac{1}{3^n}\right)\right\} + i\sin\left\{\frac{\pi}{2}\left(1 - \frac{1}{3^n}\right)\right\}\right]$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i \qquad \left[\because \lim_{n\to\infty}\frac{1}{3^n} = 0\right]$$

EXAMPLE 19 *If* $x_n = \cos\frac{\pi}{2^n} + i\sin\frac{\pi}{2^n}$, *prove that* $x_1 x_2 x_3 \ldots x_\infty = -1$.

SOLUTION We have,

$$x_1 x_2 \ldots x_n = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^3}\right)\ldots\left(\cos\frac{\pi}{2^n} + i\sin\frac{\pi}{2^n}\right)$$

$$\Rightarrow \quad x_1 x_2 \dots x_n = \cos\left\{\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+\dots+\frac{\pi}{2^n}\right\}+i\sin\left\{\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+\dots+\frac{\pi}{2^n}\right\}$$

$$\Rightarrow \quad x_1 x_2 \dots x_n = \cos\left\{\frac{\frac{\pi}{2}\left(1-\frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)}\right\}+i\sin\left\{\frac{\frac{\pi}{2}\left(1-\frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)}\right\}$$

$$\Rightarrow \quad x_1 x_2 \dots x_n = \cos\left\{\pi\left(1-\frac{1}{2^n}\right)\right\}+i\sin\left\{\pi\left(1-\frac{1}{2^n}\right)\right\}$$

$$\therefore \quad x_1 x_2 x_3 \dots x_\infty = \lim_{n\to\infty}(x_1 x_2 x_3 \dots x_n) = \lim_{n\to\infty}\cos\left\{\pi\left(1-\frac{1}{2^n}\right)\right\}+i\sin\left\{\pi\left(1-\frac{1}{2^n}\right)\right\}$$

$$= \cos\pi + i\sin\pi = -1$$

EXAMPLE 20 *Let z_1 and z_2 be two complex numbers such that $\overline{z}_1 + i\,\overline{z}_2 = 0$ and $\arg(z_1 z_2) = \pi$. Then, find $\arg(z_1)$.* [NCERT EXEMPLAR]

SOLUTION It is given that

$$\overline{z}_1 + i\,\overline{z}_2 = 0$$

$$\Rightarrow \quad \overline{z}_1 = -i\,\overline{z}_2$$

$$\Rightarrow \quad \overline{\left(\overline{z}_1\right)} = \overline{\left(-i\overline{z}_2\right)} \qquad \text{[Taking conjugate of both sides]}$$

$$\Rightarrow \quad \overline{\left(\overline{z}_1\right)} = \overline{\left(-i\overline{z}_2\right)}$$

$$\Rightarrow \quad z_1 = i z_2$$

$$\Rightarrow \quad z_2 = -i z_1$$

$$\Rightarrow \quad \arg(z_2) = \arg(-i z_1)$$

$$\Rightarrow \quad \arg(z_2) = \arg(-i) + \arg(z_1)$$

$$\Rightarrow \quad \arg(z_2) = -\frac{\pi}{2} + \arg(z_1) \qquad \dots(i)$$

It is also given that

$$\arg(z_1 z_2) = \pi$$

$$\Rightarrow \quad \arg(z_1) + \arg(z_2) = \pi$$

$$\Rightarrow \quad \arg(z_1) - \frac{\pi}{2} + \arg(z_1) = \pi \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 2\arg(z_1) = \frac{3\pi}{2}$$

$$\Rightarrow \quad \arg(z_1) = \frac{3\pi}{4}$$

EXAMPLE 21 *If z_1 and z_2 both satisfy $z + \overline{z} = 2|z-1|$ and arg $(z_1 - z_2) = \frac{\pi}{4}$, then find $\mathrm{Im}(z_1 + z_2)$.* [NCERT EXEMPLAR]

SOLUTION Let $z_1 = x_1 + iy$ and $z_2 = x_2 + i y_2$.

It is given that z_1 and z_2 satisfy $z + \overline{z} = 2|z-1|$.

$$\therefore \quad z_1 + \overline{z}_1 = 2|z_1 - 1| \text{ and } z_2 + \overline{z}_2 = 2|z_2 - 1|$$

$$\Rightarrow \quad 2x_1 = 2|(x_1-1) + i y_1| \text{ and } 2x_2 = 2|(x_2-1) + i y_2|$$

$$\Rightarrow \quad x_1 = \sqrt{(x_1-1)^2 + y_1^2} \quad \text{and } x_2 = \sqrt{(x_2-1)^2 + y_2^2}$$

$\Rightarrow \quad x_1^2 = (x_1 - 1)^2 + y_1^2$ and $x_2^2 = (x_2 - 1)^2 + y_2^2$

$\Rightarrow \quad 2x_1 = 1 + y_1^2$ ...(i) and $2x_2 = 1 + y_2^2$ ...(ii)

$\Rightarrow \quad 2(x_1 - x_2) = y_1^2 - y_2^2$ [Subtracting (ii) from (i)]

$\Rightarrow \quad 2\left(\dfrac{x_1 - x_2}{y_1 - y_2}\right) = y_1 + y_2$ ...(iii)

Now, $z_1 = x_1 + i\, y_1$ and $z_2 = x_2 + i\, y_2$

$\Rightarrow \quad z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

It is given that $\arg(z_1 - z_2) = \dfrac{\pi}{4}$.

$$\arg(z_1 - z_2) = \frac{\pi}{4}$$

$\therefore \quad \tan\dfrac{\pi}{4} = \dfrac{y_1 - y_2}{x_1 - x_2}$

$\Rightarrow \quad 1 = \dfrac{y_1 - y_2}{x_1 - x_2}$

$\Rightarrow \quad \dfrac{x_1 - x_2}{y_1 - y_2} = 1$ ...(iv)

From (iii) and (iv), we obtain

$$2 = y_1 + y_2 \Rightarrow \text{Im}(z_1 + z_2) = 2.$$

EXERCISE 13.4

LEVEL-1

1. Find the modulus and argument of the following complex numbers and hence express each of them in the polar form:
   (i) $1 + i$ (ii) $\sqrt{3} + i$ [NCERT]
   (iii) $1 - i$ [NCERT] (iv) $\dfrac{1-i}{1+i}$
   (v) $\dfrac{1}{1+i}$ (vi) $\dfrac{1+2i}{1-3i}$
   (vii) $\sin 120° - i\cos 120°$ (viii) $\dfrac{-16}{1+i\sqrt{3}}$ [NCERT]
2. Write $(i^{25})^3$ in polar form. [NCERT EXEMPLAR]

LEVEL-2

3. Express the following complex numbers in the form $r(\cos\theta + i\sin\theta)$:
   (i) $1 + i\tan\alpha$ (ii) $\tan\alpha - i$
   (iii) $1 - \sin\alpha + i\cos\alpha$ (iv) $\dfrac{1-i}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$ [NCERT EXEMPLAR]
4. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\overline{z}_2$. [NCERT EXEMPLAR]
5. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, prove that
   $$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = 0.$$ [NCERT EXEMPLAR]

6. Express $\sin\frac{\pi}{5}+i\left(1-\cos\frac{\pi}{5}\right)$ in polar form. [NCERT EXEMPLAR]

ANSWERS

1. (i) $\sqrt{2}\,(\cos\pi/4+i\sin\pi/4)$ (ii) $2\,(\cos\pi/6+i\sin\pi/6)$

(iii) $\sqrt{2}\,(\cos\pi/4-i\sin\pi/4)$ (iv) $(\cos\pi/2-i\sin\pi/2)$

(v) $\frac{1}{\sqrt{2}}(\cos\pi/4-i\sin\pi/4)$ (vi) $\frac{1}{\sqrt{2}}(\cos 3\pi/4+i\sin 3\pi/4)$

(vii) $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$ (viii) $8\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

2. $\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}$

3. (i) $1+i\tan\alpha=\begin{cases}\sec\alpha\,(\cos\alpha+i\sin\alpha), & 0\le\alpha\frac{\pi}{2}\\ -\sec\alpha\,\{\cos(\alpha-\pi)+i\sin(\alpha-\pi)\}, & \frac{\pi}{2}<\alpha\le\pi\end{cases}$

(ii) $\tan\alpha-i=\begin{cases}\sec\alpha\left\{\cos\left(\alpha-\frac{\pi}{2}\right)+i\sin\left(\alpha-\frac{\pi}{2}\right)\right\}, & 0\le\alpha<\frac{\pi}{2}\\ -\sec\alpha\left\{\cos\left(\frac{\pi}{2}+\alpha\right)+i\sin\left(\frac{\pi}{2}+\alpha\right)\right\}, & \frac{\pi}{2}<\alpha\le\pi\end{cases}$

(iii) $(1-\sin\alpha)+i\cos\alpha=\begin{cases}\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right\}, & \text{if } 0\le\alpha<\frac{\pi}{2}\\ -\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)+i\sin\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)\right\}, & \text{if } \frac{\pi}{2}<\alpha<\frac{3\pi}{2}\\ -\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)+i\sin\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)\right\}, & \text{if } \frac{3\pi}{2}<\alpha<2\pi\end{cases}$

(iv) $\dfrac{1-i}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}=\sqrt{2}\left(\cos\frac{7\pi}{12}-i\sin\frac{7\pi}{12}\right)$

6. $2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)$

HINTS TO NCERT & SELECTED PROBLEMS

1. (ii) Let $z=\sqrt{3}+i$. Then, $|z|=\sqrt{(\sqrt{3})^2+(1)^2}=2$.

Let θ be the argument of z and α be the acute angle given by $\tan\alpha=\frac{|\text{Im}(z)|}{|\text{Re}(z)|}$. Then,

$$\tan\alpha=\frac{1}{\sqrt{3}}\Rightarrow\alpha=\frac{\pi}{6}$$

Clearly, z lies in the first quadrant. So, $\arg(z)=\alpha=\frac{\pi}{6}$.

(iii) Let $z=1-i$. Then, $|z|=\sqrt{1^2+(-1)^2}=\sqrt{2}$

Let α be the acute angle given by $\tan\alpha=\frac{|\text{Im}(z)|}{|\text{Re}(z)|}$. Then,

$$\tan\alpha=\frac{|-1|}{|1|}=1\Rightarrow\alpha=\frac{\pi}{4}$$

Clearly, z lies in the fourth quadrant. Therefore, $\arg(z) = -\alpha = -\frac{\pi}{4}$.

(viii) Let $z = \frac{-16}{1+i\sqrt{3}} = \frac{-16(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{-16(1-i\sqrt{3})}{1+3} = -4+4i\sqrt{3}$. Then,

$$|z| = \sqrt{(-4)^2+(4\sqrt{3})^2} = \sqrt{16+48} = 8$$

Let α the acute angle given by $\tan\alpha = \frac{|\text{Im}(z)|}{|\text{Re}(z)|}$. Then,

$$\tan\alpha = \frac{|4\sqrt{3}|}{|-4|} = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

Clearly, z lies in the second quadrant. Therefore, $\arg(z) = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

3. (i) Let $z = 1 + i\tan\alpha$. Clearly, z is meaningful for $\alpha \neq (2n-1)\frac{\pi}{2}, n \in Z$. Also, $\tan\alpha$ is a periodic function with period π. So, let us take α lying in the interval $[0, \pi/2) \cup (\pi/2, \pi]$.

Following cases arise:

CASE I: When $\alpha \in [0, \pi/2)$

We have, $z = 1 + i\tan\alpha$

$\therefore \quad |z| = \sqrt{1+\tan^2\alpha} = \sqrt{\sec^2\alpha} = |\sec\alpha| = \sec\alpha \qquad \left[\because \frac{\pi}{2} < \alpha < \pi \therefore \sec\alpha < 0\right]$

Let β be an acute angle given by $\tan\beta = \frac{|\text{Im}(z)|}{|\text{Re}(z)|}$. Then,

$$\tan\beta = |\tan\alpha| = \tan\alpha \Rightarrow \beta = \alpha$$

As z is represented by a point lying in first quadrant. Therefore, $\arg(z) = \beta = \alpha$.

So, the polar form of z is $\sec\alpha(\cos\alpha + i\sin\alpha)$

CASE II: When $\alpha \in (\pi/2, \pi]$

We have, $z = 1 + i\tan\alpha$

$\therefore \quad |z| = \sqrt{1+\tan^2\alpha} = |\sec\alpha| = -\sec\alpha \qquad \left[\because \frac{\pi}{2} < \alpha < \pi \therefore \sec\alpha < 0\right]$

Let β be an acute angle given by $\tan\beta = \frac{|\text{Im}(z)|}{|\text{Re}(z)|}$. Then,

$\tan\beta = |\tan\alpha| = -\tan\alpha \qquad [\because \alpha \in (\pi/2, \pi]]$

$\Rightarrow \quad \tan\beta = \tan(\pi - \alpha)$

$\Rightarrow \quad \beta = \pi - \alpha$

We observe that z is represented by a point in fourth quadrant. Therefore,

$\arg(z) = -\beta = \alpha - \pi$.

Thus, z in polar form is $-\sec\alpha\{\cos(\alpha-\pi) + i\sin(\alpha-\pi)\}$.

(ii) Let $z = \tan\alpha - i$. Since $\tan\alpha$ is periodic with period π. So, let us take $\alpha \in [0, \pi/2) \cup (\pi/2, \pi)]$.

CASE I: When $\alpha \in [0, \pi/2)$

We have, $z = \tan\alpha - i$

$\therefore \quad |z| = \sqrt{\tan^2\alpha + 1} = |\sec\alpha| = \sec\alpha$

Let β be the acute angle given by $\tan\beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = \cot\alpha = \tan(\pi/2 - \alpha)$$

$$\Rightarrow \quad \beta = \frac{\pi}{2} - \alpha$$

Clearly, $\operatorname{Re}(z) > 0$ and $\operatorname{Im}(z) < 0$. So, z lies in the fourth quadrant.

$$\therefore \quad \arg(z) = -\beta = \alpha - \frac{\pi}{2}$$

Thus, z in polar form is given by

$$z = \sec\alpha\left\{\cos\left(\alpha - \frac{\pi}{2}\right) + i\sin\left(\alpha - \frac{\pi}{2}\right)\right\}$$

<u>CASE II</u>: When $\alpha \in (\pi/2, \pi]$

$$z = \tan\alpha - i \Rightarrow |z| = \sqrt{\tan^2\alpha + 1} = |\sec\alpha| = -\sec\alpha$$

Let β be the acute angle given by $\tan\beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = -\cot\alpha = \tan\left(\alpha - \frac{\pi}{2}\right)$$

$$\Rightarrow \quad \beta = \alpha - \frac{\pi}{2}$$

Clearly, $\operatorname{Re}(z) < 0$ and $\operatorname{Im}(z) < 0$. So, z lies in third quadrant.

$$\therefore \quad \arg(z) = \pi + \beta = \frac{\pi}{2} + \alpha$$

Thus, the polar form of z is $-\sec\alpha\left\{\cos\left(\frac{\pi}{2} + \alpha\right) + i\sin\left(\frac{\pi}{2} + \alpha\right)\right\}$.

(iii) Let $z = (1 - \sin\alpha) + i\cos\alpha$. Since sine and cosine functions are periodic functions with period 2π. So, let us take α lying in the interval $[0, 2\pi]$.

Now, $z = 1 - \sin\alpha + i\cos\alpha$

$$\Rightarrow \quad |z| = \sqrt{(1-\sin\alpha)^2 + \cos^2\alpha} = \sqrt{2 - 2\sin\alpha} = \sqrt{2}\,\sqrt{1-\sin\alpha}$$

$$\Rightarrow \quad |z| = \sqrt{2}\sqrt{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2} = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right|$$

Let β be the acute angle given by $\tan\beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\beta = \frac{|\cos\alpha|}{|1-\sin\alpha|} = \left|\frac{\cos\alpha}{1-\sin\alpha}\right| = \left|\frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2}\right| = \left|\frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}\right|$$

$$\Rightarrow \quad \tan\beta = \left|\frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}}\right| = \left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right|$$

Following cases arise:

CASE I: When $0 \le \alpha < \dfrac{\pi}{2}$

In this case, we have

$$\cos\frac{\alpha}{2} > \sin\frac{\alpha}{2} \text{ and } \frac{\pi}{4}+\frac{\alpha}{2} \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$\therefore \quad |z| = \sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$$

and, $\tan\beta = \left|\tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)\right| = \tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right) \Rightarrow \beta = \dfrac{\pi}{4}+\dfrac{\alpha}{2}$

Clearly, z lies in the first quadrant. Therefore, $\arg(z) = \dfrac{\pi}{4}+\dfrac{\alpha}{2}$.

Hence, the polar form of z is

$$\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right\}$$

CASE II: When $\dfrac{\pi}{2} < \alpha < \dfrac{3\pi}{2}$

In this case, we have

$$\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2} \text{ and } \frac{\pi}{4}+\frac{\alpha}{2} \in \left(\frac{\pi}{2}, \pi\right)$$

$$\therefore \quad |z| = \sqrt{2}\left|\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)$$

and, $\tan\beta = \left|\tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)\right| = -\tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right) = \tan\left\{\pi-\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)\right\} = \tan\left(\dfrac{3\pi}{4}-\dfrac{\alpha}{2}\right)$

$$\Rightarrow \quad \beta = \frac{3\pi}{4}-\frac{\alpha}{2}$$

Since $1-\sin\alpha > 0$ and $\cos\alpha < 0$. So, z lies in fourth quadrant.

$$\therefore \quad \arg(z) = -\beta = \frac{\alpha}{2}-\frac{3\pi}{4}$$

Hence, the polar form of z is

$$-\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)+i\sin\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)\right\}$$

CASE III: When $\dfrac{3\pi}{2} < \alpha < 2\pi$

In this case, we have

$$\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2} \text{ and } \left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \in \left(\pi, \frac{5\pi}{4}\right)$$

$$\therefore \quad |z| = \sqrt{2}\left|\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)$$

and, $\tan\beta = \left|\tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)\right| = \tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right) = -\tan\left\{\pi-\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)\right\} = \tan\left(\dfrac{\alpha}{2}-\dfrac{3\pi}{4}\right)$

$$\Rightarrow \quad \beta = \frac{\alpha}{2}-\frac{3\pi}{4}$$

Clearly, $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$. So, z lies in the first quadrant.

$\therefore \quad \arg(z) = \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$

Hence, the polar form of z is

$$-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)\right\}$$

4. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$|z_1| = r_1$, $\arg(z_1) = \theta_1$, $|z_2| = \theta_2$ and $\arg(z_2) = \theta_2$

It is given that

$|z_2| = |z_1|$ and $\arg(z_1) + \arg(z_2) = \pi$

$\Rightarrow \quad r_1 = r_2$ and $\theta_1 + \theta_2 = \pi$

$\Rightarrow \quad r_1 = r_2$ and $\theta_1 = \pi - \theta_2$

$\therefore \quad z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\Rightarrow \quad z_1 = r_2\{\cos(\pi - \theta_2) + i\sin(\pi - \theta_2)\} = r_2(-\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow \quad z_1 = -r_2(\cos\theta_2 - i\sin\theta_2) = -\overline{z}_2$

5. Let $\arg(z_1) = \theta_1$ and $\arg(z_3) = \theta_2$

It is given that $z_2 = \overline{z}_1$ and $z_4 = \overline{z}_3$.

$\therefore \quad \arg(z_2) = -\theta_1$ and $\arg(z_4) = -\theta_2$

Hence,

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3) = \theta_1 + \theta_2 - \theta_1 - \theta_2 = 0.$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the values of the square root of i.
2. Write the values of the square root of $-i$.
3. If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, then write the value of $(x^2 + y^2)^2$.
4. If $\pi < \theta < 2\pi$ and $z = 1 + \cos\theta + i\sin\theta$, then write the value of $|z|$.
5. If n is any positive integer, write the value of $\frac{i^{4n+1} - i^{4n-1}}{2}$.
6. Write the value of $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$.
7. Write $1 - i$ in polar form.
8. Write $-1 + i\sqrt{3}$ in polar form.
9. Write the argument of $-i$. [NCERT]
10. Write the least positive integral value of n for which $\left(\frac{1+i}{1-i}\right)^n$ is real.
11. Find the principal argument of $(1 + i\sqrt{3})^2$.
12. Find z, if $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$.
13. If $|z - 5i| = |z + 5i|$, then find the locus of z.

14. If $\frac{(a^2+1)^2}{2a-i} = x+iy$, find the value of x^2+y^2.
15. Write the value of $\sqrt{-25} \times \sqrt{-9}$.
16. Write the sum of the series $i + i^2 + i^3 + \ldots$ upto 1000 terms.
17. Write the value of arg (z) + arg $(\bar{z})$.
18. If $|z+4| \le 3$, then find the greatest and least values of $|z+1|$.
19. For any two complex numbers z_1 and z_2 and any two real numbers a, b, find the value of $|az_1 - bz_2|^2 + |az_2 + bz_1|^2$.
20. Write the conjugate of $\frac{2-i}{(1-2i)^2}$.
21. If $n \in N$, then find the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$.
22. Find the real value of a for which $3i^3 - 2ai^2 + (1-a)i + 5$ is real.
23. If $|z| = 2$ and $\arg(z) = \frac{\pi}{4}$, find z.
24. Write the argument of $(1+\sqrt{3})(1+i)(\cos\theta + i\sin\theta)$.

ANSWERS

1. $\pm\frac{1}{\sqrt{2}}(1+i)$
2. $\pm\frac{1}{\sqrt{2}}(1-i)$
3. $\frac{a^2+b^2}{c^2+d^2}$
4. $-2\cos\frac{\theta}{2}$
5. i
6. -2
7. $\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$
8. $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$
9. $\frac{3\pi}{2}$ or $-\frac{\pi}{2}$
10. 2
11. $\frac{2\pi}{3}$
12. $-2\sqrt{3}+2i$
13. Real axis
14. $\frac{(a^2+1)^4}{4a^2+1}$
15. -15
16. 0
17. 0
18. 6 and 0
19. $(a^2+b^2)(|z_1|^2+|z_2|^2)$
20. $-\frac{2}{25}-\frac{11}{25}i$
21. 0
22. $a=2$
23. $\sqrt{2}(1+i)$
24. $\frac{7\pi}{12}+\theta$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The value of $(1+i)(1+i^2)(1+i^3)(1+i^4)$ is
   (a) 2 (b) 0 (c) 1 (d) i
2. If $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is a real number and $0<\theta<2\pi$, then $\theta =$
   (a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$
3. If $(1+i)(1+2i)(1+3i)\ldots(1+ni) = a+ib$, then $2\times5\times10\times\ldots\times(1+n^2)$ is equal to
   (a) $\sqrt{a^2+b^2}$ (b) $\sqrt{a^2-b^2}$ (c) a^2+b^2 (d) a^2-b^2 (e) $a+b$
4. If $\sqrt{a+ib} = x+iy$, then possible value of $\sqrt{a-ib}$ is

(a) x^2+y^2 (b) $\sqrt{x^2+y^2}$ (c) $x+iy$ (d) $x-iy$ (e) $\sqrt{x^2-y^2}$

5. If $z=\cos\frac{\pi}{4}+i\sin\frac{\pi}{6}$, then

(a) $|z| = 1$, $\arg(z)=\frac{\pi}{4}$ (b) $|z| = 1$, $\arg(z)=\frac{\pi}{6}$

(c) $|z| = \frac{\sqrt{3}}{2}$, $\arg(z)=\frac{5\pi}{24}$ (d) $|z| = \frac{\sqrt{3}}{2}$, $\arg(z)=\tan^{-1}\frac{1}{\sqrt{2}}$

6. The polar form of $(i^{25})^3$ is

(a) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$ (b) $\cos\pi+i\sin\pi$ (c) $\cos\pi-i\sin\pi$ (d) $\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}$

7. If $i^2=-1$, then the sum $i+i^2+i^3+\ldots$ upto 1000 terms is equal to

(a) 1 (b) -1 (c) i (d) 0

8. If $z=\frac{-2}{1+i\sqrt{3}}$, then the value of arg(z) is

(a) π (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$

9. If $a=\cos\theta+i\sin\theta$, then $\frac{1+a}{1-a}=$

(a) $\cot\frac{\theta}{2}$ (b) $\cot\theta$ (c) $i\cot\frac{\theta}{2}$ (d) $i\tan\frac{\theta}{2}$

10. If $(1+i)(1+2i)(1+3i)\ldots\ldots(1+ni)=a+ib$, then $2.5.10.17\ldots\ldots(1+n^2)=$

(a) $a-ib$ (b) a^2-b^2 (c) a^2+b^2 (d) none of these

11. If $\frac{(a^2+1)^2}{2a-i}=x+iy$, then x^2+y^2 is equal to

(a) $\frac{(a^2+1)^4}{4a^2+1}$ (b) $\frac{(a+1)^2}{4a^2+1}$ (c) $\frac{(a^2-1)^2}{(4a^2-1)^2}$ (d) none of these

12. The principal value of the amplitude of $(1+i)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{12}$ (c) $\frac{3\pi}{4}$ (d) π

13. The least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer, is

(a) 16 (b) 8 (c) 4 (d) 2

14. If z is a non-zero complex number, then $\left|\frac{|\bar{z}|^2}{z\bar{z}}\right|$ is equal to

(a) $\left|\frac{\bar{z}}{z}\right|$ (b) $|z|$ (c) $|\bar{z}|$ (d) none of these

15. If $a=1+i$, then a^2 equals

(a) $1-i$ (b) $2i$ (c) $(1+i)(1-i)$ (d) $i-1$.

16. If $(x+iy)^{1/3}=a+ib$, then $\frac{x}{a}+\frac{y}{b}=$

(a) 0 (b) 1 (c) -1 (d) none of these

17. $(\sqrt{-2})(\sqrt{-3})$ is equal to

(a) $\sqrt{6}$ (b) $-\sqrt{6}$ (c) $i\sqrt{6}$ (d) none of these

18. The argument of $\dfrac{1-i\sqrt{3}}{1+i\sqrt{3}}$ is

(a) 60° (b) 120° (c) 210° (d) 240°.

19. If $z=\left(\dfrac{1+i}{1-i}\right)$, then z^4 equals

(a) 1 (b) -1 (c) 0 (d) none of these

20. If $z=\dfrac{1+2i}{1-(1-i)^2}$, then arg (z) equals

(a) 0 (b) $\dfrac{\pi}{2}$ (c) π (d) none of these

21. If $z=\dfrac{1}{(2+3i)^2}$, then $|z|=$

(a) $\dfrac{1}{13}$ (b) $\dfrac{1}{5}$ (c) $\dfrac{1}{12}$ (d) none of these

22. If $z=\dfrac{1}{(1-i)(2+3i)}$, then $|z|=$

(a) 1 (b) $1/\sqrt{26}$ (c) $5/\sqrt{26}$ (d) none of these

23. If $z=1-\cos\theta+i\sin\theta$, then $|z|=$

(a) $2\sin\dfrac{\theta}{2}$ (b) $2\cos\dfrac{\theta}{2}$ (c) $2\left|\sin\dfrac{\theta}{2}\right|$ (d) $2\left|\cos\dfrac{\theta}{2}\right|$

24. If $x+iy=(1+i)(1+2i)(1+3i)$, then $x^2+y^2=$

(a) 0 (b) 1 (c) 100 (d) none of these

25. If $z=\dfrac{1}{1-\cos\theta-i\sin\theta}$, then Re (z) =

(a) 0 (b) $\dfrac{1}{2}$ (c) $\cot\dfrac{\theta}{2}$ (d) $\dfrac{1}{2}\cot\dfrac{\theta}{2}$

26. If $x+iy=\dfrac{3+5i}{7-6i}$, then $y=$

(a) 9/85 (b) $-9/85$ (c) 53/85 (d) none of these

27. If $\dfrac{1-ix}{1+ix}=a+ib$, then $a^2+b^2=$

(a) 1 (b) -1 (c) 0 (d) none of these

28. If θ is the amplitude of $\dfrac{a+ib}{a-ib}$, then $\tan\theta=$

(a) $\dfrac{2a}{a^2+b^2}$ (b) $\dfrac{2ab}{a^2-b^2}$ (c) $\dfrac{a^2-b^2}{a^2+b^2}$ (d) none of these

29. If $z=\dfrac{1+7i}{(2-i)^2}$, then

(a) $|z|=2$ (b) $|z|=\dfrac{1}{2}$ (c) amp (z) $=\dfrac{\pi}{4}$ (d) amp (z) $=\dfrac{3\pi}{4}$

30. The amplitude of $\dfrac{1}{i}$ is equal to

(a) 0 (b) $\dfrac{\pi}{2}$ (c) $-\dfrac{\pi}{2}$ (d) π.

31. The argument of $\frac{1-i}{1+i}$ is

(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) $\frac{5\pi}{2}$

32. The amplitude of $\frac{1+i\sqrt{3}}{\sqrt{3}+i}$ is

(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $-\frac{\pi}{6}$

33. The value of $(i^5+i^6+i^7+i^8+i^9)/(1+i)$ is

(a) $\frac{1}{2}(1+i)$ (b) $\frac{1}{2}(1-i)$ (c) 1 (d) $\frac{1}{2}$

34. $\frac{1+2i+3i^2}{1-2i+3i^2}$ equals

(a) i (b) -1 (c) $-i$ (d) 4

35. The value of $\frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1$ is

(a) -1 (b) -2 (c) -3 (d) -4

36. The value of $(1+i)^4+(1-i)^4$ is

(a) 8 (b) 4 (c) -8 (d) -4

37. If $z=a+ib$ lies in third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant if

(a) $a>b>0$ (b) $a<b<0$ (c) $b<a<0$ (d) $b>a>0$

38. If $f(z)=\frac{7-z}{1-z^2}$, where $z=1+2i$, then $|f(z)|$ is

(a) $\frac{|z|}{2}$ (b) $|z|$ (c) $2|z|$ (d) none of these

39. A real value of x satisfies the equation $\frac{3-4ix}{3+4ix}=a-ib$ $(a, b \in R)$, if $a^2+b^2=$

(a) 1 (b) -1 (c) 2 (d) -2

40. The complex number z which satisfies the condition $\left|\frac{i+z}{i-z}\right|=1$ lies on

(a) circle $x^2+y^2=1$ (b) the x-axis (c) the y-axis (d) the line $x+y=1$

41. If z is a complex number, then

(a) $|z|^2>|z|^2$ (b) $|z|^2=|z|^2$ (c) $|z|^2<|z|^2$ (d) $|z|^2\geq|z|^2$

42. Which of the following is correct for any two complex numbers z_1 and z_2?

(a) $|z_1 z_2|=|z_1||z_2|$
(b) $\arg(z_1 z_2)=\arg(z_1)\arg(z_2)$
(c) $|z_1+z_2|=|z_1|+|z_2|$
(d) $|z_1+z_2|\geq|z_1|+|z_2|$

43. If the complex number $z=x+iy$ satisfies the condition $|z+1|=1$, then z lies on

(a) x-axis
(b) circle with centre $(-1, 0)$ and radius 1
(c) y-axis
(d) none of these

ANSWERS

1. (b)	2. (a)	3. (c)	4. (d)	5. (d)	6. (d)	7. (d)	8. (c)
9. (c)	10. (c)	11. (a)	12. (a)	13. (b)	14. (a)	15. (b)	16. (d)
17. (b)	18. (d)	19. (a)	20. (a)	21. (a)	22. (b)	23. (c)	24. (c)
25. (b)	26. (c)	27. (a)	28. (b)	29. (d)	30. (c)	31. (a)	32. (c)
33. (a)	34. (c)	35. (b)	36. (c)	37. (c)	38. (a)	39. (a)	40. (b)
41. (b)	42. (a)	43. (b)					

SUMMARY

1. $\sqrt{-1}$ is an imaginary quantity and is denoted by i which has the following properties:
$i^2 = -1,\ i^3 = -i,\ i^4 = 1$ and, $i^{\pm n} = i^{\pm k}, n \in N$
where k is the remainder when n is denoted by 4.
2. For any positive real number a, $\sqrt{-a} = i\sqrt{a}$.
3. For any two real numbers a and b, we have
$$\sqrt{a}\sqrt{b} = \begin{cases} \sqrt{ab}, & \text{if at least one of a and b is positive} \\ -\sqrt{ab}, & \text{if } a<0, b<0. \end{cases}$$
4. If a, b are real numbers, then a number $z = a + ib$ is called a complex number, real number a is known as the real part of z and b is known as its imaginary part. We write $a = \text{Re}(z)$, $b = \text{Im}(z)$.
A complex number z is purely real iff $\text{Im}(z) = 0$ and z is purely imaginary iff $\text{Re}(z) = 0$
5. For any two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we define
Addition: $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
Subtraction: $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$
Multiplication: $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$
Reciprocal: $\dfrac{1}{z_1} = \dfrac{a_1}{a_1^2 + b_1^2} - i\dfrac{b_1}{a_1^2 + b_1^2}$
Division: $\dfrac{z_1}{z_2} = z_1\left(\dfrac{1}{z_2}\right) = (a_1 + ib_1)\left(\dfrac{a_2}{a_2^2 + b_2^2} - i\dfrac{b_2}{a_2^2 + b_2^2}\right) = \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\dfrac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$
Addition is commutative and associative. Complex number $0 = 0 + i0$ is the identity element for addition and every complex number $z = a + ib$ has its additive inverse $-z = -a - ib$.
Multiplication is also commutative and associative. Complex number $1 = 1 + 0i$ is the identity element for multiplication. Every non-zero complex number $z = a + ib$ has its multiplicative inverse $1/z$ (also known as reciprocal of z) such that $\dfrac{1}{z} = \dfrac{a - ib}{a^2 + b^2} = \dfrac{\bar{z}}{|z|^2}$.
6. The conjugate of a complex number $z = a + ib$ is denoted by $\bar{z}$ and is equal to $a - ib$.
For any three complex numbers z, z_1, z_2, we have
(i) $(\bar{\bar{z}}) = z$
(ii) $z + \bar{z} = 2\,Re(z)$
(iii) $z - \bar{z} = 2i\,Im(z)$
(iv) $z = \bar{z} \Leftrightarrow z$ is purely real
(v) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary
(vi) $z\bar{z} = \{Re(z)\}^2 + \{Im(z)\}^2 = |z|^2$
(vii) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
(viii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(ix) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

7. The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$$

If z, z_1, z_2 are three complex numbers, then

(i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\text{Re}(z) = \text{Im}(z) = 0$ (ii) $|z| = |\bar{z}| = |-z|$

(iii) $-|z| \leq \text{Re}(z) \leq |z|$; $-|z| \leq \text{Im}(z) \leq |z|$ (iv) $z\bar{z} = |z|^2$

(v) $|\text{Im}(z^n)| \leq n|\text{Im}(z)||z|^{n-1}, n \in N$ (vi) $|\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2}|z|$

8. A complex number $z = x + iy$ can be represented by a point $P(x, y)$ (see Fig. 13.8) on the plane which is known as the Argand or Gaussian or Complex plane. The length of the line segment OP is called the modulus of z and is denoted by $|z|$.

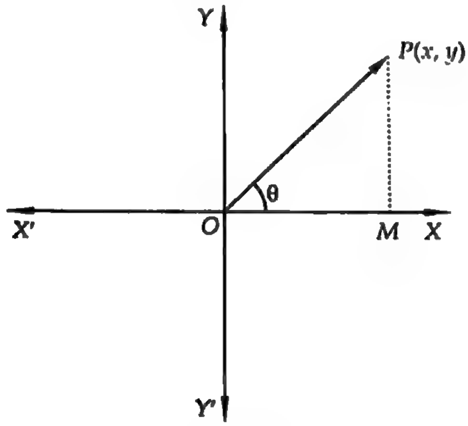


Fig. 13.8

Clearly, $|z| = \sqrt{x^2 + y^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$

The angle θ which OP makes with the positive direction of x-axis in anti-clockwise sense is called the argument or amplitude of z and is denoted by *arg* (z) or amp (z).

Clearly, $\tan\theta = \dfrac{\text{Im}(z)}{\text{Re}(z)}$.

Let $OP = r$ and $\angle XOP = \theta$. Then, $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta)$

This is known as the polar form of complex number z. The Euler's notations are

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$\therefore \quad z = r(\cos\theta + i\sin\theta)$

or, $\quad z = re^{i\theta}$, which is known as the Eulerian form of z.

CHAPTER 14

QUADRATIC EQUATIONS

14.1 INTRODUCTION

In earlier classes, we have studied about quadratic equations with real coefficients and real roots only. In this chapter, we shall study about quadratic equations with real coefficients and complex roots. We shall also discuss quadratic equations with complex coefficients and their solutions in the complex number system. But, let us first recall some definitions and results.

14.2 SOME USEFUL DEFINITIONS AND RESULTS

REAL POLYNOMIAL *Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x is a real variable. Then,*

$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ *is called a real polynomial of real variable x with real coefficients.*

For example, $x^2 - 4x + 3$, $2x^3 - 6x^2 + 11x - 5$ etc. are real polynomials.

COMPLEX POLYNOMIAL *If $a_0, a_1, a_2, \ldots, a_n$ are complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is called a complex polynomial or a polynomial of complex variable with complex coefficients.*

For example, $2x^2 - (3+7i)x + (9i-3)$, $x^3 - 5i x^2 + (1-2i)x + (3+4i)$ etc are complex polynomials.

DEGREE OF A POLYNOMIAL *A polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, real or complex, is a polynomial of degree n, if $a_n \neq 0$.*

The polynomials $2x^3 - 7x^2 + x + 5$, $(3-2i)x^2 - ix + 5$ are polynomials of degree 3 and 2 respectively.

A polynomial of second degree is generally called a quadratic polynomial and polynomials of degree 3 and 4 are known as *cubic* and *biquadratic* polynomials.

POLYNOMIAL EQUATION *If $f(x)$ is a polynomial, then $f(x) = 0$ is called a polynomial equation.*

If $f(x)$ is a quadratic polynomial, then $f(x) = 0$ is called a quadratic equation. The general form of a *quadratic equation* is $ax^2 + bx + c = 0$, $a \neq 0$.

Here, x is the variable and a, b, c are called coefficients real or imaginary.

ROOTS OF AN EQUATION *The values of the variable satisfying a given equation are called its roots.*

Thus, $x = \alpha$, is a root of the equation $f(x) = 0$, if $f(\alpha) = 0$.

For example, $x = 1$ is a root of the equation $x^3 - 6x^2 + 11x - 6 = 0$, because

$$1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 1 - 6 + 11 - 6 = 0$$

Similarly, $x = \omega$ and $x = \omega^2$ are roots of the equation $x^2 + x + 1 = 0$ as they satisfy it.

SOLUTION SET *The set of all roots of an equation, in a given domain, is called the solution set of the equation.*

For example, the set {1, 2, 3} is the solution set of the equation $x^3 - 6x^2 + 11x - 6 = 0$.

Solving an equation means finding its solution set. In other words, solving an equation is the process of obtaining its all roots.

IDENTITY *An expression involving equality and a variable is called an identity, if it is satisfied by every value of the variable.*

For example, $x^2 - 9 = (x-3)(x+3)$ is an identity as it is satisfied by every value of x.

and, $\dfrac{(x-a)(x-b)}{(c-a)(c-b)} + \dfrac{(x-b)(x-c)}{(a-b)(a-c)} + \dfrac{(x-c)(x-a)}{(b-c)(b-a)} = 1$ is also an identity as it holds good for all values of x.

FUNDAMENTAL THEOREM OF ALGEBRA *Every polynomial equation $f(x) = 0$ has at least one root, real or imaginary (complex).*

Thus, $x^7 - 3x^5 + 2x^2 = x + 2 = 0$ has at least one root. But, $f(x) = \sqrt{x} + 3 = 0$ has no root as this equation is not a polynomial equation. Fundamental theorem does not apply on this equation.

The fundamental theorem guarantees for one root of a polynomial equation. The following theorem states about the exact number of roots of a polynomial equation.

THEOREM *Every polynomial equation $f(x) = 0$ of degree n has exactly n roots real or imaginary.*

14.3 QUADRAT

The gener $ax^2 + bx + c = 0$, $a \neq 0$ where a, b, c are numbers (real

or co

T of roots of a quadratic equation.

two roots.

f the quadratic equation $ax^2 + bx + c = 0$,

l satisfy this equation.

...(i)

...(ii)

...(iii)

[$\because$ α and β are distinct $\therefore$ $\alpha - \beta \neq 0$] ...(iv)

Subtracting (iii) from (ii), we obtain

$$a(\beta^2 - \gamma^2) + b(\beta - \gamma) = 0$$

$$\Rightarrow \quad (\beta - \gamma)\{a(\beta + \gamma) + b\} = 0$$

$$\Rightarrow \quad a(\beta + \gamma) + b = 0 \qquad [\because \beta \text{ and } \gamma \text{ are distinct } \therefore \beta - \gamma \neq 0] \quad ...(v)$$

Subtracting (v) from (iv), we get : $a(\alpha - \gamma) = 0$. But, this is not possible, because α and γ are distinct and $a \neq 0$. So, their product cannot be zero.

Thus, the assumption that a quadratic equation has three distinct real roots is wrong.

Hence, a quadratic equation cannot have more than 2 roots.

Q.E.D.

REMARK *It follows from the above theorem that if a quadratic equation is satisfied by more than two values of x, then it is satisfied by every value of x and so it is an identity.*

14.4 QUADRATIC EQUATIONS WITH REAL COEFFICIENTS

In earlier classes, we have solved quadratic equations with real coefficients and real roots either by factorization or by using Sridharacharya's formula. In this section, we shall mainly concentrate on quadratic equations with real coefficients and complex roots.

Consider the quadratic equation

$$ax^2 + bx + c = 0 \qquad ...(i)$$

where $a, b, c \in R$ and $a \neq 0$.

Multiplying both sides of (i) by a, we get

$$a^2 x^2 + abx + ac = 0$$

$$\Rightarrow \quad a^2 x^2 + abx + \frac{b^2}{4} = \frac{b^2}{4} - ac$$

$$\Rightarrow \quad \left(ax + \frac{b}{2}\right)^2 = \frac{b^2 - 4ac}{4}$$

$$\Rightarrow \quad ax + \frac{b}{2} = \pm \frac{\sqrt{b^2 - 4ac}}{2}$$

$$\Rightarrow \quad ax = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4ac}}{2} \Rightarrow ax = \frac{-b \pm \sqrt{b^2 - 4ac}}{2} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$ has two roots, say α and β, given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Now, if we look at these roots, we observe that the roots depend upon the value of the quantity $b^2 - 4ac$. This quantity is generally denoted by D and is known as the *discriminant* of the quadratic equation (i). We also observe the following results:

RESULT I *If $b^2 - 4ac = 0$ i.e. $D = 0$, then $\alpha = \beta = -\frac{b}{2a}$.*

Thus, if $b^2 - 4ac = 0$, then the quadratic equation has real and equal roots each equal to $-b/2a$.

RESULT II *If a, b, c are rational numbers and $b^2 - 4ac$ is positive and a perfect square, then $\sqrt{b^2 - 4ac}$ is a rational number and hence α and β are rational and unequal.*

Thus, if $a, b, c \in Q$ and $b^2 - 4ac$ is positive and a perfect square, then roots are rational and unequal. If $a, b, c \in R$ and $b^2 - 4ac$ is positive and a perfect square, then roots are real and distinct.

RESULT III *If $b^2 - 4ac > 0$ i.e. $D > 0$ but it is not a perfect square, then roots are irrational and unequal.*

REMARK *If $a, b, c \in Q$ and $b^2 - 4ac$ is positive but not a perfect square, then roots are irrational and they always occur in conjugate pair like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, if a, b, c are irrational numbers and $b^2 - 4ac$ is positive but not a perfect square, then the roots may not occur in conjugate pairs. For example, the roots of the equation $x^2 - (5 + \sqrt{2})\, x + 5\sqrt{2} = 0$ are 5 and $\sqrt{2}$ which do not form a conjugate pair.*

RESULT IV *If $b^2 - 4ac < 0$ i.e. $D < 0$, then $4ac - b^2 > 0$ and so the roots are imaginary and are given by*

$$\alpha = \frac{-b + i\sqrt{4ac - b^2}}{2a} \text{ and } \beta = \frac{-b - i\sqrt{4ac - b^2}}{2a}$$

Clearly, α and β are complex conjugate of each other i.e. $\alpha = \overline{\beta}$ and $\overline{\alpha} = \beta$.

REMARK *If $b^2 - 4ac < 0$, then the roots are complex conjugate of each other. In fact, complex roots of an equation with real coefficients always occur in conjugate pairs like $2 + 3i$ and $2 - 3i$. However, this may not be true in case of equations with complex coefficients. For example, $x^2 - 2ix - 1 = 0$ has both roots equal to i.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the equation $4x^2 + 9 = 0$ by factorization method.*

SOLUTION We have,

$$4x^2 + 9 = 0$$
$$\Rightarrow \quad 4x^2 - 9i^2 = 0$$
$$\Rightarrow \quad (2x)^2 - (3i)^2 = 0$$
$$\Rightarrow \quad (2x + 3i)(2x - 3i) = 0$$
$$\Rightarrow \quad 2x + 3i = 0 \text{ or, } 2x - 3i = 0$$
$$\Rightarrow \quad x = -\frac{3}{2}i, \text{ or, } x = \frac{3}{2}i$$

Hence, the roots of the given equation are $\frac{3}{2}i$ and $-\frac{3}{2}i$.

EXAMPLE 2 *Solve the equation $x^2 - 4x + 13 = 0$ by factorization method.*

SOLUTION We have,

$$x^2 -$$
$$\Rightarrow \quad x^2 -$$
$$\Rightarrow \quad ($$
$$\Rightarrow \quad (x$$
$$\Rightarrow \quad (x -$$
$$\Rightarrow \quad \{(x - 2) - 3i\}\{(x -$$
$$\Rightarrow \quad (x - 2 - 3i)(x - 2 +$$
$$\Rightarrow \quad x - 2 - 3i = 0, \text{ or } x - 2 + 3i = 0$$
$$\Rightarrow \quad x = 2 + 3i, \text{ or } x = 2 - 3i$$

Hence, the roots of the given equation are $2 + 3i$ and $2 - 3i$.

EXAMPLE 3 *Solve the equation $9x^2 - 12x + 20 = 0$ by factorization method only.*

SOLUTION We have,

$$9x^2 - 12x + 20 = 0$$
$$\Rightarrow \quad 9x^2 - 12x + 4 + 16 = 0$$
$$\Rightarrow \quad (3x - 2)^2 + 16 = 0$$
$$\Rightarrow \quad (3x - 2)^2 - 16i^2 = 0$$
$$\Rightarrow \quad \{(3x - 2) + 4i\}\{(3x - 2) - 4i\} = 0$$
$$\Rightarrow \quad (3x - 2 + 4i)(3x - 2 - 4i) = 0$$
$$\Rightarrow \quad 3x - 2 + 4i = 0, \text{ or } 3x - 2 - 4i = 0$$
$$\Rightarrow \quad 3x = 2 - 4i, \text{ or } 3x = 2 + 4i$$

$$\Rightarrow \quad x = \frac{2}{3} - \frac{4}{3}i \text{ or } x = \frac{2}{3} + \frac{4}{3}i$$

Hence, the roots of the given equation are $\frac{2}{3} - \frac{4}{3}i$ and $\frac{2}{3} + \frac{4}{3}i$.

EXAMPLE 4 *Solve the quadratic equation $2x^2 - 4x + 3 = 0$ by using the general expressions for the roots of a quadratic equation.*

SOLUTION Comparing the given equation with the general form $ax^2 + bx + c = 0$, we get

$$a = 2, b = -4 \text{ and } c = 3$$

Substituting the values of a, b, c in $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$, we get

$$\alpha = \frac{4 + \sqrt{16 - 24}}{4} \quad \text{and,} \quad \beta = \frac{4 - \sqrt{16 - 24}}{4}$$

$$\Rightarrow \quad \alpha = \frac{4 + \sqrt{-8}}{4} \quad \text{and,} \quad \beta = \frac{4 - \sqrt{-8}}{4}$$

$$\Rightarrow \quad \alpha = \frac{4 + 2\sqrt{2}i}{4} \quad \text{and,} \quad \beta = \frac{4 - 2\sqrt{2}i}{4}$$

$$\Rightarrow \quad \alpha = 1 + \frac{1}{\sqrt{2}}i \quad \text{and,} \quad \beta = 1 - \frac{1}{\sqrt{2}}i$$

Hence, the roots of the given equation are $1 + \frac{1}{\sqrt{2}}i$ and $1 - \frac{1}{\sqrt{2}}i$.

EXAMPLE 5 *Solve the equation $25x^2 - 30x + 11 = 0$ by using the general expression for the roots of a quadratic equation.*

SOLUTION Comparing the given equation with the general form of the quadratic equation $ax^2 + bx + c = 0$, we get: $a = 25, b = -30$ and $c = 11$.

Substituting these values in $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$, we get

$$\alpha = \frac{30 + \sqrt{900 - 1100}}{50} \quad \text{and} \quad \beta = \frac{30 - \sqrt{900 - 1100}}{50}$$

$$\Rightarrow \quad \alpha = \frac{30 + \sqrt{-200}}{50} \quad \text{and} \quad \beta = \frac{30 - \sqrt{-200}}{50}$$

$$\Rightarrow \quad \alpha = \frac{30 + 10i\sqrt{2}}{50} \quad \text{and} \quad \beta = \frac{30 - 10i\sqrt{2}}{50}$$

$$\Rightarrow \quad \alpha = \frac{3}{5} + \frac{\sqrt{2}}{5}i \quad \text{and} \quad \beta = \frac{3}{5} - \frac{\sqrt{2}}{5}i$$

Hence, the roots of the given equation are $\frac{3}{5} \pm \frac{\sqrt{2}}{5}i$.

EXERCISE 14.1

LEVEL-1

Solve the following quadratic equations by factorization method only (1-5):

1. $x^2 + 1 = 0$
2. $9x^2 + 4 = 0$
3. $x^2 + 2x + 5 = 0$
4. $4x^2 - 12x + 25 = 0$
5. $x^2 + x + 1 = 0$

[NCERT]

Solve the following quadratics (6-18):

6. $4x^2 + 1 = 0$
7. $x^2 - 4x + 7 = 0$
8. $x^2 + 2x + 2 = 0$
9. $5x^2 - 6x + 2 = 0$
10. $21x^2 + 9x + 1 = 0$
11. $x^2 - x + 1 = 0$
12. $x^2 + x + 1 = 0$ [NCERT]
13. $17x^2 - 8x + 1 = 0$
14. $27x^2 - 10x + 1 = 0$ [NCERT]
15. $17x^2 + 28x + 12 = 0$
16. $21x^2 - 28x + 10 = 0$ [NCERT]
17. $8x^2 - 9x + 3 = 0$
18. $13x^2 + 7x + 1 = 0$
19. $2x^2 + x + 1 = 0$
20. $\sqrt{3}x^2 - \sqrt{2}\,x + 3\sqrt{3} = 0$ [NCERT]
21. $\sqrt{2}\,x^2 + x + \sqrt{2} = 0$ [NCERT]
22. $x^2 + x + \frac{1}{\sqrt{2}} = 0$ [NCERT]
23. $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$ [NCERT]
24. $\sqrt{5}x^2 + x + \sqrt{5} = 0$ [NCERT]
25. $-x^2 + x - 2 = 0$ [NCERT]
26. $x^2 - 2x + \frac{3}{2} = 0$ [NCERT]
27. $3x^2 - 4x + \frac{20}{3} = 0$ [NCERT]

ANSWERS

1. $i, -i$
2. $\frac{2}{3}i, \frac{-2}{3}i$
3. $-1 + 2i, -1 - 2i$
4. $\frac{3}{2} + 2i, \frac{3}{2} - 2i$
5. $-\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
6. $\frac{1}{2}i, \frac{-1}{2}i$
7. $2 \pm \sqrt{3}i$
8. $-1 \pm i$
9. $\frac{3}{5} \pm \frac{1}{5}i$
10. $\frac{-3}{14} \pm \frac{i\sqrt{3}}{42}$
11. $\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$
12. $-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$
13. $\frac{4}{17} \pm \frac{1}{17}i$
14. $\frac{5}{27} \pm \frac{\sqrt{2}}{27}i$
15. $\frac{-14}{17} \pm \frac{2\sqrt{2}}{17}i$
16. $\frac{2}{3} \pm \frac{\sqrt{14}}{21}i$
17. $\frac{9}{16} \pm \frac{\sqrt{15}}{16}i$
18. $\frac{-7}{26} \pm \frac{\sqrt{3}}{26}i$
19. $\frac{-1 \pm \sqrt{7}i}{4}$
20. $\frac{\sqrt{2} \pm \sqrt{34}i}{2\sqrt{3}}$
21. $\frac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$
22. $\frac{-1 \pm \sqrt{2\sqrt{2} - 1}\, i}{2}$
23. $\frac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$
24. $\frac{-1 \pm \sqrt{19}i}{2\sqrt{5}}$
25. $\frac{-1 \pm \sqrt{7}i}{-2}$
26. $1 \pm \frac{1}{\sqrt{2}}i$
27. $\frac{2}{3} \pm \frac{4}{3}i$

HINTS TO NCERT & SELECTED PROBLEMS

5. We have,

$$x^2 + x + 1 = 0$$

$$\Rightarrow \quad x^2 + 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = 0$$

$$\Rightarrow \quad \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = 0$$

$$\Rightarrow \quad \left(x+\frac{1}{2}\right)^2 - \frac{3}{4}i^2 = 0$$

$$\Rightarrow \quad \left(x+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}i\right)^2 = 0$$

$$\Rightarrow \quad \left(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(x+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = 0$$

$$\Rightarrow \quad \left(x+\frac{1+\sqrt{3}i}{2}\right)\left(x+\frac{1-\sqrt{3}i}{2}\right) = 0$$

$$\Rightarrow \quad x+\frac{1+\sqrt{3}i}{2} = 0 \text{ or, } x+\frac{1-\sqrt{3}i}{2} = 0$$

$$\Rightarrow \quad x = -\frac{1+i\sqrt{3}}{2} \text{ or, } x = \frac{-1+i\sqrt{3}}{2}$$

12. We have, $x^2 + x + 1 = 0$

Comparing the given equation with the general form $ax^2 + bx + c = 0$, we get

$$a = 1, b = 1 \text{ and } c = 1$$

Substituting the values of a, b, c in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{-1+\sqrt{1-4}}{2} \text{ and } \beta = \frac{-1-\sqrt{1-4}}{2}$$

$$\Rightarrow \quad \alpha = \frac{-1+i\sqrt{3}}{2} \text{ and } \beta = \frac{-1-i\sqrt{3}}{2}$$

14. We have, $27x^2 - 10x + 1 = 0$

Comparing this equation with $ax^2 + bx + c = 0$, we get: $a = 27, b = -10, c = 1$

Substituting the values of a, b, c in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\Rightarrow \quad \alpha = \frac{10+\sqrt{100-108}}{54} \text{ and } \beta = \frac{10-\sqrt{100-108}}{54}$$

$$\Rightarrow \quad \alpha = \frac{10+\sqrt{-8}}{54} \text{ and } \beta = \frac{10-\sqrt{-8}}{54}$$

$$\Rightarrow \quad \alpha = \frac{5+i\sqrt{2}}{27} \text{ and } \beta = \frac{5-i\sqrt{2}}{27}$$

16. The given equation is $21x^2 - 28x + 10 = 0$. Comparing this equation with $ax^2 + bx + c = 0$, we get: $a = 21, b = -28, c = 10$

Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{28+\sqrt{784-840}}{42} \text{ and } \beta = \frac{28-\sqrt{784-840}}{42}$$

$$\Rightarrow \quad \alpha = \frac{28+\sqrt{-56}}{42} \text{ and } \beta = \frac{28-\sqrt{-56}}{42}$$

$$\Rightarrow \quad \alpha = \frac{2}{3}+\frac{i\sqrt{14}}{21} \text{ and } \beta = \frac{2}{3}-\frac{i\sqrt{14}}{21}$$

20. The given equation is $\sqrt{3}x^2-\sqrt{2}x+3\sqrt{3}=0$. Comparing this equation with $ax^2+bx+c=0$, we get: $a=\sqrt{3}, b=-\sqrt{2}, c=3\sqrt{3}$.

Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}\text{, we get}$$

$$\alpha = \frac{\sqrt{2}+\sqrt{2-36}}{2\sqrt{3}} \text{ and } \beta = \frac{\sqrt{2}-\sqrt{2-36}}{2\sqrt{3}}$$

$$\Rightarrow \quad \alpha = \frac{\sqrt{2}+i\sqrt{34}}{2\sqrt{3}} \text{ and } \beta = \frac{\sqrt{2}-i\sqrt{34}}{2\sqrt{3}}$$

21. The given equation is $\sqrt{2}x^2+x+\sqrt{2}=0$. Comparing this equation with $ax^2+bx+c=0$, we get: $a=\sqrt{2}, b=1, c=\sqrt{2}$.

Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}\text{, we get}$$

$$\alpha = \frac{-1+\sqrt{1-8}}{2\sqrt{2}} \text{ and } \beta = \frac{-1-\sqrt{1-8}}{2\sqrt{2}}$$

$$\Rightarrow \quad \alpha = \frac{-1+i\sqrt{7}}{2\sqrt{2}} \text{ and } \beta = \frac{-1-i\sqrt{7}}{2\sqrt{2}}$$

22. The given equation is $x^2+x+\frac{1}{\sqrt{2}}=0$. Comparing this equation with $ax^2+bx+c=0$, we get: $a=1, b=1, c=\frac{1}{\sqrt{2}}$.

Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}\text{, we get}$$

$$\Rightarrow \quad \alpha = \frac{-1+\sqrt{1-2\sqrt{2}}}{2} \text{ and } \beta = \frac{-1-\sqrt{1-2\sqrt{2}}}{2}$$

$$\Rightarrow \quad \alpha = \frac{-1+i\sqrt{2\sqrt{2}-1}}{2} \text{ and } \beta = \frac{-1-i\sqrt{2\sqrt{2}-1}}{2}$$

23. The given equation is $x^2+\frac{x}{\sqrt{2}}+1=0$. Comparing this equation with $ax^2+bx+c=0$, we get: $a=1, b=\frac{1}{\sqrt{2}}$ and $c=1$.

Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}\text{, we get}$$

$$\alpha = \frac{-\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{2} - 4}}{2} \text{ and } \beta = \frac{-\frac{1}{\sqrt{2}} - \sqrt{\frac{1}{2} - 4}}{2}$$

$$\Rightarrow \quad \alpha = \frac{-1 + i\sqrt{7}}{2\sqrt{2}} \text{ and } \beta = \frac{-1 - i\sqrt{7}}{2\sqrt{2}}$$

24. The given equation is $\sqrt{5}x^2 + x + \sqrt{5} = 0$. Comparing this equation with $ax^2 + bx + c$, we get $a = \sqrt{5}$, $b = 1$ and $c = \sqrt{5}$. Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{-1 + \sqrt{1 - 20}}{2\sqrt{5}} \text{ and } \beta = \frac{-1 - \sqrt{1 - 20}}{2\sqrt{5}}$$

$$\Rightarrow \quad \alpha = \frac{-1 + i\sqrt{19}}{2\sqrt{5}} \text{ and } \beta = \frac{-1 - i\sqrt{19}}{2\sqrt{5}}$$

25. The given equation is $-x^2 + x - 2 = 0$. Comparing this equation with $ax^2 + bx + c$, we get: $a = -1$, $b = 1$ and $c = -2$. Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\Rightarrow \quad \alpha = \frac{-1 + \sqrt{1 - 8}}{-2} \text{ and } \beta = \frac{-1 - \sqrt{1 - 8}}{-2}$$

$$\Rightarrow \quad \alpha = \frac{-1 + i\sqrt{7}}{-2} \text{ and } \beta = \frac{-1 - i\sqrt{7}}{-2}$$

$$\Rightarrow \quad \alpha = \frac{1}{2} - i\frac{\sqrt{7}}{2} \text{ and } \beta = \frac{1}{2} + i\frac{\sqrt{7}}{2}$$

26. The given equation is $x^2 - 2x + \frac{3}{2} = 0$. Comparing this equation with $ax^2 + bx + c = 0$, we get: $a = 1$, $b = -2$ and $c = 3/2$.
Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad \alpha = \frac{2 + \sqrt{4 - 6}}{2} \text{ and } \beta = \frac{2 - \sqrt{4 - 6}}{2}$$

$$\Rightarrow \quad \alpha = 1 + \frac{i}{\sqrt{2}} \text{ and } \beta = 1 - \frac{i}{\sqrt{2}}$$

27. The given equation is $3x^2 - 4x + \frac{20}{3} = 0$. Com,paring this equation with $ax^2 + bx + c = 0$, we get: $a = 3$, $b = -4$ and $c = \frac{20}{3}$. Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{4 + \sqrt{16 - 80}}{6} \text{ and } \beta = \frac{4 - \sqrt{16 - 80}}{6}$$

$$\Rightarrow \quad \alpha = \frac{4+8i}{6} \text{ and } \beta = \frac{4-8i}{6}$$

$$\Rightarrow \quad \alpha = \frac{2}{3}+\frac{4}{3}i \text{ and } \beta = \frac{2}{3}-\frac{4}{3}i$$

14.5 QUADRATIC EQUATIONS WITH COMPLEX COEFFICIENTS

Consider the quadratic equation

$$ax^2+bx+c=0 \qquad ...(i)$$

where a, b, c are complex numbers and $a \neq 0$.

Proceeding as in section 14.4, we obtain that the roots of equation (i) are given by

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and, } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

These roots are complex as a, b, c are complex numbers.

Since the order relation is not defined in case of complex numbers. Therefore, we cannot assign positive or negative sign to the discriminate $D = b^2 - 4ac$. However, equation (i) has complex roots which are equal, if $D = b^2 - 4ac = 0$ and unequal roots if $D = b^2 - 4ac \neq 0$.

REMARK *In case of quadratic equations with real coefficients imaginary (complex) roots always occur in conjugate pairs. However, it is not true for quadratic equations with complex coefficients. For example, the equation $4x^2 - 4ix - 1 = 0$ has both roots equal to $\frac{1}{2}i$.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the following quadratic equations by factorization method:*

(i) $x^2 - 5ix - 6 = 0$ (ii) $x^2 + 4ix - 4 = 0$

SOLUTION (i) The given equation is

$$x^2 - 5ix - 6 = 0$$

$$\Rightarrow \quad x^2 - 5ix + 6i^2 = 0$$

$$\Rightarrow \quad x^2 - 3ix - 2ix + 6i^2 = 0$$

$$\Rightarrow \quad x(x-3i) - 2i(x-3i) = 0$$

$$\Rightarrow \quad (x-3i)(x-2i) = 0$$

$$\Rightarrow \quad x - 3i = 0, x - 2i = 0$$

$$\Rightarrow \quad x = 3i, x = 2i$$

Hence, the roots of the given equation are $3i$ and $2i$.

(ii) We have,

$$x^2 + 4ix - 4 = 0$$

$$\Rightarrow \quad x^2 + 4ix + 4i^2 = 0$$

$$\Rightarrow \quad (x+2i)^2 = 0$$

$$\Rightarrow \quad x + 2i = 0 \quad \text{(twice)}$$

$$\Rightarrow \quad x = -2i, -2i$$

Hence, both the roots of the equation are equal to $-2i$.

EXAMPLE 2 *Solve the following equations by factorization method*

(i) $x^2 - \sqrt{2}i\,x + 12 = 0$ (ii) $3x^2 + 7ix + 6 = 0$ (iii) $x^2 - (3\sqrt{2} + 2i)\,x + 6\sqrt{2}i = 0$

SOLUTION (i) We have,

$$x^2 - \sqrt{2}\,ix + 12 = 0$$

$$\Rightarrow \quad x^2 - 3\sqrt{2}i\,x + 2\sqrt{2}\,i\,x - 12i^2 = 0$$

$$\Rightarrow \quad x\,(x - 3\sqrt{2}i) + 2\sqrt{2}i\,(x - 3\sqrt{2}i) = 0$$

$$\Rightarrow \quad (x - 3\sqrt{2}\,i)\,(x + 2\sqrt{2}\,i) = 0$$

$$\Rightarrow \quad x - 3\sqrt{2}i = 0 \text{ or, } x + 2\sqrt{2}\,i = 0$$

$$\Rightarrow \quad x = 3\sqrt{2}\,i \text{ or, } x = -2\sqrt{2}\,i$$

Hence, the roots of the given equation are $-2\sqrt{2}\,i$ and $3\sqrt{2}\,i$.

(ii) $$3x^2 + 7\,ix + 6 = 0$$

$$\Rightarrow \quad 3x^2 + 9ix - 2ix - 6i^2 = 0$$

$$\Rightarrow \quad 3x\,(x + 3i) - 2i\,(x + 3i) = 0$$

$$\Rightarrow \quad (x + 3i)\,(3x - 2i) = 0$$

$$\Rightarrow \quad x + 3i = 0 \text{ or, } 3x - 2i = 0$$

$$\Rightarrow \quad x = -3i \text{ or, } x = \frac{2}{3}i$$

Hence, the roots of the given equation are $-3i$ and $\frac{2}{3}i$.

(iii) $$x^2 - (3\sqrt{2} + 2i)\,x + 6\sqrt{2}\,i = 0$$

$$\Rightarrow \quad (x^2 - 3\sqrt{2}x) - (2ix - 6\sqrt{2}i) = 0$$

$$\Rightarrow \quad x\,(x - 3\sqrt{2}) - 2i\,(x - 3\sqrt{2}) = 0$$

$$\Rightarrow \quad (x - 2i)\,(x - 3\sqrt{2}) = 0$$

$$\Rightarrow \quad x - 2i = 0, x - 3\sqrt{2} = 0$$

$$\Rightarrow \quad x = 2i \text{ or, } 3\sqrt{2}$$

Hence, the roots of the given equation are $2i$ and $3\sqrt{2}$.

EXAMPLE 3 *Solve the following quadratic equations by using the general expressions for the roots of a quadratic equation:*

(i) $x^2 - (3\sqrt{2} - 2i)\,x - 6\sqrt{2}\,i = 0$ (ii) $2x^2 + 3i\,x + 2 = 0$

SOLUTION (i) On comparing the given equation with the general equation $ax^2 + bx + c = 0$, we get: $a = 1, b = -(3\sqrt{2} - 2i)$ and $c = -6\sqrt{2}i$.

Substituting the values of a, b, c in $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$, we get

$$\alpha = \frac{(3\sqrt{2} - 2i) + \sqrt{(3\sqrt{2} - 2i)^2 + 24\sqrt{2}i}}{2}, \text{ and } \beta = \frac{(3\sqrt{2} - 2i) - \sqrt{(3\sqrt{2} - 2i)^2 + 24\sqrt{2}i}}{2}$$

$$\Rightarrow \quad \alpha = \frac{(3\sqrt{2} - 2i) + \sqrt{(3\sqrt{2} + 2i)^2}}{2}, \quad \text{and} \quad \beta = \frac{(3\sqrt{2} - 2i) - \sqrt{13\sqrt{2} + 2i)^2}}{2}$$

$$\Rightarrow \quad \alpha = \frac{3\sqrt{2} - 2i + 3\sqrt{2} + 2i}{2}, \quad \text{and} \quad \beta = \frac{(3\sqrt{2} - 2i) - (3\sqrt{2} + 2i)}{2}$$

$$\Rightarrow \quad \alpha = 3\sqrt{2}, \beta = -2i$$

Hence, the roots of the given equation are $3\sqrt{2}$ and $-2i$.

(ii) On comparing the given equation with the general equation $ax^2+bx+c=0$, we get $a=2, b=3i$ and $c=2$.

Substituting these values of a, b, c in $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$, we get

$$\alpha = \frac{-3i+\sqrt{9i^2-16}}{4} \quad \text{and} \quad \beta = \frac{-3i-\sqrt{9i^2-16}}{4}$$

$$\Rightarrow \quad \alpha = \frac{-3i+\sqrt{-25}}{4} \quad \text{and} \quad \beta = \frac{-3i-\sqrt{-25}}{4}$$

$$\Rightarrow \quad \alpha = \frac{-3i+5i}{4} \quad \text{and} \quad \beta = \frac{-3i-5i}{4}$$

$$\Rightarrow \quad \alpha = \frac{i}{2} \quad \text{and} \quad \beta = -2i$$

Hence, the roots of the given equation are $\frac{i}{2}$ and $-2i$.

LEVEL-2

EXAMPLE 4 *Solve:* $2x^2-(3+7i)\,x-(3-9i) = 0.$

SOLUTION On comparing the given equation with the general form $ax^2+bx+c=0$, we obtain $a=2, b=-(3+7i), c=-(3-9i)$.

Substituting the values of a, b, c in $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$, we get

$$\alpha = \frac{(3+7i)+\sqrt{(3+7i)^2+8(3-9i)}}{4} \quad \text{and, } \beta = \frac{(3+7i)-\sqrt{(3+7i)^2+8(3-9i)}}{4}$$

$$\Rightarrow \alpha = \frac{(3+7i)+\sqrt{9-49+42i+24-72i}}{4} \quad \text{and, } \beta = \frac{(3+7i)-\sqrt{9-49+42i+24-72i}}{4}$$

$$\Rightarrow \alpha = \frac{3+7i+\sqrt{-16-30i}}{4} \quad \text{and, } \beta = \frac{3+7i-\sqrt{-16-30i}}{4} \qquad \text{...(i)}$$

Let us now find $\sqrt{-16-30\,i}$.

Let $a+ib = \sqrt{-16-30i}$. Then,

$$a+ib = \sqrt{-16-30i}$$

$$\Rightarrow \quad a^2-b^2+2i\,ab = -16-30i$$

$$\Rightarrow \quad a^2-b^2 = -16 \qquad \text{...(ii)}$$

and, $\quad 2\,ab = -30 \qquad \text{...(iii)}$

$$\therefore \quad (a^2+b^2)^2 = (a^2-b^2)^2+4\,a^2\,b^2$$

$$\Rightarrow \quad (a^2+b^2)^2 = 256+900 = 1156$$

$$\Rightarrow \quad a^2+b^2 = 34$$

Now, $\quad a^2-b^2 = -16$ and $a^2+b^2 = 34$

$$\Rightarrow \quad a^2 = 9 \text{ and } b^2 = 25$$

$$\Rightarrow \quad a = \pm 3 \text{ and } b = \pm 5$$

From (iii), we find that a and b are of opposite signs.

$\therefore \quad a = 3$ and $b = -5$ or, $a = -3$ and $b = 5$.

Hence, $\sqrt{-16-30i} = 3-5i$ or, $-3+5i$.

Substituting either of these values in (i), we get

$$\alpha = \frac{(3+7i)+(3-5i)}{4} \text{ and, } \beta = \frac{(3+7i)-(3-5i)}{4}$$

$$\Rightarrow \quad \alpha = \frac{3}{2}+\frac{1}{2}i \text{ and, } \beta = 3i$$

Hence, the roots of the given equation are $\frac{3}{2}+\frac{1}{2}i$ and $3i$

EXAMPLE 5 *Solve:* $x^2-(7-i)x+(18-i)=0$ *over* C.

SOLUTION Comparing the given equation with the general form $ax^2+bx+c=0$, we get $a=1, b=-(7-i)$ and $c=18-i$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and, } \quad \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{(7-i)+\sqrt{(7-i)^2-4(18-i)}}{2}, \quad \beta = \frac{(7-i)-\sqrt{(7-i)^2-4(18-i)}}{2}$$

$$\Rightarrow \quad \alpha = \frac{(7-i)+\sqrt{-24-10i}}{2}, \quad \beta = \frac{(7-i)-\sqrt{-24-10i}}{2} \quad \text{...(i)}$$

Let us now find $\sqrt{-24-10i}$.

Let, $a+ib=\sqrt{-24-10i}$. Then,

$$(a+ib)^2=-24-10i$$

$$\Rightarrow \quad (a^2-b^2)+2i\,ab=-24-10i$$

$$\Rightarrow \quad a^2-b^2=-24 \quad \text{...(ii)}$$

and, $\quad 2ab=-10 \quad$ ...(iii)

Now, $\quad (a^2+b^2)^2 = (a^2-b^2)^2+4a^2b^2$

$$\Rightarrow \quad (a^2+b^2)^2 = 576+100 = 676 \Rightarrow a^2+b^2 = 26 \quad \text{...(iv)}$$

Solving (ii) and (iii), we get $a=\pm 1$ and $b=\pm 5$.

From (iii), we find that ab is negative.

$\therefore \quad a=1, b=-5$ or, $a=-1, b=5$.

$\therefore \quad a+ib = 1-5i$ or, $-1+5i$

Hence, $\sqrt{-24-10i} = \pm(1-5i)$

Substituting either of these values in (i), we get

$$\alpha = \frac{7-i+1-5i}{2} = 4-3i \text{ and } \beta = \frac{(7-i)-(1-5i)}{2} = 3+2i$$

Hence, the roots of the given equation are $4-3i$ and $3+2i$.

EXERCISE 14.2

1. Solving the following quadratic equations by factorization method:
   (i) $x^2+10ix-21 = 0$ (ii) $x^2+(1-2i)x-2i = 0$
   (iii) $x^2-(2\sqrt{3}+3i)x+6\sqrt{3}i = 0$ (iv) $6x^2-17ix-12 = 0$
2. Solve the following quadratic equations:
   (i) $x^2-(3\sqrt{2}+2i)x+6\sqrt{2}i = 0$ (ii) $x^2-(5-i)x+(18+i)=0$
   (iii) $(2+i)x^2-(5-i)x+2(1-i)=0$ (iv) $x^2-(2+i)x-(1-7i)=0$
   (v) $ix^2-4x-4i = 0$ (vi) $x^2+4ix-4 = 0$

(vii) $2x^2 + \sqrt{15}\, ix - i = 0$ [NCERT] (viii) $x^2 - x + (1+i) = 0$

(ix) $ix^2 - x + 12i = 0$ [NCERT] (x) $x^2 - (3\sqrt{2} - 2i)\, x - \sqrt{2}\, i = 0$

(xi) $x^2 - (\sqrt{2} + i)\, x + \sqrt{2}i = 0$ [NCERT] (xii) $2x^2 - (3 + 7i)\, x + (9i - 3) = 0$

ANSWERS

1. (i) $-3i, -7i$ (ii) $-1, 2i$ (iii) $2\sqrt{3}, 3i$ (iv) $\frac{3}{2}i, \frac{4}{3}i$

2. (i) $3\sqrt{2}, 2i$ (ii) $3-4i, 2+3i$ (iii) $1-i, \frac{4}{5} - \frac{2}{5}i$ (iv) $3-i, -1+2i$

(v) $-2i, -2i$ (vi) $-2i, -2i$ (vii) $\frac{1+(4-\sqrt{15})\,i}{4}, \frac{-1-(\sqrt{15}+4)\,i}{4}$

(viii) $1-i, i$ (ix) $-4i, 3i$ (x) $\frac{3\sqrt{2}-2i}{2} \pm \frac{4-\sqrt{2}i}{2}$

(xi) $\sqrt{2}, i$ (xii) $\frac{3+i}{2}, 3i$

HINTS TO NCERT & SELECTED PROBLEMS

2. (vii) The given equation is $2x^2 + \sqrt{15}\, i\, x - i = 0$.

Comparing this equation with the standard equation $ax^2 + bx + c = 0$, we get

$a = 2, b = \sqrt{15}i$ and $c = -i$

Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and, } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{-\sqrt{15}\, i + \sqrt{-15 + 8i}}{4} \text{ and, } \beta = \frac{-\sqrt{15}\, i - \sqrt{-15 + 8\, i}}{4}$$

Let $\sqrt{-15+8i} = a + ib$. Then,

$-15 + 8i = (a+ib)^2$

$\Rightarrow -15 + 8i = a^2 - b^2 + 2i\, ab$

$\Rightarrow a^2 - b^2 = -15$ and $2ab = 8$

Now, $(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2 b^2$

$\Rightarrow (a^2 + b^2)^2 = (-15)^2 + 64 = 289$

$\Rightarrow a^2 + b^2 = 17$

Solving $a^2 - b^2 = -15$ and $a^2 + b^2 = 17$, we get

$a^2 = 1$ and $b^2 = 16$

$\Rightarrow a = \pm 1$ and $b = \pm 4$

$\Rightarrow a = 1, b = 4$ or $a = -1, b = -4$ [$\because ab = 4 > 0 \therefore a$ and b are of the same sign]

$\therefore \sqrt{-15+8i} = 1 + 4i, -1 - 4i$

When $\sqrt{-15+8i} = 1 + 4i$:

$$\alpha = \frac{\sqrt{15}i + 1 + 4i}{4} = \frac{1 + (\sqrt{15} + 4)\,i}{4} \text{ and } \beta = \frac{-\sqrt{15}i - (1 + 4i)}{2} = -\frac{1 - (\sqrt{5} + 4)\,i}{2}$$

When $\sqrt{-15 + 8i} = -1 - 4i$:

$$\alpha = \frac{\sqrt{15}\,i - 1 - 4i}{2} = \frac{-1 - (\sqrt{15} + 4)\,i}{2} \text{ and } \beta = \frac{-\sqrt{15}i + 1 + 4i}{2} = \frac{1 + (4 - \sqrt{15})\,i}{2}$$

(ix) The given equation is $i\,x^2 - x + 12i = 0$.

Comparing this equation with the standard equation $ax^2 + bx + c = 0$, we get $a = i, b = -1$ and $c = 12i$.

Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{1 + \sqrt{1 + 48}}{2i} \text{ and } \beta = \frac{1 - \sqrt{1 + 48}}{2i}$$

$$\Rightarrow \quad \alpha = \frac{1 + 7}{2i} \text{ and } \beta = \frac{1 - 7}{2i}$$

$$\Rightarrow \quad \alpha = \frac{4}{i} \text{ and } \beta = -\frac{3}{i} \Rightarrow \alpha = 0 - 4i \text{ and } \beta = 3i$$

(xi) The given equation is $x^2 - (\sqrt{2} + i)\,x + \sqrt{2}\,i = 0$.

Comparing this equation with the standard equation $ax^2 + bx + c = 0$, we get $a = 1, b = -(\sqrt{2} + i)$ and $c = \sqrt{2}\,i$.

Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{(\sqrt{2} + i) + \sqrt{(\sqrt{2} + i)^2 - 4\sqrt{2}\,i}}{2} \text{ and } \beta = \frac{(\sqrt{2} + i) - \sqrt{(\sqrt{2} + i)^2 - 4\sqrt{2}\,i}}{2}$$

$$\Rightarrow \quad \alpha = \frac{\sqrt{2} + i + \sqrt{(\sqrt{2} - i)^2}}{2} \text{ and } \beta = \frac{(\sqrt{2} + i) - \sqrt{(\sqrt{2} - i)^2}}{2}$$

$$\Rightarrow \quad \alpha = \frac{\sqrt{2} + i + \sqrt{2} - i}{2} \text{ and } \beta = \frac{(\sqrt{2} + i) - (\sqrt{2} - i)}{2}$$

$$\Rightarrow \quad \alpha = \sqrt{2} \text{ and } \beta = i$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the number of real roots of the equation $(x - 1)^2 + (x - 2)^2 + (x - 3)^2 = 0$.
2. If a and b are roots of the equation $x^2 - px + q = 0$, then write the value of $\frac{1}{a} + \frac{1}{b}$.
3. If roots α, β of the equation $x^2 - px + 16 = 0$ satisfy the relation $\alpha^2 + \beta^2 = 9$, then write the value of p.
4. If $2 + \sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, then write the values of p and q.

5. If the difference between the roots of the equation $x^2 + ax + 8 = 0$ is 2, write the values of a.
6. Write the roots of the equation $(a-b)x^2 + (b-c)x + (c-a) = 0$.
7. If a and b are roots of the equation $x^2 - x + 1 = 0$, then write the value of $a^2 + b^2$.
8. Write the number of quadratic equations, with real roots, which do not change by squaring their roots.
9. If α, β are roots of the equation $x^2 + lx + m = 0$, write an equation whose roots are $-\frac{1}{\alpha}$ and $-\frac{1}{\beta}$.
10. If α, β are roots of the equation $x^2 - a(x+1) - c = 0$, then write the value of $(1+\alpha)(1+\beta)$.

ANSWERS

1. No real root 2. $\frac{1}{q}$ 3. ± 8 4. $p = -4, q = 1$ 5. ± 8 6. $1, \frac{c-a}{a-b}$
7. -1 8. 3 9. $mx^2 - lx + 1 = 0$ 10. $1 - c$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The complete set of values of k, for which the quadratic equation $x^2 - kx + k + 2 = 0$ has equal roots, consists of
   (a) $2 + \sqrt{12}$ (b) $2 \pm \sqrt{12}$ (c) $2 - \sqrt{12}$ (d) $-2 - \sqrt{2}$
2. For the equation $|x|^2 + |x| - 6 = 0$, the sum of the real roots is
   (a) 1 (b) 0 (c) 2 (d) none of these
3. If a, b are the roots of the equation $x^2 + x + 1 = 0$, then $a^2 + b^2 =$
   (a) 1 (b) 2 (c) -1 (d) 3
4. If α, β are roots of the equation $4x^2 + 3x + 7 = 0$, then $1/\alpha + 1/\beta$ is equal to
   (a) $7/3$ (b) $-7/3$ (c) $3/7$ (d) $-3/7$
5. The values of x satisfying $\log_3 (x^2 + 4x + 12) = 2$ are
   (a) $2, -4$ (b) $1, -3$ (c) $-1, 3$ (d) $-1, -3$
6. The number of real roots of the equation $(x^2 + 2x)^2 - (x+1)^2 - 55 = 0$ is
   (a) 2 (b) 1 (c) 4 (d) none of these
7. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then $\frac{1}{a\alpha + b} + \frac{1}{a\beta + b} =$
   (a) c/ab (b) a/bc (c) b/ac (d) none of these
8. If α, β are the roots of the equation $x^2 + px + 1 = 0$; γ, δ the roots of the equation $x^2 + qx + 1 = 0$, then $(\alpha - \gamma)(\alpha + \delta)(\beta - \gamma)(\beta + \delta) =$
   (a) $q^2 - p^2$ (b) $p^2 - q^2$ (c) $p^2 + q^2$ (d) none of these
9. The number of real solutions of $|2x - x^2 - 3| = 1$ is
   (a) 0 (b) 2 (c) 3 (d) 4

10. The number of solutions of $x^2 + |x-1| = 1$ is

(a) 0 (b) 1 (c) 2 (d) 3

11. If x is real and $k = \dfrac{x^2 - x + 1}{x^2 + x + 1}$, then

(a) $k \in [1/3, 3]$ (b) $k \geq 3$ (c) $k \leq 1/3$ (d) none of these

12. If the roots of $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ is

(a) 0 (b) 1 (c) 2 (d) none of these

13. The value of a such that $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$ may have a common root is

(a) 0 (b) 12 (c) 24 (d) 32.

14. The values of k for which the quadratic equation $kx^2 + 1 = kx + 3x - 11x^2$ has real and equal roots are

(a) $-11, -3,$ (b) $5, 7$ (c) $5, -7$ (d) none of these

15. If the equations $x^2 + 2x + 3\lambda = 0$ and $2x^2 + 3x + 5\lambda = 0$ have a non-zero common roots, then $\lambda =$

(a) 1 (b) -1 (c) 3 (d) none of these

16. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, the value of q is

(a) $49/4$ (b) $4/49$ (c) 4 (d) none of these

17. The value of p and q $(p \neq 0, q \neq 0)$ for which p, q are the roots of the equation $x^2 + px + q\ ab = 0$ are

(a) $p = 1, q = -2$ (b) $p = -1, q = -2$
(c) $p = -1, q = 2$ (d) $p = 1, q = 2$

18. The set of all values of m for which both the roots of the equation $x^2 - (m+1)x + m + 4 = 0$ are real and negative, is

(a) $(-\infty, -3] \cup [5, \infty)$ (b) $[-3, 5]$
(c) $(-4, -3]$ (d) $(-3, -1]$

19. The number of roots of the equation $\dfrac{(x+2)(x-5)}{(x-3)(x+6)} = \dfrac{x-2}{x+4}$ is

(a) 0 (b) 1 (c) 2 (d) 3

20. If α and β are the roots of $4x^2 + 3x + 7 = 0$, then the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ is

(a) $\dfrac{4}{7}$ (b) $-\dfrac{3}{7}$ (c) $\dfrac{3}{7}$ (d) $-\dfrac{3}{4}$

21. If, α, β are the roots of the equation $x^2 + px + q = 0$, then $-\dfrac{1}{\alpha}, -\dfrac{1}{\beta}$ are the roots of the equation

(a) $x^2 - px + q = 0$ (b) $x^2 + px + q = 0$
(c) $qx^2 + px + 1 = 0$ (d) $qx^2 - px + 1 = 0$

22. If the difference of the roots of $x^2 - px + q = 0$ is unity, then

(a) $p^2 + 4q = 1$ (b) $p^2 - 4q = 1$
(c) $p^2 + 4q^2 = (1 + 2q)^2$ (d) $4p^2 + q^2 = (1 + 2p)^2$

23. If α, β are the roots of the equation $x^2 - p(x+1) - c = 0$, then $(\alpha+1)(\beta+1) =$

(a) c (b) $c-1$ (c) $1-c$ (d) none of these

24. The least value of k which makes the roots of the equation $x^2 + 5x + k = 0$ imaginary is

(a) 4 (b) 5 (c) 6 (d) 7

25. The equation of the smallest degree with real coefficients having $1+i$ as one of the roots is

(a) $x^2 + x + 1 = 0$ (b) $x^2 - 2x + 2 = 0$

(c) $x^2 + 2x + 2 = 0$ (d) $x^2 + 2x - 2 = 0$

ANSWERS

1. (b)	2. (b)	3. (c)	4. (d)	5. (d)	6. (b)	7. (c)	8. (a)
9. (b)	10. (a)	11. (a)	12. (b)	13. (c)	14. (c)	15. (b)	16. (a)
17. (c)	18. (a)	19. (b)	20. (b)	21. (d)	22. (b), (c)		23. (c)
24. (d)	25. (b)						

SUMMARY

1. Fundamental Theorem of Algebra: Every polynomial equation $f(x) = 0$ has at least one root, real or imaginary (complex).
2. Every polynomial equation $f(x) = 0$ of degree n has exactly n roots real or imaginary.
3. A quadratic equation cannot have more than two roots.
4. If $ax^2 + bx + c = 0, a \neq 0$ is a quadratic equation with real coefficients, then its roots α and β given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and, } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or, } \alpha = \frac{-b + \sqrt{D}}{2a} \text{ and, } \beta = \frac{-b - \sqrt{D}}{2a}$$

where $D = b^2 - 4ac$ is as the discriminant of the equation.

(i) If $D = 0$, then $\alpha = \beta = -\dfrac{b}{2a}$

So, the equation has real and equal roots each equal to $-\dfrac{b}{2a}$.

(ii) If $a, b, c \in Q$ and D is positive and a perfect square, then roots are rational and unequal.

(iii) If $a, b, c \in R$ and D is positive and a perfect square, then the roots are real and distinct.

(iv) If $D > 0$ but it is not a perfect square, then roots are irrational and unequal.

(v) If $D < 0$, then the roots are imaginary and are given by

$$\alpha = \frac{-b + i\sqrt{4ac - b^2}}{2a} \text{ and } \beta = \frac{-b - i\sqrt{4ac - b^2}}{2a}$$

(vi) If $a = 1, b, c \in I$ and the roots are rational numbers, then these roots must be integers.

(vii) If a quadratic equation in x has more than two roots, then it is an identity in x that is $a = b = c = 0$.

(viii) Complex roots of an equation with real coefficients always occur in pairs. However, this may not be true in case of equations with complex coefficients. For example, $x^2 - 2ix - 1 = 0$ has both roots equal to i.

(ix) Surd root of an equation with rational coefficients always occur in pairs like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, Surd roots of an equation with irrational coefficients may not occur in pairs. For example, $x^2 - 2\sqrt{3}\,x + 3 = 0$ has both roots equal to $\sqrt{3}$.

CHAPTER 15

LINEAR INEQUATIONS

15.1 INTRODUCTION

In this chapter, we will study linear inequations in one and two variables. The knowledge of linear inequations is very helpful in solving problems in Science, Mathematics, Engineering, Linear Programming etc.

15.2 INEQUATIONS

In earlier classes, we have studied equations in one and two variables. An equation is defined as a statement involving variable (s) and the sign of equality (=). Similarly, we define the term inequation as follows:

INEQUATION *A statement involving variable (s) and the sign of inequality viz, >, <, ≥ or ≤ is called an inequation or an inequality.*

An inequation may contain one or more variables. Also, it may be linear or quadratic or cubic etc.

Following are some examples of inequations:

(i) $3x-2<0$ (ii) $2x+3\leq 0$ (iii) $5x-3>0$

(iv) $4x+5\geq 0$ (v) $2x+3y<1$ (vi) $5x+4y\leq 3$

(vii) $4x-6y>5$ (viii) $2x+5y\geq 4$ (ix) $2x^2+3x+4>0$

(x) $x^2-3x+2\geq 0$ (xi) $x^2+3x+2<0$ (xii) $x^2-5x+4\leq 0$

(xiii) $x^3-6x^2+11x-6>0$ (xiv) $x^3+6x^2+11x+6\leq 0$

LINEAR INEQUATION IN ONE VARIABLE *Let a be a non-zero real number and x be a variable. Then inequations of the form $ax+b<0$, $ax+b\leq 0$, $ax+b>0$ and $ax+b\geq 0$ are known as linear inequations in one variable x.*

For example, $9x-15>0$, $5x-4\geq 0$, $3x+2<0$ and $2x-3\leq 0$ are linear inequations in one variable.

LINEAR INEQUATIONS IN TWO VARIABLES *Let a, b be non-zero real numbers and x, y be variables. Then inequations of the form $ax+by<c$, $ax+by\leq c$, $ax+by>c$ and $ax+by\geq c$ are known as linear inequations in two variables x* and *y.*

For example, $2x+3y\leq 6$, $3x-2y\geq 12$, $x+y<4$, $2x+y\geq 6$ are linear inequations in two variables x and y.

QUADRATIC INEQUATION *Let a be a non-zero real number. Then an inequation of the form $ax^2+bx+c<0$, or $ax^2+bx+c\leq 0$, or $ax^2+bx+c>0$, or $ax^2+bx+c\geq 0$ is known as a quadratic inequation.*

For example, $x^2+x-6<0$, $x^2-3x+2\geq 0$, $2x^2+3x+1>0$ and $x^2-5x+4\leq 0$ are quadratic inequations.

In this chapter, we shall study linear inequations in one and two variables only.

15.3 SOLUTIONS OF AN INEQUATION

DEFINITION *A solution of an inequation is the value (s) of the variable (s) that makes it a true statement.*

Consider the inequation $\frac{3-2x}{5} < \frac{x}{3} - 4$.

Left hand side (LHS) of this inequation is $\frac{3-2x}{5}$ and right hand side (RHS) is $\frac{x}{3} - 4$.

We observe that:

For $x = 9$, we have

$$\text{LHS} = \frac{3 - 2 \times 9}{5} = -3 \text{ and, RHS} = \frac{9}{3} - 4 = -1$$

Clearly, $-3 < -1$

$\Rightarrow$ LHS < RHS, which is true.

So, $x = 9$ is a solution of the given inequation.

For $x = 6$, we have

$$\text{LHS} = \frac{3 - 2 \times 6}{5} = -\frac{9}{5} \text{ and RHS} = \frac{6}{3} - 4 = -2$$

Because, $-\frac{9}{5} < -2$ is not true. So, $x = 6$ is not a solution of the given inequation.

We can verify that any real number greater than 7 is a solution of the given inequation.

Let us now consider the inequation $x^2 + 1 < 0$.

We know that

$x^2 \geq 0$ for all $x \in R$

$\therefore$ $x^2 + 1 \geq 1$ for all $x \in R$

$\Rightarrow$ $x^2 + 1 \nless 0$ for any $x \in R$.

So, there is no real value of x which makes the given inequation a true statement. Hence, it has no solution.

It follows from the above discussion that an inequation may or may not have a solution. However, if an inequation has a solution it may have infinitely many solutions.

SOLVING AN INEQUATION *It is the process of obtaining all possible solutions of an inequation.*

SOLUTION SET *The set of all posible solutions of an inequation is known as its solution set.*

For example, the solution set of the inequation $x^2 + 1 \geq 0$ is the set R of all real numbers whereas the solution set of the inequation $x^2 + 1 < 0$ is the null set ϕ

15.4 SOLVING LINEAR INEQUATIONS IN ONE VARIABLE

As mentioned in the previous section that solving an inequation is the process of obtaining its all possible solutions. In the process of solving an inequation, we use mathematical simplifications which are governed by the following rules:

<u>RULE 1</u> *Same number may be added to (or subtracted from) both sides of an inequation without changing the sign of inequality.*

<u>RULE 2</u> *Both sides of an inequation can be multiplied (or divided) by the same positive real number without changing the sign of inequality. However, the sign of inequality is reversed when both sides of an inequation are multiplied or divided by a negative number.*

<u>RULE 3</u> *Any term of an inequation may be taken to the other side with its sign changed without affecting the sign of inequality.*

A linear inequation in one variable is of the form

$$ax + b < 0 \text{ or, } ax + b \leq 0 \text{ or, } ax + b > 0 \text{ or, } ax + b \geq 0.$$

We follow the following algorithm to solve a linear inequation in one variable.

ALGORITHM

STEP I *Obtain the linear inequation.*

STEP II *Collect all terms involving the variable on one side of the inequation and the constant terms on the other side.*

STEP III *Simplify both sides of inequality in their simplest forms to reduce the inequation in the form* $ax < b$, or $ax \leq b$, or $ax > b$, or $ax \geq b$

STEP IV *Solve the inequation obtained in step III by dividing both sides of the inequation by the coefficient of the variable.*

STEP V *Write the solution set obtained in step IV in the form of an interval on the real line.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **SOLVING EQUATIONS OF THE FORM:** $ax + b > cx + d$, or, $ax + b \geq cx + d$, or, $ax + b < cx + d$ or, $ax + b \leq cx + d$

EXAMPLE 1 *Solve the following linear inequations:*

(i) $2x - 4 \leq 0$ (ii) $-3x + 12 < 0$ (iii) $4x - 12 \geq 0$ (iv) $7x + 9 > 30$

SOLUTION (i) We have,

$$2x - 4 \leq 0$$

$\Rightarrow \quad (2x - 4) + 4 \leq 0 + 4$ [Adding 4 on both sides]

$\Rightarrow \quad 2x \leq 4$

$\Rightarrow \quad \dfrac{2x}{2} \leq \dfrac{4}{2}$

$\Rightarrow \quad x \leq 2$

Hence, any real number less than or equal to 2 is a solution of the given inequation.

These solutions can be graphed on real line as shown in Fig. 15.1

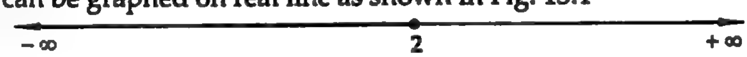


Fig. 15.1

The solution set of the given inequation is $(-\infty, 2]$

(ii) We have,

$$-3x + 12 < 0$$

$\Rightarrow \quad -3x < -12$ [Transposing 12 on right side]

$\Rightarrow \quad \dfrac{-3x}{-3} > \dfrac{-12}{-3}$ [Dividing both sides by -3]

$\Rightarrow \quad x > 4$

Thus, any real number greater than 4 is a solution of the given inequation.

Hence, the solution set of the given inequation is $(4, \infty)$. This solution set can be graphed on real line as shown in Fig. 15.2

Fig. 15.2

(iii) We have,

$$4x - 12 \geq 0$$

$\Rightarrow \quad 4x \geq 12$ [Transposing 12 on RHS]

$\Rightarrow \quad \dfrac{4x}{4} \geq \dfrac{12}{4}$ [Dividing both sides by 4]

$\Rightarrow \quad x \geq 3$

$\Rightarrow \quad x \in [3, \infty)$

Hence, the solution set of the given inequation is $[3, \infty)$. This solution set can be graphed on real line as shown in Fig. 15.3

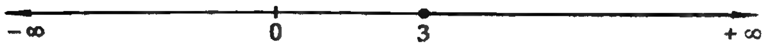


Fig. 15.3

(iv) We have,

$7x + 9 > 30$

$\Rightarrow \quad 7x > 30 - 9$

$\Rightarrow \quad 7x > 21$ [Transposing 9 on RHS]

$\Rightarrow \quad \dfrac{7x}{7} > \dfrac{21}{7}$

$\Rightarrow \quad x > 3$

$\Rightarrow \quad x \in (3, \infty)$

Hence, $(3, \infty)$ is the solution set of the given inequation. This can be graphed on real line as shown in Fig. 15.4.

Fig. 15.4

EXAMPLE 2 *Solve:* $5x - 3 < 3x + 1$ *when (i) x is a real number (ii) x is integer number (iii) x is a natural number.*

SOLUTION We have,

$5x - 3 < 3x + 1$

$\Rightarrow \quad 5x - 3x < 3 + 1$ [Transposing $3x$ on LHS and -3 on RHS]

$\Rightarrow \quad 2x < 4$

$\Rightarrow \quad \dfrac{2x}{2} < \dfrac{4}{2}$ $\left[\text{Multiplying both sides by } \dfrac{1}{2}\right]$

$\Rightarrow \quad x < 2$

(i) If $x \in R$, then

$x < 2 \Rightarrow x \in (-\infty, 2)$

Hence, the solution set is $(-\infty, 2)$ as shown in Fig. 15.5.

Fig. 15.5

(ii) If $x \in Z$, then

$x < 2 \Rightarrow x = 1, 0, -1, -2, -3, -4, \ldots$

So, the solution set is $\{\ldots -4, -3, -2, -1, 0, 1\}$

(iii) If $x \in N$, then

$x < 2 \Rightarrow x = 1$

So, the solution set is $\{1\}$.

EXAMPLE 3 *Solve the following equations:*

(i) $3x + 17 \le 2(1 - x)$ (ii) $2(2x + 3) - 10 \le 6(x - 2)$

SOLUTION (i) We have,

$$3x + 17 \le 2(1 - x)$$

$\Rightarrow \quad 3x + 17 \le 2 - 2x$

$\Rightarrow \quad 3x + 2x \le 2 - 17$ [Transposing $-2x$ to LHS and 17 to RHS]

$\Rightarrow \quad 5x \le -15$

$\Rightarrow \quad \dfrac{5x}{5} \le \dfrac{-15}{5}$

$\Rightarrow \quad x \le -3$

$\Rightarrow \quad x \in (-\infty, -3]$

Hence, the solution set of the given inequation is $(-\infty, -3]$, which can be graphed on real line as shown in Fig. 15.6.

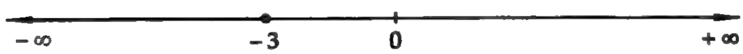


Fig. 15.6

(ii)We have,

$$2(2x + 3) - 10 \le 6(x - 2)$$

$\Rightarrow \quad 4x + 6 - 10 \le 6x - 12$

$\Rightarrow \quad 4x - 4 \le 6x - 12$

$\Rightarrow \quad 4x - 6x \le -12 + 4$ [Transposing -4 to RHS and $6x$ to LHS]

$\Rightarrow \quad -2x \le -8$

$\Rightarrow \quad \dfrac{-2x}{-2} \ge \dfrac{-8}{-2}$

$\Rightarrow \quad x \ge 4$

$\Rightarrow \quad x \in [4, \infty)$

Hence, the solution set of the given inequation is $[4, \infty)$ which can be graphed on real line as shown in Fig. 15.7.

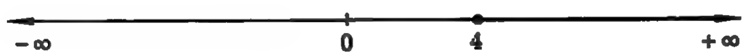


Fig. 15.7

EXAMPLE 4 *Solve the following inequations:*

(i) $\dfrac{2x - 3}{4} + 9 \ge 3 + \dfrac{4x}{3}$ (ii) $\dfrac{5x - 2}{3} - \dfrac{7x - 3}{5} > \dfrac{x}{4}$

(iii) $\dfrac{1}{2}\left(\dfrac{3}{5}x + 4\right) \ge \dfrac{1}{3}(x - 6)$ (iv) $\dfrac{3(x - 2)}{5} \ge \dfrac{5(2 - x)}{3}$

SOLUTION (i) We have,

$$\frac{2x - 3}{4} + 9 \ge 3 + \frac{4x}{3}$$

$\Rightarrow \quad \dfrac{2x - 3}{4} - \dfrac{4x}{3} \ge 3 - 9$ [Transposing $\dfrac{4x}{3}$ to LHS and 9 to RHS]

$\Rightarrow \quad \dfrac{3(2x - 3) - 16x}{12} \ge -6$

$\Rightarrow \quad \dfrac{6x - 9 - 16x}{12} \ge -6$

$$\Rightarrow \quad \frac{-9-10x}{12} \geq -6$$

$$\Rightarrow \quad -9-10x \geq -72 \qquad \text{[Multiplying both sides by 12]}$$

$$\Rightarrow \quad -10x \geq -72+9$$

$$\Rightarrow \quad -10x \geq -63$$

$$\Rightarrow \quad \frac{-10x}{-10} \leq \frac{-63}{-10}$$

$$\Rightarrow \quad x \leq \frac{63}{10}$$

$$\Rightarrow \quad x \in (-\infty, 63/10]$$

Hence, the solution set of the given inequation is $(-\infty, 63/10]$. This can be graphed on real line as shown in Fig. 15.8.

Fig. 15.8

(ii) We have,

$$\frac{5x-2}{3} - \frac{7x-3}{5} > \frac{x}{4}$$

$$\Rightarrow \quad \frac{5(5x-2)-3(7x-3)}{15} > \frac{x}{4}$$

$$\Rightarrow \quad \frac{25x-10-21x+9}{15} > \frac{x}{4}$$

$$\Rightarrow \quad \frac{4x-1}{15} > \frac{x}{4}$$

$$\Rightarrow \quad 4(4x-1) > 15x \qquad \text{[Multiplying both sides by 60 i.e. lcm of 15 and 4]}$$

$$\Rightarrow \quad 16x-4 > 15x$$

$$\Rightarrow \quad 16x-15x > 4 \qquad \text{[Transposing } 15x \text{ to LHS and } -4 \text{ to RHS]}$$

$$\Rightarrow \quad x > 4$$

$$\Rightarrow \quad x \in (4, \infty)$$

Hence, the solution set of the given inequation is $(4, \infty)$. This can be graphed on the real line as shown in Fig. 15.9.

Fig. 15.9

(iii) We have,

$$\frac{1}{2}\left(\frac{3}{5}x+4\right) \geq \frac{1}{3}(x-6)$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{3x+20}{5}\right) \geq \frac{1}{3}(x-6)$$

$$\Rightarrow \quad \frac{3x+20}{10} \geq \frac{x-6}{3}$$

$$\Rightarrow \quad 3(3x+20) \geq 10(x-6) \qquad \text{[Multiplying both sides by 30 i.e. the lcm of 3 and 10]}$$

$$\Rightarrow \quad 9x+60 \geq 10x-60$$

$$\Rightarrow \quad 9x-10x \geq -60-60 \qquad \text{[Transposing } 10x \text{ on LHS and 60 on RHS]}$$

$$\Rightarrow \quad -x \geq -120$$

$$\Rightarrow \quad x \leq 120 \qquad \text{[Multiplying both sides by } -1]$$

$$\Rightarrow \quad x \in (-\infty, 120]$$

Hence, the solution set of the given in equation is $(-\infty, 120]$ which can be graphed on real line as shown in Fig. 15.10.

Fig. 15.10

(iv) We have,

$$\frac{3(x-2)}{5} \geq \frac{5(2-x)}{3}$$

$$\Rightarrow \quad \frac{3x-6}{5} \geq \frac{10-5x}{3}$$

$$\Rightarrow \quad 3(3x-6) \geq 5(10-5x) \qquad \text{[Multiplying both sides by 15 i.e. the lcm of 5 and 3]}$$

$$\Rightarrow \quad 9x - 18 \geq 50 - 25x$$

$$\Rightarrow \quad 9x + 25x \geq 50 + 18 \qquad \text{[Transposing } -25x \text{ to LHS and 18 to RHS]}$$

$$\Rightarrow \quad 34x \geq 68$$

$$\Rightarrow \quad \frac{34x}{34} \geq \frac{68}{34}$$

$$\Rightarrow \quad x \geq 2$$

$$\Rightarrow \quad x \in [2, \infty)$$

Hence, $[2, \infty)$ is the solution set of the given inequation. This solution set can be graphed on real line as shown in Fig. 15.11.

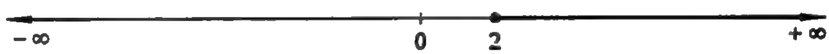


Fig. 15.11

EXAMPLE 5 *Solve the following inequations:*

(i) $\dfrac{1}{x-2} < 0$ (ii) $\dfrac{x+1}{x+2} \geq 1$

SOLUTION (i) We have,

$$\frac{1}{x-2} < 0$$

$$\Rightarrow \quad x - 2 < 0 \qquad \left[\because \frac{a}{b} < 0 \text{ and } a > 0 \Rightarrow b < 0\right]$$

$$\Rightarrow \quad x < 2$$

$$\Rightarrow \quad x \in (-\infty, 2)$$

Hence, the solution set of the given inequation is $(-\infty, 2)$.

(ii) We have,

$$\frac{x+1}{x+2} \geq 1$$

$$\Rightarrow \quad \frac{x+1}{x+2} - 1 \geq 0$$

$$\Rightarrow \quad \frac{x+1-x-2}{x+2} \geq 0$$

$\Rightarrow \quad \dfrac{-1}{x+2} \geq 0$

$\Rightarrow \quad x+2<0 \qquad \left[\because \dfrac{a}{b}>0 \text{ and } a<0 \Rightarrow b<0\right]$

$\Rightarrow \quad x<-2$

$\Rightarrow \quad x \in(-\infty,-2)$

Hence, the solution set of the given inequation is $(-\infty,-2)$.

Type II **EQUATIONS OF THE FORM**

$$\frac{ax+b}{cx+d}>k, \text{ or } \frac{ax+b}{cx+d} \geq k, \text{ or } \frac{ax+b}{cx+d}<k, \text{ or } \frac{ax+b}{cx+d} \leq k$$

In order to solve this type of inequation, we use the following algorithm.

ALGORITHM

STEP I *Obtain the inequation.*

STEP II *Transpose all terms on LHS.*

STEP III *Simplify LHS of the inequation obtained in step II to obtain an inequation of the form*

$$\frac{px+q}{rx+s}>0, \text{ or } \frac{px+q}{rx+s} \geq 0, \text{ or } \frac{px+q}{rx+s}<0, \text{ or } \frac{px+q}{rx+s} \leq 0.$$

STEP IV *Make coefficient x positive in numerator and denominator if they are not.*

STEP V *Equate numerator and denominator separately to zero and obtain the values of x. These values of x are generally called critical points.*

STEP VI *Plot the critical points obtained in step V on real line. These points will divide the real line in three regions.*

STEP VII *In the right most region the expression on LHS of the inequation obtained in step IV will be positive and in other regions it will be alternatively negative and positive. So, mark positive sign in the right most region and then mark alternatively negative and positive signs in other regions.*

STEP VIII *Select appropriate region on the basis of the sign of the inequation obtained in step IV. Write these regions in the form of intervals to obtain the desired solution sets of the given inequation.*

EXAMPLE 6 *Solve the following linear inequations:*

(i) $\dfrac{x-3}{x-5}>0$ (ii) $\dfrac{x-2}{x+5}>2$

SOLUTION (i) We have,

$$\frac{x-3}{x-5}>0 \qquad \text{...(i)}$$

Equating $x-3$ and $x-5$ to zero, we obtain $x = 3, 5$ as critical points. Plot these points on real line as shown in Fig. 15.12. The real line is divided into three regions. In the right most region the expression on LHS of (i) is positive and in the remaining two regions it is alternatively negative and positive as shown in Fig. 15.12.

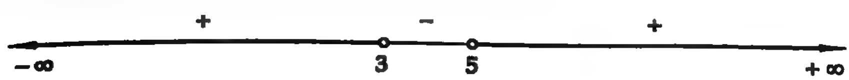


Fig. 15.12

Since the expression in (i) is positive, so the solution set of the given inequation is the union of regions containing positive signs. Hence, from Fig. 15.12

$$\frac{x-3}{x-5}>0 \Rightarrow x \in(-\infty, 3) \cup(5, \infty)$$

Hence, the solution set of the given inequation is $(-\infty, 3) \cup (5, \infty)$ as shown in Fig. 15.12.

(ii) We have,

$$\frac{x-2}{x+5} > 2$$

$$\Rightarrow \quad \frac{x-2}{x+5} - 2 > 0$$

$$\Rightarrow \quad \frac{x-2-2(x+5)}{x+5} > 0$$

$$\Rightarrow \quad \frac{x-2-2x-10}{x+5} > 0$$

$$\Rightarrow \quad \frac{-x-12}{x+5} > 0$$

$$\Rightarrow \quad \frac{x+12}{x+5} < 0 \qquad \begin{bmatrix}\text{Multiplying by } -1 \text{ to make coefficient of} \\ x \text{ positive in the expression in numerator}\end{bmatrix} \quad \text{...(i)}$$

On equating $x+12$ and $x+5$ to zero, we obtain $x = -12, -5$ as critical points. These points are plotted on number line as shown in Fig. 15.12. The real line is divided into three regions and the signs of LHS of inequation (i) are marked. Since the inequation in (i) possesses less than sign which means that LHS of the inequation is negative. So, the solution set of the given inequation is the union of the regions ontaining negative sign in Fig. 15.13. Hence, the solution set of the given inequation is $(-12, -5)$.

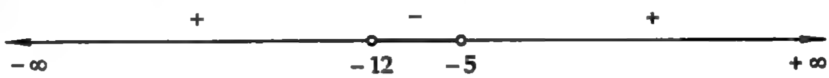


Fig. 15.13

EXAMPLE 7 *Solve the following inequations:*

(i) $\frac{2x+4}{x-1} \geq 5$ (ii) $\frac{x+3}{x-2} \leq 2$

SOLUTION (i) We have,

$$\frac{2x+4}{x-1} \geq 5$$

$$\Rightarrow \quad \frac{2x+4}{x-1} - 5 \geq 0$$

$$\Rightarrow \quad \frac{2x+4-5(x-1)}{x-1} \geq 0$$

$$\Rightarrow \quad \frac{2x+4-5x+5}{x-1} \geq 0$$

$$\Rightarrow \quad \frac{-3x+9}{x-1} \geq 0$$

$$\Rightarrow \quad \frac{3x-9}{x-1} \leq 0 \qquad \text{[Multiplying both sides by } -1\text{]}$$

$$\Rightarrow \quad \frac{3(x-3)}{(x-1)} \leq 0$$

$$\Rightarrow \quad \frac{x-3}{x-1} \leq 0 \qquad \text{[Dividing both sides by 3]}$$

$$\Rightarrow \quad 1 < x \leq 3 \qquad \text{[See Fig. 15.14]}$$

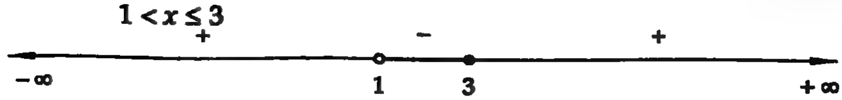


Fig. 15.14

$\Rightarrow \qquad x \in (1, 3]$

Hence, the solution set of the given inequation is $(1, 3]$.

(ii) We have,

$$\frac{x+3}{x-2} \leq 2$$

$$\Rightarrow \quad \frac{x+3}{x-2} - 2 \leq 0$$

$$\Rightarrow \quad \frac{x+3-2x+4}{x-2} \leq 0$$

$$\Rightarrow \quad \frac{-x+7}{x-2} \leq 0$$

$$\Rightarrow \quad \frac{x-7}{x-2} \geq 0 \qquad \text{[Multiplying both sides by } -1]$$

$$\Rightarrow \quad x \in (-\infty, 2) \cup [7, \infty) \qquad \text{[See Fig. 15.15]}$$

Fig. 15.15

Hence, the solution set of the given inequation is $(-\infty, 2) \cup [7, \infty)$.

EXERCISE 15.1

LEVEL-1

Solve the following linear inequations in R.

1. Solve: $12x < 50$, when
   (i) $x \in R$ (ii) $x \in Z$ (iii) $x \in N$
2. Solve: $-4x > 30$, when
   (i) $x \in R$ (ii) $x \in Z$ (iii) $x \in N$
3. Solve: $4x - 2 < 8$, when
   (i) $x \in R$ (ii) $x \in Z$ (iii) $x \in N$
4. $3x - 7 > x + 1$
5. $x + 5 > 4x - 10$
6. $3x + 9 \geq -x + 19$
7. $2(3 - x) \geq \frac{x}{5} + 4$
8. $\frac{3x-2}{5} \leq \frac{4x-3}{2}$
9. $-(x-3) + 4 < 5 - 2x$
10. $\frac{x}{5} < \frac{3x-2}{4} - \frac{5x-3}{5}$
11. $\frac{2(x-1)}{5} \leq \frac{3(2+x)}{7}$
12. $\frac{5x}{2} + \frac{3x}{4} \geq \frac{39}{4}$
13. $\frac{x-1}{3} + 4 < \frac{x-5}{5} - 2$
14. $\frac{2x+3}{4} - 3 < \frac{x-4}{3} - 2$
15. $\frac{5-2x}{3} < \frac{x}{6} - 5$
16. $\frac{4+2x}{3} \geq \frac{x}{2} - 3$
17. $\frac{2x+3}{5} - 2 < \frac{3(x-2)}{5}$
18. $x - 2 \leq \frac{5x+8}{3}$
19. $\frac{6x-5}{4x+1} < 0$
20. $\frac{2x-3}{3x-7} > 0$
21. $\frac{3}{x-2} < 1$
22. $\frac{1}{x-1} \leq 2$
23. $\frac{4x+3}{2x-5} < 6$
24. $\frac{5x-6}{x+6} < 1$
25. $\frac{5x+8}{4-x} < 2$
26. $\frac{x-1}{x+3} > 2$
27. $\frac{7x-5}{8x+3} > 4$
28. $\frac{x}{x-5} > \frac{1}{2}$

ANSWERS

1. (i) $(-\infty, 25/6)$ (ii) $\{\ldots -3, -2, -1, 0, 1, 2, 3, 4\}$ (iii) $\{1, 2, 3, 4\}$
2. (ii) $(-\infty, -15/2)$ (ii) $\{\ldots\ldots\ldots, -9, -8\}$ (iii) ϕ
3. (i) $(-\infty, 5/2)$ (ii) $\{\ldots\ldots, -2, -1, 0, 1, 2\}$ (iii) $\{1, 2\}$
4. $(4, \infty)$
5. $(-\infty, 5)$
6. $[5/2, \infty)$
7. $(-\infty, 10/11]$
8. $[11/14, \infty)$
9. $(-\infty, -2)$
10. $(-\infty, 2/9)$
11. $[-44, \infty)$
12. $[3, \infty)$
13. $(-\infty, -50)$
14. $(-\infty, -13/2)$
15. $(8, \infty)$
16. $[-26, \infty)$
17. $(-1, \infty)$
18. $[-7, \infty)$
19. $(-1/4, 5/6)$
20. $(-\infty, 3/2) \cup (7/3, \infty)$
21. $(-\infty, 2) \cup (5, \infty)$
22. $(-\infty, 1) \cup [3/2, \infty)$
23. $(-\infty, 5/2) \cup (33/8, \infty)$
24. $(-6, 3)$
25. $(-\infty, 0) \cup (4, \infty)$
26. $(-7, -3)$
27. $(-17/25, -3/8)$
28. $(-\infty, -5) \cup (5, \infty)$

15.5 SOLUTION OF SYSTEM OF LINEAR INEQUATIONS IN ONE VARIABLE

In the previous section, we have learnt how to solve a linear inequation in one variable. In this section, we shall use it to solve a system of linear inequations in one variable. Recall that the solution set of a linear inequation is the set of all points on real line satisfying the given inequation. Therefore, the solution set of a system of linear inequations in one variable is the intersection of the solution sets of the linear inequations in the given system.

We use the following algorithm to solve a system of linear inequations in one variable.

ALGORITHM

STEP I *Obtain the system of linear inequations.*

STEP II *Solve each inequation and obtain their solution sets. Also, represent them on real time.*

STEP III *Find the intersection of the solution sets obtained in step II by taking the help of the graphical representation of the solution sets in step II.*

STEP IV *The set obtained in step III is the required solution set of the given system of inequations.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the following system of linear inequations:*

$$3x - 6 \geq 0$$
$$4x - 10 \leq 6$$

SOLUTION The given system of inequations is

$$3x - 6 \geq 0 \qquad \ldots(i)$$
$$4x - 10 \leq 6 \qquad \ldots(ii)$$

Now, $3x - 6 \geq 0 \Rightarrow 3x \geq 6 \Rightarrow \frac{3x}{3} \geq \frac{6}{3} \Rightarrow x \geq 2$

$\therefore$ Solution set of inequation (i) is $[2, \infty)$

and, $4x - 10 \leq 6 \Rightarrow 4x \leq 16 \Rightarrow x \leq 4$

$\therefore$ Solution set of inequation (ii) is $(-\infty, 4]$

Fig. 15.16(i)

Fig. 15.16(ii)

The solution sets of inequations (i) and (ii) are represented graphically on real line in Figs. 15.16 (i) and (ii) respectively.

Clearly, the intersection of these solution sets is the set [2, 4].

Hence, the solution set of the given system of inequations is the interval [2, 4].

EXAMPLE 2 *Solve the following system of inequations:*

$$\frac{5x}{4}+\frac{3x}{8}>\frac{39}{8}$$

$$\frac{2x-1}{12}-\frac{x-1}{3}<\frac{3x+1}{4}$$

SOLUTION The given system of inequation is

$$\frac{5x}{4}+\frac{3x}{8}>\frac{39}{8} \qquad \text{...(i)}$$

$$\frac{2x-1}{12}-\frac{x-1}{3}<\frac{3x+1}{4} \qquad \text{...(ii)}$$

Now, $\frac{5x}{4}+\frac{3x}{8}>\frac{39}{8}$

$\Rightarrow \quad \frac{10x+3x}{8}>\frac{39}{8}$

$\Rightarrow \quad 13x > 39$

$\Rightarrow \quad x > 3$

$\Rightarrow \quad x \in (3, \infty)$

So, the solution set of inequation (i) is the interval $(3, \infty)$.

and, $\frac{2x-1}{12}-\frac{x-1}{3}<\frac{3x+1}{4}$

$\Rightarrow \quad \frac{(2x-1)-4(x-1)}{12}<\frac{3x+1}{4}$

$\Rightarrow \quad \frac{-2x+3}{12}<\frac{3x+1}{4}$

$\Rightarrow \quad -2x+3<3(3x+1)$ [Multiplying both sides by 12 i.e. the l.c.m. of 12 and 4]

$\Rightarrow \quad -2x+3<9x+3$

$\Rightarrow \quad -2x-9x<3-3$

$\Rightarrow \quad -11x<0$

$\Rightarrow \quad x>0$

$\Rightarrow \quad x \in (0, \infty)$

So, the solution set of inequation (ii) is the interval $(0, \infty)$. Let us now represent the solution sets of inequations (i) and (ii) on real line. These solution sets are graphed on real line in Figs. 15.17 (i) and 15.17 (ii) respectively.

From Figs. 15.17 (i) and (ii), we observe that the intersection of the solution sets of inequations (i) and (ii) is interval $(3, \infty)$ represented by common thick line.

Hence, the solution set of the given system of inequations is the interval $(3, \infty)$.

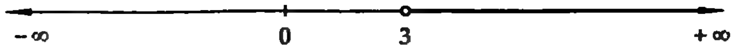


Fig. 15.17(i)

Fig. 15.17(ii)

EXAMPLE 3 *Solve the following system of inequations:* $2(2x+3)-10<6(x-2)$

$$\frac{2x-3}{4}+6 \geq 2+\frac{4x}{3}$$

SOLUTION The given system of inequations is

$$2(2x+3)-10<6(x-2) \quad \text{...(i)}$$

$$\frac{2x-3}{4}+6 \geq 2+\frac{4x}{3} \quad \text{...(ii)}$$

Now, $2(2x+3)-10<6(x-2)$

$\Rightarrow \quad 4x+6-10<6x-12$

$\Rightarrow \quad 4x-4<6x-12$

$\Rightarrow \quad 4x-6x<4-12$

$\Rightarrow \quad -2x<-8$

$\Rightarrow \quad x>4$

$\Rightarrow \quad x \in (4, \infty)$

So, the solution set of the first inequation is the interval $(4, \infty)$.

and, $\dfrac{2x-3}{4}+6 \geq 2+\dfrac{4x}{3}$

$\Rightarrow \quad \dfrac{2x-3+24}{4} \geq \dfrac{6+4x}{3}$

$\Rightarrow \quad \dfrac{2x+21}{4} \geq \dfrac{4x+6}{3}$

$\Rightarrow \quad 3(2x+21) \geq 4(4x+6)$

$\Rightarrow \quad 6x+63 \geq 16x+24$

$\Rightarrow \quad 6x-16x \geq 24-63$

$\Rightarrow \quad -10x \geq -39$

$\Rightarrow \quad x \leq \dfrac{39}{10}$

$\Rightarrow \quad x \leq 3.9$

$\Rightarrow \quad x \in (-\infty, 3.9]$

So, the solution set of inequation (ii) is the interval $(-\infty, 3.9]$.

The solution sets of inequations (i) and (ii) are graphed on real line in Figs. 15.18 (i) and (ii) respectively.

We observe that there is no common solution of the two inequations. So, the given system of inequations has no solution.

Fig. 15.18 (i)

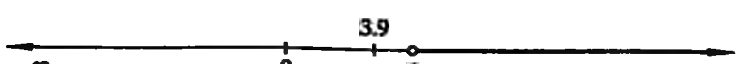


Fig. 15.18 (ii)

EXAMPLE 4 *Solve:* $-11 \leq 4x - 3 \leq 13$

SOLUTION We have,

$$-11 \geq 4x - 3 \geq 13 \Leftrightarrow -11 \geq 4x - 3 \text{ and } 4x - 3 \geq 13$$

Thus, we have two inequations and we wish to solve them simultaneously. Instead of solving these inequations by using the method discussed in first three examples, let us solve them directly in a different way as given below.

We have,

$$-11 \leq 4x - 3 \leq 13$$

$$\Rightarrow \quad -11 + 3 \leq 4x - 3 + 3 \leq 13 + 3 \qquad \text{[Adding 3 throughout]}$$

$$\Rightarrow \quad -8 \leq 4x \leq 16$$

$$\Rightarrow \quad \frac{-8}{4} \leq x \leq \frac{16}{4} \qquad \text{[Dividing by 4 throughout]}$$

$$\Rightarrow \quad -2 \leq x \leq 4$$

$$\Rightarrow \quad x \in [-2, 4]$$

Hence, the interval $[-2, 4]$ is the solution set of the given system of inequations.

EXAMPLE 5 *Solve:* $-5 \leq \dfrac{2-3x}{4} \leq 9$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$-5 \leq \frac{2-3x}{4} \leq 9$$

$$\Rightarrow \quad -5 \times 4 \leq \frac{2-3x}{4} \times 4 \leq 9 \times 4 \qquad \text{[Multiplying throughout by 4]}$$

$$\Rightarrow \quad -20 \leq 2 - 3x \leq 36$$

$$\Rightarrow \quad -20 - 2 \leq 2 - 3x - 2 \leq 36 - 2 \qquad \text{[Subtracting 2 throughout]}$$

$$\Rightarrow \quad -22 \leq -3x \leq 34$$

$$\Rightarrow \quad \frac{-22}{-3} \geq \frac{-3x}{-3} \geq \frac{34}{-3} \qquad \text{[Dividing throughout by } -3\text{]}$$

$$\Rightarrow \quad \frac{22}{3} \geq x \geq \frac{-34}{3}$$

$$\Rightarrow \quad \frac{-34}{3} \leq x \leq \frac{22}{3}$$

$$\Rightarrow \quad x \in [-34/3, 22/3]$$

Hence, the interval $[-34/3, 22/3]$ is the solution set of the given system of inequations.

EXAMPLE 6 *Solve the system of inequations:* $\dfrac{x}{2x+1} \geq \dfrac{1}{4}$, $\dfrac{6x}{4x-1} < \dfrac{1}{2}$

SOLUTION The given system of inequations is

$$\frac{x}{2x+1} \geq \frac{1}{4} \qquad \text{...(i)}$$

$$\frac{6x}{4x-1} < \frac{1}{2} \qquad \text{...(ii)}$$

Now,

$$\frac{x}{2x+1} \geq \frac{1}{4}$$

$$\Rightarrow \quad \frac{x}{2x+1} - \frac{1}{4} \geq 0$$

$$\Rightarrow \quad \frac{4x - (2x+1)}{4(2x+1)} \geq 0$$

$\Rightarrow \quad \dfrac{2x-1}{2x+1} \geq 0$ [Multiplying both sides by 4]

$\Rightarrow \quad x \in (-\infty, -1/2) \cup [1/2, \infty)$ [See Fig. 15.19 (i)]

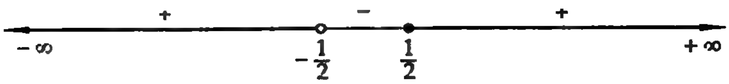


Fig. 15.19(i)

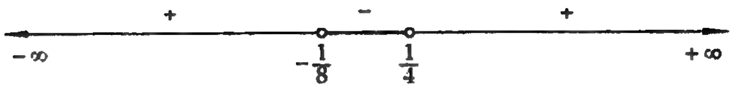


Fig. 15.19(ii)

Thus, the solution set of inequation (i) is $(-\infty, -1/2) \cup [1/2, \infty)$ ... (iii)

And, $\quad \dfrac{6x}{4x-1} < \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{6x}{4x-1} - \dfrac{1}{2} < 0$

$\Rightarrow \quad \dfrac{12x-(4x-1)}{2(4x-1)} < 0$

$\Rightarrow \quad \dfrac{8x+1}{2(4x-1)} < 0$

$\Rightarrow \quad \dfrac{8x+1}{4x-1} < 0$ [Multiplying both sides by 2]

$\Rightarrow \quad x \in (-1/8, 1/4)$ [See Fig. 15.19 (ii)]

Thus, the solution set of inequation (ii) is

$$(-1/8, 1/4) \quad \text{...(iv)}$$

It is evident from Fig. 15.19 that the intersection of (iii) and (iv) is the null set.

Hence, the given system of equations has no solution.

EXERCISE 15.2

LEVEL-1

Solve each of the following system of equations in R.

1. $x+3>0,\ 2x<14$
2. $2x-7>5-x,\ 11-5x\leq 1$
3. $x-2>0,\ 3x<18$
4. $2x+6\geq 0,\ 4x-7<0$
5. $3x-6>0,\ 2x-5>0$
6. $2x-3<7,\ 2x>-4$
7. $2x+5\leq 0,\ x-3\leq 0$
8. $5x-1<24,\ 5x+1>-24$
9. $3x-1\geq 5,\ x+2>-1$
10. $11-5x>-4,\ 4x+13\leq -11$
11. $4x-1\leq 0,\ 3-4x<0$
12. $x+5>2(x+1),\ 2-x<3(x+2)$
13. $2(x-6)<3x-7,\ 11-2x<6-x$
14. $5x-7<3(x+3),\ 1-\dfrac{3x}{2}\geq x-4$
15. $\dfrac{2x-3}{4}-2\geq\dfrac{4x}{3}-6,\ 2(2x+3)<6(x-2)+10$
16. $\dfrac{7x-1}{2}<-3,\ \dfrac{3x+8}{5}+11<0$
17. $\dfrac{2x+1}{7x-1}>5,\ \dfrac{x+7}{x-8}>2$
18. $0<\dfrac{-x}{2}<3$
19. $10\leq -5(x-2)<20$
20. $-5<2x-3<5$

21. $\frac{4}{x+1} \le 3 \le \frac{6}{x+1}, x > 0$ [NCERT EXEMPLAR]

ANSWERS

1. $(-3, 7)$	2. $(4, \infty)$	3. $(2, 6)$	4. $[-3, 7/4)$	
5. $(5/2, \infty)$	6. $(-2 5)$	7. $(-\infty, -5/2]$	8. $(-5, 5)$	
9. $[2, \infty)$	10. $(-\infty, -6]$	11. No Solution	12. $(-1, 3)$	
13. $(5, \infty)$	14. $(-\infty, 2]$	15. No Solution	16. $(-\infty, -21)$	
17. No Solution	18. $(-6, 0)$	19. $(-2, 0]$	20. $(-1, 4)$	21. $[1/3, 1]$

15.5.1 SOME IMPORTANT RESULTS

In this sub-section, let us discuss some results on inequations involving modulus of the variable. We state and prove these results as theorems.

THEOREM 1 *If a is a positive real number, then*

(i) $|x| < a \Leftrightarrow -a < x < a$ i.e. $x \in (-a, a)$

(ii) $|x| \le a \Leftrightarrow -a \le x \le a$ i.e. $x \in [-a, a]$

Fig. 15.20(i)

Fig. 15.20(ii)

PROOF (i) We know that: $|x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}$

So, we consider the following cases:

CASE I When $x \ge 0$: In this case, $|x| = x$.

$\therefore \quad |x| < a \Rightarrow x < a$

Thus, in this case the solution set of the given inequation is given by

$x \ge 0$ and $x < a \Rightarrow 0 \le x < a$ ...(i)

CASE II When $x < 0$: In this case, $|x| = -x$.

$\therefore \quad |x| < a \Rightarrow -x < a \Rightarrow x > -a$

Thus, in this case the solution set of the given inequation is given by

$x < 0$ and $x > -a \Rightarrow -a < x < 0$ ...(ii)

Combining (i) and (ii), we get

$|x| < a \Leftrightarrow -a < x < 0$ or, $0 \le x < a \Leftrightarrow -a < x < a$.

(ii) Proceeding exactly as in (i), we get

$|x| \le a \Rightarrow -a \le x \le a$.

THEOREM 2 *If a is a positive real number, then*

(i) $|x| > a \Leftrightarrow x < -a$ or $x > a$

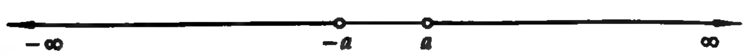


Fig. 15.21(i)

(ii) $|x| \ge a \Leftrightarrow x \le -a$ or $x \ge a$

Fig. 15.21(ii)

PROOF CASE I When $x > 0$: In this case, $|x| = x$

$\therefore \quad |x| > a \Rightarrow x > a$

Thus, in this case the solution set of the inequation $|x| > a$ is given by

$x > 0$ and $x > a \Rightarrow x > a$ $\quad [\because a > 0]$ ...(i)

CASE II *When* $x < 0$: In this case, $|x| = -x$

$\therefore \quad |x| > a \Rightarrow -x > a \Rightarrow x < -a$

Thus, in this case the solution set of the given inequation is given by

$x < 0$ and $x < -a \Rightarrow x < -a$ $\quad [\because a > 0]$ ...(ii)

Combining (i) and (ii), we get

$|x| > a \Leftrightarrow x < -a$ or $x > a$

(ii) Proceeding as in (i), we get

$|x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$.

THEOREM 3 *Let r be a positive real number and a be a fixed real number. Then,*

(i) $|x - a| < r \Leftrightarrow a - r < x < a + r$ i.e. $x \in (a - r, a + r)$

(ii) $|x - a| \leq r \Leftrightarrow a - r \leq x \leq a + r$ i.e. $x \in [a - r, a + r]$

(iii) $|x - a| > r \Leftrightarrow x < a - r$, or $x > a + r$

(iv) $|x - a| \geq r \Leftrightarrow x \leq a - r$, or $x \geq a + r$

PROOF (i) Using Theorem 1, we obtain

$|x - a| < r \Leftrightarrow -r < x - a < r \Leftrightarrow a - r < x - a + a < a + r \Leftrightarrow a - r < x < a + r$

(ii) Using Theorem 1 (ii), we obtain

$|x - a| \leq r \Leftrightarrow -r \leq x - a \leq r \Leftrightarrow a - r \leq x - a + a \leq a + r \Leftrightarrow a - r \leq x \leq a + r$

(iii) Using Theorem 2(i), we obtain

$|x - a| > r \Leftrightarrow x - a < r$, or $x - a > r \Leftrightarrow x < a - r$, or $x > a + r$

(iv) Using Theorem 2 (ii), we obtain,

$|x - a| \geq r \Leftrightarrow x - a \leq -r$, or $x - a \geq r \Leftrightarrow x \leq a - r$, or $x \geq a + r$

NOTE: *These results may be used directly for solving linear inequations involving absolute values.*

THEOREM 4 *Let a, b be positive real numbers. Then*

(i) $a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$

(ii) $a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

(iii) $a \leq |x - c| \leq b \Leftrightarrow x \in [-b + c, -a + c] \cup [a + c, b + c]$

(iv) $a < |x - c| < b \Leftrightarrow x \in (-b + c, -a + c) \cup (a + c, b + c)$

PROOF (i) $a < |x| < b \Leftrightarrow |x| > a$ and $|x| < b \Leftrightarrow (x < -a$ or $x > a)$ and $(-b < x < b)$

$\Leftrightarrow x \in (-b, -a) \cup (a, b)$

Similarly, we can prove other results.

ILLUSTRATIVE EXAMPLES

LEVEL-2

EXAMPLE 1 $|3x - 2| \leq \frac{1}{2}$

SOLUTION We know that: $|x - a| \leq r \Leftrightarrow a - r \leq x \leq a + r$

$\therefore \quad |3x - 2| \leq \frac{1}{2} \Leftrightarrow 2 - \frac{1}{2} \leq 3x \leq 2 + \frac{1}{2} \Leftrightarrow \frac{3}{2} \leq 3x \leq \frac{5}{2} \Leftrightarrow \frac{1}{2} \leq x \leq \frac{5}{6} \Leftrightarrow x \in [1/2, 5/6]$

Hence, the solution set of the given inequation is the interval [1/2, 5/6].

EXAMPLE 2 *Solve:* $|x - 2| \geq 5$

SOLUTION We know that: $|x - a| \geq r \Leftrightarrow x \leq a - r$, or $x \geq a + r$

$\therefore \quad |x - 2| \geq 5$

$\Leftrightarrow \quad x \leq 2-5$, or $x \geq 2+5$

$\Leftrightarrow \quad x \leq -3$ or $x \geq 7 \Leftrightarrow x \in (-\infty, -3]$ or $x \in [7, \infty) \Leftrightarrow x \in (-\infty, -3] \cup [7, \infty)$

Hence the solution set of the given inequation is $(-\infty, -3] \cup [7, \infty)$

EXAMPLE 3 *Solve* : $1 \leq |x - 2| \leq 3$

SOLUTION We know that :

$$a \leq |x-c| \leq b \Leftrightarrow x \in [-b+c, \ -a+c] \cup [a+c, \ b+c]$$

$\therefore \quad 1 \leq |x - 2| \leq 3 \Leftrightarrow x \in [-3+2, -1+2] \cup [1+2, 3+2] \Leftrightarrow x \in [-1, 1] \cup [3, 5]$

Hence, the solution set of the given inequation is $[-1, 1] \cup [3, 5]$.

EXAMPLE 4 *Solve the following system of inequations:* $|x-1| \leq 5, \quad |x| \geq 2$ [NCERT EXEMPLAR]

SOLUTION The given system of inequations is

$$|x-1| \leq 5 \qquad \text{...(i)}$$

$$|x| \geq 2 \qquad \text{...(ii)}$$

Now, $\quad |x-1| \leq 5$

$\Rightarrow \quad 1-5 \leq x \leq 1+5 \qquad [\because |x-a| \leq r \Leftrightarrow a-r \leq x \leq a+r]$

$\Rightarrow \quad -4 \leq x \leq 6 \Rightarrow x \in [-4, 6]$

Thus, the solution set of (i) is the interval $x \in [-4, 6]$.

and, $\quad |x| \geq 2 \Leftrightarrow x \leq -2$, or $x \geq 2 \qquad [\because |x| \geq a \Rightarrow x \leq -a$ or $x \leq a]$

$\qquad \Leftrightarrow x \in (-\infty, -2] \cup [2, \infty)$

Thus, the solution set of (ii) is $(-\infty, -2] \cup [2, \infty)$.

The solution sets of inequations (i) and (ii) are represented graphically in Figures 15.22 (i) and 15.22 (ii) respectively. The intersection of these two is $[-4, -2] \cup [2, 6]$

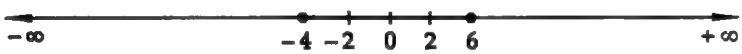


Fig. 15.22(i)

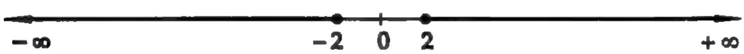


Fig. 15.22(ii)

Hence, the solution set of the given system of inequations is $[-4, -2] \cup [2, 6]$.

EXAMPLE 5 *Solve* : $\dfrac{|x|-1}{|x|-2} \geq 0, \ x \in R, \ x \neq \pm 2$.

SOLUTION We have,

$$\frac{|x|-1}{|x|-2} \geq 0$$

Fig. 15.23

$\Rightarrow \quad \dfrac{y-1}{y-2} \geq 0$, where $y = |x|$

$\Rightarrow \quad y \leq 1$ or $y > 2 \qquad$ [See Fig. 15.23]

$\Rightarrow \quad |x| \leq 1$ or $|x| > 2$

$\Rightarrow \quad (-1 \leq x \leq 1)$ or $(x < -2$ or $x > 2)$

$\Rightarrow \quad x \in [-1, 1]$ or $x \in (-\infty, -2) \cup (2, \infty)$

$\Rightarrow \quad x \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$

Hence, the solution set of the given inequation is $[-1, 1] \cup (-\infty, -2) \cup (2, \infty)$.

EXAMPLE 6 *Solve:* $\dfrac{-1}{|x|-2} \geq 1$, *where* $x \in R, x \neq \pm 2$

SOLUTION We have, $\dfrac{-1}{|x|-2} \geq 1$

$$\Rightarrow \quad \frac{-1}{|x|-2} - 1 \geq 0$$

$$\Rightarrow \quad \frac{-1-(|x|-2)}{|x|-2} \geq 0$$

$$\Rightarrow \quad \frac{1-|x|}{|x|-2} \geq 0$$

$$\Rightarrow \quad \frac{|x|-1}{|x|-2} \leq 0$$

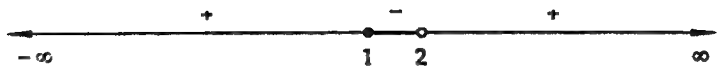


Fig. 15.24

$$\Rightarrow \quad \frac{y-1}{y-2} \leq 0, \text{ where } y = |x|$$

$\Rightarrow \quad 1 \leq y < 2$ [See Fig. 15.24]

$\Rightarrow \quad 1 \leq |x| < 2$ [$\because \ y = |x|$]

$\Rightarrow \quad x \in (-2, -1] \cup [1, 2)$ [$\because \ a < |x| \leq b \Leftrightarrow x \in [-b, -a) \cup (a, b]$]

Hence, the solution set of the given inequation is $(-2, -1] \cup [1, 2)$

EXAMPLE 7 *Solve the inequation:* $\left|\dfrac{2}{x-4}\right| > 1, \ x \neq 4.$

SOLUTION We have,

$$\left|\frac{2}{x-4}\right| > 1 \quad x \neq 4$$

$\Rightarrow \quad \dfrac{2}{|x-4|} > 1$ $\left[\because \ \left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}\right]$

$\Rightarrow \quad 2 > |x-4|$ [$\because \ |x-4| > 0$ for all $x \neq 4$]

$\Rightarrow \quad 4-2 < x < 4+2$ [$\because \ |x-a| < r \Leftrightarrow a-r < x < a+r$]

$\Rightarrow \quad 2 < x < 6$

$\Rightarrow \quad x \in (2, 6)$

But, $\quad x \neq 4.$

Hence, the solution set of the given inequation is $(2, 4) \cup (4, 6)$.

EXAMPLE 8 *Solve:* $\dfrac{|x+3|+x}{x+2} > 1$ **[NCERT EXEMPLAR]**

SOLUTION We have, $\dfrac{|x+3|+x}{x+2} > 1.$

Clearly, LHS of this inequation is meaningful for $x \neq -2$.

Now, $\quad \dfrac{|x+3|+x}{x+2} > 1$

$$\Rightarrow \quad \frac{|x+3|+x}{x+2} - 1 > 0$$

$$\Rightarrow \quad \frac{|x+3|+x-x-2}{x+2} > 0$$

$$\Rightarrow \quad \frac{|x+3|-2}{x+2} > 0.$$

Now two cases arise:

CASE I When $x+3\geq 0$ *i.e.* $x\geq -3$: In this case, $|x+3|=x+3$.

$\therefore \quad \dfrac{|x+3|-2}{x+2}>0$

$\Rightarrow \quad \dfrac{x+3-2}{x+2}>0$

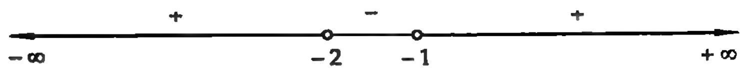


Fig. 15.25

$\Rightarrow \quad \dfrac{x+1}{x+2}>0$

$\Rightarrow \quad x\in(-\infty,-2)\cup(-1,\infty)$ [See Fig. 15.25]

But, $x\geq -3$. Therefore, the solution set of the given inequation in this case is $[-3,-2)\cup(-1,\infty)$.

CASE II *When* $x+3<0$ *i.e.* $x<-3$: In this case, $|x+3|=-(x+3)$.

$\therefore \quad \dfrac{|x+3|-2}{x+2}>0$

$\Rightarrow \quad \dfrac{-(x+3)-2}{x+2}>0$

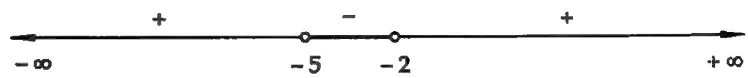


Fig. 15.26

$\Rightarrow \quad \dfrac{-(x+5)}{x+2}>0$

$\Rightarrow \quad \dfrac{x+5}{x+2}<0$

$\Rightarrow \quad x\in(-5-2)$ [See Fig. 15.26]

But, $x<-3$. Therefore, the solution set of the given inequation in this case is the interval $(-5,-3)$.

From Case I and Case II, we obtain that the solution set of the given inequation is

$$[-3,-2)\cup(-1,\infty)\cup(-5,-3) = (-5,-2)\cup(-1,\infty).$$

EXAMPLE 9 *Solve:* $|x-1|+|x-2|\geq 4$

SOLUTION On the LHS of the given inequation there are two terms both containing modulus. By equating the expressions within the modulus to zero, we get $x=1,2$ as critical points. These points divide real line in three parts viz. $(-\infty,1]$, $[1,2]$ and $[2,\infty)$. So, we consider the following three cases.

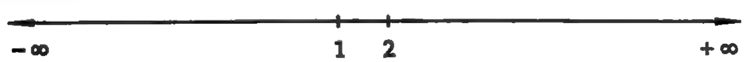


Fig. 15.27

CASE I *When* $-\infty<x<1$: In this case, we have $|x-1|=-(x-1)$ and $|x-2|=-(x-2)$

$\therefore \quad |x-1|+|x-2|\geq 4$

$\Rightarrow \quad -(x-1)-(x-2)\geq 4$

$\Rightarrow \quad -2x+3\geq 4$

$\Rightarrow \quad -2x\geq 1$

$\Rightarrow \quad x\leq -\dfrac{1}{2}$

But, $-\infty<x<1$. Therefore, in this case the solution set of the given inequation is $(-\infty,-1/2]$

CASE II When $1\leq x<2$: In this case, we have $|x-1|=(x-1)$ and $|x-2|=-(x-2)$

$\therefore \quad |x-1|+|x-2|\geq 4$

$\Rightarrow \quad x-1-(x-2) \geq 4$

$\Rightarrow \quad 1 \geq 4$, which is an absurd result.

So, the given inequation has no solution for $x \in [1, 2)$.

CASE III When $x \geq 2$: In this case, we have $|x-1| = x-1$ and $|x-2| = x-2$

$$\begin{aligned} \therefore \quad |x-1|+|x-2| &\geq 4 \\ \Rightarrow \quad x-1+x-2 &\geq 4 \\ \Rightarrow \quad 2x-3 &\geq 4 \\ \Rightarrow \quad 2x &\geq 7 \\ \Rightarrow \quad x &\geq \frac{7}{2} \end{aligned}$$

But, $x > 2$. Therefore, in this case the solution set of the given inequation is $[7/2, \infty)$.

Combining Case I and Case II, we obtain that the solution set of the given inequation is

$$(-\infty, -1/2] \cup [7/2, \infty)$$

EXAMPLE 10 *Solve:* $\dfrac{|x-1|}{x+2} < 1$.

SOLUTION We have,

$$\frac{|x-1|}{x+2} < 1 \Rightarrow \frac{|x-1|}{x+2} - 1 < 0 \Rightarrow \frac{|x-1|-(x+2)}{x+2} < 0$$

Now the following cases arise.

CASE I When $x-1 \geq 0$ *i.e.* $x \geq 1$: In this case, we have $|x-1| = x-1$

$$\begin{aligned} \therefore \quad \frac{|x-1|-(x+2)}{x+2} &< 0 \\ \Rightarrow \quad \frac{(x-1)-(x+2)}{x+2} &< 0 \\ \Rightarrow \quad \frac{-3}{x+2} &< 0 \\ \Rightarrow \quad x+2 &> 0 \qquad \left[\because \frac{a}{b} < \text{ and } a < 0 \Rightarrow b > 0\right] \\ \Rightarrow \quad x &> -2 \end{aligned}$$

But, $x \geq 1$. Therefore, $x > -2$ and $x \geq 1$. implies that $x \geq 1$. Thus, in this case the solution set of the given inequation is $[1, \infty)$.

CASE II When $x-1 < 0$ *i.e.* $x < 1$: In this case, we have $|x-1| = -(x-1)$.

$$\begin{aligned} \therefore \quad \frac{|x-1|-(x+2)}{x+2} &< 0 \\ \Rightarrow \quad \frac{-(x-1)-(x+2)}{x+2} &< 0 \\ \Rightarrow \quad -\frac{2x+1}{x+2} &< 0 \\ \Rightarrow \quad \frac{2x+1}{x+2} &> 0 \end{aligned}$$

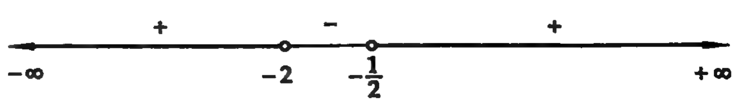


Fig. 15.28

$\Rightarrow \quad x \in (-\infty \;\; -2) \cup (-1/2, \infty)$ [See Fig. 15.28]

But, $x < 1$. Therefore, $x \in (-\infty, -2) \cup (-1/2, \infty)$ and $x < 1$ implies that $x \in (-\infty, -2) \cup \left(-\dfrac{1}{2}, 1\right)$.

Thus, in this case the solution set of the given inequation is $(-\infty, -2) \cup (-1/2, 1)$.

Combining Case I and Case II, we obtain that the solution set of the given inequation is

$$(-\infty, -2) \cup (-1/2, \infty)$$

EXERCISE 15.3

LEVEL-2

Solve each of the following system of equations in R.

1. $\left|x+\frac{1}{3}\right|>\frac{8}{3}$
2. $|4-x|+1<3$
3. $\left|\frac{3x-4}{2}\right| \le \frac{5}{12}$
4. $\frac{|x-2|}{x-2}>0$
5. $\frac{1}{|x|-3}<\frac{1}{2}$
6. $\frac{|x+2|-x}{x}<2$
7. $\left|\frac{2x-1}{x-1}\right|>2$
8. $|x-1|+|x-2|+|x-3| \ge 6$
9. $\frac{|x-2|-1}{|x-2|-2} \le 0$ [NCERT EXEMPLAR]
10. $\frac{1}{|x|-3} \le \frac{1}{2}$ [NCERT EXEMPLAR]
11. $|x+1|+|x|>3$ [NCERT EXEMPLAR]
12. $1 \le |x-2| \le 3$
13. $|3-4x| \ge 9$ [NCERT EXEMPLAR]

ANSWERS

1. $(-\infty, -3) \in (7/3, \infty)$
2. $(2, 6)$
3. $[19/18, 29/18]$
4. $(2, \infty)$
5. $(-\infty, -5) \cup (-3, 3) \cup (5, \infty)$
6. $(-\infty, 0) \cup (1, \infty)$
7. $(3/4, 1) \cup (1, \infty)$
8. $(-\infty, 0] \cup [4, \infty)$
9. $(0, 1] \cup [3, 4)$
10. $(-\infty, -5] \cup (-3, 3) \cup [5, \infty)$
11. $(-\infty, -2) \cup (1, \infty)$
12. $[-1, 1] \cup [3, 5]$
13. $(-\infty, -3/2] \cup [3, \infty)$

15.6 SOME APPLICATIONS OF LINEAR IN EQUATIONS IN ONE VARIABLE

In this section, we shall utilize the knowledge of solving linear in equations in one variable in solving different problems from various fields such as science, engineering, economics etc.

Following examples will illustrate the same

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find all pairs of consecutive odd positive integers, both of which are smaller than* 18, *such that their sum is more than* 20.

SOLUTION Let x be the smaller of the two consecutive odd positive integers. Then, the other odd integer is $x+2$.

It is given that both the integers are smaller than 18 and their sum is more than 20. Therefore,

$$x+2<18 \text{ and, } x+(x+2)>20$$

$$\Rightarrow \quad x<16 \text{ and } 2x+2>20$$

$$\Rightarrow \quad x<16 \text{ and } 2x>18$$

$$\Rightarrow \quad x<16 \text{ and } x>9 \Rightarrow 9<x<16 \Rightarrow x=11, 13, 15 \quad [\because x \text{ is an odd integer}]$$

Hence, the required pairs of odd integers are (11, 13), (13, 15) and (15, 17).

EXAMPLE 2 *Find all pairs of consecutive even positive integers, both of which are larger than 8, such that their sum is less than 25.*

SOLUTION Let x be the smaller of the two consecutive even positive integers. Then, the other even integer is $x + 2$.

It is sgiven that both the integers are larger than 8 and their sum is less than 25. Therefore,

$$x > 8 \text{ and } x + x + 2 < 25$$
$$\Rightarrow \quad x > 8 \text{ and } 2x + 2 < 25$$
$$\Rightarrow \quad x > 8 \text{ and } 2x < 23$$
$$\Rightarrow \quad x > 8 \text{ and } x < \frac{23}{2} \Rightarrow 8 < x < \frac{23}{2} \Rightarrow x = 10 \qquad [\because x \text{ is an even integer}]$$

Hence, the required pair of even integers is (10, 12).

EXAMPLE 3 *The cost and revenue functions of a product are given by* $C(x) = 2x + 400$ *and* $R(x) = 6x + 20$ *respectively, where* x *is the number of items produced by the manufacturer. How many items the manufacturer must sell to realize some profit?*

SOLUTION We know that: Profit = Revenue – Cost. Therefore, to earn some profit, we must have

$$\text{Revenue} > \text{Cost}$$
$$\Rightarrow \quad 6x + 20 > 2x + 400$$
$$\Rightarrow \quad 6x - 2x > 400 - 20 \Rightarrow 4x > 380 \Rightarrow x > \frac{380}{4} = 95$$

Hence, the manufacturer must sell more than 95 items to realize some profit.

EXAMPLE 4 *IQ of a person is given by the formula:* $IQ = \frac{MA}{CA} \times 100$, *where MA is mental age and CA is chronological age. If* $80 \leq IQ \leq 140$ *for a group of 12 year children, find the range of their mental age.*

SOLUTION We have: $CA = 12$ years

$$\therefore \quad IQ = \frac{MA}{CA} \times 100 \Rightarrow IQ = \frac{MA}{12} \times 100 = \frac{25}{3} MA$$

Now, $80 \leq IQ \leq 140$

$$\Rightarrow \quad 80 \leq \frac{25}{3} MA \leq 140$$
$$\Rightarrow \quad 240 \leq 25\, MA \leq 420 \Rightarrow \frac{240}{25} \leq MA \leq \frac{420}{25} \Rightarrow 9.6 \leq MA \leq 16.8$$

EXAMPLE 5 *In the first four papers each of 100 marks, Rishi got 95, 72, 73, 83 marks. If he wants an average of greater than or equal to 75 marks and less than 80 marks, find the range of marks he should score in the fifth paper.*

SOLUTION Suppose scores x marks in the fifth paper. Then,

$$75 \leq \frac{95 + 72 + 73 + 83 + x}{5} < 80$$
$$\Rightarrow \quad 75 \leq \frac{323 + x}{5} < 80 \Rightarrow 375 < 323 + x < 400 \Rightarrow 52 < x < 77$$

Hence, Rishi must score between 52 and 77 marks.

EXAMPLE 6 *A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18% ?*

SOLUTION Let x litres of 30% acid solution be added to 600 litres of 12% solution of acid. Then,

Total quantity of mixture $= (600 + x)$ litres

Total acid content in the $(600 + x)$ litres of mixture $= \frac{30x}{100} + \frac{12}{100} \times 600$

It is given that acid content in the resulting mixture must be more than 15% and less than 18%.

$\therefore \quad 15\% \text{ of } (600 + x) < \left(\frac{30x}{100} + \frac{12}{100} \times 600\right) < 18\% \text{ of } (600 + x)$

$\Rightarrow \quad \frac{15}{100} \times (600 + x) < \frac{30x}{100} + \frac{12}{100} \times 600 < \frac{18}{100} \times (600 + x)$

$\Rightarrow \quad 15(600 + x) < 30x + 12 \times 600 < 18(600 + x)$ [Multiplying through out by 100]

$\Rightarrow \quad 9000 + 15x < 30x + 7200 < 10800 + 18x$

$\Rightarrow \quad 9000 + 15x < 30x + 7200$ and $30x + 7200 < 10800 + 18x$

$\Rightarrow \quad 9000 - 7200 < 30x - 15x$ and $30x - 18x < 10800 - 7200$

$\Rightarrow \quad 1800 < 15x$ and $12x < 3600$

$\Rightarrow \quad 15x > 1800$ and $12x < 3600$

$\Rightarrow \quad x > 120$ and $x < 300$

$\Rightarrow \quad 120 < x < 300$

Hence, the number of litres of the 30% solution of acid must be more than 120 but less than 300.

EXAMPLE 7 *A man wants to cut three lengths from a single piece of board of length 91 cm. The second length is to be 3 cm longer than the shortest and third length is to be twice as long as the shortest. What are the possible lengths for the shortest board if third piece is to be at least 5 cm longer than the second?*

SOLUTION Let the length of the shortest piece be x cm. Then, the lengths of the second and third piece are $x + 3$ cm and $2x$ cm respectively. Then,

$x + (x + 3) + 2x \leq 91$ and $2x \geq (x + 3) + 5$

$\Rightarrow \quad 4x + 3 \leq 91$ and $2x \geq x + 8$

$\Rightarrow \quad 4x \leq 88$ and $x \geq 8 \Rightarrow x \leq 22$ and $x \geq 8 \Rightarrow 8 \leq x \leq 22.$

Hence, the shortest piece must be at least 8 cm long but not more than 22 cm long.

EXERCISE 15.4

LEVEL-1

1. Find all pairs of consecutive odd positive integers, both of which are smaller than 10, such that their sum is more than 11.
2. Find all pairs of consecutive odd natural number, both of which are larger than 10, such that their sum is less than 40.
3. Find all pairs of consecutive even positive integers, both of which are larger than 5, such that their sum is less than 23.
4. The marks scored by Rohit in two tests were 65 and 70. Find the minimum marks he should score in the third test to have an average of at least 65 marks.
5. A solution is to be kept between 86° and 95°F. What is the range of temperature in degree Celsius, if the Celsius (C)/Fahrenheit (F) conversion formula is given by $F = \frac{9}{5}C + 32.$
6. A solution is to be kept between 30°C and 35°C. What is the range of temperature in degee Fahrenheit?
7. To receive grade 'A' in a cource, one must obtain an average of 90 marks or more in five papers each of 100 marks. If Shikha scored 87, 95, 92 and 94 marks in first four papers, find the minimum marks that she must score in the last paper to get grade 'A' in the course.
8. A company manufactures cassettes and its cost and revenue functions for a week are $C = 300 + \frac{3}{2}x$ and $R = 2x$ respectively, where x is the number of cassettes produced and sold in a week. How many cassettes must be sold for the company to realize a profit?

9. The longest side of a triasngle is three times the shortest side and the third side is 2 cm shorter than the longest side if the perimeter of the triangles at least 61 cm, Find the minimum length of the shortest-side.
10. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?
11. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If there are 640 litres of the 8% solution, how many litres of 2% solution will have to be added?
12. The water acidity in a pool is considered normal when the average pH reading of three daily measurements is between 7.2 and 7.8. If the first two pH reading are 7.48 and 7.85, find the range of pH value for the third reading that will result in the acidity level being normal.

ANSWERS

1. (5,7), (7, 9)
2. (11, 13), (13, 15), (15, 17), (17, 19)
3. (6, 8), (8, 10), (10, 12)
4. 60
5. Between 30°C and 35°C
6. Between 86°F and 95°F
7. 82 marks
8. More than 600
9. 9 cm
10. More than 562.5 litres but less than 900 litres
11. More than 320 litres but less than 1280 litres
12. Between 6.27 and 8.07

15.7 GRAPHICAL SOLUTION OF LINEAR INEQUATIONS IN TWO VARIABLES

If a, b, c are real numbers, then the equation $ax + by + c = 0$ is called a linear equation in two variables x and y whereas the inequalities $ax + by \leq c$, $ax + by \geq c$, $ax + by < c$ and $ax + by > c$ are called linear inequations in two variables x and y.

We have studied in coordinate geometry that the graph of the equation $ax + by = c$ is a straight line which divides the xy-plane into two parts which are represented by $ax + by \leq c$ and $ax + by \geq c$. These two parts are known as the closed half-spaces. The regions represented by $ax + by < c$ and $ax + by > c$ are known as the open half spaces. In set theoretical notations, the set $\{(x, y) : ax + by = c\}$ is the straight line, sets $\{(x, y) : ax + by \leq c\}$ and $\{(x, y) : ax + by \geq c\}$ are closed half spaces and the sets $\{(x, y) : ax + by < c\}$ and $\{(x, y) : ax + by > c\}$ are open half-spaces. These half spaces are also known as the solution sets of the corresponding inequations.

In order to find the solution set of a linear inequation in two variables, we follow the following algorithm.

ALGORITHM

STEP I *Convert the given inequation, say $ax + by \leq c$, into the equation $ax + by = c$ which represents a straight line in xy-plane.*

STEP II *Put $y = 0$ in the equation obtained in step I to get the point where the line meets with x-axis. Similarly, put $x = 0$ to obtain a point where the line meets with y-axis.*

STEP III *Join the points obtained in step II to obtain the graph of the line obtained from the given inequation. In case of a strict inequality i.e. $ax + by < c$ or $ax + by > c$, draw the dotted line, otherwise mark it thick line.*

STEP IV *Choose a point, if possible (0, 0), not lying on this line : Substitute its coordinates in the inequation. If the inequation is satisfied, then shade the portion of the plane which contains the chosen point; otherwise shade the portion which does not contain the chosen point.*

STEP V *The shaded region obtained in step IV represents the desired solution set.*

REMARK *In case of the inequalities $ax + by \leq c$ and $ax + by > c$ points on the line are also a part of the shaded region while in case of inequalities $ax + by < c$ and $ax + by > c$ points on the line $ax + by = c$ are not in the shaded region.*

The following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the following inequations graphically:*

(i) $2x + 3y \leq 6$ (ii) $2x - y \geq 1$ (iii) $x \geq 2$ (iv) $y \leq -3$

SOLUTION (i) Converting the given inequation into equation, we obtain $2x + 3y = 6$.

Putting $y = 0$ and $x = 0$ respectively in this equation, we get $x = 3$ and $y = 2$. So, this line meets x-axis at $A(3, 0)$ and y-axis at $B(0, 2)$. We plot these points and join them by a thick line. This line divides the xy-plane in two parts. To determine the region represented by the given inequality consider the point $O(0, 0)$. Clearly, (0, 0) satisfies the inequality. So, the region containing the origin is represented by the given inequation as shown in Fig. 15.29. This region represents the solution set of the given inequations.

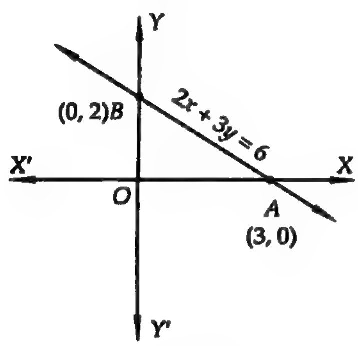


Fig. 15.29

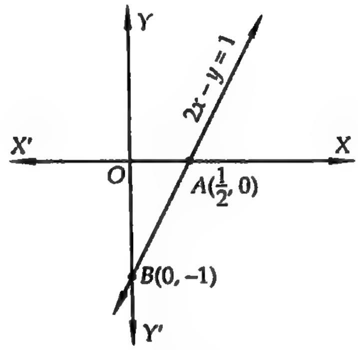


Fig. 15.30

(ii) Converting the given inequation into equation we obtain $2x - y = 1$. This line meets x and y-axes at $A(1/2, 0)$ and $B(0, -1)$ respectively. Joining these points by a thick line we obtain the line passing through A and B as shown in Fig. 15.30. This line divides the xy-plane into two regions viz. one lying above it and the other lying below it. Consider the point $O(0, 0)$. Clearly, (0, 0) does not satisfy the inequation $2x - y \geq 1$. So, the region not containing the origin is represented by the given inequation as shown in Fig. 15.30. Clearly it represents the solution set of the given inequation.

(iii) We have $x \geq 2$. Converting the inequation into equation, we obtain $x = 2$. Clearly, it is a line parallel to y-axis at a distance of 2 units from it. This line divides the xy-plane into two parts viz. one part on the LHS of $x = 2$ and the other on its RHS. We find that the point (0, 0) does not satisfy the inequation $x \geq 2$. So, the region represented by the given equation is the shaded region shown in Fig. 15.31. The shaded region is the required solution set of the given inequation.

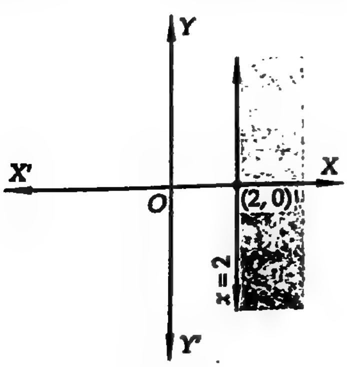


Fig. 15.31

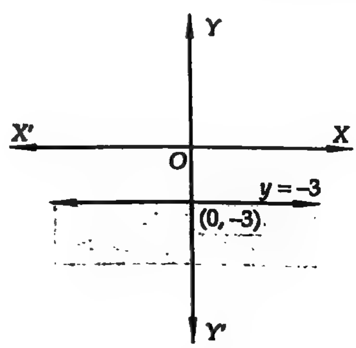


Fig. 15.32

(iv) We have $y \leq -3$. Converting the given inequation into equation we obtain $y = -3$. Clearly, it is a line parallel to x-axis at a distance of 3 units below it. The line $y = -3$ divides the xy-plane into two regions one below it and the other above it. Consider the point O (0, 0). We find that (0, 0) does not satisfy the inequation $y \leq -3$. So, the region represented by the given inequation is the region not containing the origin as show in Fig. 15.32. Clearly, it is the solution set of the given inequation.

EXAMPLE 2 *Solve the following inequations graphically:*

(i) $|x| \leq 3$ (ii) $|y - x| \leq 3$ (iii) $|x - y| \geq 1$

SOLUTION (i) Converting the given inequatino into equation, we obtain $x = 3$. This equation represents a line parallel to y-axis at a distance of 3 units from it. The line given by $x = 3$ divides the xy-plane into two regions. Clearly, the point O (0, 0) satisfies $x \leq 3$. So, the graph of $x \leq 3$ is as shown in Fig. 15.33. The shaded region represents the solution set of this inequation.

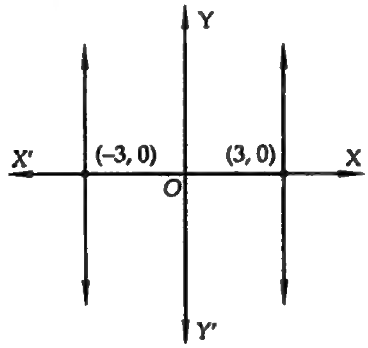


Fig. 15.33

(ii) We have, $|y - x| \leq 3$. This inequation is equivalent to

$$-3 \leq y - x \leq 3 \qquad [\because |x| \leq a \Leftrightarrow -a \leq x \leq a]$$

$$\Leftrightarrow \quad -3 \leq y - x \text{ and } y - x \leq 3$$

$$\Leftrightarrow \quad x - y - 3 \leq 0 \text{ and } x - y + 3 \geq 0$$

The region represented by $|y - x| \leq 3$ is the region common to the regions represented by $x - y - 3 \leq 0$ and $x - y + 3 \geq 0$ as shown in Fig. 15.34. This shaded region represents the solution set of the given inequation.

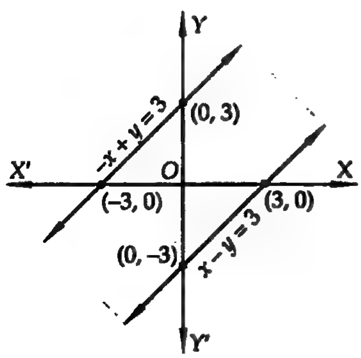


Fig. 15.34

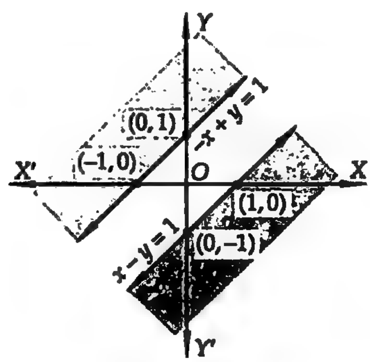


Fig. 15.35

(iii) We have,

$$|x - y| \geq 1 \Leftrightarrow x - y \geq 1 \text{ or } x - y \leq -1 \Leftrightarrow x - y - 1 > 0 \text{ or } x - y + 1 \leq 0$$

The required region is the union of regions represented by $x - y - 1 \geq 0$ and $x - y + 1 \leq 0$ as shown in Fig. 15.35. The shaded region represents the solution set of the given inequation.

EXERCISE 15.5

LEVEL-1

Represent to solution set of each of the following inequations graphically in two dimensional plane:

1. $x + 2y - y \leq 0$
2. $x + 2y \geq 6$
3. $x + 2 \geq 0$
4. $x - 2y < 0$
5. $-3x + 2y \leq 6$
6. $x \leq 8 - 4y$
7. $0 \leq 2x - 5y + 10$
8. $3y > 6 - 2x$
9. $y > 2x - 8$
10. $3x - 2y \leq x + y - 8$

15.8 SOLUTION OF SIMULTANEOUS LINEAR INEQUATIONS IN TWO VARIABLE

In this section, we will discuss the technique of finding the solution set of simultaneous linear inequations. Solving simultaneous linear inequations means finding the set of points (x, y) for which all the constraints are satisfied. Note that the solution set of simultaneous linear inequations may be an empty set or it may be the region bounded by the straight lines corresponding to linear inequations or it may be an unbounded region with straight line boundaries.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE SOLUTION SET REPRESENTED BY SIMULTANEOUS LINEAR INEQUATIONS**

EXAMPLE 1 *Exhibit graphically the solution set of the linear inequations*

$$3x + 4y \leq 12, \quad 4x + 3y \leq 12, \quad x \geq 0, \quad y \geq 0$$

SOLUTION Converting the inequations into equations, the inequations reduce to

$$3x + 4y = 12, 4x + 3y = 12, x = 0 \text{ and } y = 0.$$

Region Represented by $3x + 4y \leq 12$: The line $3x + 4y = 12$ meets the coordinate axes at A (4, 0) and B (0, 3). Draw a thick line joining A and B. We find that (0, 0) satisfies inequation $3x + 4y \leq 12$. So, the portion containing the origin represents the solution set of the inequation $3x + 4y \leq 12$.

Region Represented by $4x + 3y \leq 12$: The line $4x + 3y = 12$ meets the x and y-axes at $A_1(3, 0)$ and $B_1(0, 4)$ respectively. Join these two points by a thick line. Clearly, the region containing the origin is represented by the inequation $4x + 3y \leq 12$.

Region Represented by $x \geq 0$ and $y \geq 0$: Clearly, $x \geq 0$ and $y \geq 0$ represent the first quadrant.

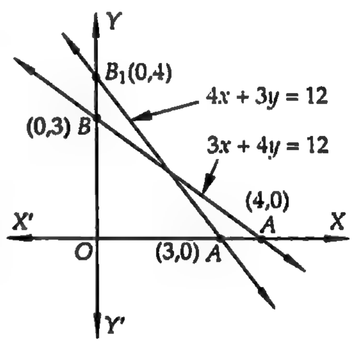


Fig. 15.36

Hence, the shaded region given in Fig. 15.36 represents the solution set of the given linear inequations.

EXAMPLE 2 *Exhibit graphically the solution set of the linear inequations*

$$x + y \leq 5, \quad 4x + y \geq 4, \quad x + 5y \geq 5, \quad x \leq 4, \quad y \leq 3$$

SOLUTION Converting the inequations into equations, we obtain

$$x + y = 5, \quad 4x + y = 4, \quad x + 5y = 5, \quad x = 4, \quad y = 3$$

Region Represented by $x + y \leq 5$: The line $x + y = 5$ meets the coordinate axes at $A(5, 0)$ and $B(0, 5)$ respectively. Join these points by a thick line. Clearly, (0, 0) satisfies the inequality $x + y \leq 5$. So, the portion containing the origin represents the solution set of the inequation $x + y \leq 5$.

Region Represented by $4x + y \geq 4$: The line $4x + y = 4$ meets the coordinate axes at $A_1(1, 0)$ and B_1 (0, 4)respectively. Join these points by a thick line. Clearly, (0, 0) does not satisfy the inequation $4x + y \geq 4$. So, the portion not containing the origin is represented by the inequation $4x + y \geq 4$.

Region Represented by $x+5y \geq 5$: The line $x+5y \geq 5$ meets the coordinate axes at A (5, 0) and B_2 (0, 1) respectively. Join these two points by a thick line. We find that (0, 0) does not satisfy the inequation $x+5y \geq 5$. So, the portion not containing the origin is represented by the given inequation.

Region Represented by $x \leq 4$: Clearly, $x=4$ is a line parallel to y-axis at a distance of 4 units from the origin. Since (0, 0) satisfies the inequation $x \leq 4$. So, the portion lying on the left side of $x=4$ is the region represented by $x \leq 4$.

Region Represented by $y \leq 3$: Clearly, $y=3$ is a line parallel to x-axis at a distance 3 from it. Since (0, 0) satisfies $y \leq 3$. So, the portion containing the origin is represented by the given inequation.

The common region of the above five regions represents the solution set of the given linear constaints as shown in Fig. 15.37.

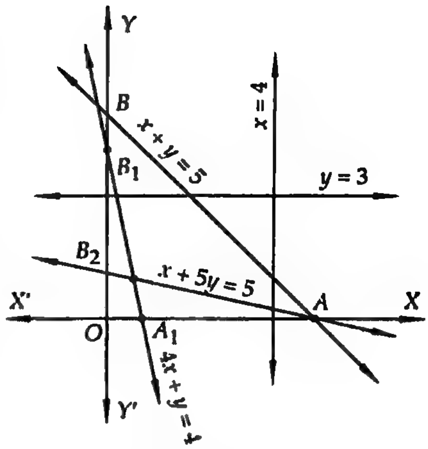


Fig. 15.37

EXAMPLE 3 *Draw the diagram of the solution set of the linear inequations* $3x+4y \geq 12$, $y \geq 1$, $x \geq 0$.

SOLUTION Converting the inequations into equations, we get $3x+4y=12$, $y=1$, $x=0$

Region Represented by $3x+4y \geq 12$: The line $3x+4y=12$ meets the coordinate axes at A (4, 0) and B(0, 3) joining these points by a thick line we get the graph of $3x+4y=12$. Since (0, 0) does not satisfy the inequation $3x+4y \geq 12$. So, the portion not containing the origin is represented by the inequation $3x+4y \geq 12$.

Region Represented by $y \geq 1$: The line $y=1$ is parallel to x-axis at a unit distance from it. Since (0, 0) does not satisfy the inequation $y \geq 1$. So, the region lying above the line $y=1$ is represented by $y \geq 1$.

Region Represented by $x \geq 0$: Clearly, $x \geq 0$ represents the region lying on the right side of y-axis.

The solution set of the given linear constraints is the intersection of the above regions as shown in Fig. 15.38.

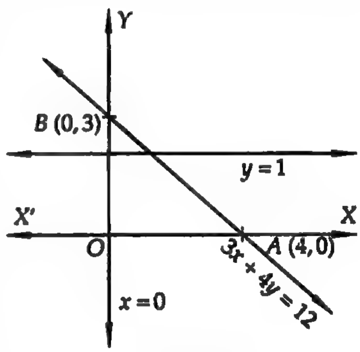


Fig. 15.38

Type II **ON FINDING THE LINEAR INEQUATIONS WHEN THEIR SOLUTION SET IS GIVEN**

EXAMPLE 4 *Find the linear inequations for which the shaded area in Fig. 15.39 is the solution set.*

SOLUTION Consider the line $x+2y=8$. We observe that the shaded region and the origin are on the same side of the line $x+2y=8$ and (0, 0) satisfies the linear constraint $x+2y \leq 8$. So, we must have one inequations as $x+2y \leq 8$.

Now consider the line $2x+y=2$. We find that the shaded region and the origin are on the opposite sides of the line $2x+y=2$ and (0, 0) does not satisfy the inequation $2x+y \geq 2$. So, the second inequations is $2x+y \geq 2$.

Finally, consider the line $x-y=1$. We observe that the shaded region and the origin are on the same side of the line $x-y=1$. We observe that the shaded region and the origin are on the same side of the line $x-y=1$ and (0, 0) satisfies $x-y \leq 1$. So, the third constraint is $x-y \leq 1$.

We also notice that the shaded region is above x-axis and is on the right side of y-axis. So, we must have $x \geq 0$ and $y \geq 0$.

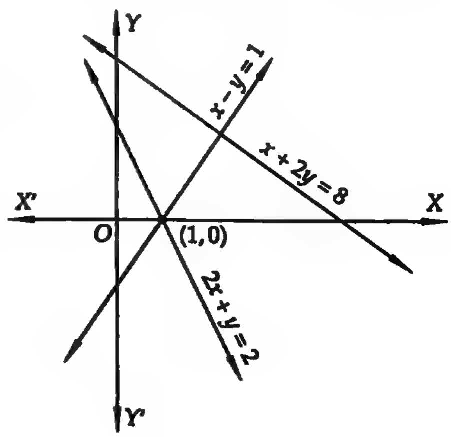


Fig. 15.39

Thus, the linear inequations corresponding to the given solution set are

$$x + 2y \le 8,\ 2x + y \ge 2,\ x - y \le 1,\ x \ge 0,\ y \ge 0$$

EXAMPLE 5 *Find the linear inequations for which the shaded region in Fig. 15.40 is the solution set.*

SOLUTION Consider the line $2x + 3y = 3$. We observe that the shaded region and the origin lie on the opposite side of this line and (0, 0) satisfies $2x + 3y \le 3$. Therefore, we must have $2x + 3y \ge 3$ as the linear inequations corresponding to the line $2x + 3y = 3$.

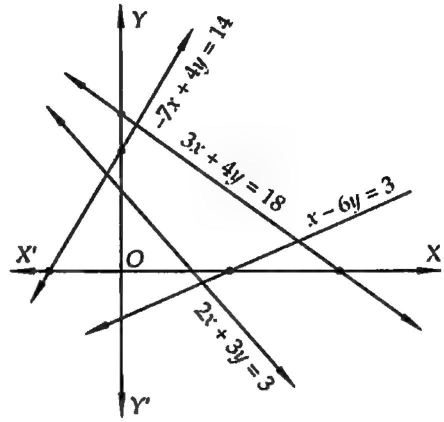


Fig. 15.40

Consider the line $3x + 4y = 18$. Clearly, the shaded region and the origin lie on the same side of this line and (0, 0) satisfies the inequation $3x + 4y \le 18$. So, we must have $3x + 4y \le 18$ as the linear inequations corresponding to $3x + 4y = 18$.

Consider the line $x - 6y = 3$. It is evident from the figure that the origin and the shaded region lie on the same side of this line and (0, 0) satisfies $x - 6y \le 3$. So, $x - 6y \le 3$ is the corresponding inequations.

Consider the line $-7x + 4y = 14$. We find that the shaded region and the origin are on the same side of this line and (0, 0) satisfies the inequations $-7x + 4y \le 14$. So, the corresponding linear inequations is $-7x + 4y \le 14$.

Also, the shaded region is in first quadrant only. So, we must have $x \ge 0$ and $y \ge 0$.

Thus, the linear inequations comprising the given solution set are

$$2x + 3y \ge 3,\ 3x + 4y \le 18,\ -7x + 4y \le 14,\ x - 6y \le 3,\ x \ge 0,\ y \ge 0$$

EXERCISE 15.6

LEVEL-1

1. Solve the following systems of linear inequations graphically:
   (i) $2x + 3y \le 6,\ 3x + 2y \le 6,\ x \ge 0,\ y \ge 0$ (ii) $2x + 3y \le 6,\ x + 4y \le 4,\ x \ge 0,\ y \ge 0$
   (iii) $x - y \le 1,\ x + 2y \le 8,\ 2x + y \ge 2,\ x \ge 0,\ y \ge 0$
   (iv) $x + y \ge 1,\ 7x + 9y \le 63,\ x \le 6,\ y \le 5,\ x \ge 0,\ y \ge 0$
   (v) $2x + 3y \le 35,\ y \ge 3,\ x \ge 2,\ x \ge 0,\ y \ge 0$
2. Show that the solution set of the following linear inequations is empty set:
   (i) $x - 2y \ge 0,\ 2x - y \le -2,\ x \ge 0,\ y \ge 0$ (ii) $x + 2y \le 3,\ 3x + 4y \ge 12,\ y \ge 1,\ x \ge 0,\ y \ge 0$
3. Find the linear inequations for which the shaded area in Fig. 15.41 is the solution set. Draw the diagram of the solution set of the linear inequations:

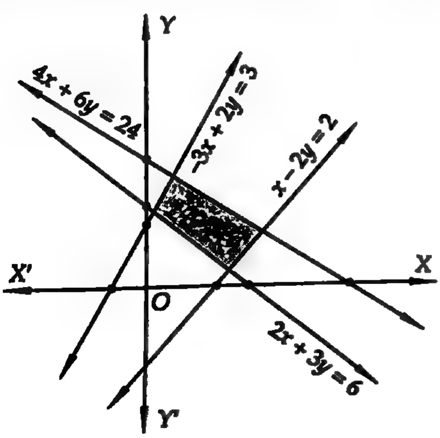


Fig. 15.41

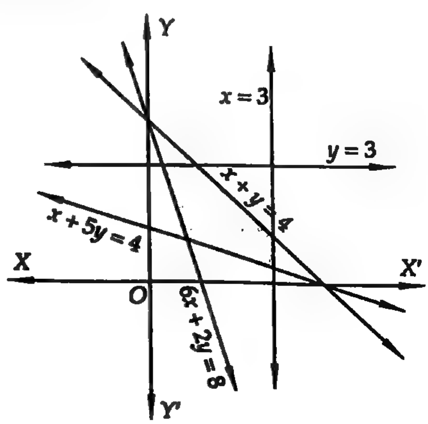


Fig. 15.42

4. Find the linear inequations for which the solution set is the shaded region given in Fig. 15.42.
5. Show that the solution set of the following linear in equations is an unbounded set:
   $x + y \geq 9,\ 3x + y \geq 12,\ x \geq 0,\ y \geq 0.$
6. Solve the following systems of inequations graphically:
   (i) $2x + y \geq 8,\ x + 2y \geq 8,\ x + y \leq 6$
   (ii) $12x + 12y \leq 840,\ 3x + 6y \leq 300,\ 8x + 4y \leq 480\quad x \geq 0, y \geq 0$
   (iii) $x + 2y \leq 40,\ 3x + y \geq 30,\ 4x + 3y \geq 60,\ x \geq 0,\ y \geq 0$
   (iv) $5x + y \geq 10,\ 2x + 2y \geq 12,\ x + 4y \geq 12,\ x \geq 0,\ y \geq 0$
7. Show that the following system of linear equations has no solution:
   $x + 2y \leq 3,\ 3x + 4y \geq 12,\ x \geq 0,\ y \geq 1.$
8. Show that the solution set of the following system of linear inequalities is an unbounded region $2x + y \geq 8,\ x + 2y \geq 10,\ x \geq 0,\ y \geq 0.$

ANSWERS

3. $2x + 3y \geq 6, 4x + 6y \leq 24, -3x + 2y \leq 3, x - 2y \leq 2, x \geq 0, y \geq 0,$
4. $x + y \leq 4, y \leq 3, x \leq 3, x + 5y \geq 4, 6x + 2y \geq 8, x \geq 0, y \geq 0$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the solution set of the inequation $\dfrac{x^2}{x-2} > 0.$
2. Write the solution set of the inequation $x + \dfrac{1}{x} \geq 2.$
3. Write the set of values of x satisfying the inequation $(x^2 - 2x + 1)(x - 4) \geq 0.$
4. Write the solution set of the equation $|2 - x| = x - 2.$
5. Write the set of values of x satisfying $|x - 1| \leq 3$ and $|x - 1| \leq 1.$
6. Write the solution set of the inequation $\left|\dfrac{1}{x} - 2\right| < 4.$
7. Write the number of integral solutions of $\dfrac{x+2}{x^2+1} > \dfrac{1}{2}.$
8. Write the set of values of x satisfying the inequations $5x + 2 < 3x + 8$ and $\dfrac{x+2}{x-1} < 4.$
9. Write the solution set of $\left|x + \dfrac{1}{x}\right| > 2.$
10. Write the solution set of the inequation $|x - 1| \geq |x - 3|.$

ANSWERS

1. $[2, \infty)$	2. $(0, \infty)$	3. $(-\infty, 4)$	4. $(2, \infty)$
5. $[2, 4]$	6. $(-\infty, -1/2) \cup (1/6, \infty)$		7. 3
8. $(-\infty, 1) \cup (2, 3)$	9. $R - \{-1, 0, 1\}$	10. $[2, \infty)$	

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $x < 7$, then
   (a) $-x < -7$ (b) $-x \leq -7$ (c) $-x > -7$ (d) $-x \geq -7$

2. If $-3x+17<-13$, then
   (a) $x\in(10,\infty)$ (b) $x\in[10,\infty)$ (c) $x\in(-\infty,10]$ (d) $x\in[-10,10)$
3. Given that x, y and b are real numbers and $x<y$, $b>0$, then
   (a) $\frac{x}{b}<\frac{y}{b}$ (b) $\frac{x}{b}\le\frac{y}{b}$ (c) $\frac{x}{b}>\frac{y}{b}$ (d) $\frac{x}{b}\ge\frac{y}{b}$
4. If x is a real number and $|x|<5$, then
   (a) $x\ge5$ (b) $-5<x<5$ (c) $x\le-5$ (d) $-5\le x\le5$
5. If x and a are real numbers such that $a>0$ and $|x|>a$, then
   (a) $x\in(-a,\infty)$ (b) $x\in[-\infty,a]$ (c) $x\in(-a,a)$ (d) $x\in(-\infty,-a)\cup(a,\infty)$
6. If $|x-1|>5$, then
   (a) $x\in(-4,6)$ (b) $x\in[-4,6]$
   (c) $x\in(-\infty,-4)\cup(6,\infty)$ (d) $x\in(-\infty,-4)\cup[6,\infty)$
7. If $|x+2|\le9$, then
   (a) $x\in(-7,11)$ (b) $x\in[-11,7]$
   (c) $x\in(-\infty,-7)\cup(11,\infty)$ (d) $x\in(-\infty,-7)\cup[11,\infty)$
8. The inequality representing the following graph is
   (a) $|x|<3$ (b) $|x|\le3$ (c) $|x|>3$ (d) $|x|\ge3$

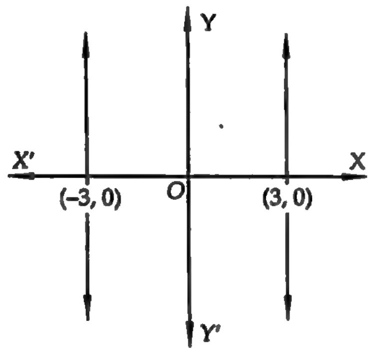


Fig. 15.43

9. The linear inequality representing the solution set given in Fig. 15.44 is
   (a) $|x|<5$ (b) $|x|>5$ (c) $|x|\ge5$ (d) $|x|\le5$

Fig. 15.44

10. The solution set of the inequation $|x+2|\le5$ is
    (a) $(-7,5)$ (b) $[-7,3]$ (c) $[-5,5]$ (d) $(-7,3)$
11. If $\frac{|x-2|}{x-2}\ge0$, then
    (a) $x\in[2,\infty)$ (b) $x\in(2,\infty)$ (c) $x\in(-\infty,2)$ (d) $x\in(-\infty,2]$
12. If $|x+3|\ge10$, then
    (a) $x\in(-13,7]$ (b) $x\in(-13,7)$
    (c) $x\in(-\infty,-13)\cup(7,\infty)$ (d) $x\in(-\infty,-13]\cup[7,\infty)$

ANSWERS

1. (c) 2. (a) 3. (a) 4. (b) 5. (d) 6. (c) 7. (b) 8. (b)
9. (c) 10. (b) 11. (b) 12. (d)

CHAPTER 16

PERMUTATIONS

16.1 THE FACTORIAL

In this section, we shall introduce the term and notation of factorial which will be often used in this chapter and the next three chapters.

FACTORIAL *The continued product of first n natural numbers is called the "n factorial" and is denoted by n! or $\lfloor n$.*

i.e $\quad n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n.$

Thus, $\quad 3! = 1 \times 2 \times 3 = 6;\ 4! = 1 \times 2 \times 3 \times 4 = 24,\ 5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ etc.

Clearly, $n!$ is defined for positive integers only.

ZERO FACTORIAL *As we will require zero factorial in the later sections of this chapter and it does not make any sense to define it as the product of the integers from 1 to zero. So, we define 0! = 1.*

NOTE *Factorials of proper fractions or negative integers are not defined. Factorial n is defined only for whole numbers.*

DEDUCTION We have,

$$n! = 1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = [1 \times 2 \times 3 \times 4 \ldots \times (n-1)]\, n = [(n-1)\,!]\, n = n \times (n-1)!$$

Thus, $\quad n! = n \times (n-1)!$

Similarly,

$$n! = n\,(n-1)\,(n-2)! = n\,(n-1)\,(n-2)\,(n-3)! = n\,(n-1)\,(n-2)\,(n-3)\,(n-4)! \text{ and so on.}$$

For example, $8! = 8\,(7!),\quad 5! = 5\,(4!)\quad$ and $\quad 2! = 2\,(1!)$

Following examples will illustrate the use of this property of factorial n.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Compute:* (i) $\dfrac{20!}{18!}$ $\qquad$ (ii) $\dfrac{10!}{6!\,4!}$

SOLUTION (i) We have,

$$\frac{20!}{18!} = \frac{20\,(19!)}{18!} = \frac{20 \times 19 \times 18!}{18!} \qquad [\because n! = n \times (n-1)!]$$

$$= 20 \times 19 = 380$$

(ii) $$\frac{10!}{6!\,4!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6! \times (4 \times 3 \times 2 \times 1)} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

EXAMPLE 2 *Convert the following products into factorials:*

(i) $6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$ $\qquad$ (ii) $2 \cdot 4 \cdot 6 \cdot 8 \cdot 10$

SOLUTION (i) $$6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{10!}{5!}$$

(ii) $$2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 = (2 \times 1)\,(2 \times 2)\,(2 \times 3)\,(2 \times 4)\,(2 \times 5) = 2^5 \times (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5) = 2^5 \times 5!$$

EXAMPLE 3 *Find the LCM of 4!, 5! and 6!*

SOLUTION We have, $5! = 5 \times 4!$ and $6! = 6 \times 5 \times 4!$

$\therefore$ L.C.M. of $4!, 5!, 6! =$ L.C.M. $\{4!, 5 \times 4!, 6 \times 5 \times 4!\} = (4!) \times 5 \times 6 = 6! = 720$

EXAMPLE 4 *If* $\dfrac{1}{9!} + \dfrac{1}{10!} = \dfrac{x}{11!}$*, find x.*

SOLUTION We have,

$$\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$$

$$\Rightarrow \quad \frac{1}{9!} + \frac{1}{10 \times 9!} = \frac{x}{11 \times 10 \times 9!}$$

$$\Rightarrow \quad \frac{1}{9!}\left(1 + \frac{1}{10}\right) = \left(\frac{x}{11 \times 10}\right) \times \frac{1}{9!}$$

$$\Rightarrow \quad 1 + \frac{1}{10} = \frac{x}{11 \times 10} \Rightarrow \frac{11}{10} = \frac{x}{11 \times 10} \Rightarrow x = 11 \times 11 = 121.$$

<u>ALITER</u> We have,

$$\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$$

Multiplying both sides by the LCM of 9!, 10! and 11! i.e. by 11!, we obtain

$$\frac{11!}{9!} + \frac{11!}{10!} = \frac{x}{11!} \times 11!$$

$$\Rightarrow \quad \frac{11 \times 10 \times 9!}{9!} + \frac{11 \times 10!}{10!} = x$$

$$\Rightarrow \quad 11 \times 10 + 11 = x$$

$$\Rightarrow \quad x = 121.$$

EXAMPLE 5 *Find n, if:*

(i) $(n+2)! = 2550 \times n!$ (ii) $(n+1)! = 12 \times (n-1)!$

SOLUTION (i) We have,

$$(n+2)! = 2550 \times n!$$

$$\Rightarrow \quad (n+2)(n+1) \times n! = 2550 \times n!$$

$$\Rightarrow \quad (n+2) \times (n+1) = 2550$$

$$\Rightarrow \quad (n+2)(n+1) = 51 \times 50$$ [Expressing 2550 as the product of two consecutive natural numbers]

$$\Rightarrow \quad n+2 = 51 \text{ or } n+1 = 50$$ [By comparing]

$$\Rightarrow \quad n = 49$$

(ii) We have,

$$(n+1)! = 12 \times (n-1)!$$

$$\Rightarrow \quad (n+1) \times n \times (n-1)! = 12 \times (n-1)!$$

$$\Rightarrow \quad n(n+1) = 12$$

$$\Rightarrow \quad (n+1)n = 4 \times 3 \Rightarrow n = 3$$ [By comparing]

EXAMPLE 6 *If* $\frac{n!}{2!(n-2)!}$ and $\frac{n!}{4!(n-4)!}$ *are in the ratio 2 : 1, find the value of n.*

SOLUTION We have,

$$\frac{n!}{2!(n-2)!} : \frac{n!}{4!(n-4)!} = 2:1$$

$$\Rightarrow \quad \frac{n!}{2!(n-2)!} \times \frac{4!(n-4)!}{n!} = \frac{2}{1}$$

$$\Rightarrow \quad \frac{4!(n-4)!}{2!(n-2)\times(n-3)\times(n-4)!} = \frac{2}{1}$$

$$\Rightarrow \quad \frac{4\times 3\times 2!}{2!(n-2)(n-3)} = \frac{2}{1}$$

$$\Rightarrow \quad (n-2)(n-3) = 6 \Rightarrow (n-2)(n-3) = 3\times 2 \Rightarrow n-2=3 \text{ and } n-3=2 \Rightarrow n=5$$

EXAMPLE 7 *Prove that:* $\frac{(2n)!}{n!} = \left\{1\cdot 3\cdot 5 \ldots (2n-1)\right\} 2^n$.

SOLUTION We have,

$$\frac{(2n)!}{n!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\ldots(2n-2)(2n-1)(2n)}{n!}$$

$$= \frac{\{1\cdot 3\cdot 5\cdot 7\ldots(2n-1)\}\cdot\{2\cdot 4\cdot 6\cdot 8\ldots(2n-2)(2n)\}}{n!}$$

$$= \frac{\{1\cdot 3\cdot 5\cdot 7\ldots(2n-1)\}\, 2^n\, \{1\cdot 2\cdot 3\cdot 4\ldots(n-1)\, n\}}{n!}$$

$$= \frac{\{1\cdot 3\cdot 5\cdot 7\ldots(2n-1)\}\cdot 2^n\cdot n!}{n!} = \{1\cdot 3\cdot 5\cdot 7\ldots(2n-1)\}\, 2^n$$

LEVEL-2

EXAMPLE 8 *Prove that* $(n!+1)$ *is not divisible by any natural number between 2 and n.*

SOLUTION Let m be divisible by k and r be any natural number between 1 and k. If $m+r$ is divided by k, then we obtain r as the remainder.

We have, $n! = 1\cdot 2\cdot 3\cdot 4\ldots(n-1)\cdot n$.

Therefore, $n!$ is divisible by every natural number between 2 and n. So, $(n!+1)$, when divided by any natural number between 2 and n, leaves 1 as the remainder.

Hence, $(n!+1)$ is not divisible by any natural number between 2 and n.

EXAMPLE 9 *Prove the inequalities* $(n!)^2 \le n^n(n!) < (2n)!$ *for all positive integers n.*

SOLUTION Clearly,

$$(n!)^2 = (n!)(n!) = (1\cdot 2\cdot 3\cdot 4\ldots(n-1)\,n)(n!)$$

Now,

$$\left.\begin{array}{r} 1\le n \\ 2\le n \\ 3\le n \\ \vdots\ \vdots\ \vdots \\ (n-1)\le n \\ n\le n \end{array}\right\} \Rightarrow 1\cdot 2\cdot(n-1)\,n \le \underbrace{n\cdot n\cdot n\ldots n}_{n-\text{times}}$$

$$\Rightarrow \quad n! \le n^n \Rightarrow (n!)(n!) \le n^n(n!) \Rightarrow (n!)^2 \le n^n(n!) \qquad \ldots(i)$$

We have, $(2n)! = 1\cdot 2\cdot 3\ldots(n-1)\,n\,(n+1)(n+2)\ldots(2n-1)(2n)$

$\Rightarrow \quad (2n)! = n!(n+1)(n+2)\dots(2n-1)(2n)$

Now, $\left.\begin{array}{r} n+1>n \\ n+2>n \\ n+3>n \\ \vdots \quad \vdots \quad \vdots \\ n+(n-1)>n \\ n+n>n \end{array}\right\} \Rightarrow (n+1)(n+2)(n+3)\dots(2n-1)(2n)>n^n$

$\Rightarrow \quad n!(n+1)(n+2)\dots(2n-1)(2n)>n!\,n^n$

$\Rightarrow \quad (2n)!>n!\ n^n \Rightarrow n!\,n^n<(2n)!$ ...(ii)

From (i) and (ii), we get $(n!)^2 \le n^n (n!) < (2n)!$

EXAMPLE 10 *Prove that 33! is divisible by 2^{15}. What is the largest integer n such that 33! is divisible by 2^n?*

SOLUTION Let $E_2(n)$ denote the index of 2 in n. Then,

$$E_2(33!) = E_2(1.2.3.4.5.6.....32.33)$$
$$\Rightarrow E_2(33!) = E_2(2.4.6.8.....30.32)$$
$$\Rightarrow E_2(33!) = 16+E_2(1.2.3......15.16)$$
$$\Rightarrow E_2(33!) = 16+E_2(2.4.6....14.16)$$
$$\Rightarrow E_2(33!) = 16+8+E_2(1.2.3\dots 8)$$
$$\Rightarrow E_2(33!) = 16+8+E_2(2.4.6.8)$$
$$\Rightarrow E_2(33!) = 16+8+4+E_2(1.2.3.4)$$
$$\Rightarrow E_2(33!) = 16+8+4+E_2(2.4) = 16+8+4+3 = 31.$$

Thus, exponent of 2 in 33! is 31 i.e. $33! = 2^{31} \times$ an integer

This shows that 33! is divisible by 2^{15} and the largest integer n such that 33! is divisible by 2^n is 31.

EXERCISE 16.1

LEVEL-1

1. Compute:
 (i) $\dfrac{30!}{28!}$ (ii) $\dfrac{11!-10!}{9!}$ (iii) L.C.M. (6!, 7!, 8!)

2. Prove that $\dfrac{1}{9!}+\dfrac{1}{10!}+\dfrac{1}{11!}=\dfrac{122}{11!}$

3. Find x in each of the following:
 (i) $\dfrac{1}{4!}+\dfrac{1}{5!}=\dfrac{x}{6!}$ (ii) $\dfrac{x}{10!}=\dfrac{1}{8!}+\dfrac{1}{9!}$ [NCERT] (iii) $\dfrac{1}{6!}+\dfrac{1}{7!}=\dfrac{x}{8!}$ [NCERT]

4. Convert the following products into factorials:
 (i) $5\cdot6\cdot7\cdot8\cdot9\cdot10$ (ii) $3\cdot6\cdot9\cdot12\cdot15\cdot18$
 (iii) $(n+1)(n+2)(n+3)\dots(2n)$ (iv) $1\cdot3\cdot5\cdot7\cdot9\dots(2n-1)$

5. Which of the following are true:
 (i) $(2+3)!=2!+3!$ (ii) $(2\times 3)!=2!\times 3!$

6. Prove that : $n!(n+2)=n!+(n+1)!$

7. If $(n+2)!=60[(n-1)!]$, find n.

8. If $(n+1)!=90[(n-1)!]$, find n.

9. If $(n+3)! = 56\,[(n+1)!]$, find n.

10. If $\dfrac{(2n)!}{3!(2n-3)!}$ and $\dfrac{n!}{2!(n-2)!}$ are in the ratio 44 : 3, find n.

11. Prove that:

(i) $\dfrac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-(r-1))$ (ii) $\dfrac{n!}{(n-r)!r!} + \dfrac{n!}{(n-r+1)!(r-1)!} = \dfrac{(n+1)!}{r!(n-r+1)!}$

12. Prove that: $\dfrac{(2n+1)!}{n!} = 2^n \left\{1 \cdot 3 \cdot 5 \ldots (2n-1)(2n+1)\right\}$

ANSWERS

1. (i) 870 (ii) 100 (iii) 8! 3. (i) 36 (ii) 100 (iii) 64 4. (i) $\dfrac{10!}{4!}$ (ii) $3^6\,(6!)$ (iii) $\dfrac{(2n)!}{n!}$ (iv) $\dfrac{(2n)!}{2^n\, n!}$ 5.(i) False (ii) False 7. 3 8. 9 9. 5 10., 6

HINTS TO NCERT & SELECTED PROBLEMS

3. (i) We have,

$$\frac{1}{4!} + \frac{1}{5!} = \frac{x}{6!}$$

$$\Rightarrow \frac{6!}{4!} + \frac{6!}{5!} = x \qquad \text{[Multiplying both sides by 6!]}$$

$$\Rightarrow \frac{6 \times 5 \times 4!}{4!} + \frac{6 \times 5!}{5!} = x \Rightarrow 6 \times 5 + 6 = x \Rightarrow x = 36$$

(ii) We have,

$$\frac{x}{10!} = \frac{1}{8!} + \frac{1}{9!}$$

$$\Rightarrow x = \frac{10!}{8!} + \frac{10!}{9!} \qquad \text{[Multiplying both sides by 10!]}$$

$$\Rightarrow x = \frac{10 \times 9 \times 8!}{8!} + \frac{10 \times 9!}{9!}$$

$$\Rightarrow x = 10 \times 9 + 10 = 100$$

(iii) We have,

$$\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$$

$$\Rightarrow \frac{8!}{6!} + \frac{8!}{7!} = x \qquad \text{[Multiplying both sides by 8!]}$$

$$\Rightarrow \frac{8 \times 7 \times 6!}{6!} + \frac{8 \times 7!}{7!} = x$$

$$\Rightarrow 8 \times 7 + 8 = x$$

$$\Rightarrow x = 64$$

7. $(n+2)! = 60(n-1)!$

$\Rightarrow (n+2)(n+1)(n)(n-1)! = 60 \times (n-1)!$

$\Rightarrow (n+2)(n+1)(n) = 5 \times 4 \times 3$ [Expressing 60 as the product of three consecutive integers]

$\Rightarrow n = 3$ [On comparing two sides]

8. $(n+1)! = 90(n-1)!$

$\Rightarrow n+1(n)(n-1)! = 90(n-1)!$

$\Rightarrow (n+1)n = 10 \times 9$ [Writing 90 as the product of consecutive integers]

$\Rightarrow n = 9$

9. $(n+3)! = 56\,(n+1)!$

$\Rightarrow \quad (n+3)(n+2)(n+1)! = 56\,(n+1)!$

$\Rightarrow \quad (n+3)(n+2) = 8 \times 7$ [Writing 56 as the product of consecutive integers]

$\Rightarrow \quad n+2 = 7 \Rightarrow n = 5$

16.2 FUNDAMENTAL PRINCIPLES OF COUNTING

In this section, we shall discuss two fundamental principles viz. principle of addition and principle of multiplication. These two principles will enable us to understand permutations and combinations. In fact these two principles form the base of permutations and combinations.

FUNDAMENTAL PRINCIPLE OF MULTIPLICATION *If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways; then the two jobs in succession can be completed in $m \times n$ ways.*

EXPLANATION If the first job is performed in any one of the m ways, we can associate with this any one of the n ways of performing the second job: and thus there are n ways of performing the two jobs without considering more than one way of performing the first; and so corresponding to each of the m ways of performing the first job, we have n ways of performing the second job. Hence, the number of ways in which the two jobs can be performed is $m \times n$.

ILLUSTRATION 1 *In a class there are 10 boys and 8 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make this selection ?*

SOLUTION Here the teacher is to perform two jobs:

(i) selecting a boy among 10 boys, and (ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore by the fundamental principle of multiplication, the required number of ways is $10 \times 8 = 80$.

REMARK *The above principle can be extended for any finite number of jobs as stated below:*

If there are n jobs $J_1, J_2, \dots J_n$ such that job J_i can be performed independently in m_i ways; $i = 1, 2, \dots, n$. Then the total number of ways in which all the jobs can be performed is $m_1 \times m_2 \times m_3 \times \dots \times m_n$.

FUNDAMENTAL PRINCIPLE OF ADDITION *If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m+n)$ ways.*

ILLUSTRATION 2 *In a class there are 10 boys and 8 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways the teacher can make this selection?*

SOLUTION Here the teacher is to perform either of the following two jobs :

(i) selecting a boy among 10 boys. or, (ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore, by fundamental principle of addition either of the two jobs can be performed in $(10 + 8) = 18$ ways. Hence, the teacher can make the selection of either a boy or a girl in 18 ways.

DIFFERENCE BETWEEN THE TWO PRINCIPLES *As we have discussed in the principle of multiplication a job is divided or decomposed into a number of sub-jobs which are unconnected to each other and the job is said to be performed if each sub-job is performed. While in the principle of addition there are a number of independent jobs and we have to perform one of them. So, the total number of ways of completing any one of the sub-jobs is the sum of the number of ways of completing each sub-jobs.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *There are 3 candidates for a Classical, 5 for a Mathematical, and 4 for a Natural science scholarship.*

(i) In how many ways can these scholarships be awarded?

(ii) In how many ways one of these scholarships be awarded?

SOLUTION Clearly, Classical scholarship can be awarded to any one of the three candidates. So, there are 3 ways of awarding the Classical scholarship.

Similarly, Mathematical and Natural science scholarships can be awarded in 5 and 4 ways respectively. So, by Fundamental Principle of multiplication,

Number of ways of awarding three scholarships $= 3 \times 5 \times 4 = 60$

By Fundamental Principle of addition,

Number of way of awarding one of the three scholarships $= 3 + 5 + 4 = 12$

EXAMPLE 2 *A room has 6 doors. In how many ways can a man enter the room through one door and come out through a different door ?*

SOLUTION Clearly, a person can enter the room through any one of the six doors. So, there are six ways of entering into the room. After entering into the room, the man can come out through any one of the remaining five doors. So, he can come out through a different door in 5 ways.

Hence, the number of ways in which a man can enter a room through one door and come out through a different door $= 6 \times 5 = 30$.

EXAMPLE 3 *The flag of a newly formed forum is in the form of three blocks, each to be coloured differently. If there are six different colours on the whole to choose from, how many such designs are possible ?*

SOLUTION Since there are six colours to choose from, therefore, first block can be coloured in 6 ways. Now, the second block can be coloured by any one of the remaining colours in five ways. So, there are five ways to colour the second block.

After colouring first two blocks only four colours are left. The third block can now be coloured by any one of the remaining four colours. So, there are four ways to colour the third block.

Hence, by the fundamental principle of multiplication, the number of flag-designs is $6 \times 5 \times 4 = 120$.

EXAMPLE 4 *Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, when* [NCERT]

(i) *the repetition of the letters is not allowed.* (ii) *the repetition of the letters is allowed.*

SOLUTION (i) The total number of words is same as the number of ways of filling in 4 vacant places by the 4 letters. The first place can be filled in 4 different ways by any one of the 4 letters R, O, S, E. Since the repetition of letters is not allowed. Therefore, the second place can be filled in by any one of the remaining 3 letters in 3 different ways, following which the third place can be filled in by the remaining 2 letters in 2 different ways; following which the fourth place can be filled in by the remaining one letter in one way. Thus, by the fundamental principle of counting the required number of ways is $4 \times 3 \times 2 \times 1 = 24$.

Hence, required number of words = 24.

(ii) If the repetition of the letters is allowed, then each of the 4 vacant places can be filled in succession in 4 different ways.

Hence, required number of words $= 4 \times 4 \times 4 \times 4 = 256$.

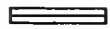
EXAMPLE 5 *Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?* [NCERT]

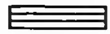
SOLUTION The total number of signals is equal to the number of ways of filling in 2 vacant places in succession by four flags of different colours. The upper cacant place can be filled in 4 different ways by any one of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by any one of the remaining the different flags.

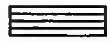
Hence, by the fundamental principle of multiplication, the required number of signals is $4 \times 3 = 12$.

EXAMPLE 6 *Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.* [NCERT]

SOLUTION Since a signal may consist of either 2 flags, 3 flags, 4 flags or 5 flags. Therefore,

Total number of signals = Number of 2 flags signals 

+ Nnumber of 3 flags signals 

+ Number of 4 flags signals 

+ Number of 5 flags signals 

$$= 5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1$$
$$= 20 + 60 + 120 + 120 = 320$$

EXAMPLE 7 *In a monthly test, the teacher decides that there will be three questions, one from each of Exercises 7, 8 and 9 of the text book. If there are 12 questions in Exercise 7, 18 in Exercise 8 and 9 in Exercise 9, in how many ways can three questions be selected ?*

SOLUTION There are 12 questions in exercise 7. So, one question from exercise 7 can be selected in 12 ways. Exercise 8 contains 18 questions. So, second question can be selected in 18 ways. There are 9 questions in exercise 9. So, third question can be selected in 9 ways. Hence, three questions can be selected in $12 \times 18 \times 9 = 1944$ ways.

EXAMPLE 8 *How many words (with or without meaning) of three distinct letters of the English alphabets are there ?*

SOLUTION Here we have to fill up three places by distinct letters of the English alphabets. Since there are 26 letters of the English alphabet, the first place can be filled by any of these letters. So, there are 26 ways of filling up the first place. Now, the second place can be filled up by any of the remaining 25 letters. So, there are 25 ways of filling up the second place. After filling up the first two places only 24 letters are left to fill up the third place. So, the third place can be filled in 24 ways.

Hence, the required number of words $= 26 \times 25 \times 24 = 15600$

EXAMPLE 9 *There are 6 multiple choice questions in an examination. How many sequence of answers are possible, if the first three questions have 4 choices each and the next three have 5 each ?*

SOLUTION Here we have to perform 6 jobs of answering 6 multiple choice questions. Each one of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences $= 4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$

EXAMPLE 10 *Find the total number of ways of answering 5 objective type questions, each question having 4 choices.*

SOLUTION Since each question can be answered in 4 ways. So, the total number of ways of answering 5 questions is $4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

EXAMPLE 11 *How many three-digit numbers can be formed without using the digits 0, 2, 3, 4, 5 and 6 ?*

SOLUTION We have to determine the total number of three digit numbers formed by using the digits 1, 7, 8, 9. Clearly, the repetition of digits is allowed.

A three digit number has three places viz. units's, ten's and hundred's. Unit's place can be filled by any of the digits 1, 7, 8, 9. So, unit's place can be filled in 4 ways. Similarly, each one of the ten's and hundred's place can be filled in 4 ways.

$\therefore$ Total number of required numbers $= 4 \times 4 \times 4 = 64$.

EXAMPLE 12 *How many numbers are there between 100 and 1000 in which all the digits are distinct ?*

SOLUTION A number between 100 and 1000 has three digits. So, we have to form all possible 3-digit numbers with distinct digits. We cannot have 0 at the hundred's place. So, the hundred's place can be filled with any of the 9 digits 1, 2, 3, ..., 9. So, there are 9 ways of filling the hundred's place.

Now, 9 digits are left including 0. So, ten's place can be filled with any of the remaining 9 digits in 9 ways. Now, the unit's place can be filled with in any of the remaining 8 digits. So, there are 8 ways of filling the unit's place.

Hence, the total number of required numbers $= 9 \times 9 \times 8 = 648$.

EXAMPLE 13 *How many numbers are there between 100 and 1000 such that every digit is either 2 or 9 ?*

SOLUTION Every number between 100 and 1000 consists of three digits. So, we have to determine the total number of three digit numbers such that every digit is either 2 or 9.

Clearly, each one of the unit's, ten's and hundred's place can be filled in 2 ways.

So, the total number of required numbers $= 2 \times 2 \times 2 = 8$.

EXAMPLE 14 *How many numbers are there between 100 and 1000 such that 7 is in the unit's place.*

SOLUTION Every number between 100 and 1000 is a three digit number. So, we have to form 3-digit numbers with 7 at the unit's place by using the digits 0, 1, 2, ..., 9. Clearly, repetition of digits is allowed. The hundred's place can be filled with any of the digits from 1 to 9 (zero cannot be there at hundred's place). So, hundred's place can be filled in 9 ways. Now, the ten's place can be filled with any of the digits from 0 to 9. So, ten's place can be filled in 10 ways. Since all the numbers have digit 7 at the unit's place, so, unit's place can be filled in only one way. Hence, by the fundamental principle of counting the total number of numbers between 100 and 1000 having 7 at the unit's place $= 9 \times 10 \times 1 = 90$.

EXAMPLE 15 *A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has three servants to carry the cards ?*

SOLUTION Since a card can be sent by any one of the three servants, so the number of ways of sending the invitation card to the first friend = 3. Similarly, invitation cards can be sent to each of the six friends in 3 ways.

So, the required number of ways $= 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 = 729$.

EXAMPLE 16 *How many three-digit numbers more than 600 can be formed by using the digits 2, 3, 4, 6, 7.*

SOLUTION Clearly, repetition of digits is allowed. Since a three-digit number greater than 600 will have 6 or 7 at hundred's place. So, hundred's place can be filled in 2 ways. Each of the ten's and one's place can be filled in 5 ways.

Hence, total number of required numbers $= 2 \times 5 \times 5 = 50$.

EXAMPLE 17 *How many numbers between 3000 and 4000 can be formed from the digits 3, 4, 5, 6, 7 and 8, no digit being repeated in any number ?*

SOLUTION Clearly, a number between 3000 and 4000 must have 3 at thousand's place. So, thousand's place can be filled in only one way. Now, hundred's place can be filled in 5 ways. Since repetition of digits is not allowed so ten's and one's places can be filled in 4 and 3 ways respectively.

So, total number of required numbers $= 1 \times 5 \times 4 \times 3 = 60$.

EXAMPLE 18 *How many numbers divisible by 5 and lying between 4000 and 5000 can be formed from the digits 4, 5, 6, 7 and 8.*

SOLUTION Clearly, a number between 4000 and 5000 must have 4 at thousand's place. Since the number is divisible by 5 it must have 5 at unit's place. Now, each of the remaining places (viz. hundred's and ten's) can be filled in 5 ways.

Hence, total number of required numbers $= 1 \times 5 \times 5 \times 1 = 25$.

EXAMPLE 19 *How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5 if (i) repetition of digits is not allowed (ii) repetition of digits is allowed ?*

SOLUTION (i) In a four-digit number 0 cannot appear in the thousand's place. So, thousand's place can be filled in 5 ways. (viz. 1, 2, 3, 4, 5). Since repetition of digits is not allowed and 0 can be used at hundred's place, so hundred's place can be filled in 5 ways.

Now, any one of the remaining four digits can be used to fill up ten's place. So, ten's place can be filled in 4 ways. One's place can be filled from the remaining three digits in 3 ways.

Hence, the required number of numbers $= 5 \times 5 \times 4 \times 3 = 300$.

(ii) For a four-digit number we have to fill up four places and 0 cannot appear in the thousand's place. So, thousand's place can be filled in 5 ways. Since repetition of digits is allowed, so each of the remaining three places viz. hundred's, ten's and one's can be filled in 6 ways.

Hence, the required number of numbers $= 5 \times 6 \times 6 \times 6 = 1080$..

EXAMPLE 20 *How many numbers greater than 1000, but not greater than 4000 can be formed with the digits 0, 1, 2, 3, 4 if : (i) repetition of digits is allowed ? (ii) repetition of digits is not allowed ?*

SOLUTION (i) Every number between 1000 and 4000 is a four digit number. In thousand's place we can put either 1 or 2 or 3 but not 4. So, thousand's place can be filled in 3 ways. Since repetition of digits is allowed, so each of the hundred's, ten's and one's place can be filled in 5 ways. So, total number of numbers between 1000 and 4000, including 1000 and excluding 4000 is $3 \times 5 \times 5 \times 5 = 375$. But, we have to find the total number of numbers greater than 1000 but not greater than 4000.

Hence, required number of numbers $= 375 + 1$ (for 4000) $- 1$ (for 1000) $= 375$.

(ii) As discussed above thousand's place can be filled in 3 ways. Since repetition of digits is not allowed, so, hundred's place can be filled from the remaining digits in 4 ways. Now, three digits are left, so ten's place can be filled in 3 ways. One's place can be filled in 2 ways.

Hence, required number of numbers $= 3 \times 4 \times 3 \times 2 = 72$.

EXAMPLE 21 *How many three digit odd numbers can be formed by using the digits 1, 2, 3, 4, 5, 6 if:* [NCERT]

(i) the repetition of digits is not allowed ? (ii) the repetition of digits is allowed ?

SOLUTION For a number to be odd, we must have 1, 3 or 5 at the unit's place. So, there are 3 ways of filling the unit's place.

(i) Since the repetition of digits is not allowed, the ten's place can be filled with any of the remaining 5 digits in 5 ways. Now, four digits are left. So, hundred's place can be filled in 4 ways.

So, required number of numbers $= 3 \times 5 \times 4 = 60$

(ii) Since the repetition of digits is allowed, so each of the ten's and hundred's place can be filled in 6 ways.

Hence, required number of numbers $= 3 \times 6 \times 6 = 108$.

EXAMPLE 22 *How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?* [NCERT]

SOLUTION For a number to be even, we must have 2, 4 or 6 at the unit's place. So, there are 3 ways to fill in the unit's place. Since digits can be repeated, so each of the ten's and hundred's place can be filled in 6 ways.

EXAMPLE 23 *How many numbers of 3 digits can be formed with the digits 1, 2, 3, 4, 5 when digits may be repeated ?*

SOLUTION The unit's place can be filled in 5 ways. Since, the repetition of digits is allowed, therefore ten's place can be filled in 5 ways and hundred's place can also be filled in 5 ways.

Therefore, by the fundamental principle of counting, the required number of three digit numbers $= 5 \times 5 \times 5 = 125$.

EXAMPLE 24 *Find the number of numbers of 5 digits that can be formed with the digits 0, 1, 2, 3, 4 if the digits can be repeated in the same number.*

SOLUTION In a five digit number 0 cannot be put in ten thousand's place. So, the number of ways of filling up the ten thousand's place = 4.

Since the repetition of digits is allowed, therefore each of the other places can be filled in 5 ways.

So, the required number of numbers $= 4 \times 5 \times 5 \times 5 \times 5 = 2500$.

EXAMPLE 25 *How many 4-digit numbers are there, when a digit may be repeated any number of times ?*

SOLUTION In a four digit number 0 cannot be placed at thousand's place. So, thousand's place can be filled with any digit from 1 to 9. Thus, thousand's place can be filled in 9 ways.

Since repetition of digits is allowed, therefore each of the remaining 3 places can be filled in 10 ways by using the digits from 0 to 9.

Hence, the required number of numbers $= 9 \times 10 \times 10 \times 10 = 9000$.

EXAMPLE 26 *How many three-letter words can be formed using a, b, c, d, e if : (i) repetition is not allowed (ii) repetition is allowed ?*

SOLUTION (i) Clearly, the total number of three-letter words is equal to the number of ways of filling three places. First place can be filled in 5 ways. Now, four letters are left. So, the second place can be filled in 4 ways. Since the repetition of letters is not allowed, so the third place can be filled from any one of the remaining 3 digits in 3 ways.

Hence, total number of words $= 5 \times 4 \times 3 = 60$.

(ii) In this case repetition of letters is allowed, so each of the three places can be filled in 5 ways.

Hence, total number of words $= 5 \times 5 \times 5 = 125$.

EXAMPLE 27 *In how many ways can the following prizes be given away to a class of 30 students, first and second in Mathematics, first and second in Physics, first in Chemistry and first in English ?*

SOLUTION Here we have to give prizes in four subjects and the process of distributing prizes can be completed by giving prizes in the four subjects.

First and second prizes can be given in Mathematics in (30×29) ways.

First and second prizes can be given in Physics in (30×29) ways.

First prize can be given in Chemistry in 30 ways.

First prize can be given in English in 30 ways.

Hence, the number of ways to give prizes in all the four subjects

$$= (30 \times 29) \times (30 \times 29) \times 30 \times 30 = 6.8121 \times 10^8$$

EXAMPLE 28 *In how many ways 5 rings of different types can be worn in 4 fingers ?*

SOLUTION The first ring can be worn in any of the 4 fingers. So, there are 4 ways of wearing it. Similarly, each one of the other rings can be worn in 4 ways.

Hence, the requisite number of ways $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

EXAMPLE 29 *In how many ways can 5 letters be posted in 4 letter boxes ?*

SOLUTION Since each letter can be posted in any one of the four letter boxes. So, a letter can be posted in 4 ways. Since there are 5 letters and each letter can be posted in 4 ways. So, total number of ways in which all the five letters can be posted is $4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

LEVEL-2

EXAMPLE 30 *Five persons entered the lift cabin on the ground floor of an 8-floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. Find the total number of ways in which each of the five persons can leave the cabin (i) at any one of the 7 floors (ii) at different floors.*

SOLUTION Suppose A_1, A_2, A_3, A_4, A_5 are five persons.

(i) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Similarly, each of A_2, A_3, A_4, A_5 can leave the cabin in 7 ways. Thus, the total number of ways in which each of the five persons can leave the cabin at any of the seven floors is $7 \times 7 \times 7 \times 7 \times 7 = 7^5$.

(ii) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Now, A_2 can leave the cabin at any of the remaining 6 floors. So, A_2 can leave the cabin in 6 ways. Similarly, A_3, A_4, and A_5 can leave the cabin in 5, 4 and 3 ways respectively. Thus, the total number of ways in which each of the five persons can leave the cabin at different floors is $7 \times 6 \times 5 \times 4 \times 3 = 2520$.

EXAMPLE 31 *A mint prepares metallic calendars specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of February calendars should it prepare to serve for all the possibilities in the future years ?*

SOLUTION The mint has to perform two jobs, viz.

(i) selecting the number of days in the February month (there can be 28 days or 29 days), and
(ii) selecting the first day of the February month.

The first job can be completed in 2 ways while the second can be performed in 7 ways by selecting any one of the seven days of a week.

Thus, the required number of plates $= 2 \times 7 = 14$.

EXAMPLE 32 *For a set of five true/false questions, no student has written all correct answers, and no two students have given the same sequence of answers. What is the maximum number of students in the class, for this to be possible ?*

SOLUTION Since a true/false type question can be answered in 2 ways either by marking it true or false. So, there are 2 ways of answering each of the 5 questions.

$\therefore$ Total number of different sequences of answers $= 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$.

Out of these 32 sequences of answers there is only one sequence of answering all the five questions correctly. But no student has written all the correct answers and different students have given different sequences of answers.

$\therefore$ Maximum number of students in the class

= Number of sequences except one sequence in which all answers are correct $= 32 - 1 = 31$

EXAMPLE 33 *How many numbers are there between 100 and 1000 such that at least one of their digits is 7?*

SOLUTION Clearly, a number between 100 and 1000 has 3-digits

$\therefore$ Total number of 3-digit numbers having at least one of their digits as 7

= (Total number of three-digit numbers) – (Total number of 3-digit numbers in which 7 does not appear at all)

Total number of three-digit numbers : We have to form three-digit numbers by using the digits 0, 1, 2, 3, ..., 9. Clearly, hundred's place can be filled in 9 ways and each of the ten's and one's place can be filled in 10 ways.

So, total number of 3-digit number $= 9 \times 10 \times 10 = 900$.

Total number of three-digit number in which 7 does not appear at all : Here we have to form three-digit numbers by using the digits 0 to 9, except 7. So, hundred's place can be filled in 8

ways and each of the ten's and one's place can be filled in 9 ways. So, total number of three-digit numbers in which 7 does not appear at all is $8 \times 9 \times 9$.

Hence, total number of 3-digit numbers having at least one of their digits as 7 is

$$9 \times 10 \times 10 - 8 \times 9 \times 9 = 252.$$

EXAMPLE 34 *How many numbers are there between 100 and 1000 which have exactly one of their digits as 7?*

SOLUTION A number between 100 and 1000 contains 3-digits. So, we have to form 3-digit numbers having exactly one of their digits as 7. Such type of numbers can be divided into three types:

(i) Those numbers that have 7 in the unit's place but not in any other place.
(ii) Those numbers that have 7 in the ten's place but not in any other place.
(iii) Those numbers that have 7 in the hundred's place but not in any other place.

Required number of numbers is the total number of these three types of numbers.

We shall now count these three types of numbers separately.

(i) *Those three-digit numbers that have 7 in the unit's place but not in any other place.*

The hundred's place can have any one of the digits from 0 to 9 except 0 and 7. So, hundred's place can be filled in 8 ways. The ten's place can have any one of the digits from 0 to 9 except 7. So, the number of ways the ten's place can be filled is 9. The unit's place has 7. So, it can be filled in only one way.

Thus, there are $8 \times 9 \times 1 = 72$ numbers of the first kind.

(ii) *Those three-digit numbers that have 7 in the ten's place but not in any other place.*

The number of ways to fill the hundred's place $= 8$

(by any one of the digits from $1, 2, 3, 4, 5, 6, 8, 9$)

The number of ways to fill the ten's place $= 1$ (by 7 only)

The number of ways to fill the one's place $= 9$ (by any one of the digits $0, 1, 2, 3, 4, 5, 6, 8, 9$)

Thus, there are $8 \times 1 \times 9 = 72$ numbers of the second kind.

(iii) *Those three-digit numbers that have 7 in the hundred's place but not at any other place.*

In this case, the hundred's place can be filled only in one way and each of the ten's and one's place can be filled in 9 ways.

So, there are $1 \times 9 \times 9 = 81$ numbers of the third kind.

Hence, the total number of required type of numbers $= 72 + 72 + 81 = 225$.

EXAMPLE 35 *A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. What is the total number of signals that can be made ?*

SOLUTION Since each arm can be kept in 4 positions and a signal is possible when all the 5 arms are simultaneously placed in positions.

$\therefore$ Total number of ways of placing the arms $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

But, this includes one inadmissible case, when all the arms are in the position of rest and then no signal can be made.

Hence, required number of signals $= (4^5 - 1) = 1023$.

EXAMPLE 36 *In how many ways can 3 prizes be distributed among 4 boys, when*

(i) no boy gets more than one prize ? (ii) a boy may get any number of prizes ? (iii) no boy gets all the prizes ?

SOLUTION (i) The first prize can be given away in 4 ways as it may be given to any one of the 4 boys. The second prize can be given away in 3 ways, because the boy who got the first prize cannot receive the second prize. The third prize can be given away to anyone of the remaining 2 boys in 2 ways. So, the number of ways in which all the prizes can be given away $= 4 \times 3 \times 2 = 24$.

ALITER The total number of ways is the number of arrangements of 4 taken 3 at a time. So, the requisite number of ways = ${}^4P_3 = 4! = 24$.

(ii) The first prize can be given away in 4 ways as it may be given to anyone of the 4 boys. The second prize can also be given away in 4 ways, since it may be obtained by the boy who has already received a prize. Similarly, third prize can be given away in 4 ways.

Hence, the number of ways in which all the prizes can be given away $= 4 \times 4 \times 4 = 4^3 = 64$.

(iii) Since any one of the 4 boys may get all the prizes. So, the number of ways in which a boy gets all the 3 prizes is 4.

So, the number of ways in which a boy does not get all the prizes $= 64 - 4 = 60$.

EXAMPLE 37 *Find the total number of ways in which n distinct objects can be put into two different boxes.*

SOLUTION Let the two boxes be B_1 and B_2. We observe that there are two choices for each of the n objects. Therefore, by fundamental principle of counting

Total number of ways $= \underbrace{2 \times 2 \times \dots \times 2}_{n-\text{times}} = 2^n$

EXAMPLE 38 *Find the total number of ways in which n-distinct objects can be put into two different boxes so that no box remains empty.*

SOLUTION Each object can be put either in box B_1 (say) or in box B_2 (say). So, there are two choices for each of the n objects. Therefore, the number of choices for n distinct objects is $\underbrace{2 \times 2 \times \dots \times 2}_{n-\text{times}} = 2^n$. Two of these choices correspond to either the first or the second box being empty. Thus, there are $2^n - 2$ ways in which neither box is empty.

EXAMPLE 39 *By using the digits 0, 1, 2, 3, 4 and 5 (repetitions not allowed) numbers are formed by using any number of digits. Find the total number of non-zero numbers that can be formed.*

SOLUTION Required number of numbers

= Number of 1 digit number + No. of 2 digit numbers + ... + Number of 6 digit numbers

$= 5 + 5 \times 5 + 5 \times 5 \times 4 + 5 \times 5 \times 4 \times 3 + 5 \times 5 \times 4 \times 3 \times 2 + 5 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 5 + 25 + 100 + 300 + 600 + 600 = 1630.$

EXERCISE 16.2

LEVEL-1

1. In a class there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent the class in a function. In how many ways can the teacher make this selection?
2. A person wants to buy one fountain pen, one ball pen and one pencil from a stationery shop. If there are 10 fountain pen varieties, 12 ball pen varieties and 5 pencil varieties, in how many ways can he select these articles?
3. From Goa to Bombay there are two roots; air, and sea. From Bombay to Delhi there are three routes; air, rail and road. From Goa to Delhi via Bombay, how many kinds of routes are there?
4. A mint prepares metallic calenders specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of calendars should it prepare to serve for all the possibilities in future years?
5. There are four parcels and five post-offices. In how many different ways can the parcels be sent by registered post?

6. A coin is tossed five times and outcomes are recorded. How many possible outcomes are there?
7. In how many ways can an examinee answer a set of ten true/false type questions?
8. A letter lock consists of three rings each marked with 10 different letters. In how many ways it is possible to make an unsuccessful attempt to open the lock?
9. There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 2 each?
10. There are 5 books on Mathematics and 6 books on Physics in a book shop. In how many ways can a student buy : (i) a Mathematics book and a Physics book (ii) either a Mathematics book or a Physics book?
11. Given 7 flags of different colours, how many different signals can be generated if a signal requires the use of two flags, one below the other? [NCERT]
12. A team consists of 6 boys and 4 girls and other has 5 boys and 3 girls. How many single matches can be arranged between the two teams when a boy plays against a boy and a girl plays against a girl?
13. Twelve students compete in a race. In how many ways first three prizes be given?
14. How many A.P.'s with 10 terms are there whose first term is in the set {1, 2, 3} and whose common difference is in the set {1, 2, 3, 4, 5}?
15. From among the 36 teachers in a college, one principal, one vice-principal and the teacher-incharge are to be appointed. In how many ways can this be done?
16. How many three-digit numbers are there with no digit repeated?
17. How many three-digit numbers are there?
18. How many three-digit odd numbers are there?
19. How many different five-digit number licence plates can be made if
    (i) first digit cannot be zero and the repetition of digits is not allowed,
    (ii) the first-digit cannot be zero, but the repetition of digits is allowed?
20. How many four-digit numbers can be formed with the digits 3, 5, 7, 8, 9 which are greater than 7000, if repetition of digits is not allowed?
21. How many four-digit numbers can be formed with the digits 3, 5, 7, 8, 9 which are greater than 8000, if repetition of digits is not allowed?
22. In how many ways can six persons be seated in a row?
23. How many 9-digit numbers of different digits can be formed?
24. How many odd numbers less than 1000 can be formed by using the digits 0, 3, 5, 7 when repetition of digits is not allowed?
25. How many 3-digit numbers are there, with distinct digits, with each digit odd?
26. How many different numbers of six digits each can be formed from the digits 4, 5, 6, 7, 8, 9 when repetition of digits is not allowed?
27. How many different numbers of six digits can be formed from the digits 3, 1, 7, 0, 9, 5 when repetition of digits is not allowed?
28. How many four digit different numbers, greater than 5000 can be formed with the digits 1, 2, 5, 9, 0 when repetition of digits is not allowed?
29. Serial numbers for an item produced in a factory are to be made using two letters followed by four digits (0 to 9). If the letters are to be taken from six letters of English alphabet

without repetition and the digits are also not repeated in a serial number, how many serial numbers are possible?

30. A number lock on a suitcase has 3 wheels each labelled with ten digits 0 to 9. If opening of the lock is a particular sequence of three digits with no repeats, how many such sequences will be possible? Also, find the number of unsuccessful attempts to open the lock.
31. A customer forgets a four-digit code for an Automatic Teller Machine (ATM) in a bank. However, he remembers that this code consists of digits 3, 5, 6 and 9. Find the largest possible number of trials necessary to obtain the correct code.
32. In how many ways can three jobs I, II and III be assigned to three persons A, B and C if one person is assigned only one job and all are capable of doing each job?
33. How many four digit natural numbers not exceeding 4321 can be formed with the digits 1, 2, 3 and 4, if the digits can repeat?
34. How many numbers of six digits can be formed from the digits 0, 1, 3, 5, 7 and 9 when no digit is repeated? How many of them are divisible by 10?
35. If three six faced die each marked with numbers 1 to 6 on six faces, are thrown find the total number of possible outcomes.
36. A coin is tossed three times and the outcomes are recorded. How many possible outcomes are there ? How many possible outcomes if the coin is tossed four times? Five times? n times?
37. How many numbers of four digits can be formed with the digits 1, 2, 3, 4, 5 if the digits can be repeated in the same number?
38. How many three digit numbers can be formed by using the digits 0, 1, 3, 5, 7 while each digit may be repeated any number of times?
39. How many natural numbers less than 1000 can be formed from the digits 0, 1, 2, 3, 4, 5 when a digit may be repeated any number of times?
40. How many five digit telephone numbers can be constructed using the digits 0 to 9. If each number starts with 67 and no digit appears more than once? [NCERT]

LEVEL-2

41. Find the number of ways in which 8 distinct toys can be distributed among 5 children.
42. Find the number of ways in which one can post 5 letters in 7 letter boxes.
43. Three dice are rolled. Find the number of possible outcomes in which at least one die shows 5.
44. Find the total number of ways in which 20 balls can be put into 5 boxes so that first box contains just one ball.
45. In how many ways can 5 different balls be distributed among three boxes?
46. In how many ways can 7 letters be posted in 4 letter boxes?
47. In how many ways can 4 prizes be distributed among 5 students, when
    (i) no student gets more than one prize?
    (ii) a student may get any number of prizes?
    (iii) no student gets all the prizes?
48. There are 10 lamps in a hall. Each one of them can be switched on independently. Find the number of ways in which the hall can be illuminated. [NCERT]

ANSWERS

1. 378	2. 600	3. 6	4. 14	5. 625	6. 32	7. 1024	8. 999
9. 512	10. (i) 30	(ii) 11	11. 42	12. 42	13. 1320	14. 15	15. 42840
16. 648	17. 900	18. 450	19. (i) 27216	(ii) 90000		20. 72	21. 48
22. 720	23. 9(9!)	24. 21	25. 60	26. 720	27. 600	28. 48	29. 151200
30. 720, 719		31. 24	32. 6	33. 229	34. 600, 120		35. 216
36. 8, 16, 2^n		37. 625	38. 100	39. 215	40. 336	41. 5^8	42. 7^5
43. 91	44. 20×4^{19}		45. 243	46. 4^7	47. (i) 5!	(ii) 625	(iii) 620
48. $2^{10}-1$.							

HINTS TO NCERT & SELECTED PROBLEMS

1. No. of ways $= 27 \times 14$.
2. Required number of ways $= 10 \times 12 \times 5 = 600$.
3. No of routes $= 2 \times 3 = 6$.
4. Total number of calendars $= 7 \times 2 = 14$.
5. Since a parcel can be sent to any one of the five post offices. So, required number of ways $= 5 \times 5 \times 5 \times 5 = 5^4$.
6. Since toss of each coin can result in 2 ways. So, required no. of ways $= 2 \times 2 \times 2 \times 2 \times 2 = 2^5$.
8. Required no. of ways $= 10 \times 10 \times 10 - 1$.
9. Each one of the first three questions can be answered in 4 ways and each one of the next three questions can be answered in 2 ways. So, total no. of sequences of answers $= 4 \times 4 \times 4 \times 2 \times 2 \times 2$.
11. Required no. of signals $= 7 \times 6$.
12. A boy can be selected from the first team in 6 ways, and from the second in 5 ways. So, no. of single matches between the boys of two teams $= 6 \times 5 = 30$. Similarly, the no. of single matches between the girls of two teams $= 4 \times 3 = 12$. So, total number of matches $= 30 + 12 = 42$.
13. Required no. of ways $= 12 \times 11 \times 10$.
14. There are 3 ways to choose the first term and corresponding to each such way there are 5 ways of selecting the common difference. So, required no. of A.P.'s $= 3 \times 5$.
15. Required no. of ways $= 36 \times 35 \times 34$.
16. The total no. of required numbers $= 9 \times 9 \times 8$.
17. The total no. of required numbers $= 9 \times 10 \times 10$.
18. The total no. of required number $= 9 \times 10 \times 5$.
19. (i) Required no. of licence plates $= 9 \times 9 \times 8 \times 7 \times 6$
    (ii) Required no. of licence plates $= 9 \times 10 \times 10 \times 10 \times 10$.
20. Required no. of numbers $= 3 \times 4 \times 3 \times 2$.
21. Required no. of numbers $= 2 \times 4 \times 3 \times 2$.
22. Required no. of ways $= 6 \times 5 \times 4 \times 3 \times 2 \times 1$.
23. Required no. of numbers $= 9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$.
24. An odd number less than 1000 may be a one-digit number, two-digit number or a three-digit number. So, required no. of numbers is
    $$3\,(\text{one-digit nos.}) + 2 \times 3\,(\text{two-digit nos.}) + 2 \times 2 \times 3\,(3\text{-digit nos.}).$$
25. Required no. of numbers $= 5 \times 4 \times 3$.

26. Required no. of numbers $= 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

27. Required no. of numbers $= 5 \times 5 \times 4 \times 3 \times 2 \times 1$.

28. Required no. of numbers $= 2 \times 4 \times 3 \times 2$.

29. Here we have to perform 6 jobs. So, required number of serial numbers i
$6 \times 5 \times 10 \times 9 \times 8 \times 7$

30. Required number of sequences $= 10 \times 9 \times 8$.
Also, total number of unsuccessful attempts $= 10 \times 9 \times 8 - 1$

31. Number of trials $= 4 \times 3 \times 2 \times 1$

32. Required number of ways $= 3 \times 2 \times 1$

36. Since a toss of a coin can result in a head or a tail. Therefore, if a coin is tossed n-times, then the total number of outcomes is $\underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{n\text{-times}} = 2^n$

41. Each toy can be distributed in 5 ways.
So, total number of ways $= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^8$

42. Each letter can be posted in any one of the 7 letter boxes.
So, required number of ways $= 7 \times 7 \times 7 \times 7 \times 7 = 7^5$

43. Required number of possible outcomes
= Total number of possible outcomes – Number of possible out- comes in which 5 does not appear on any dice.
$= 6^3 - 5^3 = 216 - 125 = 91$.

44. One ball can be put in first box in 20 ways because we can put any one of the twenty balls in the first box. Now, remaining 19 balls are to be put into remaining 4 boxes. This can be done in 4^{19} ways, because there are 4 choices for each ball. Hence, the required number of ways $= 20 \times 4^{19}$.

16.3 PERMUTATIONS

Each of the arrangements which can be made by taking some or all of a number of things is called a permutation.

For example, if there are three objects, then the permutations of these objects, taking two at a time, are

ab, ba, bc, cb, ac, ca

So, the number of permutations of three different things taken two at a time is 6.

NOTE It should be noted that in permutations the order of arrangement is taken into account; when the order is changed, a different permutation is obtained.

ILLUSTRATION 1 *Write down all the permutations of the set of three letters A, B, C.*

SOLUTION The permutations of three letters A, B, C taking all at a time are :

ABC, ACB, BCA, BAC, CBA, CAB.

Clearly, there are 6 permutations.

ILLUSTRATION 2 *Write down all the permutations of the vowels A, E, I, O, U in English alphabets taking three at a time, and starting with A.*

SOLUTION The permutations of vowels A, E, I, O, U taking three at a time, and starting with A are:

AEI, AIE, AEO, AOE, AEU, AUE, AIO, AOI, AIU, AUI, AOU, AUO

Clearly, there are 12 permutations.

ILLUSTRATION 3 *Write down all the permutations of letters A, B, C, D taking three at a time.*

SOLUTION The desired permutations are :

ABC	ABD	BCD	ACD
ACB	ADB	BDC	ADC
BCA	BDA	CBD	CAD
BAC	BAD	CDB	CDA
CAB	DAB	DCB	DAC
CBA	DBA	DBC	DCA

Clearly, there are 24 permutations. These permutations are obtained by first selecting three letters out of 4 and then arranging them in all possible ways.

A NOTATION *If n and r are positive integers such that $1 \le r \le n$, then the number of all permutations of n distinct things, taken r at a time is denoted by the symbol $P(n, r)$ or ${}^nP_r$.*

Thus,

$${}^nP_r \text{ or, } P(n, r) = \text{Total number of permutations of n distinct things, taken } r \text{ at a time.}$$

In illustration 3, we have seen that there are 24 permutations, on a set of 4 letters, taken 3 at a time. Therefore, as per our notation, we have ${}^4P_3 = 24$ or, $P(4, 3) = 24$.

THEOREM 1 *Let r and n be positive integers such that $1 \le r \le n$. Then the number of all permutations of n distinct things taken r at a time is given by $n(n-1)(n-2)(n-3)\dots(n-(r-1))$.*

i.e. $\quad P(n, r) = {}^nP_r = n(n-1)(n-2)\dots(n-(r-1))$.

PROOF The number of permutations of n distinct things, taken r at a time, is same as the number of ways in which we can fill up r-places when we have n different things at our disposal.

The first place can be filled in n ways, for any one of the n things can be used to fill up the first place. Having filled it, there are $(n-1)$ things left and any one of these $(n-1)$ things can be used to fill up the second place. So, the second place can be filled in $(n-1)$ ways. Hence, by the fundamental principle of counting, the first two places can be filled in n $(n-1)$ ways. When the first two places are filled, there are $(n-2)$ places left, so that the third place can be filled from the remaining $(n-2)$ things in $(n-2)$ ways. Therefore, the first three places can be filled in $n(n-1)(n-2)$ ways. Continuing in this manner, we find that the first $(r-1)$ places can be filled in $n(n-1)(n-2)\dots(n-(r-2))$ ways. After filling up first $(r-1)$ places, exactly $n-(r-1) = n-r+1$ things are left. So, the rth place can be filled in $(n-(r-1))$ ways. Hence, the r places can be filled in $n(n-1)(n-2)\dots(n-(r-1))$ ways.

Hence, the total number of permutations of n distinct things, taken r at a time is

$$n(n-1)(n-2)(n-3)\dots(n-(r-1)).$$

Thus, $\quad P(n, r) = n(n-1)(n-2)(n-3)\dots(n-(r-1))$.

THEOREM 2 *Prove that:* $P(n,r) = {}^nP_r = \dfrac{n!}{(n-r)!}$.

PROOF We have,

$$P(n, r) = n(n-1)(n-2)(n-3)\dots(n-(r-1))$$

$$\Rightarrow \quad P(n, r) = \frac{n(n-1)(n-2)(n-3)\dots(n-(r-1))(n-r)(n-(r+1))\dots 3.2.1}{(n-r)(n-(r+1))\dots 3.2.1}$$

$$\Rightarrow \quad P(n, r) = \frac{n!}{(n-r)!}$$

THEOREM 3 *The number of all permutations of n distinct things, taken all at a time is n!.*

PROOF The number of all permutations of n distinct things, taken all at a time is same as the number of ways of filling n places when we have n distinct things at our disposal.

Proceeding as in theorem 1, we have

$$P(n, n) = n(n-1)(n-2)(n-3)\ldots(n-(n-1)) = n(n-1)(n-2)(n-3)\ldots 3.2.1 = n!$$

THEOREM 4 *Prove that $0! = 1$.*

PROOF We have,

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$\Rightarrow \quad P(n, n) = \frac{n!}{0!} \qquad [\text{Putting } r = n]$$

$$\Rightarrow \quad n! = \frac{n!}{0!} \qquad [\because P(n, n) = n! \text{ (See Theorem 3)}]$$

$$\Rightarrow \quad 0! = \frac{n!}{n!} = 1.$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS BASED UPON THE VALUE OF ^nP_r OR $P(n, r)$**

EXAMPLE 1 *Evaluate the following:*

(i) $^5P_3$ (ii) $P(15, 3)$ (iii) $P(5, 5)$

SOLUTION (i) $^5P_3 = \dfrac{5!}{(5-3)!}$ $\qquad \left[\because\ ^nP_r = \dfrac{n!}{(n-r)!}\right]$

$$\Rightarrow \quad ^5P_3 = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2!}{2!} = 60$$

(ii) $$P(15, 3) = \frac{15!}{(15-3)!} = \frac{15!}{12!} = \frac{15 \times 14 \times 13 \times 12!}{12!} = 2730$$

(iii) $$P(5, 5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5! = 120.$$

Type II **ON FINDING THE VALUE OF REQUIRED UN-KNOWN WHEN A RELATION CONNECTING $P(n, r)$ IS GIVEN**

EXAMPLE 2 *If $2.P(5, 3) = P(n, 4)$, find n.*

SOLUTION We have,

$$2.P(5, 3) = P(n, 4)$$

$$\Rightarrow \quad P(n, 4) = 2.P(5, 3)$$

$$\Rightarrow \quad \frac{n!}{(n-4)!} = 2\left(\frac{5!}{(5-3)!}\right)$$

$$\Rightarrow \quad \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!} = \frac{2(5!)}{2!}$$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5!$$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5 \times 4 \times 3 \times 2 \times 1$$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5(5-1)(5-2)(5-3)$$

$$\Rightarrow \quad n = 5 \qquad [\text{By comparing two sides}]$$

EXAMPLE 3 *If* $P(n, 4) = 20 \times P(n, 2)$, *find* n.

SOLUTION We have,

$$P(n, 4) = 20 \times P(n, 2)$$

$$\Rightarrow \quad \frac{n!}{(n-4)!} = 20 \times \frac{n!}{(n-2)!}$$

$$\Rightarrow \quad (n-2)! = 20 \times (n-4)!$$

$$\Rightarrow \quad (n-2)(n-3)(n-4)! = 20 \times (n-4)!$$

$$\Rightarrow \quad (n-2)(n-3) = 20$$

$$\Rightarrow \quad (n-2)(n-3) = 5 \times 4$$

$$\Rightarrow \quad n-3 = 4$$

$$\Rightarrow \quad n = 7 \qquad \text{[By comparing two sides]}$$

EXAMPLE 4 *If* $P(5, r) = 2. P(6, r-1)$, *find* r. **[NCERT]**

SOLUTION We have,

$$P(5, r) = 2 . P(6, r-1)$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = 2. \frac{6!}{(6-(r-1))!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{2 \times 6 \times 5!}{(7-r)!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{12 \times 5!}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow \quad 1 = \frac{12}{(7-r)(6-r)}$$

$$\Rightarrow \quad (7-r)(6-r) = 12$$

$$\Rightarrow \quad (7-2)(6-r) = 4 \times 3 \qquad \text{[By comparing]}$$

$$\Rightarrow \quad 7-r = 4$$

$$\Rightarrow \quad r = 3$$

EXAMPLE 5 *If* ${}^{10}P_r = 5040$, *find the value of* r.

SOLUTION We have,

$${}^{10}P_r = 5040$$

$$\Rightarrow \quad \frac{10!}{(10-r)!} = 10 \times 504$$

$$\Rightarrow \quad \frac{10!}{(10-r)!} = 10 \times 9 \times 8 \times 7$$

$$\Rightarrow \quad \frac{10!}{(10-r)!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6!}$$

$$\Rightarrow \quad \frac{10!}{(10-r)!} = \frac{10!}{6!} \Rightarrow (10-r)! = 6! \Rightarrow 10-r = 6 \Rightarrow r = 4.$$

EXAMPLE 6 *If* $P(n-1, 3) : P(n, 4) = 1 : 9$, find n.

SOLUTION We have,

$$P(n-1, 3) : P(n, 4) = 1 : 9$$

$$\Rightarrow \quad \frac{P(n-1, 3)}{P(n, 4)} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{\frac{(n-1)!}{(n-1-3)!}}{\frac{n!}{(n-4)!}} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{(n-1)!}{(n-4)!} \times \frac{(n-4)!}{n!} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{n!} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{n \cdot (n-1)!} = \frac{1}{9} \Rightarrow n = 9$$

EXAMPLE 7 *If* ${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_r$, *find r.*

SOLUTION We have,

$${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_r$$

$$\Rightarrow \quad \frac{9!}{(9-5)!} + 5 \cdot \frac{9!}{(9-4)!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{9!}{4!} + 5 \cdot \frac{9!}{5!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{9!}{4!} + \frac{9!}{4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad 2 \times \frac{9!}{4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{5 \times 2 \times 9!}{5 \times 4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{10 \times 9!}{5!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{10!}{5!} = \frac{10!}{(10-r)!} \Rightarrow (10-r)! = 5! \Rightarrow 10 - r = 5 \Rightarrow r = 5$$

EXAMPLE 8 *If* ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30\,800 : 1$, *find r.*

SOLUTION We have,

$${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30\,800 : 1$$

$$\Rightarrow \quad \frac{56!}{(56-r-6)!} : \frac{54!}{(54-r-3)!} = \frac{30800}{1}$$

$$\Rightarrow \quad \frac{56!}{(50-r)!} : \frac{54!}{(51-r)!} = 30800 : 1$$

$$\Rightarrow \quad \frac{56!}{(50-r)!} \times \frac{(51-r)!}{54!} = \frac{30800}{1}$$

$$\Rightarrow \quad \frac{56 \times 55 \times 54!}{(50-r)!} \times \frac{(51-r) \times (50-r)!}{54!} = \frac{30800}{1}$$

$$\Rightarrow \quad 56 \times 55 \times (51-r) = 30800 \Rightarrow (51-r) = 10 \Rightarrow r = 41.$$

EXAMPLE 9 *If* ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$, *find n.*

SOLUTION We have,

$${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$$

$$\Rightarrow \quad \frac{{}^{2n+1}P_{n-1}}{{}^{2n-1}P_n} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)(2n)(2n-1)!}{(n+2)(n+1)n(n-1)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{2(2n+1)}{(n+2)(n+1)} = \frac{3}{5}$$

$$\Rightarrow \quad 10(2n+1) = 3(n+2)(n+1)$$

$$\Rightarrow \quad 3n^2 + 9n + 6 = 20n + 10$$

$$\Rightarrow \quad 3n^2 - 11n - 4 = 0 \Rightarrow (n-4)(3n+1) = 0 \Rightarrow n = 4 \qquad [\because n \neq -1/3]$$

EXAMPLE 10 *If* ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$, *find* r.

SOLUTION We have,

$$ {}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$$

$$\Rightarrow \quad \frac{22!}{(21-r)!} : \frac{20!}{(18-r)!} = 11:52$$

$$\Rightarrow \quad \frac{22!}{(21-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22 \times 21 \times 20!}{(21-r)(20-r)(19-r)\cdot(18-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22 \times 21}{(21-r)(20-r)(19-r)} = \frac{11}{52}$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 2 \times 21 \times 52$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 2 \times 3 \times 7 \times 4 \times 13$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 12 \times 13 \times 14$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = (21-7)(20-7)(19-7)$$

$$\Rightarrow \quad r = 7$$

Type III **ON PROVING RESULTS RELATED TO** $P(n, r)$ **or** ${}^nP_r$

EXAMPLE 11 *Prove the following:*

(i) $P(n, n) = 2\,P(n, n-2)$ (ii) $P(n, n) = P(n, n-1)$

(iii) $P(n, r) = P(n-1, r) + r\,.\,P(n-1, r-1)$ (iv) $P(n, r) = n \cdot P(n-1, r-1)$

SOLUTION (i) $2P(n, n-2) = 2\dfrac{n!}{(n-(n-2))!} = 2\left(\dfrac{n!}{2!}\right) = n! = P(n, n)$

(ii) $P(n, n-1) = \dfrac{n!}{(n-(n-1))!} = \dfrac{n!}{1!} = n! = P(n, n)$

(iii) $P(n-1, r) + r \cdot P(n-1, r-1) = \dfrac{(n-1)!}{(n-1-r)!} + r \cdot \dfrac{(n-1)!}{((n-1)-(r-1))!}$

$$= \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)!} = \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left\{1 + \frac{r}{n-r}\right\} = \frac{(n-1)!}{(n-r-1)!}\left(\frac{n-r+r}{n-r}\right)$$

$$= \frac{(n-1)!}{(n-r-1)!} \cdot \frac{n}{n-r} = \frac{n!}{(n-r)!} = P(n, r)$$

(iv) $n \cdot P(n-1, r-1) = n \dfrac{(n-1)!}{((n-1)-(r-1))!} = \dfrac{n!}{(n-r)!} = P(n, r)$

Type III PRACTICAL PROBLEMS ON PERMUTATIONS

NOTE ALITER 2 of each of the following examples should be done after studying permutations and combinations.

EXAMPLE 12 *In how many ways three different rings can be worn in four fingers with at most one in each finger?*

SOLUTION The total number of ways is same as the number of arrangements of 4 fingers, taken 3 at a time.

So, required number of ways $= {}^4P_3 = \dfrac{4!}{(4-3)!} = \dfrac{4!}{1!} = 4! = 24.$

ALITER 1 Let R_1, R_2, R_3 be three rings. Since R_1 can be put in any one of the four fingers. So, there are four ways in which R_1 can be worn. Now, R_2 can be worn in any one of the remaining three fingers in 3 ways. In the remaining 2 fingers ring R_3 can be worn in 2 ways. So, by the fundamental principle of counting the total number of ways in which three different rings can be worn in four fingers is $4 \times 3 \times 2 = 24$.

ALITER 2 Out of 4 fingers, 3 fingers can be chosen in ${}^4C_3$ ways. Now, three rings can be worn in the selected three fingers in 3! ways. Hence, three rings can be worn in four fingers in ${}^4C_3 \times 3! = 24$ ways.

EXAMPLE 13 *Seven athletes are participating in a race. In how many ways can the first three prizes be won?*

SOLUTION The total number of ways in which first three prizes can be won is the number of arrangements of seven different things taken 3 at a time.

So, required number of ways $= {}^7P_3 = \dfrac{7!}{(7-3)!} = \dfrac{7!}{4!} = \dfrac{7 \times 6 \times 5 \times 4!}{4!} = 210.$

ALITER 1 First prize can be won in seven ways. Second prize can be won by any one of the remaining six athletes in 6 ways. Now, five athletes are left. So, third prize can be won by any one of the remaining 5 athletes in 5 ways.

Hence, by the fundamental principle of counting, the required number of ways $= 7 \times 6 \times 5 = 210$.

ALITER 2 Out of 7 athletes, 3 can be chosen for prize in ${}^7C_3$ ways. Now, three prizes can be given to three chosen athletes in 3! ways.

$\therefore$ Numbers of ways in which 3 prizes can be won $= {}^7C_3 \times 3! = 210$

EXAMPLE 14 *How many different signals can be made by 5 flags from 8 flags of different colours?*

SOLUTION The total number of signals is the number of arrangements of 8 flags by taking 5 flags at a time.

Hence, required number of signals $= {}^8P_5 = \dfrac{8!}{(8-5)!} = \dfrac{8!}{3!} = \dfrac{8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!} = 6720$

EXAMPLE 15 *In how many ways can 6 persons stand in a queue?*

SOLUTION The number of ways in which 6 persons can stand in a queue is same as the number of arrangements of 6 different things taken all at a time.

Hence, the required number of ways $= {}^6P_6 = 6! = 720.$

EXAMPLE 16 *It is required to seat 8 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?*

SOLUTION In all 12 persons are to be seated in a row and in the row of 12 positions there are exactly 6 even places viz second, fourth, sixth, eighth, tenth and twelfth. It is given that four women are to occupy 4 places out of these six even places. This can be done in ${}^6P_4$ ways (ways of arranging 6 women in 4 positions). The remaining 8 positions can be filled by the 8 men in ${}^8P_8$ ways. So, by the fundamental principle of counting, the number of seating arrangements as required, is ${}^6P_4 \times {}^8P_8 = 360 \times 40320 = 14515200$.

ALITER 1 In all 12 persons are to be seated in a row and in the row of 12 positions there are exactly 6 even places viz. 2nd, 4th, 6th, 8th and 12th. It is given that 4 women are to occupy any 4 places out of these six positions. This can be done in ${}^6C_4 \times 4!$ ways. The remaining 8 positions are to be occupied by 8 men. This can be done in ${}^8C_8 \times 8!$ ways.

Hence, total number of seating arrangements $= ({}^6C_4 \times 4!) \times ({}^8C_8 \times 8!)$

$$= 360 \times 40320 = 14515200.$$

EXAMPLE 17 *Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them ?*

SOLUTION The total number of ways in which three men can wear 4 coats is the number of arrangements of 4 different coats taken 3 at a time. So, three men can wear 4 coats in ${}^4P_3$ ways. Similarly, 5 waist coats and 6 caps can be worn by three men in ${}^5P_3$ and ${}^6P_3$ ways respectively. Hence, by the fundamental principle of counting, the required number of ways as desired

$$= {}^4P_3 \times {}^5P_3 \times {}^6P_3 = (4!) \times (5 \times 4 \times 3) \times (6 \times 5 \times 4) = 172800$$

EXAMPLE 18 *How many different signals can be given using any number of flags from 5 flags of different colours ?*

SOLUTION The signals can be made by using at a time one or two or three or four or five flags. The total number of signals when r flags are used at a time from 5 flags is equal to the number of arrangements of 5, taking r at a time i.e. ${}^5P_r$. Since r can take values 1, 2, 3, 4, 5. Hence, by the fundamental principle of addition, the total number of signals

$$= {}^5P_1 + {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$$

$$= 5 + 5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1 = 5 + 20 + 60 + 120 + 120 = 325$$

EXAMPLE 19 *How many numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed ?*

SOLUTION Every number lying between 100 and 1000 is a three digit number. Therefore, we have to find the number of permutations of five digits 1, 2, 3, 4, 5 taken three at a time.

Hence, the required number of numbers $= {}^5P_3 = \dfrac{5!}{(5-3)!} = \dfrac{5!}{2!} = 5 \times 4 \times 3 = 60$

EXAMPLE 20 *How many four digit numbers are there with distinct digits ?*

SOLUTION The total number of arrangements of ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 taking 4 at a time is ${}^{10}P_4$. But, these arrangements also include those numbers which have 0 at thousand's place. Such numbers are not four digit numbers. When 0 is fixed at thousand's place, we have to arrange remaining 9 digits by taking 3 at a time. The number of such arrangements is ${}^9P_3$.

So, the total number of numbers having 0 at thousand's place $= {}^9P_3$.

Hence, the total number of four digit numbers $= {}^{10}P_4 - {}^9P_3 = 5040 - 504 = 4536$.

EXAMPLE 21 *In how many ways 7 pictures can be hung from 5 picture nails on a wall ?*

SOLUTION The number of ways in which 7 pictures can be hung from 5 picture nails on a wall is same as the number of arrangements of 7 things, taking 5 at a time.

Hence, the required number $= {}^7P_5 = \frac{7!}{(7-5)!} = \frac{7!}{2!} = 2520.$

EXAMPLE 22 *Determine the number of natural numbers smaller than 10^4, in the decimal notation of which all the digits are distinct.*

SOLUTION The required natural numbers consist of 4 digits, 3 digits, 2 digits and one digit.

Total number of 4 digit natural numbers with distinct digits $= {}^{10}P_4 - {}^9P_3$

Total number of 3 digit natural numbers with distinct digits $= {}^{10}P_3 - {}^9P_2$

Total number of 2 digit natural numbers with distinct digits $= {}^{10}P_2 - {}^9P_1$

Total number of one digit natural numbers = 9

Hence, the required number of natural numbers $= ({}^{10}P_4 - {}^9P_3) + ({}^{10}P_3 - {}^9P_2) + ({}^{10}P_2 - {}^9P_1) + 9$

$= 9 \times 9 \times 8 \times 7 + 9 \times 9 \times 8 + 9 \times 9 + 9 = 5274..$

EXAMPLE 23 *How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once.*

SOLUTION There are eight letters in the word 'EQUATION'. So, the total number of words is equal to the number of arrangements of these letters, taken all at a time. The number of such arrangements is ${}^8P_8 = 8!$ Hence, the total number of words = 8!

EXAMPLE 24 *How many 4-letter words, with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?*

SOLUTION There are 10 letters in the word 'LOGARITHMS'.

So, the number of 4 - letter word = Number of arrangements of 10 letters, taken 4 at a time

$= {}^{10}P_4 = 5040.$

LEVEL-2

EXAMPLE 25 *Prove that if $r \le s \le n$, then $P(n, s)$ is divisible by $P(n, r)$.*

SOLUTION Let $s = r + k$ where $0 \le k \le s - r$. Then,

$$P(n, s) = \frac{n!}{(n-s)!} = n(n-1)(n-2)\dots(n-(s-1))$$

$$\Rightarrow \quad P(n, s) = n(n-1)(n-2)\dots\{n-(r+k-1)\}$$

$$\Rightarrow \quad P(n, s) = n(n-1)(n-2)\dots\{n-(r-1)\}(n-r)\{n-(r+1)\}\dots\{n-(r+k-1)\}$$

$$\Rightarrow \quad P(n, s) = \{n(n-1)(n-2)\dots n-(r-1)\}\{(n-r)(n-(r+1))\dots(n-(r+k-1))\}$$

$$\Rightarrow \quad P(n, s) = P(n, r)\cdot\{(n-r)(n-(r+1))\dots(n-(r+k-1))\}$$

$$\left[\because P(n, r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)\dots(n-(r-1))\right]$$

$$\Rightarrow \quad P(n, s) = P(n, r)\cdot\{(n-r)(n-(r+1))\dots(n-(r+k-1))\}$$

$\Rightarrow$ $P(n, s)$ is divisible by $P(n, r)$.

EXAMPLE 26 *If P_m stands for ${}^mP_m$, then prove that:*

$$1 + 1\cdot P_1 + 2\cdot P_2 + 3\cdot P_3 + \dots + n\cdot P_n = (n+1)!$$

SOLUTION We have, $P_m = {}^mP_m = m!$

So, $\quad 1 + 1\cdot P_1 + 2\cdot P_2 + 3\cdot P_3 + \dots + n\cdot P_n$

$$= 1 + 1 + 2\cdot 2! + 3\cdot 3! + 4\cdot 4! + \ldots + n\cdot n!$$

$$= 1 + \sum_{r=1}^{n} r\cdot r!$$

$$= 1 + \sum_{r=1}^{n} \{(r+1) - 1\}\, r!$$

$$= 1 + \sum_{r=1}^{n} [(r+1)\, r! - r!]$$

$$= 1 + \sum_{r=1}^{n} [(r+1)! - r!]$$

$$= 1 + [(2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots + ((n+1)! - n!)] = 1 + ((n+1)! - 1!) = (n+1)!$$

EXAMPLE 27 *In an examination hall there are four rows of chairs. Each row has 8 chairs one behind the other. There are two classes sitting for the examination with 16 students in each class. It is desired that in each row, all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many ways can these 32 students be seated?*

SOLUTION Let the two classes be C_1 and C_2 and the four rows be R_1, R_2, R_3, R_4. There are 16 students in each class. So, there are 32 students. According to the given conditions there are two different ways in which 32 students can be seated:

	R_1	R_2	R_3	R_4
I	C_1	C_2	C_1	C_2
II	C_2	C_1	C_2	C_1

Since the seating arrangement can be completed by using any one of these two ways. So, by the fundamental principle of addition,

Total number of seating arrangements = No. of arrangement in I case + No. of arrangements in II case.

In case I, 16 students of class C_1 can be seated in R_1 and R_3 in ${}^{16}P_8 \times 8! = 16!$ ways. And 16 students of class C_2 can be seated in R_2 and R_4 in ${}^{16}P_8 \times 8! = 16!$ ways

$\therefore$ Number of seating arrangements in case $I = 16! \times 16!$

Similarly, Number of seating arrangements in case $II = 16! \times 16!$

Hence, Total number of seating arrangements = $(16! \times 16!) + (16! \times 16!) = 2\,(16! \times 16!)$

EXAMPLE 28 *Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. Determine the number of words which have at least one letter repeated.*

SOLUTION The number of 5-letter words which can be formed from 10 letters when one or more of its letters is repeated $= 10 \times 10 \times 10 \times 10 \times 10 = 10^5$.

The number of 5-letter words which can be formed when none of their letters is repeated

= Number of arrangements of 10 letters by taking 5 at a time $= {}^{10}P_5 = 30240$

Hence, the number of 5-letter words which have at least one of their letters repeated is $10^5 - 30240 = 69760$.

EXAMPLE 29 *Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.*

SOLUTION The total number of numbers formed with the digits 2, 3, 4, 5 taken all at a time

= Number of arrangement of 4 digits, taken all at a time $= {}^4P_4 = 4! = 24$.

To find the sum of these 24 numbers, we will find the sum of digits at unit's, ten's, hundred's and thousand's places in all these numbers.

Consider the digits in the unit's places in all these numbers. Each of the digits 2, 3, 4, 5 occurs in $3!(=6)$ times in the unit's place.

So, total for the digits in the unit's place in all the numbers $= (2+3+4+5)\times 3! = 84$.

Since each of the digits 2, 3, 4, 5 occurs $3!$ times in any one of the remaining places.

So, the sum of the digits in the ten's, hundred's and thousand's places in all the numbers

$$= (2+3+4+5)\times 3! = 84.$$

Hence, the sum of all the numbers $= 84\,(10^0 + 10^1 + 10^2 + 10^3) = 93324$.

EXERCISE 16.3

LEVEL-1

1. Evaluate each of the following:
   (i) ${}^8P_3$ (ii) ${}^{10}P_4$ (iii) ${}^6P_6$ (iv) $P(6, 4)$
2. If $P(5, r) = P(6, r-1)$, find r. [NCERT]
3. If $5\,P(4, n) = 6.\,P(5, n-1)$, find n.
4. If $P(n, 5) = 20\,.\,P(n, 3)$, find n
5. If ${}^nP_4 = 360$, find the value of n.
6. If $P(9, r) = 3024$, find r.
7. If $P(11, r) = P(12, r-1)$ find r.
8. If $P(n, 4) = 12\,.\,P(n, 2)$, find n.
9. If $P(n-1, 3) : P(n, 4) = 1 : 9$, find n. [NCERT]
10. If $P(2n-1, n) : P(2n+1, n-1) = 22 : 7$ find n.
11. If $P(n, 5) : P(n, 3) = 2 : 1$, find n.
12. Prove that: $1\,.\,P(1, 1) + 2\,.\,P(2, 2) + 3\,.\,P(3, 3) + \ldots + n\,.\,P(n, n) = P(n+1, n+1) - 1$.
13. If $P(15, r-1) : P(16, r-2) = 3 : 4$, find r.
14. If ${}^{n+5}P_{n+1} = \dfrac{11\,(n-1)}{2}\,{}^{n+3}P_n$, find n.
15. In how many ways can five children stand in a queue?
16. From among the 36 teachers in a school, one principal and one vice-principal are to be appointed. In how many ways can this be done?
17. Four letters E, K, S and V, one in each, were purchased from a plastic warehouse. How many ordered pairs of letters, to be used as initials, can be formed from them?
18. Four books, one each in Chemistry, Physics, Biology and Mathematics, are to be arranged in a shelf. In how many ways can this be done?
19. Find the number of different 4-letter words, with or without meanings, that can be formed from the letters of the word 'NUMBER'.
20. How many three-digit numbers are there, with distinct digits, with each digit odd?
21. How many words, with or without meaning, can be formed by using all the letters of the word 'DELHI', using each letter exactly once?
22. How many words, with or without meaning, can be formed by using the letters of the word 'TRIANGLE'?

23. There are two works each of 3 volumes and two works each of 2 volumes; In how many ways can the 10 books be placed on a shelf so that the volumes of the same work are not separated?

24. There are 6 items in column A and 6 items in column B. A student is asked to match each item in column A with an item in column B. How many possible, correct or incorrect, answers are there to this question?

25. How many three-digit numbers are there, with no digit repeated?

26. How many 6-digit telephone numbers can be constructed with digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if each number starts with 35 and no digit appears more than once?

27. In how many ways can 6 boys and 5 girls be arranged for a group photograph if the girls are to sit on chairs in a row and the boys are to stand in a row behind them?

28. If a denotes the number of permutations of $(x+2)$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $x-11$ things taken all at a time such that $a = 182\,bc$, find the value of x.

29. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated? [NCERT]

30. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 5, 6, 7, if no digits is repeated? [NCERT]

31. Find the numbers of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, if no digit is repreated? Howmany of these will be even? [NCERT]

32. All the letters of the word 'EAMCOT' are arranged in different possible ways. Find the number of arrangements in which no two vowels are adjacent to each other.

ANSWERS

1. (i) 336	(ii) 5040	(iii) 720	(iv) 360
2. 4	3. 3	4. 8	5. 6
6. 4	7. 9	8. 6	9. 9
10. 10	11. 5	13. 14	14. 6, 7
15. 120	16. 1260	17. 12	18. 24
19. 360	20. 60	21. 120	22. 8!
23. 3456	24. 720	25. 648	26. 1680
27. 86400	28. 12	29. 504	30. 90
31. 120, 48	32. 144		

HINTS TO NCERT & SELECTED PROBLEMS

2. We have

$$P(5, r) = P(6, r-1)$$

$$\Rightarrow \frac{5!}{(5-r)!} = \frac{6!}{\{6-(r-1)\}!}$$

$$\Rightarrow \frac{5!}{(5-r)!} = \frac{6 \times 5!}{(7-r)!}$$

$$\Rightarrow \frac{1}{(5-r)!} = \frac{6}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow 1 = \frac{6}{(7-r)(6-r)} \Rightarrow (7-r)(6-r) = 3 \times 2 \Rightarrow 7-r = 3 \Rightarrow r = 4$$

9. $P(n-1, 3) : P(n, 4) = 1 : 9 \Rightarrow \frac{P(n-1, 3)}{P(n, 4)} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{(n-4)!} \times \frac{(n-4)!}{n!} = \frac{1}{9} \Rightarrow \frac{1}{n} = \frac{1}{9} \Rightarrow n = 9.$

15. The total no. of ways = No. of arrangements of 5 things, taken all at a time = ${}^5P_5$.

16. Total no of ways = No. of arrangements of 36 things taken two at a time = ${}^{36}P_2$.

17. The total no. of ordered pairs = No. of arrangements of 4 letters, taken two at a time = ${}^4P_2$.

18. No. of ways = No. of arrangements of 4 books, taken all at a time = ${}^4P_4$.

19. Total no. of words = No. of arrangements of 6 letters, taken 4 at a time = ${}^6P_4$.

20. Required number of numbers = Number of arrangements of digits 1, 3, 5, 7, 9 by taking 3 at a time = ${}^5P_3$.

23. Let $\frac{W_{11}, W_{12}, W_{13}}{W_1}; \frac{W_{21}, W_{22}, W_{23}}{W_2}; \frac{W_{31}, W_{32}}{W_3}; \frac{W_{41}, W_{42}}{W_4}$ be 4 works. These 4 works can be arranged in 4! ways. Now, volumes of each work can be arranged in the following ways: $W_1 \rightarrow 3!$ ways; $W_2 \rightarrow 3!$ ways, $W_3 \rightarrow 2!$ ways, $W_4 \rightarrow 2!$ ways.

Hence, total no. of ways to arrange all books $= 4!(3! \times 3! \times 2! \times 2!) = 3456$.

24. Each answer to the given question is an arrangement of the 6 items of column B keeping the order of items in column A fixed. Hence, the total number of answers = Number of arrangements of 6 items in column $B = {}^6P_6 = 6!$.

25. Total number of three digit numbers with distinct digits $= {}^{10}P_3 - {}^9P_2$.

26. Required number of telephone numbers $= {}^8P_4$.

27. Five girls can sit on chairs in a row in ${}^5P_5 = 5!$ ways. Also, 6 boys can stand behind them in a row in ${}^6P_6 = 6!$ ways. Hence, the total number of ways $= 5! \times 6!$

31. The total number of 4 digit numbers formed by using the digits 1, 2, 3, 4, 5 is same as the number of arrangements of 5 digits taken 4 at a time.

So, required number of numbers $= {}^5P_4 = \frac{5!}{(5-4)!} = 120$

An even number will have 2 or 4 at its unit's place. So, unit's place can be filled in 2 ways and the remaining three places (tens, hundreds and thousands) can be filled with remaining 4 digits in ${}^4P_3$ ways. Hence, total number of 4 digit even numbers formed by using the given digits is ${}^4P_3 \times 2 = 48$.

16.4 PERMUTATIONS UNDER CERTAIN CONDITIONS

In this section, we shall discuss permutations where either repetitions of items are allowed or distinction between some of the items are ignored or a particular item occurs in every arrangement etc. Such type of permutations are known as permutations under certain conditions as discussed below.

THEOREM 1 *Prove that the number of all permutations of n different objects taken r at a time, when a particular object is to be always included in each arrangement, is* $r \cdot {}^{n-1}P_{r-1}$.

PROOF Here we have to find the number of ways in which r places can be filled with n given objects such that a particular object occurs in each arrangement. Suppose the particular object is

placed at the first place. Then, the remaining $(n-1)$ places can be filled with remaining $(r-1)$ objects in ${}^{n-1}P_{r-1}$ ways. Similarly, by fixing the particular object at the second, third, fourth, ..., rth places, we find that the number of permutations in each case is ${}^{n-1}P_{r-1}$.

Hence, by the fundamental principle of addition,
The required number of permutations $= {}^{n-1}P_{r-1} + {}^{n-1}P_{r-1} + \dots + {}^{n-1}P_{r-1} = r \cdot {}^{n-1}P_{r-1}$.

Q.E.D.

THEOREM 2 *Prove that the number of permutations of n distinct objects taken r at a time, when a particular object is never taken in each arrangement, is ${}^{n-1}P_r$.*

PROOF Since one particular object out of n given objects is never taken. So, we have to determine the number of ways in which r places can be filled with $(n-1)$ distinct objects.
Clearly, the number of such arrangement is ${}^{n-1}P_r$.

Q.E.D.

THEOREM 3 *Prove that the number of permutations of n different objects taken r at a time in which two specified objects always occur together is $2!(r-1)\,{}^{n-2}P_{r-2}$.*

PROOF First let us leave out the two specified objects. Then the number of permutations of the remaining $(n-2)$ objects, taken $(r-2)$at a time, is ${}^{n-2}P_{r-2}$. Now, we consider two specified objects temporarily as a single object and add it to each of these ${}^{n-2}P_{r-2}$ permutations which can be done in $(r-1)$ ways. Thus, the number of permutations becomes $(r-1)\,{}^{n-2}P_{r-2}$. But two specified things can be put together in $2!$ ways.
Hence, the required number of permutations is $2! \cdot (r-1) \cdot {}^{n-2}P_{r-2}$.

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *In how many ways can the letters of the word PENCIL be arranged so that (i) N is always next to E ? (ii) N and E are always together ?*

SOLUTION (i) Let us keep EN together and consider it as one letter. Now, we have 5 letters which can be arranged in a row in ${}^5P_5 = 5! = 120$ ways. Hence, the total number of ways in which N is always next to E is 120.

(ii) Keeping E and N together and considering it as one letter, we have 5 letters which can be arranged in ${}^5P_5 = 5!$ ways. But, E and N can be put together 2! ways (viz. EN, NE).
Hence, the total number of ways $= 5! \times 2! = 240$.

EXAMPLE 2 *How many different words can be formed with the letters of the word EQUATION so that*
(i) *the words begin with E ?* (ii) *the words begin with E and end with N ?*
(iii) *the words begin and end with a consonant ?*

SOLUTION Clearly, the given word contains 8 letters out of which 5 are vowels and 3 consonants.

(i) Since all words must begin with E. So, we fix E at the first place. Now, remaining 7 letters can be arranged in ${}^7P_7 = 7!$ ways.

So, total number of words $= 7!$

(ii) Since all words must begin with E and end with N. So, we fix E at the first place and N at the last place. Now, remaining 6 letters can be arranged in ${}^6P_6 = 6!$ ways.

Hence, the required number of words $= {}^6P_6 = 6!$

(iii) There are 3 consonants and all words should begin and end with a consonant. So, first and last places can be filled with 3 consonants in ${}^3P_2$ ways. Now, the remaining 6 places are to be filled up with the remaining 6 letters in ${}^6P_6$ ways.

Hence, the required number of words $= {}^3P_2 \times {}^6P_6 = 6 \times 720 = 4320$

EXAMPLE 3 *How many words can be formed from the letters of the word, 'TRIANGLE' ? How many of these will begin with T and end with E ?*

SOLUTION There are 8 letters in the word 'TRIANGLE'. The total number of words formed with these 8 letters is the number of arrangements of 8 items, taken all at a time, which is equal to ${}^8P_8 = 8! = 40320$. If we fix up T in the beginning and E at the end, then the remaining 6 letters can be arranged in ${}^6P_6 = 6!$ ways.

So, the total number of words which begin with T and end with $E = 6! = 720$.

EXAMPLE 4 *How many words can be formed with the letters of the word 'ORDINATE' so that vowels occupy odd places?*

SOLUTION There are 4 vowels and 4 consonants in the word 'ORDINATE'. We have to arrange 8 letters in a row such that vowels occupy odd places. There are 4 odd places viz. 1, 3, 5, 7. Four vowels can be arranged in these 4 odd places in 4! ways. Remaining 4 even places viz. 2, 4, 6, 8 are to be occupied by the 4 consonants. This can be done in 4! ways. Hence, the total number of words in which vowels occupy odd places $= 4! \times 4! = 576$.

EXAMPLE 5 *In how many ways 5 boys and 3 girls can be seated in a row so that no two girls are together?*

SOLUTION The 5 boys can be seated in a row in ${}^5P_5 = 5!$ ways. In each of these arrangements 6 places are created, shown by the cross-marks, as given below:

$$\times B \times B \times B \times B \times B \times$$

Since no two girls are to sit together, so we may arrange 3 girls in 6 places. This can be done in ${}^6P_3$ ways i.e. 3 girls can be seated in ${}^6P_3$ ways.

Hence, the total number of seating arrangements $= {}^5P_5 \times {}^6P_3 = 5! \times 6 \times 5 \times 4 = 14400$.

EXAMPLE 6 *In how many ways can the letters of the word 'DELHI' be arranged so that the vowels occupy only even places?*

SOLUTION There are 5 distinct letters in the word 'DELHI'. We wish to find the total number of arrangements of these 5 letters so that vowels occupy only even places. There are two vowels E and I and 2 even places viz 2^{nd} and 4^{th}. These two vowels can be arranged in the two even places in 2! ways. The remaining three letters (D, L, H) can be arranged in 3 places (viz Ist 3rd, 5th) in 3! ways. Hence, by the fundamental principle of counting the total number of arrangements $= 3! \times 2! = 12$

EXAMPLE 7 *How many words can be formed from the letters of the word 'DAUGHTER' so that*

(i) *the vowels always come together?* (ii) *the vowels never come together?* [NCERT]

SOLUTION There are 8 letters in the word 'DAUGHTER', including 3 vowels (A, U, E) and 5 consonants (D, G, H, T, R).

(i) Considering three vowels as one letter, we have 6 letters which can be arranged in ${}^6P_6 = 6!$ ways. But, corresponding each way of these arrangements, the vowels A, U, E can be put together in 3! ways.

Hence, required number of words $= 6! \times 3! = 720 \times 6 = 4320$

(ii) The total number of words formed by using all the eight letters of the word 'DAUGHTER' is ${}^8P_8 = 8! = 40320$.

So, the total number of words in which vowels are never together

= Total number of words – Number of words in which vowels are always together

$= 40320 - 4320 = 36000$

EXAMPLE 8 *In how many ways can 9 examination papers be arranged so that the best and the worst papers are never together?*

SOLUTION The number of arrangements in which the best and the worst papers never come together can be obtained by subtracting from the total number of arrangements, the number of arrangements in which the best and worst come together.

The total number of arrangements of 9 papers $= {}^9P_9 = 9!$

Considering the best and the worst papers as one paper, we have 8 papers which can be arranged in ${}^8P_8 = 8!$ ways. But, the best and worst papers can be put together in $2!$ ways. So, the number of permutations in which the best and the worst papers can be put together $= (2! \times 8!)$.

Hence, the number of ways in which the best and the worst papers never come together $= 9! - 2! \times 8! = 9 \times 8! - 2 \times 8! = 7 \times 8! = 282240$.

EXAMPLE 9 *In how many ways can 5 children be arranged in a row such that*

(i) *two of them, Ram and Shyam, are always together?*

(ii) *two of them, Ram and Shyam, are never together?*

SOLUTION There are five children including Ram and Shyam.

(i) Considering Ram and Shyam as one child, there are four children. They can be arranged in a row in 4! ways. But Ram and Shyam can be arranged together in 2! ways.

Hence, the required number of arrangements $= 4! \times 2! = 48$.

(ii) Total number of arrangements of 5 children in a row $= 5! = 120$.

$\therefore$ Total number of arrangements in which Ram and Shyam are never together

= Total number of arrangements – Number of arrangements in which Ram and Shyam are together

$= 120 - 48 = 72$.

EXAMPLE 10 *A code word is to consist of two distinct English alphabets followed by two distinct numbers from 1 to 9. For example, CA 23 is a code word. How many such code words are there? How many of them end with an even integer ?*

SOLUTION There are 26 English alphabets. So, first two places in the code word can be filled in ${}^{26}P_2$ ways. In last two places we have to use two distinct numbers from 1 to 9. So, last two places can be filled in ${}^9P_2$ ways. Hence, by the fundamental principle of counting, the total number of code words $= {}^{26}P_2 \times {}^9P_2 = 650 \times 72 = 46800$.

Number of code words ending with an even integer.

In this case, the code word can have any of the numbers 2, 4, 6, 8 at the extreme right position. So, the extreme right position can be filled in 4 ways. Now, next left position can be filled with any one of the remaining 8 digits in 8 ways and the two extreme left positions can be filled by two English alphabets in ${}^{26}P_2$ ways.

Hence, the total number of code words which end with an even integer $= 4 \times 8 \times {}^{26}P_2$

$= 4 \times 8 \times 650 = 20800$.

EXAMPLE 11 *The Principal wants to arrange 5 students on the platform such that the boy 'SALIM' occupies the second position and such that the girl, 'SITA' is always adjacent to the girl 'RITA'. How many such arrangements are possible ?*

SOLUTION Since SALIM occupies the second position and the two girls RITA and SITA are always adjacent to each other. So, none of these two girls can occupy the first seat. Thus, first seat can be occupied by any one of the remaining two students in 2 ways. Second seat can be occupied by SALIM in only one way.

Now, in the remaining three seats SITA and RITA can be seated in the following four ways:

	I	II	III	IV	V
1.	×	SALIM	SITA	RITA	×
2.	×	SALIM	RITA	SITA	×
3.	×	SALIM	×	SITA	RITA
4.	×	SALIM	×	RITA	SITA

Now, only one seat is left which can be occupied by the 5th student in one way.

Hence, the number of required type of arrangements $= 2 \times 4 \times 1 = 8$.

EXAMPLE 12 *How many numbers between 400 and 1000 can be formed with the digits 0, 2, 3, 4, 5, 6 if no digit is repeated in the same number ?*

SOLUTION Number between 400 and 1000 consist of three digits with digit at hundred's place greater than or equal to 4. Hundred's place can be filled, by using the digits 4, 5, 6 in 3 ways. Now, ten's and unit's places can be filled by the remaining 5 digits in ${}^5P_2$ ways.

Hence, the required number of numbers $= 3 \times {}^5P_2 = 3 \times \frac{5!}{3!} = 3 \times 20 = 60.$

EXAMPLE 13 *In a class of 10 students there are 3 girls A, B, C. In how many different ways can they be arranged in a row such that no two of the three girls are consecutive.*

SOLUTION There are 7 boys and 3 girls. Seven boys can be arranged in a row in ${}^7P_7 = 7!$ ways. Now, we have 8 places in which we can arrange 3 girls in ${}^8P_3$ ways.

Hence, by the fundamental principle of counting, the number of arrangements $= 7! \times {}^8P_3$

$$= 7! \times 336.$$

LEVEL-2

EXAMPLE 14 *When a group photograph is taken, all the seven teachers should be in the first row and all the twenty students should be in the second row. If the two corners of the second row are reserved for the two tallest students, interchangeable only between them, and if the middle seat of the front row is reserved for the Principal, how many arrangements are possible?*

SOLUTION Since the middle seat of the front row is reserved for the Principal, the remaining 6 teachers can be arranged in the front row in ${}^6P_6 = 6!$ ways.

The two corners of the second row are reserved for the two tallest students. They can occupy these two places in 2! ways. The remaining 18 seats may be occupied by the remaining 18 students in 18! ways.

Hence, by the fundamental principle of counting, the total number of arrangements
$= 6! \times (18! \times 2!) = 18! \times 1440.$

EXAMPLE 15 *How many even numbers are there with three digits such that if 5 is one of the digits, then 7 is the next digit?*

SOLUTION We have to determine the total number of even numbers formed by using the given condition. So, at unit's place we can use one of the digits 0, 2, 4, 6, 8. If 5 is at ten's place then, as per the given condition, 7 should be at unit's place. In such a case the number will not be an even number. So, 5 cannot be at ten's and one's places. Hence, 5 can be only at hundred's place.

Now two cases arise.

CASE I *When 5 is at hundred's place:*

If 5 is at hundred's place, then 7 will be at ten's place. So, unit's place can be filled in 5 ways by using the digits 0, 2, 4, 6, 8.

So, total number of even numbers $= 1 \times 1 \times 5 = 5$.

CASE II *When 5 is not at hundred's place:*

In this case, hundred's place can be filled in 8 ways (0 and 5 cannot be used at hundred's place). In ten's place we can use any one of the ten digits except 5. So, ten's place can be filled in 9 ways. At unit's place we have to use one of the even digits 0, 2, 4, 6, 8. So, units place can be filled in 5 ways.

So, total number of even numbers $= 8 \times 9 \times 5 = 360$

Hence, the total number of required even numbers $= 360 + 5 = 365$.

EXAMPLE 16 *How many four digit numbers divisible by 4 can be made with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed ?*

SOLUTION Recall that a number is divisible by 4 if the number formed by the last two digits is divisible by 4. The digits at unit's and ten's places can be arranged as follows:

Th	H	T	O
×	×	1	2
×	×	2	4
×	×	3	2
×	×	5	2

Now, corresponding each such way the remaining three digits at thousand's and hundred's places can be arranged in ${}^3P_2$ ways.

Hence, the required number of numbers $= {}^3P_2 \times 4 = 3! \times 4 = 24$.

EXAMPLE 17 *Find the number of ways in which 5 boys and 5 girls be seated in a row so that*

(i) *No two girls may sit together.* (ii) *All thgirls sit together and all the boys sit together.*

(iii) *All the girls are never together.*

SOLUTION (i) 5 boys can be seated in a row in ${}^5P_5 = 5!$ ways. Now, in the 6 gaps 5 girls can be arranged in ${}^6P_5$ ways.

Hence, the number of ways in which no two girls sit together $= 5! \times {}^6P_5 = 5! \times 6!$

(ii) The two groups of girls and boys can be arranged in 2! ways. 5 girls can be arranged among themselves in 5! ways. Similarly, 5 boys can be arranged among themselves in 5! ways. Hence, by the fundamental principle of counting, the total number of requisite seating arrangements $= 2!(5! \times 5!) = 2(5!)^2$.

(iii) The total number of ways in which all the girls are never together

= Total number of arrangements – Total number of arrangements in which all the girls are always together

$= 10! - 5! \times 6!$

EXAMPLE 18 *Five boys and five girls form a line with the boys and girls alternating. Find the number of ways of making the line.*

SOLUTION 5 boys can be arranged in a line in $^5P_5 = 5!$ ways. Since the boys and girls are alternating. So, corresponding each of the 5! ways of arrangements of 5 boys we obtain 5 places marked by cross as shown below:

(i) $B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times$ (ii) $\times B_1 \times B_2 \times B_3 \times B_4 \times B_5$.

Clearly, 5 girls can be arranged in 5 places marked by cross in (5! + 5!) ways.

Hence, the total number of ways of making the line $= 5! \times (5! + 5!) = 2\,(5!)^2$

EXAMPLE 19 *In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls sit together in a back row on adjacent seats ?*

SOLUTION Total number of persons = 3 girls + 9 boys = 12.

Total number of numbered seats $= 2 \times 3 + 4 \times 2 = 14$

So, total number of ways in which 12 persons can be seated on 14 seats

= Number of arrangements or 14 seats by taking 12 at a time $= {}^{14}P_{12}$.

Three girls can be seated together in a back row on adjacent seats in the following ways:

1, 2, 3 or 2, 3, 4 of first van

and, 1, 2, 3 or 2, 3, 4 of second one.

In each way the three girls can interchange among themselves in 3! ways. So, the total number of ways in which three girls can be seated together in a back row on adjacent seats $= 4 \times 3!$

Now, 9 boys are to be seated on remaining 11 seats, which can be done in ${}^{11}P_9$ ways.

Hence, by the fundamental principle of counting, the total number of seating arrangements is ${}^{11}P_9 \times 4 \times 3!$.

EXAMPLE 20 *A tea party is arranged for 16 persons along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular and two on the other side. In how many ways can they be seated?*

SOLUTION Let the two sides be A and B. Assume that four persons wish to sit on side A. Four persons who wish to sit on side A can be accommodated on eight chairs in ${}^8P_4$ ways and two persons who wish to sit on side B can be accommodated on 8 chairs in ${}^8P_2$ ways. Now, 10 persons are left, who can sit on 10 chairs on both the sides of the table in 10! ways.

Hence, the total number of ways in which 16 persons can be seated $= {}^8P_4 \times {}^8P_2 \times 10!$

EXERCISE 16.4

LEVEL-1

1. In how many ways can the letters of the word 'FAILURE' be arranged so that the consonants may occupy only odd positions?
2. In how many ways can the letters of the word 'STRANGE' be arranged so that
   (i) the vowels come together ? (ii) the vowels never come together? and
   (iii) the vowels occupy only the odd places?
3. How many words can be formed from the letters of the word 'SUNDAY'? How many of these begin with D?

4. How many words can be formed out of the letters of the word, 'ORIENTAL', so that the vowels always occupy the odd places ?
5. How many different words can be formed with the letters of word 'SUNDAY'? How many of the words begin with N? How many begin with N and end in Y?
6. How many different words can be formed from the letters of the word 'GANESHPURI' ? In how many of these words:
   (i) the letter G always occupies the first place?
   (ii) the letters P and I respectively occupy first and last place?
   (iii) the vowels are always together?
   (iv) the vowels always occupy even places?
7. How many permutations can be formed by the letters of the word, 'VOWELS', when
   (i) there is no restriction on letters? (ii) each word begins with E?
   (iii) each word begins with O and ends with L? (iv) all vowels come together?
   (v) all consonants come together?
8. How many words can be formed out of the letters of the word 'ARTICLE', so that vowels occupy even places?
9. In how many ways can a lawn tennis mixed double be made up from seven married couples if no husband and wife play in the same set?
10. m men and n women are to be seated in a row so that no two women sit together. If $m > n$ then show that the number of ways in which they can be seated as $\dfrac{m!\,(m+1)!}{(m-n+1)\,!}$.
11. How many words (with or without dictionary meaning) can be made from the letters in the word MONDAY, assuming that no letter is repeated, if
   (i) 4 letters are used at a time? (ii) all letters are used at a time?
   (iii) all letters are used but first is vowel?
12. How many three letter words can be made using the letters of the word 'ORIENTAL'?

ANSWERS

1. 576 2. (i) 1440 (ii) 3600 (iii) 1440 3. 720, 120 4. 576
5. 720, 120, 24 6. 10! (i) 9! (ii) 8! (iii) $7! \times 4!$ (iv) $5! \times 6!$
7. (i) 720 (ii) 120 (iii) 24 (iv) 240 (v) 144 8. 144 9. 840
11. (i) 360 (ii) 720 (iii) 240 12. 336

16.5 PERMUTATIONS OF OBJECTS NOT ALL DISTINCT

So far we were discusing permutations of distinct objects (things) by taking some or all at a time. In this section, we intend to discuss the permutations of a given number of objects when objects are not all different. For example, the number of arrangements of the letters of the word MISSISSIPPI, the number of six digit numbers formed by using the digits 1, 1, 2, 3, 3, 4 etc. The following theorem is very helpful to determine the number of such arrangements.

THEOREM *The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$ is $\dfrac{n!}{p!\,q!}$.*

PROOF Let the required number of permutations be x. Consider one of these x permutations. Now, replace p alike things in this permutation by p distinct things which are also different from others. These p different things may be permuted among themselves in $p!$ ways without changing the positions of other things. Similarly, if we replace q alike things by q distinct things, which are also different from others, then they can be permuted among themselves in $q!$ ways.

Thus, if both the replacements are done simultaneously, then we find that each one of the x permutations give rise to $p! \times q!$ permutations. Therefore, x permutations give rise to $x \times p! \times q!$ permutations. Now, each of these $x \times p! \times q!$ permutations, is a permutation of n different things, taken all at a time.

$\therefore$ $x \times p! \times q! =$ Number of permutations of n different things taken all at a time $= n!$

Hence, $x = \dfrac{n!}{p!\,q!}$

Q.E.D.

REMARK 1 *The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind; ...; p_r are alike of rth kind such that $p_1 + p_2 + \dots + p_r = n$, is*

$$\frac{n!}{p_1!\,p_2!\,p_3!\dots p_r!}$$

REMARK 2 *The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is $\dfrac{n!}{p!\,q!}$.*

REMARK 3 *Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \dots p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects to the above condition is $\dfrac{(p_1 + p_2 + \dots + p_r)!}{p_1!\,p_2!\,p_3!\dots p_r!}$.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *How many different words can be formed with the letters of the word 'MISSISSIPPI'? In how many of these permutations four I's do not come together?* [NCERT]

SOLUTION There are 11 letters in the given word, of which 4 are S's, 4 are I's and 2 are P's. So, total number of words is the number of arrangements of 11 things, of which 4 are similar of one kind, 4 are similar of second kind and 2 are similar of third kind i.e. $\dfrac{11!}{4!\,4!\,2!}$.

Hence,the total number of words $= \dfrac{11!}{4!\,4!\,2!} = 34650$.

Considering 4 I's as one letter, we have 8 letters of which 4 are S's and 2 are P's. These 8 letters can be arranged in $\dfrac{8!}{4!\,2!}$ ways.

$\therefore$ Number of words in which 4 I's come together $= \dfrac{8!}{4!\,2!} = 840$.

Hence, number of words in which 4 I's do not come together $= 34650 - 840 = 33810$.

EXAMPLE 2 *How many permutations of the letters of the word 'APPLE' are there?*

SOLUTION Here there are 5 letters, two of which are of the same kind. The others are each of its own kind. So, the required number of permutations is $\dfrac{5!}{2!1!1!1!} = \dfrac{120}{2} = 60$.

EXAMPLE 3 *How many words can be formed using the letter A thrice, the letter B twice and the letter C thrice?*

SOLUTION We are given 8 letters viz. A, A, A, B, B, C, C, C. Clearly, there are 8 letters of which three are of one kind, two are of second kind and three are of third kind.

So, the total number of permutations $= \dfrac{8!}{3!\,2!\,3!} = 560$.

Hence, the requisite number of words = 560.

EXAMPLE 4 *Find the number of different permutations of the letters of the word BANANA ?*

SOLUTION Clearly, there are six letters in the word 'BANANA' of which three are alike of one kind (3 A's), two are alike of second kind (2 N's) and one of its own kind.

$\therefore$ Total number of their permutations $= \dfrac{6!}{3!\,2!\,1!} = 60.$

Hence, the requisite number of words = 60

EXAMPLE 5 (i) *How many different words can be formed with the letters of the word HARYANA?*

(ii) *How many of these begin with H and end with N?*

(iii) *In how many of these H and N are together?*

SOLUTION (i) There are 7 letters in the word 'HARYANA' of which 3 are A's and remaining all are each of its own kind.

So, total number of words $= \dfrac{7!}{3!1!1!1!1!} = \dfrac{7!}{3!} = 840.$

(ii) After fixing H in first place and N in last place, we have 5 letters out of which three are alike i.e. A's and remaining all are each of its own kind.

So, total number of words $= \dfrac{5!}{3!} = 20.$

(iii) Considering H and N together we have $7 - 2 + 1 = 6$ letters out of which three are alike i.e. A's and others are each of its own kind. These six letters can be arranged in $\dfrac{6!}{3!}$ ways. But H and N can be arranged amongst themselves in 2! ways.

Hence, the requisite number of words $= \dfrac{6!}{3!} \times 2! = 120 \times 2 = 240.$

EXAMPLE 6 *How many different words can be formed by using all the letters of the word 'ALLAHABAD' ?* [NCERT]

(i) *In how many of them vowels occupy the even positions ?*

(ii) *In how many of them both L do not come together ?*

SOLUTION There are 9 letters in the word 'ALLAHABAD' out of which 4 are A's, 2 are L's and the rest are all distinct.

So, the requisite number of words $= \dfrac{9!}{4!\,2!} = 7560.$

(i) There are 4 vowels and all are alike i.e. 4 A's. Also, there are 4 even places viz 2nd, 4th, 6th and 8th. So, these 4 even places can be occupied by 4 vowels in $\dfrac{4!}{4!} = 1$ way. Now, we are left with 5 places in which 5 letters, of which two are alike (2 L's) and other distinct, can be arranged in $\dfrac{5!}{2!}$ ways.

Hence, the total number of words in which vowels occupy the even places $= \dfrac{5!}{2!} \times \dfrac{4!}{4!} = \dfrac{5!}{2!} = 60.$

(ii) Considering both L together and treating them as one letter we have 8 letters out of which A repeats 4 times and others are distinct. These 8 letters can be arranged in $\dfrac{8!}{4!}$ ways.

So, the number of words in which both L come together $= \dfrac{8!}{4!} = 1680.$

Hence, the number of words in which both L do not come together

= Total no. of words – No. of words in which both L come together = $7560 - 1680 = 5880.$

EXAMPLE 7 *Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements* [NCERT]

(i) *do the words start with P?* (ii) *do all the vowels always occur together?*

(iii) *do all the vowels never occur together?* (iv) *do the words begin with I and end in P?*

SOLUTION In the word 'INDEPENDENCE' there are 12 letters of which 3 are N's, 4 are E's and 2 are D's. Therefore,

Total number of arrangements $= \dfrac{12!}{3!\,4!\,2!} = 1663200$

(i) After fixing the letter P at the extreme left position, there are 11 letters consisting of 3 N's, 4E's and 2D's. These 11 letters can be arranged in $\dfrac{11!}{3!\,4!\,2!} = 138600$

$\therefore$ Number of words beginning with P $= \dfrac{11!}{3!\,4!\,2!} = 138600$

(ii) There are 5 vowels in the given word of which 4 are E's and one I. These vowels can be put together in $\dfrac{5!}{4!\,1!}$ ways. Considering these 5 vowels as one letter there are 8 letters (taking 7 remaining letters) which can be arranged in $\dfrac{8!}{3!\,2!}$ ways (as there are 3 N's and 2D's). Since corresponding to each arrangement of 5 vowels there are $\dfrac{8!}{3!\,2!}$ ways of arranging remaining 7 letters and one letter formed by 5 vowels.

Hence, by fundamental principle of multiplication, the required number of arrangements is $\dfrac{8!}{3!\,2!} \times \dfrac{5!}{4!\,1!} = 16800$

(iii) The required number of arrangements

= The total number of arrangements – The number of arrangements in which all the vowels occur together

$= 1663200 - 16800 = 1646400$

(iv) Let us fix I at the extreme left end and P at the extreme right end. Now, we are left with 10 letters of which 3 are N's, 4 are E's and 2 are D's. These ten letters can be arranged in $\dfrac{10!}{4!\,3!\,2!}$ ways.

Hence, required number of arrangements $= \dfrac{10!}{4!\,3!\,2!} = 12600.$

EXAMPLE 8 *In how many ways can the letters of the word PERMUTATIONS be arranged if (i) the words start with P end with S (ii) vowels are all together.*

SOLUTION (i) There are 12 letters in the given word of which 2 are T's and the remaining are distinct. Remaining 10 letters between P and S can be arranged in $\dfrac{10!}{2!}$ ways.

$\therefore$ Total number of words starting with P and ending in S $= \dfrac{10!}{2!} = 1814400$

(ii) There are 5 vowels in the given word. These vowels can be put together in 5! ways. Considering these 5 vowels as one letter, we have 8 letters (7 remaining letters and one letter formed by 5 vowels) of which 2 are T's. These 8 letters can be arranged in $\dfrac{8!}{2!}$ ways.

Hence, by the fundamental principle of multiplication, required number of words is $5! \times \dfrac{8!}{2!} = 2419200.$

EXAMPLE 9 *How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3 ?*

SOLUTION Any number greater than a million will contain all the seven digits.

Now, we have to arrange these seven digits, out of which 2 occur twice, 3 occurs twice and the rest are distinct.

The number of such arrangements $= \dfrac{7!}{2! \times 3!} = 420.$

These arrangements also include those numbers which contain 0 at the million's place.

Keeping 0 fixed at the millionth place, we have 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct. These 6 digits can be arranged in $\dfrac{6!}{2! \times 3!} = 60$ ways.

Hence, the number of required numbers = 420 – 60 = 360.

EXAMPLE 10 *There are six periods in each working day of a school. In how many ways can one arrange 5 subjects such that each subject is allowed at least one period?*

SOLUTION Since each subject is allowed at least one period. So, we first select one subject for the left out period. This can be done in ${}^5C_1$ ways. Now, six subject can be arranged in $\dfrac{6!}{2!}$ ways.

Hence, the total number of arrangements $= {}^5C_1 \times \dfrac{6!}{2!} = 1800$

LEVEL-2

EXAMPLE 11 *How many arrangements can be made with the letters of the word 'MATHEMATICS'? In how many of them vowels are together ?*

SOLUTION There are 11 letters in the word 'MATHEMATICS' of which two are M's, two are A's, two are T's and all other are distinct. So,

$$\text{Required number of arrangements} = \frac{11!}{2! \times 2! \times 2!} = 4989600$$

There are 4 vowels viz. A, E, A, I. Considering these four vowels as one letter we have 8 letters (M, T, H, M, T, C, S and one letter obtained by combining all vowels), out of which M occurs twice, T occurs twice and the rest all different. These 8 letters can be arranged in $\dfrac{8!}{2! \times 2!}$ ways.

But, the four vowels (*A, E, A, I*) can be put together in $\dfrac{4!}{2!}$ ways.

Hence, the total number of arrangements in which vowels are always together $= \dfrac{8!}{2! \times 2!} \times \dfrac{4!}{2!}$

$$= 10080 \times 12 = 120960.$$

EXAMPLE 12 *If all the letters of the word 'AGAIN' be arranged as in a dictionary, what is the fiftieth word?* **[NCERT]**

SOLUTION In dictionary the words at each stage are arranged in alphabetical order. Starting with the letter A, and arranging the other four letters GAIN, we obtain 4! = 24 words.

Thus, there are 24 words which start with A. These are the first 24 words.

Then, starting with G, and arranging the other four letters A, A, I, N in different ways, we obtain $\dfrac{4!}{2!} = \dfrac{24}{2} = 12$ words. Thus, there are 12 words, which start with G.

Now, we start with I. The remaining 4 letters A, G, A, N can be arranged in $\dfrac{4!}{2!} = 12$ ways. So, there are 12 words, which start with I.

Thus, we have so far constructed 48 words. The 49th word is NAAGI and hence the 50th word is NAAIG.

EXAMPLE 13 *The letters of the word 'RANDOM' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'RANDOM'.*

SOLUTION In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we must therefore consider the words beginning with A, D, M, N, O, R in order. A will occur in the first place as often as there are ways of arranging the remaining 5 letters all at a time i.e. A will occur 5! times. Similarly, D, M, N, O will occur in the first place the same number of times.

$\therefore$ Number of words starting with A = 5! = 120
Number of words starting with D = 5! = 120
Number of words starting with M = 5! = 120
Number of words starting with N = 5! = 120
Number of words starting with O = 5! = 120

Number of words beginning with R is 5!, but one of these words is the word RANDOM. So, we first find the number of words beginning with RAD and RAM.
No. of words starting with RAD = 3! = 6
No. of words starting with RAM = 3! = 6
Now, the words beginning with 'RAN' must follow. There are 3! words beginning with RAN. One of these words is the word RANDOM itself.
The first word beginning with RAN is the word RANDMO and the next word is RANDOM.

$\therefore$ Rank of RANDOM $= 5 \times 120 + 2 \times 6 + 2 = 614$.

EXAMPLE 14 *If the different permutations of the word, 'EXAMINATION' are listed as in a dictionary, how many items are there in the list before the first word starting with E?* [NCERT]

SOLUTION In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we have to find the total number of words starting with A, because the very next word will start with E.
For finding the number of words starting with A, we have to find the number of arrangements of the remaining 10 letters, EXMINATION, of which there are 2 I's, 2 N's and the others each of its own kind.

The number of such arrangements $= \dfrac{10!}{2!\,2!} = 907200$.

Hence, the required number of items = 907200.

EXERCISE 16.5

LEVEL-1

1. Find the number of words formed by permuting all the letters of the following words:
   (i) INDEPENDENCE (ii) INTERMEDIATE (iii) ARRANGE
   (iv) INDIA (v) PAKISTAN (vi) RUSSIA
   (vii) SERIES (viii) EXERCISES (ix) CONSTANTINOPLE
2. In how many ways can the letters of the word 'ALGEBRA' be arranged without changing the relative order of the vowels and consonants?
3. How many words can be formed with the letters of the word 'UNIVERSITY', the vowels remaining together?
4. Find the total number of arrangements of the letters in the expression $a^3\, b^2\, c^4$ when written at full length.

5. How many words can be formed with the letters of the word 'PARALLEL' so that all L's do not come together?
6. How many words can be formed by arranging the letters of the word 'MUMBAI' so that all M's come together?
7. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?
8. How many different signals can be made from 4 red, 2 white and 3 green flags by arranging all of them vertically on a flagstaff?
9. How many number of four digits can be formed with the digits 1, 3, 3, 0?
10. In how many ways can the letters of the word 'ARRANGE' be arranged so that the two R's are never together?
11. How many different numbers, greater than 50000 can be formed with the digits 0, 1, 1, 5, 9.
12. How many words can be formed from the letters of the word 'SERIES' which start with S and end with S?
13. How many permutations of the letters of the word 'MADHUBANI' do not begin with M but end with I?
14. Find the number of numbers, greater than a million, that can be formed with the digits 2, 3, 0, 3, 4, 2, 3.
15. There are three copies each of 4 different books. In how many ways can they be arranged in a shelf?
16. How many different arrangements can be made by using all the letters in the word 'MATHEMATICS'. How many of them begin with C? How many of them begin with T?
17. A biologist studying the genetic code is interested to know the number of possible arrangements of 12 molecules in a chain. The chain contains 4 different molecules represented by the initials A (for Adenine), C (for Cytosine), G (for Guanine) and T (for Thymine) and 3 molecules of each kind. How many different such arrangements are possible?
18. In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable? [NCERT]
19. How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4? [NCERT]
20. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together? [NCERT]
21. Find the total number of permutations of the letters of the word 'INSTITUTE'. [NCERT]

LEVEL-2

22. The letters of the word 'SURITI' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'SURITI'.
23. If the letters of the word 'LATE' be permuted and the words so formed be arranged as in a dictionary, find the rank of the word LATE.
24. If the letters of the word 'MOTHER' are written in all possible orders and these words are written out as in a dictionary, find the rank of the word 'MOTHER'.
25. If the permutations of a, b, c, d, e taken all together be written down in alphabetical order as in dictionary and numbered, find the rank of the permutation debac.
26. Find the total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together.

27. In how many ways can the letters of the word "INTERMEDIATE" be arranged so that:

(i) the vowels always occupy even places?

(ii) the relative order of vowels and consonants do not alter?

28. The letters of the word 'ZENITH' are written in all possible orders. How many words are possible if all these words are written out as in a dictionary ? What is the rank of the word 'ZENITH'?

ANSWERS

1. (i) 1663200	(ii) 19958400	(iii) 1260	(iv) 60
(v) 20160	(vi) 360	(vii) 180	(viii) 30240
(ix) $\frac{14!}{24}$	**2.** 72	**3.** 60480	**4.** 1260
5. 3000	**6.** 120	**7.** 18	**8.** 1260
9. 9	**10.** 900	**11.** 24	**12.** 12
13. 17640	**14.** 360	**15.** $12!/(3!)^4$	**16.** $\frac{11!}{2!2!2!}, \frac{10!}{2!2!2!}, \frac{10!}{2!2!}$
17. 369600	**18.** 1260	**19.** 360	**20.** 151200
21. $\frac{9!}{2!3!}$	**22.** 236	**23.** 14	**24.** 309
25. 93	**26.** 35	**27.** (i) 21600	(ii) 21600
28. 616			

HINTS TO NCERT & SELECTED PROBLEMS

2. The consonants can be arranged among themselves in 4! ways and the vowels among themselves in $\frac{3!}{2!}$ ways. Hence, the required number of arrangements $= 4! \times \frac{3!}{2!} = 72$.

4. There are 3 a's, 2 b's and 4 c's. So, the total number of arrangements $= \frac{9!}{3!\,2!\,4!} = 1260$.

7. There are 4 odd digits 1, 1, 3, 3 and 4 odd places. So, odd digits can be arranged in odd places in $\frac{4!}{2!\,2!}$ ways. The remaining 3 even digits 2, 2, 4 can be arranged in 3 even places in $\frac{3!}{2!}$ ways. Hence, the requisite number of numbers $= \frac{4!}{2!\,2!} \times \frac{3!}{2!} = 18$.

8. We have to arrange 9 flags, out of which 4 are of one kind, 2 are of another kind and 3 are of third kind. So, total number of signals $= \frac{9!}{4!\,2!\,3!}$.

9. Required number of numbers $= \frac{4!}{2!} - \frac{3!}{2!}$.

11. Numbers greater than 50000 will have either 5 or 9 in the first place and will consist of 5 digits.

Number of numbers of with digit 5 at first place $= \frac{4!}{2!}$

Number of numbers with digit 9 at first place $= \frac{4!}{2!}$

Hence, the required number of numbers $= \frac{4!}{2!} + \frac{4!}{2!} = 24$.

18. Required number of ways $= \frac{(4+3+2)!}{4!\,3!\,2!} = \frac{9!}{4!\,3!\,2!} = 1260$

19. Number of numbers greater than 1000000 that can be formed by using the digits 1, 2, 0, 2, 4, 2, 4.

= Number of numbers formed by given digits – Number of numbers having 0 as left most digit

$= \frac{7!}{3!\,2!} - \frac{6!}{3!\,2!} = \frac{7!-6!}{3!\,2!} = \frac{6\times 6!}{3!\,2!} = 360$

20. Considering all S as one letter there are 10 letters containing 3A's, 2I's, 2N's, 1T, 1O which can be arranged in $\frac{10!}{3!\,2!\,2!} = 151200$ ways.

21. There are 9 letters in the word INSTITUTE containing 2I's, 3T's, 1N, 1S, 1U and 1E. These letters can be arranged in $\frac{9!}{2!\,3!} = 21040$ ways.

26. Six '+' signs can be arranged in a row in $\frac{6!}{6!} = 1$ way. Now, we are left with seven places in which four different things can be arranged in ${}^7P_4$ ways but as all the four '–' signs are identical, therefore, four '–' signs can be arranged in $\frac{{}^7P_4}{4!} = 35$ ways.

Hence, the required number of ways $= 1 \times 35 = 35$.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. In how many ways can 4 letters be posted in 5 letter boxes?
2. Write the number of 5 digit numbers that can be formed using digits 0, 1 and 2.
3. In how many ways 4 women draw water from 4 taps, if no tap remains unused?
4. Write the total number of possible outcomes in a throw of 3 dice in which at least one of the dice shows an even number.
5. Write the number of arrangements of the letters of the word BANANA in which two N's come together.
6. Write the number of ways in which 7 men and 7 women can sit on a round table such that no two women sit together.
7. Write the number of words that can be formed out of the letters of the word 'COMMITTEE'.
8. Write the number of all possible words that can be formed using the letters of the word 'MATHEMATICS'.
9. Write the number of ways in which 6 men and 5 women can dine at a round table if no two women sit together.
10. Write the number of ways in which 5 boys and 3 girls can be seated in a row so that each girl is between 2 boys.
11. Write the remainder obtained when $1!+2!+3!+\ldots+200!$ is divided by 14.
12. Write the number of numbers that can be formed using all for digits 1, 2, 3, 4.

ANSWERS

1. 5^4 2. 2×3^4 3. $4!$ 4. 189 5. 20 6. $7!\times 6!$ 7. $\dfrac{9!}{(2!)^3}$ 8. $\dfrac{11!}{2!\,2!\,2!}$

9. $6!\times 5!$ 10. 2880 11. 5 12. 24

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The number of permutations of n different things taking r at a time when 3 particular things are to be included is
   (a) $^{n-3}P_{r-3}$ (c) $^{n}P_{r-3}$ (d) $r!\,^{n-3}C_{r-3}$
2. The number of five- numbers having at least one of their digits repeated is
   (a) 90000 (c) 30240 (d) 69760
3. The number of wor ırmed out of the letters of the word "ARTICLE" so that vowels occupy ever
   (a) 574 (b) 36 (c) 754 (d) 144
4. How many numbers greater than 10 lacs be formed from 2, 3, 0, 3, 4, 2, 3 ?
   (a) 420 (b) 360 (c) 400 (d) 300
5. The number of different signals which can be given from 6 flags of different colours taking one or more at a time, is
   (a) 1958 (b) 1956 (c) 16 (d) 64
6. The number of words from the letters of the word 'BHARAT' in which B and H will never come together, is
   (a) 360 (b) 240 (c) 120 (d) none of these
7. The number of six letter words that can be formed using the letters of the word "ASSIST" in which S's alternate with other letters is
   (a) 12 (b) 24 (c) 18 (d) none of these
8. The number of arrangements of the word "DELHI" in which E precedes I is
   (a) 30 (b) 60 (c) 120 (d) 59
9. The number of ways in which the letters of the word 'CONSTANT' can be arranged without changing the relative positions of the vowels and consonants is
   (a) 360 (b) 256 (c) 444 (d) none of these
10. The number of ways to arrange the letters of the word CHEESE are
    (a) 120 (b) 240 (c) 720 (d) 6
11. Number of all four digit numbers having different digits formed of the digits 1, 2, 3, 4 and 5 and divisible by 4 is
    (a) 24 (b) 30 (c) 125 (d) 100
12. If the letters of the word KRISNA are arranged in all possible ways and these words are written out as in a dictionary, then the rank of the word KRISNA is
    (a) 324 (b) 341 (c) 359 (d) none of these
13. If in a group of n distinct objects, the number of arrangements of 4 objects is 12 times the number of arrangements of 2 objects, then the number of objects is
    (a) 10 (b) 8 (c) 6 (d) none of these

14. The number of ways in which 6 men can be arranged in a row so that three particular men are consecutive, is
(a) $4! \times 3!$ (b) $4!$ (c) $3! \times 3!$ (d) none of these

15. A 5-digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways in which this can be done is
(a) 216 (b) 600 (c) 240 (d) 3125

16. The product of r consecutive positive integers is divisible by
(a) $r!$ (b) $r!+1$ (c) $(r+1)!$ (d) none of these

17. If ${}^{k+5}P_{k+1} = \frac{11(k-1)}{2} \cdot {}^{k+3}P_k$, then the values of k are
(a) 7 and 11 (b) 6 and 7 (c) 2 and 11 (d) 2 and 6

18. The number of arrangements of the letters of the word BHARAT taking 3 at a time is
(a) 72 (b) 120 (c) 14 (d) none of these.

19. The number of words that can be made by re-arranging the letters of the word APURBA so that vowels and consonants are alternate is
(a) 18 (b) 35 (c) 36 (d) none of these

20. The number of different ways in which 8 persons can stand in a row so that between two particular persons A and B there are always two persons, is
(a) $60 \times 5!$ (b) $15 \times 4! \times 5!$ (c) $4! \times 5!$ (d) none of these

21. The number of ways in which the letters of the word ARTICLE can be arranged so that even places are always occupied by consonants is
(a) 576 (b) ${}^4C_3 \times 4!$ (c) $2 \times 4!$ (d) none of these

22. In a room there are 12 bulbs of the same wattage, each having a separate switch. The number of ways to light the room with different amounts of illumination is
(a) $12^2 - 1$ (b) 2^{12} (c) $2^{12} - 1$ (d) none of these

ANSWERS

1. (d)	2. (d)	3. (d)	4. (b)	5. (b)	6. (b)	7. (a)	8. (b)	9. (a)
10. (a)	11. (a)	12. (a)	13. (c)	14. (a)	15. (a)	16. (a)	17. (b)	18. (a)
19. (c)	20. (a)	21. (a)	22. (c)					

SUMMARY

1. The continued product of first n natural numbers is called the "n factorial" and is denoted by $\lfloor n$ or $n!$.
Thus, $n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n$
Factorials of proper fractions and negative integers are not defined.

2. $\frac{(2n)!}{n!} = 1 \cdot 3 \cdot 5 \ldots (2n-1)\, 2^n$

3. $n!+1$ is not divisible by any natural number between 2 and n.

4. Let p be a prime number and n be a natural number, if $E_p(n)$ denotes the exponent of p in n, then

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots + \left[\frac{n}{p^s}\right]$$

where s is the largest positive integer such that $p^s \le n < p^{s+1}$ and $[x]$ denotes the greatest integer less than or equal to x.

5. If n is a natural number and r is a positive integer such that $0 \le r \le n$, then ${}^nP_r = \dfrac{n!}{(n-r)!}$.

6. (i) (*Fundamental Principle of Multiplication*): If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways; then the two jobs in succession can be completed in $m \times n$ ways.

(ii) (*Fundamental Principle of Addition*) If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in (ways.

7. (i) Let r and n be positive integers such that $1 \le r \le n$. Then, the number of all permutations of n distinct items or objects taken r at a time is

$$n(n-1)(n-2)(n-3)\dots(n-(r-1))$$

(ii) The number of all permutations (arrangements) of n distinct objects taken all at a time is $n!$.

(iii) The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$, is $\dfrac{n!}{p!\,q!}$.

(iv) The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind; ...; pr are alike of rth kind such that $p_1 + p_2 + \dots + p_r = n$, is $\dfrac{n!}{p_1!\,p_2!\,p_3!\dots p_r!}$.

(v) The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is $\dfrac{n!}{p!\,q!}$.

(vi) Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \dots p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects to the above condition is $\dfrac{(p_1 + p_2 + \dots + p_r)!}{p_1!\,p_2!\,p_3!\dots p_r!}$

CHAPTER 17

COMBINATIONS

17.1 INTRODUCTION

In the previous chapter, we have studied arrangements of a certain number of objects by taking some of them or all at a time. Most of the times we are not interested in arranging the objects, but we are more concerned in selecting a number of objects from given number of objects. In other words, we do not want to specify the ordering of selected objects. For example, a company may want to select 3 persons out of 10 applicants, a student may want to choose three books from his library at a time etc.

Suppose we want to select three persons out of 4 persons A, B, C and D. We may choose A, B, C or A, B, D or A, C, D or B, C, D. Note that we have not listed A, B, C ; B, C, A; C, A, B; B, A, C; C, B, A and A, C, B separately here, because they represent the same selection A, B, C. But, they give rise to different arrangements. It is evident from the above discussion that in a selection the order in which objects are arranged is immaterial.

17.2 COMBINATIONS

COMBINATIONS *Each of the different selections made by taking some or all of a number of objects, irrespective of their arrangements is called a combination.*

ILLUSTRATION 1 *List the different combinations formed of three letters A, B, C taken two at a time.*

SOLUTION The different combinations formed of three letters A, B, C are: AB, AC, BC.

ILLUSTRATION 2 *Write all combinations of four letters A, B, C, D taken two at a time.*

SOLUTION Various combinations of two letters out of four letters A, B, C, D are:

$$AB, \quad AC, \quad AD, \quad BC, \quad BD, \quad CD.$$

DIFFERENCE BETWEEN A PERMUTATION AND COMBINATION

(i) In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.

(ii) In a combination, the ordering of the selected objects is immaterial whereas in a permutation, the ordering is essential. For example, A, B and B, A are same as combinations but different as permutations.

(iii) Practically to find the permutations of n different items, taken r at a time, we first select r items from n items and then arrange them. So, usually the number of permutations exceeds the number of combinations.

(iv) Each combination corresponds to many permutations. For example, the six permutations ABC, ACB, BCA, BAC, CBA and CAB correspond to the same combination ABC.

REMARK *Generally we use the word 'arrangements' for permutations and the word 'selections' for combinations.*

NOTATION *The number of all combinations of n objects, taken r at a time is generally denoted by $C(n, r)$ or, ${}^nC_r$ or, $\binom{n}{r}$.*

Thus, ${}^nC_r$ or $C(n, r)$ = Number of ways of selecting r objects from n objects.

Clearly, ${}^nC_r$ is defined only when n and r are non-negative integers such that $0 \le r \le n$.

THEOREM *The number of all combinations of n distinct objects, taken r at a time is given by*

$$^nC_r = \frac{n!}{(n-r)!\,r!}$$

PROOF Let the number of combinations of n distinct objects taken r at a time be x. Consider one of these x ways. There are r objects in this selection which can be arranged in $r!$ ways. Thus, each of the x combinations gives rise to $r!$ permutations. So, x combinations will give rise to $x \times (r!)$ permutations. Consequently, the number of permutations of n things, taken r at a time is $x \times (r!)$. But, this number is also equal to ${}^nP_r$.

$$\therefore \quad x\,(r!) = {}^nP_r$$

$$\Rightarrow \quad x = \frac{^nP_r}{r!} = \frac{n!}{(n-r)!\,r!} \qquad \left[\because\ {}^nP_r = \frac{n!}{(n-r)!}\right]$$

$$\Rightarrow \quad {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

Q.E.D.

REMARK 1 *We have,*

$$^nC_r = \frac{n!}{(n-r)!\,r!}$$

$$\Rightarrow \quad {}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 3.2.1}{\{(n-r)(n-r-1)\ldots 3.2.1\}\{1.2.3\ldots.r\}}$$

$$\Rightarrow \quad {}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots.r}$$

Sometimes this form of ${}^nC_r$ is also very convenient to use.

REMARK 2 *We have,* $^nC_r = \dfrac{n!}{(n-r)!\,r!}$

Putting $r = n$, we obtain

$$^nC_n = \frac{n!}{(n-n)!\,n!} = \frac{n!}{n!\,0!} = 1 \qquad [\because\ 0! = 1]$$

Putting $r = 0$, we obtain

$$^nC_0 = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!} = 1$$

Thus, $^nC_n = {}^nC_0 = 1.$

REMARK 3 *We have,*

$$^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{1}{r!}\left(\frac{n!}{(n-r)!}\right) = \frac{^nP_r}{r!}.$$

17.3 PROPERTIES OF ${}^nC_r$ OR, $C(n, r)$

In this section, we shall discuss some important properties of ${}^nC_r$.

PROPERTY 1 *For $0 \le r \le n$, we have* $^nC_r = {}^nC_{n-r}$.

PROOF We have,

$$^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!} = {}^nC_r$$

REMARK 1 *The use of this property simplifies the calculation of ${}^nC_r$ when r is large.*

For example, if we want to calculate ${}^{20}C_{19}$, by using this property, we get

$$^{20}C_{19} = {}^{20}C_{20-19} = {}^{20}C_1 = 20.$$

REMARK 2 *The above property can be restated as follows:*
If x and y are non-negative integers such that $x + y = n$, then ${}^{n}C_x = {}^{n}C_y$
This can also be stated as : ${}^{n}C_x = {}^{n}C_y \Rightarrow x = y$, or $x + y = n$

ILLUSTRATION 1 *If ${}^{n}C_7 = {}^{n}C_4$, find the value of n.*

SOLUTION We know that : ${}^{n}C_x = {}^{n}C_y \Leftrightarrow x + y = n$ or $x = y$.

$\therefore \quad {}^{n}C_7 = {}^{n}C_4 \Rightarrow n = 7 + 4 = 11$

PROPERTY 2 *Let n and r be non-negative integers such that $r \le n$. Then,* ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$.

PROOF We have,

$$
\begin{aligned}
{}^{n}C_r + {}^{n}C_{r-1} &= \frac{n!}{(n-r)!\,r!} + \frac{n!}{[n-(r-1)]!\,(r-1)!} \\
&= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!} \\
&= \frac{n!}{(n-r)!\,r\,(r-1)!} + \frac{n!}{(n-r+1)\,(n-r)!\,(r-1)!} \\
&= \frac{n!}{(n-r)!\,(r-1)!}\left\{\frac{1}{r} + \frac{1}{n-r+1}\right\} \\
&= \frac{n!}{(n-r)!\,(r-1)!}\left\{\frac{n-r+1+r}{r\,(n-r+1)}\right\} \\
&= \frac{n!\,(n+1)}{(n-r)!\,(r-1)!\,r\,(n-r+1)} \\
&= \frac{(n+1)\,n!}{(n-r+1)\,(n-r)!\,r\,(r-1)!} = \frac{(n+1)!}{(n-r+1)!\,r!} = \frac{(n+1)!}{((n+1)-r)!\,r!} = {}^{n+1}C_r.
\end{aligned}
$$

REMARK 3 *This property is known as Pascal's rule and it can also be proved by giving combinatorial arguments.*

ILLUSTRATION 2 *Find the value of the expression* ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$.

SOLUTION We have,

$$
\begin{aligned}
& {}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3 \\
&= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 \\
&= {}^{47}C_3 + {}^{47}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 \\
&= ({}^{47}C_3 + {}^{47}C_4) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 \\
&= {}^{48}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 && [\because {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r] \\
&= ({}^{48}C_3 + {}^{48}C_4) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 && [\because {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r] \\
&= {}^{49}C_4 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 \\
&= ({}^{49}C_3 + {}^{49}C_4) + {}^{50}C_3 + {}^{51}C_3 && [\because {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r] \\
&= {}^{50}C_4 + {}^{50}C_3 + {}^{51}C_4 \\
&= ({}^{50}C_3 + {}^{50}C_4) + {}^{51}C_3 && [\because {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r] \\
&= {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4 && [\because {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r]
\end{aligned}
$$

PROPERTY 3 *Let n and r be non-negative integers such that* $1 \le r \le n$. *Then,* ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$.

PROOF We have,

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

$$\Rightarrow \quad {}^nC_r = \frac{n\,(n-1)!}{\{(n-1)-(r-1)\}!\,r\,(r-1)!} = \frac{n}{r}\,\frac{(n-1)!}{\{(n-1)-(r-1)\}!\,(r-1)!} = \frac{n}{r}\,{}^{n-1}C_{r-1}$$

REMARK 4 *This property is very useful to find the value of* ${}^nC_r$.

For example, ${}^{10}C_3 = \frac{10}{3} \times {}^9C_2 = \frac{10}{3} \times \frac{9}{2} \times {}^8C_1 = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1}\,{}^7C_0 = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1} \times 1 = 120$

REMARK 5 *By using the above property, we obtain that*

$${}^nC_r = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot \frac{n-2}{r-2} \cdots \frac{n\,(r-1)}{2} \cdot \frac{n-(r-1)}{1}$$

For example, ${}^9C_4 = \frac{9}{4} \times \frac{8}{3} \times \frac{7}{2} \times \frac{6}{1} = 126.$

PROPERTY 4 If $1 \le r \le n$, then $n\,{}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$.

PROOF We have,

$$n \cdot {}^{n-1}C_{r-1} = n \cdot \frac{(n-1)!}{\{(n-1)-(r-1)\}!\,(r-1)!}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = \frac{n!}{(n-r)!\,(r-1)!}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = \frac{(n-r+1) \cdot n!}{(n-r+1)\,(n-r)!\,(r-1)!}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = (n-r+1)\left\{\frac{n!}{(n-r+1)!\,(r-1)!}\right\}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = (n-r+1)\left\{\frac{n!}{\{n-(r-1)\}!\,(r-1)!}\right\}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$$

PROPERTY 5 ${}^nC_x = {}^nC_y \Rightarrow x = y$ *or,* $x + y = n$.

PROOF We have,

$${}^nC_x = {}^nC_y$$

$$\Rightarrow \quad {}^nC_x = {}^nC_y = {}^nC_{n-y} \qquad [\because\ {}^nC_y = {}^nC_{n-y}]$$

$$\Rightarrow \quad x = y \quad \text{or} \quad x = n - y$$

$$\Rightarrow \quad x = y \quad \text{or} \quad x + y = n.$$

REMARK 6 *If* ${}^nC_x = {}^nC_y$ *and* $x \ne y$, *then* $x + y = n$.

ILLUSTRATION 3 *If* ${}^nC_{15} = {}^nC_8$, *find the value of* ${}^nC_{21}$.

SOLUTION We have,

$${}^nC_{15} = {}^nC_8 \Rightarrow n = (15+8) = 23 \qquad [{}^nC_x = {}^nC_y \Rightarrow x + y = n]$$

$$\therefore \quad {}^nC_{21} = {}^{23}C_{21} = {}^{23}C_{23-21} \qquad [\because\ {}^nC_r = {}^nC_{n-r}]$$

$$= {}^{23}C_2 = \frac{23}{2} \times \frac{22}{1} \times {}^{21}C_0 \qquad \left[\because {}^nC_r = \frac{n}{r} \times \frac{n-1}{r-1} \times {}^{n-2}C_{r-2}\right]$$

$$= \frac{23}{2} \times \frac{22}{1} \times 1 = 23 \times 11 = 253 \qquad [\because {}^nC_0 = 1]$$

ILLUSTRATION 4 *If* ${}^{10}C_x = {}^{10}C_{x+4}$, *find the value of* x.

SOLUTION We have, ${}^{10}C_x = {}^{10}C_{x+4} \Rightarrow x + x + 4 = 10 \Rightarrow 2x = 6 \Rightarrow x = 3$.

PROPERTY 6 *If n is an even natural number, then the greatest of the values*
${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ *is* ${}^nC_{n/2}$

If n is an odd natural number, then the greatest of the values
${}^nC_0, {}^nC_1, {}^nC_2, ... {}^nC_n$ is ${}^nC_{n-1/2} = {}^nC_{n+1/2}$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate the following: (i)* ${}^{10}C_8$ *(ii)* ${}^{100}C_{98}$ *(iii)* ${}^{52}C_{52}$

SOLUTION (i) ${}^{10}C_8 = {}^{10}C_{10-8}$ $\quad [\because {}^nC_r = {}^nC_{n-r}]$

$$= {}^{10}C_2 = \frac{10}{2} \times \frac{9}{1} \times {}^8C_0 \qquad \left[\because {}^nC_r = \frac{n}{r} . {}^{n-1}C_{r-1}\right]$$

$$= \frac{10}{2} \times \frac{9}{1} \times 1 \qquad [\because {}^nC_0 = 1]$$

$$= 45 \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

(ii) ${}^{100}C_{98} = {}^{100}C_{100-98}$ $\quad \left[\because {}^nC_r = {}^nC_{n-r}\right]$

$$= {}^{100}C_2 = \frac{100}{2} \times \frac{99}{1} \times {}^{98}C_0 \qquad [\because {}^nC_r = \frac{n}{r} . {}^{n-1}C_{r-1}]$$

$$= \frac{100}{2} \times \frac{99}{1} \times 1 \qquad [\because {}^nC_0 = 1]$$

$$= 4950$$

(iii) ${}^{52}C_{52} = 1$ $\quad [\because {}^nC_n = 1]$

EXAMPLE 2 *If* ${}^nC_8 = {}^nC_6$, *find* ${}^nC_2$

SOLUTION If ${}^nC_x = {}^nC_y$ and $x \neq y$, then $x + y = n$.

$\therefore \quad {}^nC_8 = {}^nC_6 \Rightarrow n = (8 + 6) = 14$

Now, $\quad {}^nC_2 = {}^{14}C_2 = \frac{14}{2} \times \frac{13}{1} \times {}^{12}C_0$ $\quad [\because {}^nC_r = \frac{n}{r} . {}^{n-1}C_{r-1}]$

$$= \frac{14}{2} \times \frac{13}{1} \times 1 = 91 \qquad [\because {}^nC_0 = 1]$$

EXAMPLE 3 *If* ${}^nP_r = 720$ and ${}^nC_r = 120$, find r.

SOLUTION We know that

$${}^nC_r = \frac{{}^nP_r}{r!}$$

$$\therefore \quad 120 = \frac{720}{r!} \qquad \left[\because {}^nC_r = 120 \text{ and } {}^nP_r = 720\right]$$

$$\Rightarrow \quad r! = 6 \Rightarrow r! = 3! \Rightarrow r = 3.$$

EXAMPLE 4 *If the ratio* ${}^{2n}C_3 : {}^{n}C_3$ *is equal to 11 : 1, find n.*

SOLUTION We have,

$$ {}^{2n}C_3 : {}^{n}C_3 = 11:1 $$

$$ \Rightarrow \quad \frac{{}^{2n}C_3}{{}^{n}C_3} = \frac{11}{1} $$

$$ \Rightarrow \quad \frac{\dfrac{(2n)!}{(2n-3)!\,3!}}{\dfrac{n!}{(n-3)!\,(3!)}} = \frac{11}{1} $$

$$ \Rightarrow \quad \frac{(2n)!}{(2n-3)!} \times \frac{(n-3)!}{n!} = \frac{11}{1} $$

$$ \Rightarrow \quad \frac{(2n)(2n-1)(2n-2)(2n-3)!}{(2n-3)!} \times \frac{(n-3)!}{n(n-1)(n-2)(n-3)!} = \frac{11}{1} $$

$$ \Rightarrow \quad \frac{(2n)(2n-1)(2n-2)}{n(n-1)(n-2)} = \frac{11}{1} $$

$$ \Rightarrow \quad \frac{4(2n-1)}{n-2} = \frac{11}{1} \Rightarrow 8n-4 = 11n-22 \Rightarrow 3n = 18 \Rightarrow n = 6 $$

EXAMPLE 5 *Prove that:* ${}^{2n}C_n = \dfrac{2^n \{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)\}}{n!}$.

SOLUTION We have,

$$ {}^{2n}C_n = \frac{2n!}{(2n-n)!\,n!} = \frac{(2n)!}{n!\,n!} $$

$$ = \frac{(2n)(2n-1)(2n-2)\ldots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{n!\,n!} $$

$$ = \frac{\{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)\}\{2 \cdot 4 \cdot 6 \cdot \ldots 2n\}}{n!\,n!} $$

$$ = \frac{\{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)\} \times 2^n \{1 \cdot 2 \cdot 3 \cdot \ldots n\}}{n!\,n!} $$

$$ = \frac{\{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)\} \times 2^n \times n!}{n!\,n!} = 2^n \frac{\{1 \cdot 3 \cdot 5 \ldots (2n-1)\}}{n!} $$

EXAMPLE 6 *If* ${}^{n+2}C_8 : {}^{n-2}P_4 = 57:16$, *find n.*

SOLUTION We have,

$$ {}^{n+2}C_8 : {}^{n-2}P_4 = 57:16 $$

$$ \Rightarrow \quad \frac{{}^{n+2}C_8}{{}^{n-2}P_4} = \frac{57}{16} $$

$$ \Rightarrow \quad \frac{(n+2)!}{8!\,(n-6)!} \times \frac{(n-6)!}{(n-2)!} = \frac{57}{16} $$

$$ \Rightarrow \quad \frac{(n+2)(n+1)\,n\,(n-1)(n-2)!}{8!} \times \frac{1}{(n-2)!} = \frac{57}{16} $$

$$ \Rightarrow \quad (n+2)(n+1)\,n\,(n-1) = \frac{57}{16} \times 8! = \frac{19 \times 3}{16} \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 $$

$$ \Rightarrow \quad (n+2)(n+1)(n-1)\,n = 143640 $$

$$ \Rightarrow \quad (n-1)\,n\,(n+1)(n+2) = 19 \times 3 \times 7 \times 6 \times 5 \times 4 \times 3 $$

$$\Rightarrow \quad (n-1)\,n\,(n+1)\,(n+2) = 19 \times (3 \times 7) \times (6 \times 3) \times (4 \times 5)$$
$$\Rightarrow \quad (n-1)\,n\,(n+1)\,(n+2) = 18 \times 19 \times 20 \times 21 \Rightarrow n-1 = 18 \Rightarrow n = 19$$

EXAMPLE 7 *If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then find ${}^rC_2$.*

SOLUTION We know that

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}$$

It is given that ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$

$$\therefore \quad \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{84}{126}$$

$$\Rightarrow \quad \frac{r+1}{n-r} = \frac{2}{3} \Rightarrow 2n - 5r = 3 \qquad \text{...(i)}$$

Replacing r by $(r-1)$ in $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}$, we get

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{r}{n-(r-1)}$$

$$\Rightarrow \quad \frac{36}{84} = \frac{r}{n-r+1} \qquad \left[\because\ {}^nC_{r-1} = 36 \text{ and } {}^nC_r = 84\right]$$

$$\Rightarrow \quad \frac{r}{n-r+1} = \frac{3}{7} \Rightarrow 3n - 10r = -3 \qquad \text{...(ii)}$$

Solving (i) and (ii), we get $r = 3$.

$$\therefore \quad {}^rC_2 = {}^3C_2 = \frac{3!}{(3-2)!\,2!} = 3.$$

NOTE *Students are advised to learn that* $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ *as it is a very useful result.*

EXAMPLE 8 *If ${}^nP_r = {}^nP_{r+1}$ and ${}^nC_r = {}^nC_{r-1}$, find the values of n and r.*

SOLUTION We have,

$${}^nP_r = {}^nP_{r+1}$$

$$\Rightarrow \quad \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!}$$

$$\Rightarrow \quad \frac{1}{(n-r)\,(n-r-1)!} = \frac{1}{(n-r-1)!}$$

$$\Rightarrow \quad n - r = 1 \qquad \text{...(i)}$$

and, $\quad {}^nC_r = {}^nC_{r-1}$

$$\Rightarrow \quad \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$\Rightarrow \quad \frac{n!}{(n-r)!\,r\,(r-1)!} = \frac{n!}{(n-r+1)\,(n-r)!\,(r-1)!}$$

$$\Rightarrow \quad \frac{1}{r} = \frac{1}{n-r+1} \Rightarrow n-r+1 = r \Rightarrow n - 2r = -1 \qquad \text{...(ii)}$$

Solving (i) and (ii), we obtain $n = 3$ and $r = 2$.

LEVEL-2

EXAMPLE 9 *Prove that the product of r consecutive positive integers is divisible by r!.*

SOLUTION Let the r consecutive positive integers be $(n+1), (n+2), (n+3), \ldots, (n+r)$. Then,

$$\begin{aligned}\text{Product} &= (n+1)(n+2)(n+3)\ldots(n+r)\\ &= \frac{n!(n+1)(n+2)(n+3)\ldots(n+r)}{n!}\\ &= \frac{1\cdot 2\cdot 3\cdot \ldots n\cdot(n+1)(n+2)\ldots(n+r)}{n!}\\ &= \frac{(n+r)!}{n!} = \frac{(n+r)!}{r!\{(n+r)-r\}!}(r!)\\ &= \left({}^{n+r}C_r\right)r!\text{, which is divisible by } r!\end{aligned}$$

[$\because$ ${}^{n+r}C_r$ is an integer]

EXERCISE 17.1

LEVEL-1

1. Evaluate the following:
   (i) ${}^{14}C_3$ (ii) ${}^{12}C_{10}$ (iii) ${}^{35}C_{35}$ (iv) ${}^{n+1}C_n$ (v) $\sum_{r=1}^{5} {}^5C_r$
2. If ${}^nC_{12} = {}^nC_5$, find the value of n.
3. If ${}^nC_4 = {}^nC_6$, find ${}^{12}C_n$.
4. If ${}^nC_{10} = {}^nC_{12}$, find ${}^{23}C_n$.
5. If ${}^{24}C_x = {}^{24}C_{2x+3}$, find x.
6. If ${}^{18}C_x = {}^{18}C_{x+2}$, find x.
7. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, find r.
8. If ${}^8C_r - {}^7C_3 = {}^7C_2$, find r.
9. If ${}^{15}C_r : {}^{15}C_{r-1} = 11:5$, find r.
10. If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57:16$, find n.
11. If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225:11$, find r.
12. If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then find n.
13. If ${}^{2n}C_3 : {}^nC_2 = 44:3$, find n.
14. If ${}^{16}C_r = {}^{16}C_{r+2}$, find ${}^rC_4$.
15. If $\alpha = {}^mC_2$, then find the value of ${}^{\alpha}C_2$.

LEVEL-2

16. Prove that the product of $2n$ consecutive negative integers is divisible by $(2n)!$
17. For all positive integers n, show that ${}^{2n}C_n + {}^{2n}C_{n-1} = \frac{1}{2}\left({}^{2n+2}C_{n+1}\right)$.
18. Prove that: ${}^{4n}C_{2n} : {}^{2n}C_n = [1\cdot 3\cdot 5\ldots(4n-1)] : [1\cdot 3\cdot 5\ldots(2n-1)]^2$.
19. Evaluate ${}^{20}C_5 + \sum_{r=2}^{5} {}^{25-r}C_4$.
20. Let r and n be positive integers such that $1 \le r \le n$. Then prove the following:
   (i) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ (ii) $n\ {}^{n-1}C_{r-1} = (n-r+1)\ {}^nC_{r-1}$
   (iii) $\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}$ (iv) ${}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$.

ANSWERS

1. (i) 364 (ii) 66 (iii) 1 (iv) $(n+1)$ (v) 31 2. 17 3. 66

4. 23 5. 7 6. 8 7. 3 8. 3, 5 9. 5 10. 19 11. 7
12. 14, 7 13. 6 14. 35 15. $\dfrac{(m+1)(m)(m-1)(m-2)}{8}$ 19. 42504

HINTS TO NCERT & SELECTED PROBLEM

16. Let $(-r), (-r-1), (-r-2), \ldots, (-r-2n+1)$ be $2n$ consecutive negative integers. Then, their product P is given by

$$P = (-1)^{2n}\, r\,(r+1)(r+2)\ldots(r+2n-1)$$

$$\Rightarrow\ P = \frac{(r-1)!(r)(r+1)\ldots(r+2n-1)}{(r-1)!}$$

$$\Rightarrow\ P = \frac{(r+2n-1)!}{(r-1)!} = \frac{(r+2n-1)!}{(r-1)!(2n)!}(2n)! = \left({}^{r+2n-1}C_{2n}\right)(2n)!$$

Clearly, P is divisible by $(2n)!$

17.4 PRACTICAL PROBLEMS ON COMBINATIONS

In this section, we intend to discuss some problems in real life where the formula for ${}^nC_r$ and its meaning can be applied.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *From a class of 32 students, 4 are to be chosen for a competition. In how many ways can this be done ?*

SOLUTION Out of 32 students, 4 students can be selected in ${}^{32}C_4$ ways.

$\therefore$ Required number of ways ${}^{32}C_4 = \dfrac{32!}{28!\,4!}$.

EXAMPLE 2 *Three gentlemen and three ladies are candidates for two vacancies. A voter has to vote for two candidates. In how many ways can one cast his vote ?*

SOLUTION Clearly, there are 6 candidates and a voter has to vote for any two of them. So, the required number of ways is the number of ways of selecting 2 out of 6 i.e. ${}^6C_2$.

Hence, the required number of ways $= {}^6C_2 = \dfrac{6!}{2!\,4!} = 15.$

EXAMPLE 3 *If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party?*

SOLUTION It is to note here that, when two persons shake hands, it is counted as one handshake, not two. So, this is a problem on combinations.

The total number of handshakes is same as the number of ways of selecting 2 persons among 12 persons i.e. ${}^{12}C_2 = \dfrac{12!}{10! \times 2!} = 66.$

EXAMPLE 4 *A question paper has two parts, Part A and Part B, each containing 10 questions. If a student has to choose 8 from Part A and 5 from Part B, in how many ways can he choose the questions ?*

SOLUTION There are 10 questions in Part A out of which 8 questions can be chosen in ${}^{10}C_8$ ways. Similarly, 5 questions can be chosen from part B containing 10 questions in ${}^{10}C_5$ ways.

Hence, the total number of ways of selecting 8 questions from part A and 5 from part B

$$= {}^{10}C_8 \times {}^{10}C_5 = \frac{10!}{8!\,2!} \times \frac{10!}{5! \times 5!} = 11340.$$

EXAMPLE 5 *In how many ways a committee of 5 members can be selected from 6 men and 5 women, consisting of 3 men and 2 women ?*

SOLUTION Three men out of 6 men can be selected in ${}^6C_3$ ways. Two women out of 5 women can be selected in ${}^5C_2$ ways. Therefore, by the fundamental principle of counting, 3 men out of 6 men and 2 women out of 5 women can be selected in

$$ {}^6C_3 \times {}^5C_2 = \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1} \right) = 200 \text{ ways.} $$

EXAMPLE 6 *In how many ways can a cricket eleven be chosen out of a batch of 15 players if*
(i) there is no restriction on the selection? (ii) a particular player is always chosen?
(iii) a particular player is never chosen?

SOLUTION (i) The total number of ways of selecting 11 players out of 15 is

$$ {}^{15}C_{11} = {}^{15}C_{15-11} = {}^{15}C_4 = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 1365 $$

(ii) If a particular player is always chosen. This means that 10 players are selected out of the remaining 14 players.

$\therefore$ Required number of ways $= {}^{14}C_{10} = {}^{14}C_{14-10} = {}^{14}C_4 = 1001$

(iii) If a particular player is never chosen. This means that 11 players are selected out of the remaining 14 players.

$\therefore$ Required number of ways $= {}^{14}C_{11} = {}^{14}C_{14-11} = {}^{14}C_3 = 364$

EXAMPLE 7 *A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least five women have to be included in a committee ? In how many of these committees (i) the women are in majority (ii) the men are in majority ?*

SOLUTION There are 9 women and 8 men. A committee of 12, consisting of at least 5 women, can be formed by choosing :

(i) 5 women and 7 men (ii) 6 women and 6 men (iii) 7 women and 5 men
(iv) 8 women and 4 men (v) 9 women and 3 men

$\therefore$ Total number of ways of forming the committee

$= {}^9C_5 \times {}^8C_7 + {}^9C_6 \times {}^8C_6 + {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3$

$= 126 \times 8 + 84 \times 28 + 36 \times 56 + 9 \times 70 + 1 \times 56 = 6062$

Clearly, women are in majority in (iii), (iv) and (v) cases as discussed above.

So, total number of committees in which women are in majority

$= {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3 = 36 \times 56 + 9 \times 70 + 1 \times 56 = 2702$

Clearly, men are in majority in only (i) case as discussed above.

So, total number of committees in which men are in majority $= {}^9C_5 \times {}^8C_7 = 126 \times 8 = 1008$.

EXAMPLE 8 *A committee of three persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?* **[NCERT]**

SOLUTION There are 5 persons (2 men and 3 women). In order consitute a committee of 3 persons we need to select three persons out of given 5 persons. This can be done in ${}^5C_3$ ways.

So, the committee can be formed in ${}^5C_3 = \frac{5!}{3!\,2!} = 10$ ways.

Now, 1 man can be selected from 2 men in ${}^2C_1$ ways and 2 women can be selected from 3 women in ${}^3C_2$ ways.

Therefore, required number of committees is ${}^2C_1 \times {}^3C_2 = 2 \times 3 = 6$

EXAMPLE 9 *What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these*

(i) *four cards are of the same suit?*
(ii) *four cards belong to four different suits?*
(iii) *four cards are face cards?*
(iv) *two are red cards and two are black cards?*
(v) *cards are of the same colour?* [NCERT]

SOLUTION Four cards can be chosen from 52 playing cards in ${}^{52}C_4$ ways.

Now, ${}^{52}C_4 = \frac{52!}{48!\,4!} = \frac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4} = 270725$

Hence, required number of ways = 270725

(i) There are four suits (diamond, spade, club and heart) of 13 cards each. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamond cards, ${}^{13}C_4$ ways of choosing 4 club cards, ${}^{13}C_4$ ways of choosing 4 spade cards and ${}^{13}C_4$ ways of choosing heart cards.

$\therefore$ Required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4 \times {}^{13}C_4 = 4 \times \frac{13!}{9!\,4!} = 2860$

(ii) There are 13 cards in each suit. Four cards drawn belong to four different suits means one card is drawn from each suit. Out of 13 diamond cards one card can be drawn in ${}^{13}C_1$ ways. Similarly, there are ${}^{13}C_1$ ways of choosing one club card, ${}^{13}C_1$ ways of choosing one spade card and ${}^{13}C_1$ ways of choosing one heart card.

$\therefore$ Number of ways of selecting one card from each suit $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$

(iii) There are 12 face cards out of which 4 cards can be chosen in ${}^{12}C_4$ ways.

$\therefore$ Required number of ways $= {}^{12}C_4 = \frac{12!}{4!\,8!} = 495$

(iv) There are 26 red cards and 26 black cards. Therefore, 2 red cards can be chosen in ${}^{26}C_2$ ways and 2 black cards can be chosen in ${}^{26}C_2$ ways. Hence, 2 red and 2 black cards can be chosen in

${}^{26}C_2 \times {}^{26}C_2 = \left(\frac{26!}{24!\,2!}\right)^2 = (325)^2 = 105625$ ways.

(v) Out of 26 red cards, 4 red cards can be chosen in ${}^{26}C_4$ ways. Similarly, 4 black cards can be chosen in $26C_4$ ways.

Hence, 4 red or 4 black cards can be chosen in ${}^{26}C_4 + {}^{26}C_4 = 2 \times {}^{26}C_4 = 2 \times \frac{26!}{4!\,22!} = 29900$ ways.

EXAMPLE 10 *Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included ?*

SOLUTION The committee can be formed in the following ways:

(i) By selecting 2 men and 1 woman
(ii) By selecting 1 man and 2 women

Now, 2 men out of 5 men and 1 woman out of 2 woman can be chosen in ${}^5C_2 \times {}^2C_1$ ways.

And, 1 man out of 5 men and 2 women out of 2 women can be chosen in ${}^5C_1 \times {}^2C_2$ ways.

$\therefore$ Total number of ways of forming the committee $= {}^5C_2 \times {}^2C_1 + {}^5C_1 \times {}^2C_2 = 20 + 5 = 25.$

EXAMPLE 11 *In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers ? Assume that the team of 11 players requires 5 batsmen, 3 all-rounder, 2 bowlers and 1 wicket keeper.*

SOLUTION The selection of team is divided into four phases:

(i) Selection of 5 batsmen out of 10. This can be done in ${}^{10}C_5$ ways.

(ii) Selection of 3 all-rounders out of 5. This can be done in ${}^5C_3$ ways.

(iii) Selection of 2 bowlers out of 8. This can be done in ${}^8C_2$ ways.

(iv) Selection of one wicket keeper out of 2. This can be done in ${}^2C_1$ ways.

e selection of team is completed by completing all the four phases.

The team can be selected in ${}^{10}C_5 \times {}^5C_3 \times {}^8C_2 \times {}^2C_1 = 141120$ ways.

MPLE 12 *A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be , when*

(i) at least two ladies are included? *(ii) at most two ladies are included ?*

LUTION (i) A committee of 5 persons, consisting of at least two ladies, can be formed in the ollowing ways:

I Selecting 2 ladies out of 4 and 3 gents out of 6. This can be done in ${}^4C_2 \times {}^6C_3$ ways.

II Selecting 3 ladies out of 4 and 2 gents out of 6. This can be done in ${}^4C_3 \times {}^6C_2$ ways.

III Selecting 4 ladies out of 4 and 1 gent out of 6. This can be done in ${}^4C_4 \times {}^6C_1$ ways.

Since the committee is formed in each case. Therefore, by the fundamental principle of addition,

The total number of ways of forming the committee $= {}^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1$

$= 120 + 60 + 6 = 186$

(ii) A committee of 5 persons, consisting of at most two ladies, can be constituted in the following ways :

I Selecting 5 gents only out of 6. This can be done in ${}^6C_5$ ways.

II Selecting 4 gents only out of 6 and one lady out of 4. This can be done in ${}^6C_4 \times {}^4C_1$ ways.

III Selecting 3 gents only out of 6 and two ladies out of 4. This can be done is ${}^6C_3 \times {}^4C_2$ ways.

Since the committee is formed in each case. So, the total number of ways of forming the committee $= {}^6C_5 + {}^6C_4 \times {}^4C_1 + {}^6C_3 \times {}^4C_2 = 6 + 60 + 120 = 186$.

EXAMPLE 13 *A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour ?*

SOLUTION The selection of 6 balls, consisting of at least two balls of each colour from 5 red and 6 white balls, can be made in the following ways :

(i) By selecting 2 red balls out of 5 and 4 white balls out of 6. This can be done in ${}^5C_2 \times {}^6C_4$ ways.

(ii) By selecting 3 red balls out of 5 and 3 white balls out of 6. This can be done in ${}^5C_3 \times {}^6C_3$ ways.

(iii) By selecting 4 red balls out of 5 and 2 white balls out of 6. This can be done in ${}^5C_4 \times {}^6C_2$ ways.

Since the selection of 6 balls can be completed in any one of the above ways.

Hence, by the fundamental principle of addition, the total number of ways to select the balls

$= {}^5C_2 \times {}^6C_4 + {}^5C_3 \times {}^6C_3 + {}^5C_4 \times {}^6C_2 = 10 \times 15 + 10 \times 20 + 5 \times 15 = 425.$

EXAMPLE 14 *For the post of 5 teachers, there are 23 applicants, 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made ?*

SOLUTION Clearly, there are 7 SC candidates and 16 other candidates. We have to select 2 out of 7 SC candidates and 3 out of 16 other candidates. This can be done in ${}^7C_2 \times {}^{16}C_3$ ways.

$\therefore$ The number of ways of making the selection $= {}^7C_2 \times {}^{16}C_3 = 11760$.

EXAMPLE 15 *How many triangles can be formed by joining the vertices of a hexagon ?*

SOLUTION There are 6 vertices of a hexagon. One triangle is formed by selecting a group of 3 vertices from given 6 vertices. This can be done in ${}^6C_3$ ways.

$\therefore$ Number of triangles $= {}^6C_3 = \dfrac{6!}{3!\,3!} = 20$.

EXAMPLE 16 *How many diagonals are there in a polygon with n sides ?*

SOLUTION A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon. Number of line segments obtained by joining the vertices of an n sided polygon taken two at a time

$=$ Number of ways of selecting 2 out of n $= {}^nC_2 = \dfrac{n(n-1)}{2}$

Out of these lines, n lines are the sides of the polygon.

$\therefore$ Number of diagonals of the polygon $= \dfrac{n(n-1)}{2} - n = \dfrac{n(n-3)}{2}$.

EXAMPLE 17 *A polygon has 44 diagonals. Find the number of its sides.*

SOLUTION Let there be n sides of the polygon. We know that the number of diagonals of n sided polygon is $\dfrac{n(n-3)}{2}$.

$$\therefore \quad \frac{n(n-3)}{2} = 44 \Rightarrow n^2 - 3n - 88 = 0 \Rightarrow (n-11)(n+8) = 0 \Rightarrow n = 11 \qquad (\because n > 0)$$

Hence, there are 11 sides of the polygon.

EXAMPLE 18 *How many chords can be drawn through 21 points on a circle?*

SOLUTION A chord is obtained by joining any two points on a circle. Therefore, total number of chords drawn through 21 points is same as the number of ways of selecting 2 points out of 21 points. This can be done in ${}^{21}C_2$ ways.

Hence, total number of chords $= {}^{21}C_2 = \dfrac{21!}{19!\,2!} = 21 \times 10 = 210$.

LEVEL-2

EXAMPLE 19 *A person wishes to make up as many different parties as he can out of his 20 friends such that each party consists of the same number of persons. How many friends should he invite ?*

SOLUTION Suppose he invites r friends at a time. Then the total number of parties is ${}^{20}C_r$. We have to find the maximum value of ${}^{20}C_r$ which is for $r = 10$, because ${}^nC_r$ is maximum for $r = n/2$, when n is even.

Hence, he should invite 10 friends at a time in order to form the maximum number of parties.

EXAMPLE 20 *If m parallel lines in plane are intersected by a family of n parallel lines. Find the number of parallelograms formed.*

SOLUTION A parallelogram is formed by choosing two straight lines from the set of m parallel lines and two straight lines from the set of n parallel lines.

Two straight lines from the set of m parallel lines can be chosen in $^{m}C_2$ ways and two straight lines from the set of n parallel lines can be chosen in $^{n}C_2$ ways.

Hence, the number of parallelograms formed = $^{m}C_2 \times {}^{n}C_2$

$$= \frac{m(m-1)}{2} \times \frac{n(n-1)}{2} = \frac{mn(m-1)(n-1)}{4}$$

EXAMPLE 21 *There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the (i) number of straight lines obtained from the pairs of these points; (ii) number of triangles that can be formed with the vertices as these points.*

SOLUTION (i) Number of straight lines formed joining the 10 points, taking 2 at a time = $^{10}C_2$

$$= \frac{10!}{2!\,8!} = 45.$$

Number of straight lines formed by joining the four points, taking 2 at a time = $^{4}C_2 = \frac{4!}{2!\,2!} = 6$

But, 4 collinear points, when joined pairwise give only one line.

$\therefore$ Required number of straight lines = 45 − 6 + 1 = 40.

(ii) Number of triangles formed by joining the points, taking 3 at a time = $^{10}C_3 = \frac{10!}{3!\,7!} = 120.$

Number of triangles formed by joining the 4 points, taken 3 at a time = $^{4}C_3 = {}^{4}C_1 = 4.$

But, 4 collinear points cannot form a triangle when taken 3 at a time.

So, Required number of triangles = 120 − 4 = 116.

EXAMPLE 22 *In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Find the number of points of intersection of the straight lines.*

SOLUTION The number of points of intersection of 37 straight lines is $^{37}C_2$. But 13 straight lines out of the given 37 straight lines pass through the same point A. Therefore instead of getting $^{13}C_2$ points, we get merely one point A. Similarly, 11 straight lines out of the given 37 straight lines intersect at point B. Therefore instead of getting $^{11}C_2$ points, we get only one point B. Hence, the number of intersection points of the lines is $^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535.$

EXAMPLE 23 *From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen ?* **[NCERT]**

SOLUTION We have the following possibilities:

(i) *Three particular students join the excursion party:* In this case, we have to choose 7 students from the remaining 22 students. This can be done in $^{22}C_7$ ways.

(ii) *Three particular students do not join the excursion party:* In this case, we have to choose 10 students from the remaining 22 students. This can be done in $^{22}C_{10}$ ways.

Hence, the required number of ways = $^{22}C_7 + {}^{22}C_{10} = 817190.$

EXAMPLE 24 *A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Chemistry Part II, unless Chemistry Part I is also borrowed. In how many ways can he choose the three books to be borrowed ?*

SOLUTION We have the following two possibilities :

(i) *When Chemistry part I is borrowed*: In this case the boy may borrow Chemistry Part II. So, he has to select now two books out of the remaining 7 books of his interest. This can be done in $^{7}C_2$ ways.

(ii) *When Chemistry part I is not borrowed* : In this case the boy does not want to borrow Chemistry Part II. So, he has to select three books from the remaining 6 books. This can be done in ${}^6C_3$ ways.

Hence, the required number of ways $= {}^7C_2 + {}^6C_3 = 21 + 20 = 41$.

EXAMPLE 25 *In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus signs are together ?*

SOLUTION The plus signs can be arranged in only one way, because all are identical, as shown below:

$$\square + \square + \square + \square + \square + \square + \square + \square$$

A blank box in the above arrangement shows available space for the minus signs. Since there are 7 plus signs, the number of blank boxes is therefore 8. The five minus signs are now to be arranged in the 8 boxes so that no two of them are together. Now, 5 boxes out of 8 can be chosen in ${}^8C_5$ ways. Since all minus signs are identical, so 5 minus signs can be arranged in 5 chosen boxes in only one way. Hence, the number of possible arrangements $= 1 \times {}^8C_5 \times 1 = 56$.

EXAMPLE 26 *In how many ways can 21 identical books on English and 19 identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together?*

SOLUTION In order that no two books on Hindi are together, we must first arrange all books in English in a row. Since all English books are identical, so they can be arranged in a row in only one way as shown below:

$$\times E \times E \times E \times E \times \ldots \times E \times E$$

Here E denotes the position of an English book and $\times$ that of a Hindi book.

Since there are 21 books on English, the number places mark $\times$ are therefore 22. Now, 19 books on Hindi are to be arranged in these 22 places so that no two of them are together. Out of 22 places 19 places for Hindi books can be chosen in ${}^{22}C_{19}$ ways. Since all books on Hindi are identical, so 19 books on Hindi can be arranged in 19 chosen places in only one way. Hence, the required number of ways $= 1 \times {}^{22}C_{19} \times 1 = 1540$.

EXERCISE 17.2

LEVEL-1

1. From a group of 15 cricket players, a team of 11 players is to be chosen. In how many ways can this be done?
2. How many different boat parties of 8, consisting of 5 boys and 3 girls, can be made from 25 boys and 10 girls?
3. In how many ways can a student choose 5 courses out of 9 courses if 2 courses are compulsory for every student?
4. In how many ways can a football team of 11 players be selected from 16 players? How many of these will (i) include 2 particular players? (ii) exclude 2 particular players?
5. There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees:
   (i) a particular professor is included. (ii) a particular student is included.
   (iii) a particular student is excluded.
6. How many different products can be obtained by multiplying two or more of the numbers 3, 5, 7, 11 (without repetition)?

7. From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition; at least including 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can the selection be made?
8. How many different selections of 4 books can be made from 10 different books, if
   (i) there is no restriction; (ii) two particular books are always selected;
   (iii) two particular books are never selected?
9. From 4 officers and 8 jawans in how many ways can 6 be chosen (i) to include exactly one officer (ii) to include at least one officer?
10. A sports team of 11 students is to be constituted, choosing at least 5 from class XI and at least 5 from class XII. If there are 20 students in each of these classes, in how many ways can the teams be constituted?
11. A student has to answer 10 questions, choosing at least 4 from each of part A and part B. If there are 6 questions in part A and 7 in part B, in how many ways can the student choose 10 questions?
12. In an examination, a student has to answer 4 questions out of 5 questions; questions 1 and 2 are however compulsory. Determine the number of ways in which the student can make the choice.
13. A candidate is required to answer 7 questions out of 12 questions which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. In how many ways can he choose the 7 questions?
14. There are 10 points in a plane of which 4 are collinear. How many different straight lines can be drawn by joining these points.
15. Find the number of diagonals of (i) a hexagon (ii) a polygon of 16 sides.
16. How many triangles can be obtained by joining 12 points, five of which are collinear ?
17. In how many ways can a committee of 5 persons be formed out of 6 men and 4 women when at least one woman has to be necessarily selected ?
18. In a village, there are 87 families of which 52 families have at most 2 children. In a rural development programme, 20 families are to be helped chosen for assistance, of which at least 18 families must have at most 2 children. In how many ways can the choice be made ?
19. A group consists of 4 girls and 7 boys. In how many ways can a tem of 5 members be selected if the team has (i) no girl? (ii) at least one boy and one girl? (iii) at least 3 girls? [NCERT]
20. A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women? [NCERT]
20. Find the number of (i) diagonals (ii) triangles formed in a decagon.
22. Determine the number of 5 cards combinations out of a deck of 52 cards if at least one of the 5 cards has to be a king ? [NCERT]
23. We wish to select 6 persons from 8, but if the person A is chosen, then B must be chosen. In how many ways can the selection be made ?
24. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls? [NCERT]
25. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour. [NCERT]
26. Determine the number of 5 cards combinations out of a deck of 52 cards if there is exactly one ace in each combination. [NCERT]

27. In how many ways can one select a circket team of eleven from 17 players in which only 5 persons can bowl if each cricket team of 11 must include exactly 4 bowlers?
28. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected. [NCERT]
29. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student? [NCERT]
30. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consits of :

    (i) exactly 3 girls? (ii) at least 3 girls? (iii) at most 3 girls? [NCERT]
31. In an examination, a question paper consists of 12 questions divided into two parts i.e., Part I and Part II, containing 5 and 7 questions, respectively. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions? [NCERT]

LEVEL-2

32. A parallelogram is cut by two sets of m lines parallel to its sides. Find the number of parallelograms thus formed.
33. Out of 18 points in a plane, no three are in the same straight line except five points which are collinear. How many (i) straight lines (ii) triangles can be formed by joining them ?

ANSWERS

1. 1365 2. 6375600 3. 35 4. 4368 (i) 2002 (ii) 364
5. 51300 (i) 10260 (ii) 7695 (iii) 43605 6. 11 7. 104874
8. (i) 210 (ii) 28 (iii) 70 9. (i) 224 (ii) 896 10. $2\left({}^{20}C_5 \times {}^{20}C_6\right)$
11. 266 12. 3 13. 780 14. 40 15. (i) 9 (ii) 104 16. 210 17. 246
18. ${}^{52}C_{18} \times {}^{35}C_2 + {}^{52}C_{19} \times {}^{35}C_1 + {}^{52}C_{20} \times {}^{35}C_0$ 19. (i) 21(ii) 441(iii) 91
20. (i) 35 (ii) 120 21. 10, 6 22. 886656 23. 22 24. 40 25. 2000
26. 778320 27. 3960 28. 200
29. 35 30. (i) 504 (ii) 588 (iii) 1630 31. 420 32. $({}^{m+2}C_2)^2$
33. (i) 144 (ii) 806

HINTS TO NCERT & SELECTED PROBLEMS

2. Required no. of boat parties = ${}^{25}C_5 \times {}^{10}C_3$.
3. Since 2 courses are compulsory. So, the student is to choose 3 courses out of the remaining 7 courses. This can be done in ${}^7C_3$ ways.
4. We have to select 11 players out of 16. So, required number of ways = ${}^{16}C_{11}$.
    (i) Since 2 particular players are always included, so, we have to select 9 players out of the remaining 14 players. This can be done in ${}^{14}C_9$ ways.
    (ii) Since 2 particular players are excluded from every selection, so, we have to select 11 players from the remaining 14 players. This can be done in ${}^{14}C_{11}$ ways.
6. Total number of products = Number of ways of selecting 2 or 3 or all out of 4 numbers 3,5,7,11

$$= {}^4C_2 + {}^4C_3 + {}^4C_4 = 6 + 4 + 1 = 11.$$

7. Since two girls who won the prizes last year are to be included in every selection. So, we have to select 8 students out of 12 boys and 8 girls, choosing at least 4 boys and at least two girls. This can be done in ${}^{12}C_6 \times {}^8C_2 + {}^{12}C_5 \times {}^8C_3 + {}^{12}C_4 \times {}^8C_4 = 104874$ ways.

9. (i) Required number of ways = ${}^4C_1 \times {}^8C_5$

(ii) Required number of ways = Total no. of ways – No. of ways of selecting no officer

$= {}^{12}C_6 - {}^8C_6.$

10. Required number of ways = ${}^{20}C_5 \times {}^{20}C_6 + {}^{20}C_6 \times {}^{20}C_5.$

11. The various possibilities are : (i) 4 from part A and 6 from part B (ii) 5 from part A and 5 from part B (iii) 6 from part A and 4 from part B.

So, the required number of ways = ${}^6C_4 \times {}^7C_6 + {}^6C_5 \times {}^7C_5 + {}^6C_6 \times {}^7C_4 = 266.$

12. Required number of ways = ${}^3C_2.$

13. Required number of ways = ${}^6C_5 \times {}^6C_2 + {}^6C_4 \times {}^6C_3 + {}^6C_3 \times {}^6C_4 + {}^6C_2 \times {}^6C_5 = 780.$

14. Number of straight lines = ${}^{10}C_2 - {}^4C_2 + 1.$

16. Number of triangles = ${}^{12}C_3 - {}^5C_3.$

18. 52 families have at most 2 children, while 35 families have more than 2 children. The selection of 20 families of which at least 18 families must have at most 2 children can be made as under:

(i) 18 families out of 52 and 2 families out of 35

or, (ii) 19 families out of 52 and 1 family out of 35

or, (iii) 20 families out of 52.

19. (i) From a group of 4 girls and 7 boys, a team of 5 consisting of no girls can be chosen in ${}^7C_5 = 21$ ways.

(ii) A team of 5 consisting of at least one boy and one girl can be chosen in

${}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1 = 441$ ways.

(iii) A team of 5 consisting of at least 3 girls can be chosen in

${}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 = 91$ ways.

21. A committee of 3 persons out of 2 men and 3 women can be constituted in ${}^5C_3 = 10$ ways.

A committee of 1 man and 2 women can be constituted in ${}^2C_1 \times {}^3C_2 = 6$ ways.

22. Required number of combinations = Total number of 5 card combinations

– Number of 5 car combinations having no king.

$= {}^{52}C_5 - {}^{48}C_5 = 886656.$

24. Number of ways of selecting team = ${}^5C_3 \times {}^4C_3 = 40.$

25. Number of ways of selecting 9 balls = ${}^6C_3 \times {}^5C_3 \times {}^5C_3 = 2000.$

26. Out of 4 aces one ace can be selected in ${}^4C_1$ ways and from the remaining 48 cards, four cards can be selected in ${}^{48}C_4$ ways. So, number of 5 cards combinations consisting of exactly one ace = ${}^4C_1 \times {}^{48}C_4 = 778320.$

27. Required number of ways = ${}^5C_4 \times {}^{12}C_7.$

28. Out of 5 black and 6 red balls, 2 black and 3 red balls can be chosen in ${}^5C_2 \times {}^6C_3 = 200$ ways.

29. Required number of ways = Number of ways of selecting 3 courses out of 7 courses.

$= {}^7C_3$ ways $= 35.$

30. (i) A committee consisting of 3 girls and 4 boys can be formed in ${}^9C_4 \times {}^4C_3 = 504$ ways.

(ii) A committee consisting of at least 3 girls can be formed in ${}^9C_4 \times {}^4C_3 + {}^9C_3 \times {}^4C_4$ $= 588$ ways.

(iii) A committee of at most 3 girls can be formed in

$$ {}^9C_7 \times {}^4C_0 + {}^9C_6 \times {}^4C_1 + {}^9C_5 \times {}^4C_2 + {}^9C_4 \times {}^4C_3 = 1632 \text{ ways.} $$

31. At least 3 questions can be selected in the following ways:

Part I	Part II
3	5
4	4
5	3

So, required number of ways $= {}^5C_3 \times {}^7C_5 + {}^5C_4 \times {}^7C_4 + {}^5C_5 \times {}^7C_3 = 420$.

32. Each set of parallel lines consists of $(m + 2)$ lines and each parallelogram is formed by choosing two lines from the first set and two straight lines from the second set.

Hence, the total number of parallelograms $= {}^{m+2}C_2 \times {}^{m+2}C_2$.

17.5 MIXED PROBLEMS ON PERMUTATIONS AND COMBINATIONS

In this section, we intend to discuss some practical problems where both permutations and combinations are used as is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed ?*

SOLUTION Three consonants out of 7 and 2 vowels out of 4 can be chosen in ${}^7C_3 \times {}^4C_2$ ways. Thus, there are ${}^7C_3 \times {}^4C_2$ groups each containing 3 consonants and 2 vowels. Since each group contains 5 letters, which can be arranged among themselves in 5! ways.

Hence, the required number of words $= ({}^7C_3 \times {}^4C_2) \times 5! = 25200$.

EXAMPLE 2 *How many four-letter words can be formed using the letters of the word 'FAILURE', so that*
(i) *F is included in each word ?* (ii) *F is not included in any word ?*

SOLUTION There are 7 letters in the word 'FAILURE'.

(i) To include F in every 4 letter word, we first select four letters from the 7 letters of the word 'FAILURE' such that F is included in every selection. This can be done by selecting three letters from the remaining 6 letters i.e. A, I, L, U, R, E in ${}^6C_3$ ways. Now, there are 4 letters in each of ${}^6C_3$ selections. Consider one of these ${}^6C_3$ selections. This selection contains 4 letters which can be arranged in 4! ways. Thus, each of ${}^6C_3$ selections provides 4! words.

Hence, the total number of words $= {}^6C_3 \times 4! = 480$.

(ii) If F is not to be included in any word, then we first select 4 letters from the remaining 6 letters. This can be done in ${}^6C_4$ ways. Now, every selection has 4 letters which can be arranged in a row in 4! ways.

Hence, the total number of words $= {}^6C_4 \times 4! = 360$.

EXAMPLE 3 *How many words with or without meaning, can be formed using all the letters of the word EQUATION at a time so that vowels and consonants occur together?* [NCERT]

SOLUTION There are 5 vowels and 3 consonants in the word EQUATION. All vowels can be put together in 5! ways and all consonants can be put together in 3! ways. Considering all vowels as one letter and all consonants as at letter, vowels and consonants can be arranged in 2! ways. Therefore, vowels and consonants can be put together in $5! \times 3! \times 2!$ ways i.e. 1440 ways.

EXAMPLE 4 *How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word 'EQUATION' so that the two consonants occur together ?*

SOLUTION There are 5 vowels and 3 consonants in the word 'EQUATION'. Three vowels out of 5 and 2 consonants out of 3 can be chosen in ${}^5C_3 \times {}^3C_2$ ways. So, there are ${}^5C_3 \times {}^3C_2$ groups each containing 3 consonants and two vowels. Now, each group contains 5 letters which are to be arranged in such a way that 2 consonants occur together. Considering 2 consonants as one letter, we have 4 letters which can be arranged in 4! ways. But two consonants can be put together in 2! ways. Therefore, 5 letters in each group can be arranged in $4! \times 2!$ ways.

Hence, the required number of words $= ({}^5C_3 \times {}^3C_2) \times 4! \times 2! = 1440$.

EXAMPLE 5 *How many words with or without meaning, each 2 of vowels and 3 consonants can be formed from the letters of the word DAUGHTER?* [NCERT]

SOLUTION There are 3 vowels and 5 consonants in the word DAUGHTER out of which 2 vowels and 3 consonants can be chosen in ${}^3C_2 \times {}^5C_3$ ways. These selected five letters can now be arranged in 5! ways.

Hence, required number of words $= {}^3C_2 \times {}^5C_3 \times 5! = 3 \times 10 \times 120 = 3600$

EXAMPLE 6 *The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet ?* [NCERT]

SOLUTION Out of 5 vowels and 21 consonants, 2 vowels and 2 consonants can be chosen in ${}^5C_2 \times {}^{21}C_2$ ways. These selected 4 letters can now be arranged in 4! ways. Therefore, by the fundamental principle of counting, required number of words is

$${}^5C_2 \times {}^{21}C_2 \times 4! = 10 \times 210 \times 24 = 50400.$$

EXAMPLE 7 *In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together?* [NCERT]

SOLUTION Since boys are to be separated. Therefore, let us first seat 5 girls. This can be done in 5! ways. For each such arrangement, three boys can be seated only at the cross marked places.

$$\times G \times G \times G \times G \times G \times$$

There are 6 crossed marked places and three boys can be seated in ${}^6C_3 \times 3!$ ways. Hence, by the fundamental principle of counting, the total number of ways is $5! \times {}^6C_3 \times 3! = 14400$.

LEVEL-2

EXAMPLE 8 *How many words can be formed by taking 4 letters at a time out of the letters of the word 'MATHEMATICS'.*

SOLUTION There are 11 letters viz. MM, AA, TT, H, E, I, C, S. All these letters are not distinct, so we cannot use ${}^nP_r$. We can choose 4 letters from the following ways:

(i) All the four distinct letters: There are 8 distinct letters viz. M, A, T, H, E, I, C, S out of which 4 can be chosen in ${}^8C_4$ ways. So, the total number of groups of 4 letters $= {}^8C_4$. Each such group has 4 letters which can be arranged in 4! ways.

Hence, the total number of words $= {}^8C_4 \times 4! = {}^8P_4 = 1680$.

(ii) Two distinct and two alike letters: There are 3 pairs of alike letters viz MM, AA, TT, out of which one pair can be chosen in ${}^3C_1$ ways. Now we have to choose two letters out of the remaining 7 different types of letters which can be done in ${}^7C_2$ ways. So, the total number of groups of 4 letters in which two are different and 2 are alike is ${}^3C_1 \times {}^7C_2$. Each such group has 4 letters of which 2 are alike and remaining two distinct and they can be arranged in $\frac{4!}{2!}$ ways.

Hence, the total number of words in which two letters are alike $= {}^3C_1 \times {}^7C_2 \times \frac{4!}{2!} = 756.$

(iii) Two alike of one kind and two alike of other kind: There are 3 pairs of 2 alike letters out of which 2 pairs can be chosen in ${}^3C_2$ ways. So, there are ${}^3C_2$ groups of 4 letters each. In each group there are 4 letters of which 2 are alike of one kind and two alike of other kind. These 4 letters can be arranged in $\frac{4!}{2!\,2!}$ ways. Hence, the total number of words in which two letters are alike of one kind and two alike of other kind $= {}^3C_2 \times \frac{4!}{2!\,2!} = 18.$

From (i), (ii) and (iii) the total number of 4 letter words $= 1680 + 756 + 18 = 2454$.

EXAMPLE 9 *Eighteen guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on the other side. Determine the number of ways in which the seating arrangement can be made.*

SOLUTION Since four particular guests want to sit on a particular side A (say) and three others on the other side B (say). So, we are left with 11 guests out of which we choose 5 for side A in ${}^{11}C_5$ ways and the remaining 6 for side B in ${}^6C_6$ ways. Hence, the number of selections for the two sides is ${}^{11}C_5 \times {}^6C_6$.

Now 9 persons on each side of the table can be arranged among themselves in 9! ways.

Hence, the total number of arrangements $= {}^{11}C_5 \times {}^6C_6 \times 9! \times 9! = \frac{11!}{6!5!} \times 9! \times 9!$

EXAMPLE 10 *How many four-letter words can be formed using the letter of the word 'INEFFECTIVE'?*

SOLUTION There are 11 letters in the word 'INEFFECTIVE'. viz. EEE, FF, II, C, T, N, V.

The four-letter words may consist of:

(i) 3 alike letters and 1 distinct letter
(ii) 2 alike letters of one kind and 2 alike letters of the second kind
(iii) 2 alike letters and 2 distinct letters
(iv) all different letters

Now we, shall discuss these four cases one by one:

(i) *3 alike letters and 1 distinct letter*: There is one set of three alike letters viz. EEE. So, three alike letters can be selected in one way. Out of the 6 different letters F, I, T, N, V, C one letter can be selected in ${}^6C_1$ ways. Thus, three alike and one different letter can be selected in $1 \times {}^6C_1 = {}^6C_1$ ways. So, there are ${}^6C_1$ groups each of which contains 3 alike letters and one different letter. These 4 letters can be arranged in $\frac{4!}{3!1!}$ ways.

Hence, the total number of words consisting of three alike and one distinct letters

$$= {}^6C_1 \times \frac{4!}{3!1!} = {}^6C_1 \times 4 = 24.$$

(ii) *2 alike letters of one kind and 2 alike letters of second kind:* There are three sets of two alike letters viz EE, FF, II. Out of these three sets two can be selected in ${}^3C_2$ ways. So, there are ${}^3C_2$ groups each of which contains 4 letters out of which 2 are alike of one type and two are alike of second type. Now, 4 letters in each group can be arranged in $\frac{4!}{2!\,2!}$ ways.

Hence, the total number of words consisting of two alike letters of one type and 2 alike letters of second type $= {}^3C_2 \times \frac{4!}{2!\,2!} = 18.$

(iii) *2 alike and 2 different letters:* Out of 3 sets of two alike letters one set can be chosen in ${}^3C_1$ ways. Now, from the remaining 6 distinct letters, 2 letters can be chosen in ${}^6C_2$ ways. Thus, 2 alike letters and 2 distinct letters can be selected in $({}^3C_1 \times {}^6C_2)$ ways. So, there are $({}^3C_1 \times {}^6C_2)$ groups of 4 letters each. Now, letters of each group can be arranged among themselves in $\frac{4!}{2!}$ ways.

Hence, the total number of words consisting of two alike letters and 2 distinct

$$= {}^3C_1 \times {}^6C_2 \times \frac{4!}{2!} = 540.$$

(iv) *All different letters:* There are 7 distinct letters E, F, I, T, N, V, C out of which 4 can be selected in ${}^7C_4$ ways. So, there are ${}^7C_4$ groups of 4 letters each. The letters in each of ${}^7C_4$ groups can be arranged in 4! ways.

So, the total number of 4 letter words in which all letters are distinct $= {}^7C_4 \times 4! = 840.$

Hence, the total number of 4-letter words $= 24 + 18 + 540 + 840 = 1422.$

EXAMPLE 11 *In how many ways can the letters of the word PERMUTATIONS be arranged if there are always 4 letters between P and S?* [NCERT]

SOLUTION There 12 letters in the given word of which 2 are T's. There can be 4 letters between P and S in one of the following ways:

(i) There are 2T's and 2 other letters from the remaining 8 letters (excluding 2T's and P and S).
(ii) One T and 3 other letters from the remaining 8 letters.
(iii) There is no T and 4 other letters.

Let us now find the number of words in each case.

(i) In the first case, 2 letters can be chosen from remaining 8 letters in ${}^8C_2$ ways. Now, 2T's and 2 other letters can be arranged between P and S in $\frac{4!}{2!}$ ways. Also, P and S can interchange their positions. So, 2T's and 2 other letters can be arranged between P and S in ${}^8C_2 \times \frac{4!}{2!} \times 2!$ ways. Considering these six letters as one letter and the remaining 6 letters can be arranged in 7! ways.

$\therefore$ Total number of words, in this case $= {}^8C_2 \times \frac{4!}{2!} \times 2! \times 7!$

(ii) In this case, 3 letters can be chosen from the remaining 8 letters in ${}^8C_3$ ways. Now, one T and 3 other letters from the remaining 8 letters can be arranged between P and S in 4! ways. Also, P and S can interchange their positions. So, one T and 3 other letters can be arranged between P and S in ${}^8C_3 \times 4! \times 2!$ ways. Considering these six letters as one letter and the remaining 6 letters can be arranged in 7! ways.

$\therefore$ Total number of words formed $= {}^8C_3 \times 4! \times 2! \times 7!$

(iii) In this case, 4 letters other than 2T's can be chosen from the remaining 8 letters in ${}^8C_4$ ways. These 4 letters can be arranged between P and S in 4! ways. Also, P and S can interchange their positions in 2! ways. Thus, 4 letters between P and S can be arranged in ${}^8C_4 \times 4! \times 2!$ ways. Taking these 6 letters as one letter with the remaining 6 letters (including 2T's), we have 7 letters which can be arranged in $\frac{7!}{2!}$ ways.

$\therefore$ Number of words formed $= {}^8C_4 \times 4! \times 2! \times \frac{7!}{2!}$

Hence, total number of words $= {}^8C_2 \times \frac{4!}{2!} \times 2! \times 7! + {}^8C_3 \times 4! \times 2! \times 7! + {}^8C_4 \times 4! \times 2! \times \frac{7!}{2!}$

$= 25401600$

EXERCISE 17.3

LEVEL-1

1. How many different words, each containing 2 vowels and 3 consonants can be formed with 5 vowels and 17 consonants?
2. There are 10 persons named $P_1, P_2, P_3 \ldots, P_{10}$. Out of 10 persons, 5 persons are to be arranged in a line such that is each arrangement P_1 must occur whereas P_4 and P_5 do not occur. Find the number of such possible arrangements.
3. How many words, with or without meaning can be formed from the letters of the word 'MONDAY', assuming that no letter is repeated, if (i) 4 letters are used at a time (ii) all letters are used at a time (iii) all letters are used but first letter is a vowel? [NCERT]
4. Find the number of permutations of n distinct things taken r together, in which 3 particular things must occur together.
5. How many words each of 3 vowels and 2 consonants canbe formed from the letters of the word INVOLUTE? [NCERT]
6. Find the number of permutations of n different things taken r at a time such that two specified things occur together? [NCERT]

LEVEL-2

7. Find the number of ways in which : (a) a selection (b) an arrangement, of four letters can be made from the letters of the word 'PROPORTION'.
8. How many words can be formed by taking 4 letters at a time from the letters of the word 'MORADABAD'?
9. A business man hosts a dinner to 21 guests. He is having 2 round tables which can accommodate 15 and 6 persons each. In how many ways can he arrange the guests?
10. Find the number of combinations and permutations of 4 letters taken from the word 'EXAMINATION'.
11. A tea party is arranged for 16 persons along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular side and two on the other side. In how many ways can they be seated?

ANSWERS

1. 816000 2. ${}^7C_4 \times 5!$ 3. (i) 360 (ii) 720 (iii) 240 4. ${}^{n-3}C_{r-3}\,(r-2)!\,3!$

5. 2880 6. $2(r-1)\,{}^{n-2}P_{r-2}$ 7. (a) 53 (b) 758 8. 626

9. ${}^{21}C_{15} \times 14! \times 5!$ 10. 2454 11. ${}^{10}C_4 \times (8!)^2$

HINTS TO NCERT & SELECTED PROBLEMS

1. 2 vowels out of 5 and 3 consonants out of 17 can be chosen in ${}^5C_2 \times {}^{17}C_3$ ways.
   Now, 5 letters in each selection can be arranged in 5! ways.
   So, total number of words $= {}^5C_2 \times {}^{17}C_3 \times 5! = 816000$
3. (i) Total number of 4 letter words formed from the letters of the word 'MONDAY"
   $= {}^6C_4 \times 4! = 360.$
   (ii) Total number of words formed by using all letters of the word 'MONDAY' $= 6! = 720$
   (iii) There are two vowels A and O. So, first place can be filled in 2 ways and the remaining 5 places can be filled in 5! ways.
   So, total number of words beginning with a vowel $= 2 \times 5! = 240$.
5. Required number of words $= {}^4C_3 \times {}^4C_2 \times 5!$
6. Out of $(n-2)$ remaining things select $(r-2)$ things in ${}^{n-2}C_{r-2}$ ways. Consider two specified things as one and mix it with $(r-2)$ selected things. Now we have $(r-1)$ things which can be arranged in $(r-1)!$ ways, but two specified things can be put together in 2! ways. Hence, required number of ways $= {}^{n-2}C_{r-2} \times (r-1)! \times 2!$
9. Total number of ways $= {}^{21}C_{15} \times {}^6C_6 \times 14! \times 5!$
11. 4 persons wish to sit on side A(say) and two on the other side B(say). So, 10 persons are left, out of which 4 persons for side A can be selected in ${}^{10}C_4$ ways and 6 persons for side B from the remaining 6 persons in ${}^6C_6$ ways. Hence, the number of selections for two sides $= {}^{10}C_4 \times {}^6C_6$. Now, 8 persons on each side can be arranged amongst themselves in 8! ways. Hence, the total number of seating arrangements $= {}^{10}C_4 \times {}^6C_6 \times 8! \times 8!$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write $\sum_{r=0}^{m} {}^{n+r}C_r$ in the simplified form.
2. If ${}^{35}C_{n+7} = {}^{35}C_{4n-2}$, then write the values of n.
3. Write the number of diagonals of an n-sided polygon.
4. Write the expression ${}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r$ in the simplest form.
5. Write the value of $\sum_{r=1}^{6} {}^{56-r}C_3 + {}^{50}C_4$.
6. There are 3 letters and 3 directed envelopes. Write the number of ways in which no letter is put in the correct envelope.
7. Write the maximum number of points of intersection of 8 straight lines in a plane.
8. Write the number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines.
9. Write the number of ways in which 5 red and 4 white balls can be drawn from a bag containing 10 red and 8 white balls.
10. Write the number of ways in which 12 boys may be divided into three groups of 4 boys each.
11. Write the total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants.

ANSWERS

1. ${}^{n+m+1}C_{n+1}$ 2. 3, 6 3. $\dfrac{n(n-3)}{2}$ 4. ${}^{n+2}C_{r+1}$ 5. ${}^{56}C_4$

6. 2 7. 28 8. 18 9. ${}^{10}C_5 \times {}^8C_4$ 10. $\dfrac{12!}{(4!)^3\, 3!}$

11. ${}^4C_2 \times 5C_3 \times 5!$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If ${}^{20}C_r = {}^{20}C_{r-10}$, then ${}^{18}C_r$ is equal to
   (a) 4896 (b) 816 (c) 1632 (d) none of these
2. If ${}^{20}C_r = {}^{20}C_{r+4}$, then ${}^rC_3$ is equal to
   (a) 54 (b) 56 (c) 58 (d) none of these
3. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, then r is equal to
   (a) 5 (b) 4 (c) 3 (d) 2
4. If ${}^{20}C_{r+1} = {}^{20}C_{r-1}$, then r is equal to
   (a) 10 (b) 11 (c) 19 (d) 12
5. If $C(n, 12) = C(n, 8)$, then $C(22, n)$ is equal to
   (a) 231 (b) 210 (c) 252 (d) 303
6. If ${}^mC_1 = {}^nC_2$, then
   (a) $2m = n$ (b) $2m = n(n+1)$ (c) $2m = n(n-1)$ (d) $2n = m(m-1)$
7. If ${}^nC_{12} = {}^nC_8$, then $n =$
   (a) 20 (b) 12 (c) 6 (d) 30
8. If ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_x$, then $x =$
   (a) r (b) $r-1$ (c) n (d) $r+1$
9. If ${}^{(a^2-a)}C_2 = {}^{(a^2-a)}C_4$, then $a =$
   (a) 2 (b) 3 (c) 4 (d) none of these
10. ${}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$ is equal to
    (a) 30 (b) 31 (c) 32 (d) 33
11. Total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants is equal to
    (a) 60 (b) 120 (c) 7200 (d) none of these
12. There are 12 points in a plane. The number of the straight lines joining any two of them when 3 of them are collinear, is
    (a) 62 (b) 63 (c) 64 (d) 65
13. Three persons enter a railway compartment. If there are 5 seats vacant, in how many ways can they take these seats ?
    (a) 60 (b) 20 (c) 15 (d) 125
14. In how many ways can a committee of 5 be made out of 6 men and 4 women containing at least one women ?

(a) 246 (b) 222 (c) 186 (d) none of these

15. There are 10 points in a plane and 4 of them are collinear. The number of straight lines joining any two of them is
(a) 45 (b) 40 (c) 39 (d) 38

16. There are 13 players of cricket, out of which 4 are bowlers. In how many ways a team of eleven be selected from them so as to include at least two bowlers ?
(a) 72 (b) 78 (c) 42 (d) none of these

17. If $C_0 + C_1 + C_2 + \ldots + C_n = 256$, then ${}^{2n}C_2$ is equal to
(a) 56 (b) 120 (c) 28 (d) 91

18. The number of ways in which a host lady can invite for a party of 8 out of 12 people of whom two do not want to attend the party together is
(a) $2 \times {}^{11}C_7 + {}^{10}C_8$ (b) ${}^{10}C_8 + {}^{11}C_7$
(c) ${}^{12}C_8 - {}^{10}C_6$ (d) none of these

19. Given 11 points, of which 5 lie on one circle, other than these 5, no 4 lie on one circle. Then the number of circles that can be drawn so that each contains at least 3 of the given points is
(a) 216 (b) 156 (c) 172 (d) none of these

20. How many different committees of 5 can be formed from 6 men and 4 women on which exact 3 men and 2 women serve ?
(a) 6 (b) 20 (c) 60 (d) 120

21. If ${}^{43}C_{r-6} = {}^{43}C_{3r+1}$, then the value of r is
(a) 12 (b) 8 (c) 6 (d) 10 (e) 14

22. The number of diagonals that can be drawn by joining the vertices of an octagon is
(a) 20 (b) 28 (c) 8 (d) 16

23. The value of $\left({}^7C_0 + {}^7C_1\right) + \left({}^7C_1 + {}^7C_2\right) + \ldots + \left({}^7C_6 + {}^7C_7\right)$ is
(a) $2^7 - 1$ (b) $2^8 - 2$ (c) $2^8 - 1$ (d) 2^8

24. Among 14 players, 5 are bowlers. In how many ways a team of 11 may be formed with at least 4 bowlers?
(a) 265 (b) 263 (c) 264 (d) 275

25. A lady gives a dinner party for six guests. The number of ways in which they may be selected from among ten friends if two of the friends will not attend the party together is
(a) 112 (b) 140 (c) 164 (d) none of these

26. If ${}^{n+1}C_3 = 2 \cdot {}^nC_2$, then $n =$
(a) 3 (b) 4 (c) 5 (d) 6

27. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is
(a) 6 (b) 9 (c) 12 (d) 18

ANSWERS

1. (b)	2. (b)	3. (c)	4. (a)	5. (a)	6. (c)	7. (a)	8. (d)
9. (b)	10. (b)	11. (c)	12. (c)	13. (a)	14. (a)	15. (b)	16. (b)
17. (b)	18. (c)	19. (b)	20. (d)	21. (a)	22. (a)	23. (b)	24. (c)
25. (b)	26. (c)	27. (d)					

SUMMARY

1. If n is a natural number and r is a non-negative integer such that $0 \leq r \leq n$, then

   (i) ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$ (ii) ${}^nC_r \times r! = {}^nP_r$

   (iii) ${}^nC_r = {}^nC_{n-r}$ (iv) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

   (v) ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n}{r} \times \dfrac{n-1}{r-1} \cdot {}^{n-2}C_{r-2} = \ldots = \dfrac{n}{r} \times \dfrac{n-1}{r-1} \times \dfrac{n-2}{r-2} \times \ldots \times \dfrac{n-(r-1)}{1}$

   (vi) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or, $x + y = n$

   (vii) If n is an even natural number, then the greatest among ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ is ${}^nC_{\frac{n}{2}}$.

   If n is an odd natural number, then the greatest among ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ is ${}^nC_{\frac{n-1}{2}}$ or, ${}^nC_{\frac{n+1}{2}}$.

2. The number of ways of selecting r items or objects from a group of n distinct items or objects is $\dfrac{n!}{(n-r)!\,r!} = {}^nC_r$.

CHAPTER 18

BINOMIAL THEOREM

18.1 INTRODUCTION

An algebraic expression containing two terms is called a binomial expression.

For example, $(a + b)$, $(2x - 3y)$, $\left(x + \frac{1}{y}\right)$, $\left(x + \frac{3}{x}\right)$, $\left(\frac{2}{x} - \frac{1}{x^2}\right)$ etc. are binomial expressions.

Similarly, an algebraic expression containing three terms is called a *trinomial*. In general, expressions containing more than two terms are known as multinomial expression.

The general form of the binomial expression is $(x + a)$ and the expansion of $(x + a)^n$, $n \in N$ is called the *binomial theorem*. This theorem was first given by **Sir Issac Newton.** It gives a formula for the expansion of the powers of a binomial expression.

In earlier classes, we have learnt that:

$$(x + a)^0 = 1$$

$$(x + a)^1 = x + a$$

$$(x + a)^2 = x^2 + 2ax + a^2$$

$$(x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$$

$$(x + a)^4 = x^4 + 4x^3 a + 6x^2 a^2 + 4x a^3 + a^4$$

We observe that the coefficients in the above expansions follow a particular pattern as given below:

Index of the binomial	*Coefficients of various terms*
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1

We also observe that each row is bounded by 1 on both sides. Any entry, except the first and last, in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right. The above pattern is known as ***Pascal's triangle.*** It has been checked that the above pattern also holds good for the coefficients in the expansions of the binomial expressions having index (exponent) greater than 4 as given below.

Index of the binomial	*Coefficients of various terms*
0	1
1	1 ∇ 1
2	1 ∇ 2 ∇ 1
3	1 ∇ 3 ∇ 3 ∇ 1
4	1 ∇ 4 ∇ 6 ∇ 4 ∇ 1
5	1 ∇ 5 ∇ 10 ∇ 10 ∇ 5 ∇ 1
6	1 ∇ 6 ∇ 15 ∇ 20 ∇ 15 ∇ 6 ∇ 1

....

....

Pascal's Triangle

Using the above Pascal's triangle, we obtain

$$(x+a)^1 = x + a$$

or, $$(x+a)^1 = {}^1C_0\, x^1 a^0 + {}^1C_1\, x^0 a^1 \qquad \left[\because\ {}^1C_0 = 1 = {}^1C_1\right]$$

$$(x+a)^2 = x^2 + 2ax + a^2$$

or, $$(x+a)^2 = {}^2C_0\, x^2 a^0 + {}^2C_1\, x^{2-1} a^1 + {}^2C_2\, x^{2-2} a^2 \qquad \left[\because\ {}^2C_0 = 1,\ {}^2C_1 = 2,\ {}^2C_2 = 1\right]$$

$$(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$$

or, $$(x+a)^3 = {}^3C_0\, x^3 a^0 + {}^3C_1\, x^{3-1} a^1 + {}^3C_2\, x^{3-2} a^2 + {}^3C_3\, x^{3-3} a^3$$

$$(x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$$

or, $$(x+a)^4 = {}^4C_0\, x^4 a^0 + {}^4C_1\, x^{4-1} a^1 + {}^4C_2\, x^{4-2} a^2 + {}^4C_3\, x^{4-3} a^3 + {}^4C_4\, x^{4-4} a^4$$

$$(x+a)^5 = {}^5C_0\, x^5 a^0 + {}^5C_1\, x^{5-1} a^1 + {}^5C_2\, x^{5-2} a^2 + {}^5C_3\, x^{5-3} a^3 + {}^5C_4\, x^{5-4} a^4 + {}^5C_5\, x^{5-5} a^5$$

By looking at the above expansions we can easily guess that the general formula would be of the form as given in the following theorem.

18.2 BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

THEOREM *If x and a are real numbers, then for all $n \in N$,*

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \dots + {}^nC_r\, x^{n-r} a^r + \dots + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x^0 a^n$$

i.e., $$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

PROOF We shall prove the theorem by using the principle of mathematical induction on n.

Let $P(n)$ be the statement:

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \dots + {}^nC_r\, x^{n-r} a^r + \dots + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x^0 a^n$$

STEP I We have, $P(1)$: $(x+a)^1 = {}^1C_0\, x^1 a^0 + {}^1C_1\, x^0 a^1$

We know that: $(x+a)^1 = x + a = {}^1C_0\, x^1 a^0 + {}^1C_1\, x^0 a^1$

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$(x+a)^m = {}^mC_0\, x^m a^0 + {}^mC_1\, x^{m-1} a^1 + {}^mC_2\, x^{m-2} a^2 + \ldots + {}^mC_{m-1}\, x^1 a^{m-1} + {}^mC_m\, x^0 a^m \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true. For this we have to show that

$$(x+a)^{m+1} = {}^{m+1}C_0\, x^{m+1} a^0 + {}^{m+1}C_1\, x^m a^1 + {}^{m+1}C_2\, x^{m-1} a^2 + \ldots + {}^{m+1}C_m\, x^1 a^m + {}^{m+1}C_{m+1}\, x^0 a^{m+1}$$

Now, $(x+a)^{m+1}$

$$= (x+a)\,.\,(x+a)^m = (x+a)\left[{}^mC_0\, x^m a^0 + {}^mC_1\, x^{m-1} a^1 + \ldots + {}^mC_r\, x^{m-r} a^r + \ldots + {}^mC_{m-1}\, x^1 a^{m-1} + {}^mC_m\, x^0 a^m\right]$$

$$= {}^mC_0\, x^{m+1} a^0 + ({}^mC_1 + {}^mC_0)\, x^m a^1 + ({}^mC_2 + {}^mC_1)\, x^{m-1} a^2 + \ldots + ({}^mC_r + {}^mC_{r-1})\, x^{m-r+1} a^r + \ldots + ({}^mC_{m-1} + {}^mC_m)\, x^1 a^m + {}^mC_m\, a^{m+1}$$

$$= {}^{m+1}C_0\, x^{m+1} a^0 + {}^{m+1}C_1\, x^m a^1 + {}^{m+1}C_2\, x^{m-1} a^2 + \ldots + {}^{m+1}C_r\, x^{(m+1)-r} a^r + \ldots + {}^{m+1}C_m\, x^1 a^m + {}^{m+1}C_{m+1}\, a^{m+1} \quad \left[\because {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r,\ r = 1, 2, 3, \ldots, m\right]$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, the theorem is true for all $n \in N$.

Q.E.D.

18.3 SOME IMPORTANT CONCLUSIONS FROM THE BINOMIAL THEOREM

In this section, we shall draw some useful conclusions from the binomial theorem.

(i) We have,

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

or,
$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots + {}^nC_r\, x^{n-r} a^r + \ldots + {}^nC_n\, x^0 a^n$$

Since r can have values from 0 to n, the total number of terms in the expansion is $(n+1)$.

(ii) The sum of the indices of x and a in each term is n.

(iii) Since ${}^nC_r = {}^nC_{n-r}$, for $r = 0, 1, 2, \ldots, n$

$$\therefore\ {}^nC_0 = {}^nC_n,\ {}^nC_1 = {}^nC_{n-1},\ {}^nC_2 = {}^nC_{n-2} = \ldots$$

So, the coefficients of terms equidistant from the beginning and end are equal. These coefficients are known as the binomial coefficients.

(iv) Replacing a by $-a$, we get

$$(x-a)^n = {}^nC_0\, x^n a^0 - {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 - {}^nC_3\, x^{n-3} a^3 + \ldots + (-1)^r\, {}^nC_r\, x^{n-r} a^r + \ldots + (-1)^n\, {}^nC_n\, x^0 a^n.$$

i.e.
$$(x-a)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r} a^r$$

Thus, the terms in the expansion of $(x-a)^n$ are alternatively positive and negative, the last term is positive or negative according as n is even or odd.

(v) Putting $x = 1$ and $a = x$ in the expansion of $(x + a)^n$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \dots + {}^nC_r\, x^r + \dots + {}^nC_n\, x^n$$

i.e. $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

This is the expansion of $(1 + x)^n$ in ascending powers of x.

(vi) Putting $a = 1$ in the expansion of $(x + a)^n$, we get

$$(1+x)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} + {}^nC_2\, x^{n-2} + \dots + {}^nC_r\, x^{n-r} + \dots + {}^nC_{n-1}\, x + {}^nC_n$$

i.e. $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}$$

This is the expansion of $(1 + x)^n$ in descending powers of x.

(vii) Putting $x = 1$ and $a = -x$ in the expansion of $(x + a)^n$, we get

$$(1-x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 - {}^nC_3\, x^3 + \dots + (-1)^r\, {}^nC_r\, x^r + \dots + (-1)^n\, {}^nC_n\, x^n.$$

i.e. $$(1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r$$

(viii) The coefficient of $(r + 1)$th term in the expansion of $(1 + x)^n$ is ${}^nC_r$.

(ix) The coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$.

(x) $$(x+a)^n + (x-a)^n = 2\left\{{}^nC_0\, x^n\, a^0 + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{n-4}\, a^4 + \dots\right\}$$

and, $$(x+a)^n - (x-a)^n = 2\left\{{}^nC_1\, x^{n-1}\, a^1 + {}^nC_3\, x^{n-3}\, a^3 + \dots\right\}$$

<u>NOTE:</u> *If n is odd than $\{(x + a)^n + (x - a)^n\}$ and $\{(x + a)^n - (x - a)^n\}$ both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$ whereas if n is even, then $\{(x + a)^n + (x - a)^n\}$ has $\left(\frac{n}{2} + 1\right)$ terms and $\left\{(x + a)^n - (x - a)^n\right\}$ has $\left(\frac{n}{2}\right)$ terms.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **DETERMINING THE NUMBER OF TERMS IN THE EXPANSIONS OF BINOMIAL AND TRINOMIAL EXPRESSIONS**

EXAMPLE 1 *Find the number of terms in the expansions of the following:*

(i) $(2x - 3y)^9$ (ii) $(1 + 5\sqrt{2}\,x)^9 + (1 - 5\sqrt{2}\,x)^9$

(iii) $(\sqrt{x} + \sqrt{y})^{10} + (\sqrt{x} - \sqrt{y})^{10}$ (iv) $(2x + 3y - 4z)^n$

(v) $[(3x + y)^8 - (3x - y)^8]$ (vi) $(1 + 2x + x^2)^{20}$

SOLUTION (i) The expansion of $(x + a)^n$ has $(n + 1)$ terms. So, the expansion of $(2x - 3y)^9$ has 10 terms.

(i) If n is odd, then the expansion of $(x + a)^n + (x - a)^n$ contains $\left(\frac{n+1}{2}\right)$ terms. So, the expansion of $(1 + 5\sqrt{2}x)^9 + (1 - 5\sqrt{2}\,x)^9$ has $\left(\frac{9+1}{2}\right) = 5$ terms.

(iii) If n is even, then the expansion of $\left\{(x+a)^n+(x-a)^n\right\}$ has $\left(\frac{n}{2}+1\right)$ terms.

So, $(\sqrt{x}+\sqrt{y})^{10}+(\sqrt{x}-\sqrt{y})^{10}$ has 6 terms.

(iv) We have,

$$(2x+3y-4z)^n=\left\{2x+(3y-4z)\right\}^n$$
$$={}^nC_0\,(2x)^n\,(3y-4z)^0+{}^nC_1\,(2x)^{n-1}\,(3y-4z)^1+{}^nC_2\,(2x)^{n-2}\,(3y-4z)^2+\dots$$
$$+{}^nC_{n-1}\,(2x)^1\,(3y-4z)^{n-1}+{}^nC_n\,(3y-4z)^n.$$

Clearly, the first term in the above expansion gives one term, second term gives two terms, third term gives three terms and so on.

So, total number of terms $=1+2+3+\dots+n+(n+1)=\frac{(n+1)(n+2)}{2}$

(v) If n is even, then $\left\{(x+a)^n-(x-a)^n\right\}$ has $\frac{n}{2}$ terms. So, $(3x+y)^8-(3x-y)^8$ has 4 terms.

(vi) We have,

$$(1+2x+x^2)^{20}=\left\{(1+x)^2\right\}^{20}=(1+x)^{40}$$

So, there are 41 terms in the expansion of $(1+2x+x^2)^{20}$

Type II **EXPANDING A GIVEN EXPRESSION USING THE BINOMIAL THEOREM**

EXAMPLE 2 *Expand $(x^2+2a)^5$ by binomial theorem.*

SOLUTION Using binomial theorem,

$$(x^2+2a)^5={}^5C_0\,(x^2)^5\,(2a)^0+{}^5C_1\,(x^2)^4\,(2a)^1+{}^5C_2\,(x^2)^3\,(2a)^2$$
$$+{}^5C_3\,(x^2)^2\,(2a)^3+{}^5C_4\,(x^2)\,(2a)^4+{}^5C_5\,(x^2)^0\,(2a)^5$$
$$=x^{10}+5\,(x^8)\,(2a)+10\,(x^6)\,(4a^2)+10\,(x^4)\,(8a^3)+5\,(x^2)\,(16\,a^4)+32\,a^5$$
$$=x^{10}+10\,x^8\,a+40\,x^6\,a^2+80\,x^4\,a^3+80\,x^2\,a^4+32\,a^5$$

EXAMPLE 3 *Expand $(2x-3y)^4$ by binomial theorem.*

SOLUTION Using binomial theorem, we obtain

$$(2x-3y)^4=\{2x+(-3y)\}^4$$
$$={}^4C_0\,(2x)^4\,(-3y)^0+{}^4C_1\,(2x)^3\,(-3y)+{}^4C_2\,(2x)^2\,(-3y)^2+{}^4C_3\,(2x)^1\,(-3y)^3+{}^4C_4\,(-3y)^4$$
$$=16\,x^4+4\,(8x^3)\,(-3y)+6\,(4x^2)\,(9y^2)+4\,(2x)\,(-27\,y^3)+81\,y^4$$
$$=16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$$

EXAMPLE 4 *By using binomial theorem, expand:*

(i) $(1+x+x^2)^3$ (ii) $(1-x+x^2)^4$

SOLUTION (i) Let $y=x+x^2$. Then, [NCERT EXEMPLAR]

$$(1+x+x^2)^3=(1+y)^3={}^3C_0+{}^3C_1\,y+{}^3C_2\,y^2+{}^3C_3\,y^3=1+3y+3y^2+y^3$$
$$=1+3\,(x+x^2)+3\,(x+x^2)^2+(x+x^2)^3$$
$$=1+3\,(x+x^2)+3\,(x^2+2x^3+x^4)+\left\{{}^3C_0\,x^3\,(x^2)^0+{}^3C_1\,x^{3-1}\,(x^2)^1\right.$$
$$\left.+{}^3C_2\,x^{3-2}\,(x^2)^2+{}^3C_3\,x^0\,(x^2)^3\right\}$$

$$= 1 + 3(x + x^2) + 3(x^2 + 2x^3 + x^4) + (x^3 + 3x^4 + 3x^5 + x^6)$$
$$= x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$$

(ii) Let $y = -x + x^2$. Then,

$$(1 - x + x^2)^4 = (1 + y)^4 = {}^4C_0 + {}^4C_1\, y + {}^4C_2\, y^2 + {}^4C_3\, y^3 + {}^4C_4\, y^4$$
$$= 1 + 4y + 6y^2 + 4y^3 + y^4 = 1 + 4(-x + x^2) + 6(-x + x^2)^2 + 4(-x + x^2)^3 + (-x + x^2)^4$$
$$= 1 - 4x(1 - x) + 6x^2(1 - x)^2 - 4x^3(1 - x)^3 + x^4(1 - x)^4$$
$$= 1 - 4x(1 - x) + 6x^2(1 - 2x + x^2) - 4x^3({}^3C_0 - {}^3C_1 x + {}^3C_2 x^2 - {}^3C_3 x^3)$$
$$+ x^4\left({}^4C_0 - {}^4C_1 x + {}^4C_2 x^2 - {}^4C_3 x^3 + {}^4C_4 x^4\right)$$
$$= 1 - 4x + 4x^2 + 6x^2(1 - 2x + x^2) - 4x^3(1 - 3x + 3x^2 - x^3) + x^4(1 - 4x + 6x^2 - 4x^3 + x^4)$$
$$= 1 - 4x + 4x^2 + 6x^2 - 12x^3 + 6x^4 - 4x^3 + 12x^4 - 12x^5 + 4x^6 + x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$$
$$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$$

EXAMPLE 5 *Using binomial theorem, expand* $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4$, $x \neq 0$.

SOLUTION We have, **[NCERT]**

$$\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4 = \left\{1 + \left(\frac{x}{2} - \frac{2}{x}\right)\right\}^4$$
$$= {}^4C_0 + {}^4C_1\left(\frac{x}{2} - \frac{2}{x}\right) + {}^4C_2\left(\frac{x}{2} - \frac{2}{x}\right)^2 + {}^4C_3\left(\frac{x}{2} - \frac{2}{x}\right)^3 + {}^4C_4\left(\frac{x}{2} - \frac{2}{x}\right)^4$$
$$= 1 + 4\left(\frac{x}{2} - \frac{2}{x}\right) + 6\left(\frac{x^2}{4} - 2 + \frac{4}{x^2}\right) + 4\left\{\frac{x^3}{8} - \frac{8}{x^3} - 3\left(\frac{x}{2} - \frac{2}{x}\right)\right\}$$
$$+ \left\{{}^4C_0\left(\frac{x}{2}\right)^4\left(-\frac{2}{x}\right)^0 + {}^4C_1\left(\frac{x}{2}\right)^3\left(-\frac{2}{x}\right) + {}^4C_2\left(\frac{x}{2}\right)^2\left(-\frac{2}{x}\right)^2\right.$$
$$\left. + {}^4C_3\left(\frac{x}{2}\right)\left(-\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^0\left(-\frac{2}{x}\right)^4\right\}$$
$$= 1 + \left(2x - \frac{8}{x}\right) + 6\left(\frac{x^2}{4} - 2 + \frac{4}{x^2}\right) + 4\left(\frac{x^3}{8} - \frac{8}{x^3} - \frac{3x}{2} - \frac{6}{x}\right)$$
$$+ \left(\frac{x^4}{16} + 4 \times \frac{x^3}{8} \times -\frac{2}{x} + 6 \times \frac{x^2}{4} \times \frac{4}{x^2} + 4 \times \frac{x}{2} \times -\frac{8}{x^3} + \frac{16}{x^4}\right)$$
$$= 1 + \left(2x - \frac{8}{x}\right) + \left(\frac{3}{2}x^2 - 12 + \frac{24}{x^2}\right) + \left(\frac{x^3}{2} - \frac{32}{x^3} - 6x + \frac{24}{x}\right) + \left(\frac{x^4}{16} - x^2 + 6 - \frac{16}{x^2} + \frac{16}{x^4}\right)$$

$$= (1-12+6)+(2x-6x)+\left(\frac{3}{2}x^2-x^2\right)+\frac{x^3}{2}+\frac{x^4}{16}+\left(\frac{-8}{x}+\frac{24}{x}\right)+\left(\frac{24}{x^2}-\frac{16}{x^2}\right)-\frac{32}{x^3}+\frac{16}{x^4}$$

$$= -5-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}+\frac{16}{x}+\frac{8}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}$$

EXAMPLE 6 *Find the expansion of* $(3x^2-2ax+3a^2)^3$ *using binomial theorem.* **[NCERT]**

SOLUTION We have,

$$(3x^2-2ax+3a^2)^3$$

$$= \left\{(3x^2-2ax)+3a^2\right\}^3$$

$$= {}^3C_0\,(3x^2-2ax)^3\,(3a^2)^0+{}^3C_1\,(3x^2-2ax)^2\,(3a^2)+{}^3C_2\,(3x^2-2ax)^1\,(3a^2)^2 + {}^3C_3\,(3x^2-2ax)^0\,(3a^2)^3$$

$$= (3x^2-2ax)^3+9a^2\,(3x^2-2ax)^2+27a^4\,(3x^2-2ax)+27a^6$$

$$= \left\{{}^3C_0\,(3x^2)^3(-2ax)^0+{}^3C_1(3x^2)^2\,(-2ax)^1+{}^3C_2\,(3x^2)\,(-2ax)^2+{}^3C_3\,(3x^2)^0\,(-2ax)^3\right\} + 9a^2\,(9x^4-12ax^3+4a^2x^2)+27a^4\,(3x^2-2ax)+27a^6$$

$$= (27x^6-54x^5a+36x^4a^2-8x^3a^3)+(81x^4a^2-108x^3a^3+36x^2a^4) + (81x^2a^4-54xa^5)+27a^6$$

$$= 27x^6-54x^5a+117x^5a^2-116x^3a^3+117x^2a^4-54xa^5+27a^6$$

EXAMPLE 7 *Using binomial theorem, expand* $\left(x+\frac{1}{y}\right)^{11}$.

SOLUTION We have,

$$\left(x+\frac{1}{y}\right)^{11} = {}^{11}C_0\,x^{11}\left(\frac{1}{y}\right)^0+{}^{11}C_1\,x^{10}\left(\frac{1}{y}\right)+{}^{11}C_2\,x^9\left(\frac{1}{y}\right)^2+{}^{11}C_3\,x^8\left(\frac{1}{y}\right)^3$$
$$+{}^{11}C_4\,x^7\left(\frac{1}{y}\right)^4+{}^{11}C_5\,x^6\left(\frac{1}{y}\right)^5+{}^{11}C_6\,x^5\left(\frac{1}{y}\right)^6+{}^{11}C_7\,x^4\left(\frac{1}{y}\right)^7+{}^{11}C_8\,x^3\left(\frac{1}{y}\right)^8$$
$$+{}^{11}C_9\,x^2\left(\frac{1}{y}\right)^9+{}^{11}C_{10}\,x\left(\frac{1}{y}\right)^{10}+{}^{11}C_{11}\left(\frac{1}{y}\right)^{11}$$
$$= x^{11}+11\frac{x^{10}}{y}+55\frac{x^9}{y^2}+165\frac{x^8}{y^3}+330\frac{x^7}{y^4}+462\frac{x^6}{y^5}+462\frac{x^5}{y^6}$$
$$+\frac{330x^4}{y^7}+\frac{165x^3}{y^8}+\frac{55x^2}{y^9}+\frac{11x}{y^{10}}+\frac{1}{y^{11}}$$

EXAMPLE 8 *Prove that* $\sum_{r=0}^{n} {}^nC_r\,3^r = 4^n$ **[NCERT]**

SOLUTION We have,

$$(1+x)^n = {}^nC_0+{}^nC_1\,x+{}^nC_2\,x^2+\ldots+{}^nC_n\,x^n$$

or, $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\;x^r$$

Putting $x = 3$ on both sides, we get

$$(1+3)^n = \sum_{r=0}^{n} {}^nC_r \ 3^r \text{ or, } 4^n = \sum_{r=0}^{n} {}^nC_r \ 3^r$$

Type III **ON APPLICATIONS OF BINOMIAL THEOREM**

EXAMPLE 9 *Find an approximation of* $(0.99)^5$ *using the first three terms of its expansion.* [NCERT]

SOLUTION We have,

$$(0.99)^5 = (1-0.01)^5 = \left(1-\frac{1}{100}\right)^5$$

$$= {}^5C_0 - {}^5C_1 \times \frac{1}{100} + {}^5C_2 \times \left(\frac{1}{100}\right)^2 - {}^5C_3\left(\frac{1}{100}\right)^3 + {}^5C_4\left(\frac{1}{100}\right)^4 - {}^5C_5\left(\frac{1}{100}\right)^5$$

$$= 1 - \frac{5}{100} + \frac{10}{10000} - \frac{10}{1000000} + \frac{5}{(100)^4} - \frac{1}{(100)^5}$$

$$= 1 - 0.05 + 0.001 \quad \text{[Neglecting fourth and other terms]}$$

$$= 0.951$$

EXAMPLE 10 *Using binomial theorem, compute the following:*

(i) $(99)^5$ (ii) $(102)^6$ (iii) $(10.1)^5$

SOLUTION (i) We have,

$$(99)^5 = (100-1)^5$$

$$= {}^5C_0 \times (100)^5 - {}^5C_1 \times (100)^4 + {}^5C_2 \times (100)^3 - {}^5C_3 \times (100)^2 + {}^5C_4 \times (100)^1 - {}^5C_5 \times (100)^0$$

$$= (100)^5 - 5 \times (100)^4 + 10 \times (100)^3 - 10 \times (100)^2 + 5 \times 100 - 1$$

$$= 10^{10} - 5 \times 10^8 + 10^7 - 10^5 + 5 \times 10^2 - 1$$

$$= (10^{10} + 10^7 + 5 \times 10^2) - (5 \times 10^8 + 10^5 + 1) = 10010000500 - 500100001 = 9509900499.$$

(ii) We have,

$$(102)^6 = (100+2)^6$$

$$= {}^6C_0 \times (100)^6 + {}^6C_1 \times (100)^5 \times 2 + {}^6C_2 \times (100)^4 \times 2^2$$
$$+ {}^6C_3 \times (100)^3 \times 2^3 + {}^6C_4 \times (100)^2 \times 2^4 + {}^6C_5 \times (100)^1 \times 2^5 + {}^6C_6 \times (100)^0 \times 2^6$$

$$= (100)^6 + 6 \times (100)^5 \times 2 + 15 \times (100)^4 \times 2^2 + 20 \times (100)^3 \times 2^3 + 15 \times (100)^2 \times 2^4$$
$$+ 6 \times (100)^1 \times 2^5 + 2^6$$

$$= 10^{12} + 12 \times 10^{10} + 6 \times 10^9 + 16 \times 10^7 + 24 \times 10^5 + 192 \times 10^2 + 64$$

$$= 1126162419264.$$

(iii) We have,

$$(10.1)^5 = (10+0.1)^5$$

$$= {}^5C_0 \times (10)^5 \times (0.1)^0 + {}^5C_1 \times (10)^4 \times (0.1) + {}^5C_2 \times (10)^3 \times (0.1)^2 + {}^5C_3 \times (10)^2 \times (0.1)^3$$
$$+ {}^5C_4 \times (10)^1 \times (0.1)^4 + {}^5C_5 \times (10)^0 \times (0.1)^5$$

$$= (10)^5 + 5 \times 10^4 \times 0.1 + 10 \times 10^3 \times (0.1)^2 + 10 \times (10)^2 \times (0.1)^3 + 5 \times 10 \times (0.1)^4 + (0.1)^5$$

$$= 10^5 + 5 \times 10^3 + 10^2 + 1 + 5 \times 0.001 + 0.00001$$

$$= 100000 + 5000 + 100 + 1 + 0.005 + 0.00001 = 105101.00501.$$

EXAMPLE 11 *Write down the binomial expansion of* $(1+x)^{n+1}$, *when* $x=8$. *Deduce that* $9^{n+1}-8n-9$ *is divisible by 64, where n is a positive integer.* [NCERT]

SOLUTION We have,

$$(1+x)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\, x + {}^{n+1}C_2\, x^2 + {}^{n+1}C_3\, x^3 + \ldots + {}^{n+1}C_{n+1}\, x^{n+1}$$

Putting $x=8$, we get

$$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\,(8)^1 + {}^{n+1}C_2(8)^2 + {}^{n+1}C_3(8)^3 + \ldots + {}^{n+1}C_{n+1}\,(8)^{n+1} \quad \ldots\text{(i)}$$

$$\Rightarrow \quad 9^{n+1} = 1 + (n+1)\times 8 + {}^{n+1}C_2\,(8)^2 + {}^{n+1}C_3\,(8)^3 + \ldots + {}^{n+1}C_{n+1}\,(8)^{n+1}$$

$$\Rightarrow \quad 9^{n+1} - 8n - 9 = (8)^2\left\{{}^{n+1}C_2 + {}^{n+1}C_3\,(8) + {}^{n+1}C_4\,(8)^2 + \ldots + {}^{n+1}C_{n+1}\,(8)^{n-1}\right\}$$

$$\Rightarrow \quad 9^{n+1} - 8n - 9 = 64 \times \text{an integer}$$

$$\Rightarrow \quad 9^{n+1} - 8n - 9 \text{ is divisible by 64.}$$

EXAMPLE 12 *Using binomial theorem, prove that* $6^n - 5n$ *always leaves the remainder 1 when divided by 25.* [NCERT]

SOLUTION We have,

$$6^n - 5n = (1+5)^n - 5n$$

$$\Rightarrow \quad 6^n - 5n = \left\{{}^nC_0 + {}^nC_1 \times (5) + {}^nC_2 \times (5)^2 + {}^nC_3 \times (5)^3 + \ldots + {}^nC_n \times (5)^n\right\} - 5n$$

$$\Rightarrow \quad 6^n - 5n = 1 + 5n + {}^nC_2 \times 5^2 + {}^nC_3 \times 5^3 + \ldots + {}^nC_n \times 5^n - 5n$$

$$\Rightarrow \quad 6^n - 5n - 1 = {}^nC_2 \times 5^2 + {}^nC_3 \times 5^3 + \ldots + {}^nC_n \times 5^n$$

$$\Rightarrow \quad 6^n - 5n - 1 = 5^2\left\{{}^nC_2 + {}^nC_3 \times 5 + {}^nC_4 \times 5^2 + \ldots + {}^nC_n \times 5^{n-2}\right\}$$

$$\Rightarrow \quad 6^n - 5n - 1 = 25 \times \text{an integer}$$

$$\Rightarrow \quad 6^n - 5n = 25 \times \text{an integer} + 1$$

$$\Rightarrow \quad 6^n - 5n \text{ leaves the remainder 1 when divided by 25.}$$

LEVEL-2

Type IV ON EXPANSION OF A BINOMIAL BY USING BINOMIAL THEOREM

EXAMPLE 13 *Using binomial theorem, expand* $\left\{(x+y)^5 + (x-y)^5\right\}$ *and hence find the value of* $\left\{(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5\right\}$.

SOLUTION We have,

$$(x+y)^5 + (x-y)^5 = 2\left\{{}^5C_0\, x^5 + {}^5C_2\, x^3 y^2 + {}^5C_4\, x^1 y^4\right\} = 2\left(x^5 + 10x^3y^2 + 5xy^4\right)$$

Putting $x=\sqrt{2}$ and $y=1$, we get

$$(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5 = 2\left\{(\sqrt{2})^5 + 10\,(\sqrt{2})^3 + 5\sqrt{2}\right\} = 2\left(4\sqrt{2} + 20\sqrt{2} + 5\sqrt{2}\right) = 58\sqrt{2}$$

EXAMPLE 14 *If O be the sum of odd terms and E that of even terms in the expansion of* $(x+a)^n$, *prove that:*

(i) $O^2 - E^2 = (x^2 - a^2)^n$ (ii) $4\,OE = (x+a)^{2n} - (x-a)^{2n}$

(iii) $2\,(O^2 + E^2) = (x+a)^{2n} + (x-a)^{2n}$

SOLUTION We have,

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots + {}^nC_{n-1}\, x\, a^{n-1} + {}^nC_n\, a^n$$

$$\Rightarrow \quad (x+a)^n = \left\{ {}^nC_0\, x^n a^0 + {}^nC_2\, x^{n-2} a^2 + \ldots \right\} + \left\{ {}^nC_1\, x^{n-1} a^1 + {}^nC_3\, x^{n-3} a^3 + \ldots \right\}$$

$$\Rightarrow \quad (x+a)^n = O + E \qquad \ldots(i)$$

and, $$(x-a)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 - {}^nC_3\, x^{n-3} a^3 + \ldots + {}^nC_{n-1}\, x\, (-1)^{n-1} a^{n-1} + {}^nC_n\, (-1)^n a^n$$

$$\Rightarrow \quad (x-a)^n = \left\{ {}^nC_0\, x^n + {}^nC_2\, x^{n-2} a^2 + \ldots \right\} - \left\{ {}^nC_1\, x^{n-1} a^1 + {}^nC_3\, x^{n-3} a^3 + \ldots \right\}$$

$$\Rightarrow \quad (x-a)^n = O - E \qquad \ldots(ii)$$

(i) Multiplying (i) and (ii), we get

$$(x+a)^n (x-a)^n = (O+E)(O-E)$$

$$\Rightarrow \quad (x^2-a^2)^n = O^2 - E^2$$

(ii) We have,

$$4\,OE = (O+E)^2 - (O-E)^2$$

$$\Rightarrow \quad 4\,OE = \left\{(x+a)^n\right\}^2 - \left\{(x-a)^n\right\}^2 \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad 4\,OE = (x+a)^{2n} - (x-a)^{2n}$$

(iii) Squaring (i) and (ii) and then adding, we get

$$(x+a)^{2n} + (x-a)^{2n} = (O+E)^2 + (O-E)^2 = 2(O^2+E^2).$$

Type V **ON APPLICATIONS OF BINOMIAL THEOREM**

EXAMPLE 15 *Which is larger* $(1.01)^{1000000}$ *or, 10,000?* **[NCERT]**

SOLUTION We have,

$(1.01)^{1000000} - 10000$

$= (1+0.01)^{1000000} - 10000$

$= {}^{1000000}C_0 + {}^{1000000}C_1\,(0.01) + {}^{1000000}C_2\,(0.01)^2 + \ldots + {}^{1000000}C_{1000000} \times (0.01)^{1000000} - 10000$

$= (1 + 1000000 \times 0.01 + \text{other positive terms}) - 10000$

$= (1 + 10000 + \text{other positive terms}) - 10000$

$= 1 + \text{other positive terms} > 0$

$\therefore \quad (1.01)^{1000000} > 10000$

EXAMPLE 16 *If a and b are distinct integers, prove that* $a^n - b^n$ *is divisible by* $(a-b)$, *whenever* $n \in N$. **[NCERT]**

SOLUTION We have,

$$a^n = \{(a-b)+b\}^n$$

$$\Rightarrow a^n = {}^nC_0\,(a-b)^n + {}^nC_1\,(a-b)^{n-1} b^1 + {}^nC_2\,(a-b)^{n-2} b^2 + \ldots + {}^nC_{n-1}\,(a-b)\, b^{n-1} + {}^nC_n\, b^n$$

$$\Rightarrow a^n - b^n = (a-b)^n + {}^nC_1\,(a-b)^{n-1} b^1 + {}^nC_2\,(a-b)^{n-2} b^2 + \ldots + {}^nC_{n-1}\,(a-b)\, b^{n-1}$$

$$\Rightarrow a^n - b^n = (a-b)\left\{(a-b)^{n-1} + {}^nC_1 (a-b)^{n-2} b + {}^nC_2 (a-b)^{n-3} b^2 + \dots + {}^nC_{n-1} b^{n-1}\right\}$$

Clearly, RHS is divisible by $(a-b)$. Hence, $a^n - b^n$ is divisible by $(a-b)$.

EXAMPLE 17 *Using binomial theorem, prove that* $(101)^{50} > 100^{50} + 99^{50}$. [NCERT EXEMPLAR]

SOLUTION Let $x = 101^{50}$ and $y = 100^{50} + 99^{50}$. Then,

$$x - y = 101^{50} - 100^{50} - 99^{50}$$

$$\Rightarrow \quad x - y = 101^{50} - 99^{50} - 100^{50}$$

$$\Rightarrow \quad x - y = (100+1)^{50} - (100-1)^{50} - 100^{50}$$

$$\Rightarrow \quad x - y = 2\left\{{}^{50}C_1 \times 100^{49} + {}^{50}C_3 \times 100^{47} + \dots + {}^{50}C_{49} \times 100\right\} - 100^{50}$$

$$\Rightarrow \quad x - y = 100^{50} + 2 \times {}^{50}C_3 \times 100^{47} + \dots + 2 \times {}^{50}C_{49} \times 100 - 100^{50}$$

$$\Rightarrow \quad x - y = 2 \times {}^{50}C_3 \times 100^{47} + \dots + 2 \times {}^{50}C_{49} \times 100$$

$$\Rightarrow \quad x - y = a \text{ positive integer}$$

$$\Rightarrow \quad x - y > 0 \Rightarrow x > y \Rightarrow 101^{50} > 100^{50} + 99^{50}$$

EXERCISE 18.1

LEVEL-1

1. Using binomial theorem, write down the expansions of the following:

(i) $(2x + 3y)^5$ (ii) $(2x - 3y)^4$ (iii) $\left(x - \frac{1}{x}\right)^6$

(iv) $(1 - 3x)^7$ (v) $\left(ax - \frac{b}{x}\right)^6$ (vi) $\left(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}}\right)^6$

(vii) $(\sqrt[3]{x} - \sqrt[3]{a})^6$ (viii) $(1 + 2x - 3x^2)^5$ (ix) $\left(x + 1 - \frac{1}{x}\right)^3$

(x) $(1 - 2x + 3x^2)^3$

2. Evaluate the following:

(i) $\left(\sqrt{x+1} + \sqrt{x-1}\right)^6 + \left(\sqrt{x+1} - \sqrt{x-1}\right)^6$ (ii) $\left(x + \sqrt{x^2-1}\right)^6 + \left(x - \sqrt{x^2-1}\right)^6$

(iii) $(1 + 2\sqrt{x})^5 + (1 - 2\sqrt{x})^5$. (iv) $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$

(v) $(3 + \sqrt{2})^5 - (3 - \sqrt{2})^5$ (vi) $(2 + \sqrt{3})^7 + (2 - \sqrt{3})^7$

(vii) $(\sqrt{3} + 1)^5 - (\sqrt{3} - 1)^5$ (viii) $(0.99)^5 + (1.01)^5$

(ix) $(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$ [NCERT]

(x) $\left\{a^2 + \sqrt{a^2-1}\right\}^4 + \left\{a^2 - \sqrt{a^2-1}\right\}^4$ [NCERT, NCERT EXEMPLAR]

3. Find $(a+b)^4 - (a-b)^4$. Hence, evaluate $\left(\sqrt{3} + \sqrt{2}\right)^4 - \left(\sqrt{3} - \sqrt{2}\right)^4$. [NCERT]

4. Find $(x+1)^6 + (x-1)^6$. Hence, or otherwise evaluate $(\sqrt{2}+1)^6 + \sqrt{2}-1)^6$. [NCERT]

5. Using binomial theorem evaluate each of the following:
   (i) $(96)^3$ [NCERT] (ii) $(102)^5$ [NCERT] (iii) $(101)^4$ [NCERT] (iv) $(98)^5$ [NCERT]
6. Using binomial theorem, prove that $2^{3n} - 7n - 1$ is divisible by 49, where $n \in N$.
7. Using binomial theorem, prove that $3^{2n+2} - 8n - 9$ is divisible by 64, $n \in N$.
8. If n is a positive integer, prove that $3^{3n} - 26n - 1$ is divisible by 676.

LEVEL-2

9. Using binomial theorem, indicate which is larger $(1.1)^{10000}$ or 1000? [NCERT]
10. Using binomial theorem determine which number is larger $(1.2)^{4000}$ or 800?
11. Find the value of $(1.01)^{10} + (1 - 0.01)^{10}$ correct to 7 places of decimal.
12. Show that $2^{4n+4} - 15n - 16$, where $n \in N$ is divisible by 225. [NCERT EXEMPLAR]

ANSWERS

1. (i) $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$
   (ii) $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$
   (iii) $x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$
   (iv) $1 - 21x + 189x^2 - 945x^3 + 2835x^4 - 5103x^5 + 5103x^6 - 2187x^7$
   (v) $a^6x^6 - 6a^5x^4b + 15a^4x^2b^2 - 20a^3b^3 + 15\frac{a^2b^4}{x^2} - \frac{6ab^5}{x^4} + \frac{b^6}{x^6}$
   (vi) $\frac{x^3}{a^3} - 6\frac{x^2}{a^2} + 15\frac{x}{a} - 20 + 15\frac{a}{x} - 6\frac{a^2}{x^2} + \frac{a^3}{x^3}$
   (vii) $x^2 - 6x^{5/3}a^{1/3} + 15x^{4/3}a^{2/3} - 20ax + 15x^{2/3}a^{4/3} - 6x^{1/3}a^{5/3} + a^2$
   (viii) $1 + 10x + 25x^2 - 40x^3 - 190x^4 + 92x^5 + 570x^6 - 360x^7 - 675x^8 + 810x^9 - 243x^{10}$
   (ix) $x^3 + 3x^2 - 5 + \frac{3}{x^2} - \frac{1}{x^3}$
   (x) $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$
2. (i) $16x(4x^2 - 3)$ (ii) $64x^6 - 96x^4 + 36x^2 - 2$ (iii) $2(1 + 40x + 80x^2)$
   (iv) 198 (v) $1178\sqrt{2}$ (vi) 10084 (vii) 152
   (viii) 2.0020001 (ix) $396\sqrt{6}$ (x) $2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$
3. $8(a^3b + ab^3)$, $40\sqrt{6}$ 4. $2(x^6 + 15x^4 + 15x^2 + 1)$, 198
5. (i) 884736 (ii) 11040808032 (iii) 104060401 (iv) 9039207968
9. $(1.1)^{10000} > 1000$ 10. 800 11. 2.0090042

HINTS TO NCERT & SELECTED PROBLEMS

2. (ix) We know that $(x+a)^n - (x-a)^n = 2\left\{{}^nC_1 x^{n-1}a^1 + {}^nC_3 x^{n-3}a^3 + ...\right\}$

$$\therefore\ (\sqrt{3}+\sqrt{2})^6-(\sqrt{3}-\sqrt{2})^6 = 2\left\{{}^6C_1(\sqrt{3})^5(\sqrt{2})^1+{}^6C_3(\sqrt{3})^3(\sqrt{2})^3+{}^6C_5(\sqrt{3})^1(\sqrt{2})^5\right\}$$

$$= 2(6\times 9\times\sqrt{6}+20\times 6\times\sqrt{6}+6\times 4\times\sqrt{6})$$

$$= 2\left(54\sqrt{6}+120\sqrt{6}+24\sqrt{6}\right) = 2\times 198\sqrt{6} = 396\sqrt{6}$$

(x) Using $(x+a)^n+(x-a)^n = 2\left\{{}^nC_0x^na^0+{}^nC_2x^{n-2}a^2+\ldots\right\}$, we get

$$\left\{a^2+\sqrt{a^2-1}\right\}^4+\left\{a^2-\sqrt{a^2-1}\right\}^4$$

$$= 2\left\{{}^4C_0(a^2)^0\left(\sqrt{a^2-1}\right)^4+{}^4C_2(a^2)^2\left(\sqrt{a^2-1}\right)^2+{}^4C_4(a^2)^4\left(\sqrt{a^2-1}\right)^0\right\}$$

$$= 2\left\{(a^2-1)^2+6a^4(a^2-1)+a^8\right\}$$

$$= 2(a^8+6a^6-5a^4-2a^2+1) = 2a^8+12a^6-10a^4-4a^2+2$$

3. Using $(x+a)^n-(x-a)^n = 2\left\{{}^nC_1x^{n-1}a^1+{}^nC_3x^{x-3}a^3+\ldots\right\}$, we get

$$(a+b)^4-(a-b)^4 = 2\left\{{}^4C_1a^3b^1+{}^4C_3a^1b^2\right\} = 2(4a^3b+4ab^3) = 8ab(a^2+b^2)$$

$$\therefore\quad (\sqrt{3}+\sqrt{2})^4-(\sqrt{3}-\sqrt{2})^4 = 8\sqrt{3}\times\sqrt{2}\left\{(\sqrt{3})^2+(\sqrt{2})^2\right\} = 40\sqrt{6}$$

4. Using $(x+a)^n+(x-a)^n = 2\left({}^nC_0x^na^0+{}^nC_2x^{x-2}a^2+\ldots\right)$ we get

$$(x+1)^6+(x-1)^6 = 2\left({}^6C_0x^6+{}^6C_2x^4+{}^6C_4x^2+{}^6C_6x^0\right) = 2(x^6+15x^4+15x^2+1)$$

Putting $x=\sqrt{2}$, we get

$$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6 = 2\left\{(\sqrt{2})^6+15(\sqrt{2})^4+15(\sqrt{2})^2+1\right\} = 2(8+60+30+1) = 198$$

5. (i) $96^3 = (100-4)^3$

$$= {}^3C_0(100)^3(4)^0-{}^3C_1(100)^2(4)^1+{}^3C_2(100)^1(4)^2-{}^3C_3(100)^0(4)^3$$

$$= 10^6-12\times 10^4+4800-64 = 1000000-120000+4800-64 = 884736$$

(ii) $(102)^5 = (100+2)^5$

$$= {}^5C_0(100)^5\,2^0+{}^5C_1(100)^4\times 2+{}^5C_2\times(100)^3\times 2^2+{}^5C_3\times(100)^2\times 2^3$$
$$+\,{}^5C_4\times(100)^1\times 2^4+{}^5C_5\times(100)^0\times 2^5$$

$$= 10^{10}+10^9+40\times 10^6+80\times 10^4+80\times 10^2+32 = 11040808032$$

(iii) $(101)^4 = (10^2+1)^4$

$$= {}^4C_0(10^2)^0+{}^4C_1(10^2)^1+{}^4C_2(10^2)^2+{}^4C_3(10^2)^3+{}^4C_4(10^2)^4$$

$$= 1+400+6\times 10^4+4\times 10^6+10^8 = 104060401$$

(ix) $(98)^5 = (100-2)^5$

$$= {}^5C_0\,(100)^5 - {}^5C_1(100)^4 \times 2 + {}^5C_2 \times (100)^3 \times 2 - {}^5C_3 \times (100)^2 \times 2^3 + {}^5C_4 \times (100)^1 \times 2^4 - {}^5C_5 \times (100)^0 \times 2^5$$

$$= 10^{10} - 10^9 + 40 \times 10^6 + 8000 - 32 = 1039207968$$

9. Using $(x+a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1} a^1 + {}^nC_3 x^{n-3} a^3 + \ldots + {}^nC_n\, x^0\, a^n$, we get

$$(1.1)^{10000} = \left(1 + \frac{1}{10}\right)^{10000}$$

$$= {}^{10000}C_0 + {}^{10000}C_1 \times \frac{1}{10} + {}^{10000}C_2 \times \left(\frac{1}{10}\right)^2 + \ldots + {}^{10000}C_{10000}\left(\frac{1}{10}\right)^{10000}$$

$$= 1 + 1000 + {}^{10000}C_2 \times \left(\frac{1}{10}\right)^2 + \ldots + {}^{10000}C_{10000}\left(\frac{1}{10}\right)^{10000}$$

$$\therefore \quad (1.1)^{10000} - 1000 = 1 + {}^{10000}C_2 \times \left(\frac{1}{10}\right)^2 + \ldots + {}^{10000}C_{10000}\left(\frac{1}{10}\right)^{10000}$$

$$\Rightarrow \quad (1.1)^{10000} - 1000 > 0$$

$$\Rightarrow \quad (1.1)^{10000} > 1000$$

12. $2^{4n+4} - 15n - 16 = 2^{4(n+1)} - 15n - 15 - 1$

$$= (2^4)^{n+1} - 15\,(n+1) - 1$$

$$= 16^{n+1} - 15\,(n+1) - 1$$

$$= (1+15)^{n+1} - 15\,(n+1) - 1$$

$$= \left\{{}^{n+1}C_0 + {}^{n+1}C_1(15) + {}^{n+1}C_2\,(15)^2 + {}^{n+1}C_3\,(15)^3 + \ldots\ldots + {}^{n+1}C_{n+1}\,(15)^{n+1}\right\} - 15\,(n+1) - 1$$

$$= \left\{1 + 15\,(n+1) + {}^{n+1}C_2\,(15)^2 + {}^{n+1}C_3\,(15)^3 + \ldots + {}^{n+1}C_{n+1}\,(15)^{n+1}\right\} - 15\,(n+1) - 1$$

$$= 225\left\{{}^{n+1}C_2 + {}^{n+1}C_3\,(15) + \ldots + {}^{n+1}C_{n+1}\,(15)^{n-1}\right\}$$

$$= 225 \times \text{A natural number.}$$

Hence, $2^{4n+4} - 15n - 16$ is divisible by 225.

18.4 GENERAL TERM AND MIDDLE TERMS IN A BINOMIAL EXPANSION

We have,

$$(x+a)^n = {}^nC_0\, x^n\, a^0 + {}^nC_1\, x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + {}^nC_n\, x^0\, a^n$$

We find that: The first term $= {}^nC_0\, x^n\, a^0$

The second term $= {}^nC_1\, x^{n-1}\, a^1$

$$\text{The third term} = {}^nC_2\, x^{n-2}\, a^2$$

$$\text{The fourth term} = {}^nC_3\, x^{n-3}\, a^3, \text{ and so on.}$$

We thus observe that the suffix of C in any term is one less than the number of terms, the index of x is n minus the suffix of C and the index of a is the same as the suffix of C.

Hence, the $(r+1)$th term is given by ${}^nC_r\, x^{n-r}\, a^r$. Thus, if T_{r+1} denotes the $(r+1)^{th}$ term, then

$$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$$

This is called the *general term*, because by giving different values to r we can determine all terms of the expansion.

Since, $(x-a)^n = \{x+(-a)\}^n$. So, the general term in the binomial expansion of $(x-a)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}(-a)^r = (-1)^r\, {}^nC_r\, x^{n-r} a^r$$

In the binomial expansion of $(1+x)^n$, the general term is given by

$$T_{r+1} = {}^nC_r\, x^r$$

In the binomial expansion of $(1-x)^n$, the general term is given by

$$T_{r+1} = (-1)^r\, {}^nC_r\, x^r$$

NOTE: *In the binomial expansion of $(x+a)^n$, the r^{th} term from the end is $((n+1)-r+1) = (n-r+2)th$ term from the beginning.*

18.4.1 MIDDLE TERMS IN A BINOMIAL EXPANSION

The binomial expansion of $(x+a)^n$ contains $(n+1)$ terms. Therefore,

(i) If n is even, then $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term.

(ii) If n is odd, then $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms are the two middle terms.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE GENERAL TERM OR AN INDICATED TERM IN THE BINOMIAL EXPANSION OF SOME GIVEN EXPRESSION**

EXAMPLE 1 *Write the general term in the expansion of $(x^2-y)^6$.* [NCERT]

SOLUTION We have, $(x^2-y)^6 = \left\{x^2+(-y)\right\}^6$

The general term in the expansion of the above binomial is given by

$$T_{r+1} = {}^6C_r\,(x^2)^{6-r}\,(-y)^r \qquad [\because T_{r+1} = {}^nC_r\, x^{n-r}\, a^r]$$

$$\Rightarrow \quad T_{r+1} = (-1)^r\, {}^6C_r\, x^{12-2r}\, y^r$$

EXAMPLE 2 *Find the 10th term in the binomial expansion of $\left(2x^2+\frac{1}{x}\right)^{12}$.*

SOLUTION We know that the $(r+1)$th term in the expansion of $(x+a)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$$

Therefore, in the expansion of $\left(2x^2 + \frac{1}{x}\right)^{12}$, the tenth term T_{10} is given by

$$T_{10} = T_{9+1} = {}^{12}C_9\,(2x^2)^{12-9}\left(\frac{1}{x}\right)^9 \qquad \left[\text{Here } n = 12, r = 9, x = 2x^2 \text{ and } a = \frac{1}{x}\right]$$

$$\Rightarrow \quad T_{10} = {}^{12}C_9\,(2x^2)^3 \times \frac{1}{x^9} = {}^{12}C_9 \times 2^3\left(\frac{1}{x^3}\right)$$

$$\Rightarrow \quad T_{10} = {}^{12}C_3\,\frac{8}{x^3} = \frac{12\times 11\times 10}{3\times 2\times 1}\times\frac{8}{x^3} = \frac{1760}{x^3} \qquad \left[\because\ {}^{12}C_9 = {}^{12}C_3\right]$$

EXAMPLE 3 *Find the 9th term in the expansion of* $\left(\frac{x}{a} - \frac{3a}{x^2}\right)^{12}$.

SOLUTION We know that the $(r+1)$th term in the expansion of $(x+a)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$$

Therefore, in the expansion of $\left(\frac{x}{a} - \frac{3a}{x^2}\right)^{12}$, the 9th term T_9 is given by

$$\Rightarrow \quad T_9 = T_{8+1} = {}^{12}C_8\left(\frac{x}{a}\right)^{12-8}\left(-\frac{3a}{x^2}\right)^8 = {}^{12}C_8\left(\frac{x}{a}\right)^4\left(-\frac{3a}{x^2}\right)^8 = {}^{12}C_4 \times 3^8 \times \frac{a^4}{x^{12}}$$

$$\Rightarrow \quad T_9 = ({}^{12}C_4\, x^{-12}\, a^4)\, 3^8$$

EXAMPLE 4 *Find the 6th term in the expansion of* $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$.

SOLUTION Clearly, $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9 = \left\{\frac{4x}{5} + \left(-\frac{5}{2x}\right)\right\}^9$

$$\therefore \quad T_6 = T_{5+1} = {}^9C_5\left(\frac{4x}{5}\right)^{9-5}\left(-\frac{5}{2x}\right)^5 \qquad [\because\ T_{r+1} = {}^nC_r\, x^{n-r}\, a^r]$$

$$\Rightarrow \quad T_6 = {}^9C_5\left(\frac{4x}{5}\right)^4(-1)^5\left(\frac{5}{2x}\right)^5 = -{}^9C_4\left(\frac{4x}{5}\right)^4\left(\frac{5}{2x}\right)^5 \qquad [\because\ {}^9C_5 = {}^9C_4]$$

$$\Rightarrow \quad T_6 = -\frac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1}\left(\frac{2^8 x^4}{5^4}\right)\left(\frac{5^5}{2^5 x^5}\right) = -\frac{5040}{x}$$

EXAMPLE 5 *Find 13th term in the expansion of* $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$. [NCERT]

SOLUTION Clearly,

$$\left(9x - \frac{1}{3\sqrt{x}}\right)^{18} = \left\{9x + \left(\frac{-1}{3\sqrt{x}}\right)\right\}^{18}$$

$$\therefore \quad T_{13} = T_{12+1} = {}^{18}C_{12}\,(9x)^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12} = {}^{18}C_{12}\,(9x)^6\left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$\Rightarrow \quad T_{13} = {}^{18}C_6 \times 9^6 \times x^6 \times \frac{1}{3^{12}\,x^6} = {}^{18}C_6 = \frac{18!}{12!\,6!} = 18564$$

EXAMPLE 6 *Find the 4th term from the end in the expansion of* $\left(\frac{3}{x^2} - \frac{x^3}{6}\right)^7$.

SOLUTION Clearly, the given expansion contains 8 terms.
So, 4th term from the end $= (8-4+1)$th $=$ 5th term from the beginning

$$\therefore \quad \text{Required term} = T_5 = T_{4+1} = {}^7C_4\left(\frac{3}{x^2}\right)^{7-4}\left(-\frac{x^3}{6}\right)^4$$

$$= {}^7C_3\left(\frac{3}{x^2}\right)^3\left(\frac{x^3}{6}\right)^4 = \frac{7\times6\times5}{3\times2\times1}\left(\frac{3^3}{x^6}\right)\left(\frac{x^{12}}{6^4}\right) = \frac{35}{48}x^6 \quad \left[\because\ {}^7C_4 = {}^7C_3\right]$$

EXAMPLE 7 *Find the 11th term from the end in the expansion of* $\left(2x - \frac{1}{x^2}\right)^{25}$.

SOLUTION Clearly, the given expansion contains 26 terms.
So, 11th term from the end = (26 − 11 + 1)th term from the beginning i.e. 16th term from the beginning

$$\therefore \quad \text{Required term} = T_{16} = T_{15+1} = {}^{25}C_{15}\,(2x)^{25-15}\left(-\frac{1}{x^2}\right)^{15}$$

$$= {}^{25}C_{15} \times 2^{10} \times x^{10} \times \frac{(-1)^{15}}{x^{30}} = -\,{}^{25}C_{15} \times \frac{2^{10}}{x^{20}}$$

EXAMPLE 8 *Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of* $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ *is* $\sqrt{6}:1$. [NCERT]

SOLUTION Clearly,

$$\text{Fifth term from the end} = (n+1-5+1)^{\text{th}} \text{ term from the beginning}$$
$$= (n-3)^{\text{th}} \text{ term from the beginning}$$

$$\text{Now,} \quad T_5 = T_{4+1} = {}^nC_4\left\{\sqrt[4]{2}\right\}^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4 = {}^nC_4 \times 2^{\frac{n-4}{4}} \times \frac{1}{3}$$

$$\text{and,} \quad T_{n-3} = T_{(n-4)+1} = {}^nC_{n-4}\left\{\sqrt[4]{2}\right\}^{n-(n-4)}\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^nC_{n-4} \times 2 \times \frac{1}{3^{\frac{n-4}{4}}}$$

It is given that

$$\frac{T_5}{T_{n-3}} = \frac{\sqrt{6}}{1}$$

$$\Rightarrow \quad \frac{{}^nC_4 \times 2^{\frac{n-4}{4}} \times \frac{1}{3}}{{}^nC_{n-4} \times 2 \times 3^{-\frac{n-4}{4}}} = \frac{\sqrt{6}}{1}$$

$$\Rightarrow \quad 2^{\frac{n-4}{4}-1} \times 3^{\frac{n-4}{4}-1} = 6^{1/2} \qquad \left[\because {}^nC_4 = {}^nC_{n-4}\right]$$

$$\Rightarrow \quad 2^{\frac{n-8}{4}} \times 3^{\frac{n-8}{4}} = 6^{1/2}$$

$$\Rightarrow \quad (2 \times 3)^{\frac{n-8}{4}} = 6^{1/2} \Rightarrow 6^{\frac{n-8}{4}} = 6^{1/2} \Rightarrow \frac{n-8}{4} = \frac{1}{2} \Rightarrow n-8 = 2 \Rightarrow n = 10$$

EXAMPLE 9 *Find a, if 17th and 18th terms in the expansion of $(2+a)^{50}$ are equal.* [NCERT]

SOLUTION We have,

$$T_{17} = T_{16+1} = {}^{50}C_{16}\,(2)^{50-16}\,a^{16} = {}^{50}C_{16} \times 2^{34} \times a^{16}$$

and, $$T_{18} = T_{17+1} = {}^{50}C_{17}\,(2)^{50-17}\,a^{17} = {}^{50}C_{17} \times 2^{33} \times a^{17}$$

It is given that 17^{th} and 18^{th} terms are equal.

i.e. $T_{17} = T_{18}$

$$\Rightarrow \quad {}^{50}C_{16} \times 2^{34} \times a^{16} = {}^{50}C_{17} \times 2^{33} \times a^{17}$$

$$\Rightarrow \quad \frac{{}^{50}C_{16}}{{}^{50}C_{17}} \times 2 = \frac{a^{17}}{a^{16}} \Rightarrow a = \frac{50!}{34!\,16!} \times \frac{33!\,17!}{50!} \times 2 = \frac{17}{34} \times 2 = 1$$

Type II **ON FINDING THE MIDDLE TERM(S)**

EXAMPLE 10 *Find the middle term in the expansion of* $\left(\frac{2}{3}x^2 - \frac{3}{2x}\right)^{20}$.

SOLUTION Here $n = 20$, which is an even number. So, $\left(\frac{20}{2}+1\right)^{th}$ term i.e. 11th term is the middle term.

Hence, the middle term $= T_{11} = T_{10+1} = {}^{20}C_{10}\left(\frac{2}{3}x^2\right)^{20-10}\left(-\frac{3}{2x}\right)^{10} = {}^{20}C_{10}\,x^{10}$

EXAMPLE 11 *Find the middle terms in the expansion of* $\left(3x - \frac{x^3}{6}\right)^7$.

SOLUTION The given expression is $\left(3x - \frac{x^3}{6}\right)^7$. Here $n = 7$, which is an odd number.

So, $\left(\frac{7+1}{2}\right)^{th}$ and $\left(\frac{7+1}{2}+1\right)^{th}$ i.e. 4th and 5th terms are two middle terms.

Now, $T_4 = T_{3+1} = {}^7C_3\,(3x)^{7-3}\left(-\frac{x^3}{6}\right)^3 = (-1)^3\;{}^7C_3\,(3x)^4\left(\frac{x^3}{6}\right)^3 = -\frac{105x^{13}}{8}$

and, $T_5 = T_{4+1} = {}^7C_4\,(3x)^{7-4}\left(-\frac{x^3}{6}\right)^4 = {}^7C_4\,(3x)^3\left(-\frac{x^3}{6}\right)^4 = \frac{35\,x^{15}}{48}$

Hence, the middle terms are $-\frac{105\,x^{13}}{8}$ and $\frac{35\,x^{15}}{48}$

EXAMPLE 12 *Show that the middle term in the expansion of* $(1+x)^{2n}$ *is*

$$\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\,2^n\cdot x^n$$ [NCERT]

SOLUTION The exponent of $(1+x)$ in $(1+x)^{2n}$ is an even number $2n$.

So, $\left(\frac{2n}{2}+1\right)^{th}$ i.e. $(n+1)^{th}$ term is the middle term in the binomial expansion of $(1+x)^{2n}$.

Now, $T_{n+1} = {}^{2n}C_n\,(1)^{2n-n}\,x^n = {}^{2n}C_n\,x^n$

$$= \frac{(2n)!}{(2n-n)!\,n!}\,x^n$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\ldots(2n-3)(2n-2)(2n-1)(2n)}{n!\,n!}\,x^n$$

$$= \frac{\{1\cdot 3\cdot 5\ldots(2n-3)\ (2n-1)\}\ \{2\cdot 4\cdot 6\ldots\ (2n-2)(2n)\}}{n!\,n!}\,x^n$$

$$= \frac{\{1\cdot 3\cdot 5\ldots(2n-3)\ (2n-1)\}\ \{1\cdot 2\cdot 3\ldots(n-1)\ (n)\}2^n}{n!\,n!}\,x^n$$

$$= \frac{\{1\cdot 3\cdot 5\ldots(2n-3)\ (2n-1)\}\ n!\cdot 2^n\cdot x^n}{n!\,n!} = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\,2^n\,x^n$$

EXAMPLE 13 *Show that the middle term in the expansion of* $\left(x-\frac{1}{x}\right)^{2n}$ *is*

$$\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\,(-2)^n$$ [NCERT EXEMPLAR]

SOLUTION The exponent in $\left(x-\frac{1}{x}\right)^{2n}$ is an even natural number. So, $\left(\frac{2n}{2}+1\right)^{th}$ i.e. $(n+1)^{th}$ term is the middle term and is given by

$$T_{n+1} = {}^{2n}C_n\,(x)^{2n-n}\left(-\frac{1}{x}\right)^n$$

$$\Rightarrow \quad T_{n+1} = \frac{(2n)!}{n!\,n!}\,x^n \times \frac{(-1)^n}{x^n}$$

$$\Rightarrow \quad T_{n+1} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\ldots(2n-1)(2n)}{n!\,n!} \times (-1)^n$$

$$\Rightarrow \quad T_{n+1} = \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\ \}\{2\cdot 4\cdot 6\ldots\ (2n-2)(2n)\}}{n!\,n!} \times (-1)^n$$

$$\Rightarrow \quad T_{n+1} = \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}\{1\cdot 2\cdot 3\ldots(n-1)\, n\}}{n!\,n!} \times (-1)^n$$

$$\Rightarrow \quad T_{n+1} = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!} \times 2^n \times (-1)^n = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!} \times (-2)^n$$

EXAMPLE 14 *Prove that the coefficient of the middle term in the expansion of* $(1+x)^{2n}$ *is equal to the sum of the coefficients of middle terms in the expansion of* $(1+x)^{2n-1}$. [NCERT]

SOLUTION As discussed in the previous example, the middle term in the expansion of $(1+x)^{2n}$ is given by $T_{n+1} = {}^{2n}C_n\, x^n$.

So, the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is ${}^{2n}C_n$.

Now, consider the expansion of $(1+x)^{2n-1}$. Here, the index $(2n-1)$ is odd.

So, $\left(\frac{(2n-1)+1}{2}\right)^{th}$ and $\left(\frac{(2n-1)+1}{2}+1\right)^{th}$ i.e. n^{th} and $(n+1)^{th}$ terms are middle terms.

Now, $T_n = T_{(n-1)+1} = {}^{2n-1}C_{n-1}\,(1)^{(2n-1)-(n-1)}\, x^{n-1} = {}^{2n-1}C_{n-1}\, x^{n-1}$

and, $T_{n+1} = {}^{2n-1}C_n\,(1)^{(2n-1)-n}\, x^n = {}^{2n-1}C_n\, x^n$

So, the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$ are ${}^{2n-1}C_{n-1}$ and ${}^{2n-1}C_n$.

$\therefore$ Sum of these coefficients $= {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$

$= {}^{(2n-1)+1}C_n$ $\quad [\because\ {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$

$= {}^{2n}C_n$

$=$ Coefficient of middle term in the expansion of $(1+x)^{2n}$

Type III **ON FINDING THE COEFFICIENT FOR A GIVEN INDEX (EXPONENT) OF THE VARIABLE**

EXAMPLE 15 *Find the coefficient of* x^{10} *in the binomial expansion of* $\left(2x^2 - \frac{3}{x}\right)^{11}$, *when* $x \neq 0$.

SOLUTION Suppose $(r+1)$th term contains x^{10} in the binomial expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$.

Now, $T_{r+1} = {}^{11}C_r\,(2x^2)^{11-r}\left(-\frac{3}{x}\right)^r = (-1)^r\, {}^{11}C_r\, 2^{(11-r)}\,.\,3^r\,.\,x^{22-3r}$ ...(i)

If T_{r+1} contains x^{10}, then

$22 - 3r = 10 \Rightarrow r = 4.$

So, $(4+1)$th i.e. 5th term contains x^{10}.

Putting $r = 4$ in (i), we get

$T_5 = (-1)^4\, {}^{11}C_4\, 2^{11-4} \times 3^4 \times x^{10} = {}^{11}C_4 \times 2^7 \times 3^4 \times x^{10}$

$\therefore$ Coefficient of $x^{10} = {}^{11}C_4 \times 2^7 \times 3^4$

EXAMPLE 16 *Find the coefficients of* x^{32} *and* x^{-17} *in the expansion of* $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

SOLUTION Suppose $(r+1)$th term involves x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

Now, $$T_{r+1} = {}^{15}C_r\,(x^4)^{15-r}\left(-\frac{1}{x^3}\right)^r = (-1)^r\ {}^{15}C_r\,x^{60-7r} \quad \text{...(i)}$$

For this term to contain x^{32}, we must have

$60 - 7r = 32 \Rightarrow r = 4.$

So, $(4+1)$th i.e. 5th term contains x^{32}.

Putting $r = 4$ in (i), we get

$T_5 = (-1)^4\ {}^{15}C_4\,x^{(60-28)} = {}^{15}C_4\,x^{32}.$

$\therefore$ Coefficient of $x^{32} = {}^{15}C_4 = 1365.$

Suppose $(s+1)$th term in the binomial expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ contains x^{-17}.

Now, $$T_{s+1} = {}^{15}C_s\,(x^4)^{15-s}\left(-\frac{1}{x^3}\right)^s = (-1)^s\ {}^{15}C_s\,x^{60-7s} \quad \text{...(ii)}$$

If this term contains x^{-17}, we must have

$60 - 7s = -17 \Rightarrow s = 11$

So, $(11+1)$th i.e. 12th term contains x^{-17}.

Putting $s = 11$ in (ii), we get

$T_{12} = (-1)^{11}\ {}^{15}C_{11}\,x^{-17} = -\ {}^{15}C_{11}\,x^{-17} = -\ {}^{15}C_4\,x^{-17}$ $[\because {}^nC_r = {}^nC_{n-r}]$

$\therefore$ Coefficient of $x^{-17} = -\ {}^{15}C_4 = -1365.$

EXAMPLE 17 *Find the coefficient of* $x^6\,y^3$ *in the expansion of* $(x+2y)^9$. **[NCERT]**

SOLUTION Suppose $x^6\,y^3$ occurs in $(r+1)^{th}$ term of the expansion of $(x+2y)^9$.

Now,

$T_{r+1} = {}^9C_r \times (x)^{9-r} \times (2y)^r = {}^9C_r \times 2^r \times x^{9-r} \times y^r$

This will contain $x^6\,y^3$, if

$9 - r = 6$ and $r = 3 \Rightarrow r = 3$

$\therefore$ Coefficient of $x^6\,y^3 = {}^9C_3 \times 2^3 = \frac{9!}{3!\,6!} \times 2^3 = \frac{9 \times 8 \times 7 \times 6!}{3! \times 6!} \times 8 = 672$

EXAMPLE 18 *Find the coefficient of* x^{40} *in the expansion of* $(1+2x+x^2)^{27}$.

SOLUTION We have,

$(1+2x+x^2)^{27} = \left\{(1+x)^2\right\}^{27} = (1+x)^{54}$

Suppose x^{40} occurs in $(r+1)^{th}$ term in the expansion of $(1+x)^{54}$.

Now, $T_{r+1} = {}^{54}C_r\,x^r$

For this term to contain x^{40}, we must have $r = 40$.

So, coefficient of $x^{40} = {}^{54}C_{40}$.

ALITER We know that the coefficient of x^r in $(1+x)^n$ is ${}^nC_r$.

$\therefore$ Coefficient of x^{40} in $(1+x)^{54}$ is ${}^{54}C_{40}$.

EXAMPLE 19 *Prove that there is no term involving x^6 in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$, where $r \neq 0$.*

SOLUTION Suppose x^6 occurs in $(r+1)^{th}$ term in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$.

Now, $T_{r+1} = {}^{11}C_r\,(2x^2)^{11-r}\left(-\frac{3}{x}\right)^r = {}^{11}C_r\,(-1)^r\,2^{11-r}\,3^r\,x^{22-3r}$ ...(i)

For this term to contain x^6, we must have

$22 - 3r = 6 \Rightarrow r = \frac{16}{3}$, which is a fraction.

But, r is a natural number. Hence, there is no term containing x^6.

EXAMPLE 20 *Find the coefficient of x^5 in the expansion of the product* $(1+2x)^6\,(1-x)^7$. [NCERT]

SOLUTION We have,

$$(1+2x)^6(1-x)^7 = \left\{1 + {}^6C_1\,(2x) + {}^6C_2\,(2x)^2 + {}^6C_3\,(2x)^3 + {}^6C_4\,(2x)^4 + {}^6C_5\,(2x)^5 + {}^6C_6\,(2x)^6\right\}$$
$$\times\left\{1 - {}^7C_1\,x + {}^7C_2\,x^2 - {}^7C_3\,x^3 + {}^7C_4\,x^4 - {}^7C_5\,x^5 + \dots\right\}$$
$$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + \dots)$$
$$\times (1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + \dots)$$

$\therefore$ Coefficient of x^5 in the product $= 1\times(-21) + 12\times 35 + 60\times(-35) + 160\times 21 + 240\times -7 + 192\times 1$

$$= -21 + 420 - 2100 + 3360 - 1680 + 192 = 171$$

Type IV ON FINDING THE TERM INDEPENDENT OF THE VARIABLE

EXAMPLE 21 *Find the term independent of x in the expansion of $\left(3x^2 - \frac{1}{2x^3}\right)^{10}$.*

SOLUTION Let $(r+1)$th term be independent of x in the given expression.

Now, $T_{r+1} = {}^{10}C_r\,(3x^2)^{10-r}\left(-\frac{1}{2x^3}\right)^r = {}^{10}C_r\,3^{10-r}\left(-\frac{1}{2}\right)^r x^{20-5r}$ ...(i)

This term will be independent of x, if

$20 - 5r = 0 \Rightarrow r = 4$

So, $(4+1)$th i.e. 5th term is independent of x. Putting $r = 4$ in (i), we get

$$T_5 = {}^{10}C_4\,3^6\left(-\frac{1}{2}\right)^4 = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}\times\frac{729}{16} = \frac{76545}{8}$$

Hence, required term $= \frac{76545}{8}$

EXAMPLE 22 *Find the term independent of x in the expansion of*

(i) $\left(x-\frac{1}{x}\right)^{12}$ (ii) $\left(2x-\frac{1}{x}\right)^{10}$

SOLUTION (i) Let $(r+1)$th term be independent of x in the given expression.

Now, $T_{r+1} = {}^{12}C_r\, x^{12-r}\left(-\frac{1}{x}\right)^r = {}^{12}C_r\,(-1)^r\, x^{12-2r}$ ...(i)

For this term to be independent of x, we must have

$12-2r=0 \Rightarrow r=6.$

So, $(6+1)$th i.e. 7th term is independent of x. Putting $r=6$ in (i), we get

$T_7 = {}^{12}C_6\,(-1)^6 = {}^{12}C_6$

Hence, required term $= {}^{12}C_6$

(ii) Let $(r+1)$th term be independent of x in the given expression.

Now, $T_{r+1} = {}^{10}C_r\,(2x)^{10-r}\left(-\frac{1}{x}\right)^r = (-1)^r\, {}^{10}C_r\, 2^{10-r}\, x^{10-2r}$ ...(i)

For this term to be independent of x, we must have

$10-2r=0 \Rightarrow r=5$

So, $(5+1)$th i.e. 6th term is independent of x. Putting $r=5$ in (i), we get

$$T_6 = (-1)^5\ {}^{10}C_5 \,.\, 2^{10-5} = -\,{}^{10}C_5\times 2^5 = -\frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1}\times 32 = -8064$$

Hence, required term $=-8064$

EXAMPLE 23 *Find the value of a so that the term independent of x in* $\left(\sqrt{x}+\frac{a}{x^2}\right)^{10}$ *is 405.*

SOLUTION Let $(r+1)^{th}$ term in the expansion of $\left(\sqrt{x}+\frac{a}{x^2}\right)^{10}$ be independent of x.

Now,

$$T_{r+1} = {}^{10}C_r\ (\sqrt{x})^{10-r}\left(\frac{a}{x^2}\right)^r = {}^{10}C_r\ x^{5-\frac{r}{2}-2r}\ a^r \quad \text{...(i)}$$

This will be independent of x, if

$5-\frac{r}{2}-2r = 0 \Rightarrow 5-\frac{5r}{2} = 0 \Rightarrow 5 = \frac{5r}{2} \Rightarrow r = 2$

Putting $r = 2$ in (i), we get: $T_3 = {}^{10}C_2\ a^2$

It is given that the term independent of x is equal to 405.

$\therefore\quad {}^{10}C_2\ a^2 = 405 \Rightarrow 45a^2 = 405 \Rightarrow a^2 = 9 \Rightarrow a = \pm 3$

Type V PROBLEMS RELATING TO COEFFICIENTS IN A BINOMIAL EXPANSION

In solving the problems relating the coefficients in the binomial expansion we generally use the following results:

(i) Coefficient of $(r+1)$th term in the binomial expansion of $(1+x)^n$ is ${}^nC_r$.

(ii) Coefficient of x^r in the binomial expansion of $(1+x)^n$ is ${}^nC_r$.

(iii) Coefficient of x^r in the expansion of $(1-x)^n$ is $(-1)^r\ {}^nC_r$.

(iv) Coefficient of $(r+1)$th term in the expansion of $(1-x)^n$ is $(-1)^r\ {}^nC_r$.

EXAMPLE 24 *In the binomial expansion of $(1+a)^{m+n}$, prove that the coefficients of a^m and a^n are equal.* **[NCERT]**

SOLUTION Let A and B be the coefficients of a^m and a^n respectively in the expansion of $(1+a)^{m+n}$. Then,

$A =$ Coefficient of a^m in the binomial expansion of $(1+a)^{m+n}$

$$\Rightarrow \quad A = {}^{m+n}C_m = \frac{(m+n)!}{m!\,n!} \qquad \text{...(i)}$$

$B =$ Coefficient of a^n in the binomial expansion of $(1+a)^{m+n}$

$$\Rightarrow \quad B = {}^{m+n}C_n = \frac{(m+n)!}{m!\,n!} \qquad \text{...(ii)}$$

Clearly, $A = B$ i.e. the coefficients of a^m and a^n in the binomial expansion of $(1+a)^{m+n}$ are equal.

EXAMPLE 25 *Prove that the coefficients of x^n in $(1+x)^{2n}$ is twice the coefficient of x^n in $(1+x)^{2n-1}$.* **[NCERT]**

SOLUTION Let A and B be the coefficients of x^n in the binomial expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively. Then,

$$A = \text{Coefficient of } x^n \text{ in } (1+x)^{2n} = {}^{2n}C_n = \frac{(2n)!}{n!\,n!} = \frac{(2n)\,(2n-1)!}{n\,(n-1)!\,n!} = 2\,\frac{(2n-1)!}{(n-1)!\,n!} \qquad \text{...(i)}$$

and,

$$B = \text{Coefficient of } x^n \text{ in } (1+x)^{2n-1} = {}^{2n-1}C_n = \frac{(2n-1)!}{(n-1)!\,n!} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$A = 2B$ i.e. Coefficient of x^n in $(1+x)^{2n} = 2 \times$ Coefficient of x^n in $(1+x)^{2n-1}$.

EXAMPLE 26 *In the binomial expansion of $(a+b)^n$, the coefficients of the fourth and thirteenth terms are equal to each other. Find n.*

SOLUTION The coefficients of the fourth and thirteenth terms in the binomial expansion of $(a+b)^n$ are ${}^nC_3$ and ${}^nC_{12}$ respectively. It is given that:

Coefficient of 4th term in $(a+b)^n$ = Coefficient of 13th term in $(a+b)^n$

$$\Rightarrow \quad {}^nC_3 = {}^nC_{12}$$

$$\Rightarrow \quad n = 15 \qquad [\because {}^nC_x = {}^nC_y \Rightarrow x = y, \text{ or } x + y = n]$$

EXAMPLE 27 *Find a positive value of m for which the coefficient of x^2 in the expansion of $(1+x)^m$ is 6.* **[NCERT]**

SOLUTION We know that the coefficient of x^r in $(1+x)^n$ is ${}^nC_r$. Therefore, coefficient of x^2 in $(1+x)^m$ is ${}^mC_2$.

It is given that the coefficient of x^2 in $(1+x)^m$ is 6.

$$\therefore \quad {}^mC_2 = 6$$

$$\Rightarrow \quad \frac{m(m-1)}{2!} = 6$$

$$\Rightarrow \quad m^2 - m = 12$$

$\Rightarrow \quad m^2 - m - 12 = 0$

$\Rightarrow \quad (m-4)(m+3) = 0$

$\Rightarrow \quad m - 4 = 0 \qquad [\because m + 3 \neq 0]$

$\Rightarrow \quad m = 4.$

EXAMPLE 28 *If the coefficients of* $(r-5)^{th}$ *and* $(2r-1)^{th}$ *terms in the expansion of* $(1+x)^{34}$ *are equal, find* r. [NCERT]

SOLUTION We know that the coefficient of r^{th} term in the expansion of $(1+x)^n$ is ${}^nC_{r-1}$. Therefore,

Coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1+x)^{34}$ are ${}^{34}C_{r-6}$ and ${}^{34}C_{2r-2}$ respectively.

It is given that these coefficients are equal

$\therefore \quad {}^{34}C_{r-6} = {}^{34}C_{2r-2}$

$\Rightarrow \quad r - 6 = 2r - 2 \text{ or, } r - 6 + 2r - 2 = 34 \qquad \left[\because {}^nC_r = {}^nC_s \Rightarrow r = s \text{ or, } r + s = n\right]$

$\Rightarrow \quad 3r - 8 = 34 \qquad [\because r - 6 = 2r - 2 \Rightarrow r = -4, \text{ which is not possible}]$

$\Rightarrow \quad 3r = 42 \Rightarrow r = 14$

Type VI **PROBLEMS BASED ON CONSECUTIVE TERMS OR CONSECUTIVE COEFFICIENTS**

If consecutive terms or coefficients of consecutive terms in the expansion of $(x+a)^n$ are given, we assume that the consecutive terms are r^{th}, $(r+1)^{\text{th}}$ and $(r+2)^{\text{th}}$ i.e. T_r, T_{r+1} and T_{r+2}.

In case of consecutive terms, we find $\dfrac{T_{r+1}}{T_r}$ and $\dfrac{T_r}{T_{r-1}}$.

It should be noted that $\dfrac{T_{r+1}}{T_r} = \dfrac{n-r+1}{r}\dfrac{a}{x}$

In case of consecutive coefficients, we find the ratios $\dfrac{r^{\text{th}} \text{ coefficient}}{(r+1)^{\text{th}} \text{ coefficient}}$ and $\dfrac{(r+1)^{th} \text{ coefficient}}{(r+2)^{\text{th}} \text{ coefficient}}$ etc. to get equations and solve them.

In computing these ratios, we may use the following results:

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} \text{ and } \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1}$$

EXAMPLE 29 *The coefficients of three consecutive terms in the expansion of* $(1+x)^n$ *are in the ratio* $1:7:42$. *Find* n. [NCERT]

SOLUTION Let the three consecutive terms be rth, $(r+1)$th and $(r+2)$th terms. Their coefficients in the expansion of $(1+x)^n$ are ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ respectively. It is given that,

$${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 1:7:42.$$

Now, $\dfrac{{}^nC_{r-1}}{{}^nC_r} = \dfrac{1}{7}$

$\Rightarrow \quad \dfrac{r}{n-r+1} = \dfrac{1}{7} \qquad \left[\because \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}\right]$

$$\Rightarrow \quad n - 8r + 1 = 0 \qquad \text{...(i)}$$

and, $$\frac{^nC_r}{^nC_{r+1}} = \frac{7}{42}$$

$$\Rightarrow \quad \frac{r+1}{n-r} = \frac{1}{6} \qquad \left[\because \frac{^nC_{r+1}}{^nC_r} = \frac{n-r}{r+1}\right]$$

$$\Rightarrow \quad n - 7r - 6 = 0 \qquad \text{...(ii)}$$

Solving (i) and (ii), we get $r = 7$ and $n = 55$.

EXAMPLE 30 *In the binomial expansion of $(1 + x)^n$, the coefficients of the fifth, sixth and seventh terms are in A.P. Find all values of n for which this can happen.*

SOLUTION The coefficients of fifth, sixth and seventh terms in the binomial expansion of $(1 + x)^n$ are $^nC_4$, $^nC_5$ and $^nC_6$ respectively. We are given that $^nC_4$, $^nC_5$ and $^nC_6$ are in A.P.

$$\therefore \quad 2\,{}^nC_5 = {}^nC_4 + {}^nC_6$$

$$\Rightarrow \quad 2 = \frac{^nC_4}{^nC_5} + \frac{^nC_6}{^nC_5} \qquad [\text{Dividing both sides by } {}^nC_5]$$

$$\Rightarrow \quad 2 = \frac{5}{n-4} + \frac{n-5}{6} \qquad \left[\because \frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad 2 = \frac{30 + (n-4)(n-5)}{6(n-4)}$$

$$\Rightarrow \quad 12n - 48 = 30 + n^2 - 9n + 20 \Rightarrow n^2 - 21n + 98 = 0 \Rightarrow (n-14)(n-7) = 0 \Rightarrow n = 7, 14.$$

EXAMPLE 31 *If the coefficients of a^{r-1}, a^r, a^{r+1} in the binomial expansion of $(1 + a)^n$ are in A.P., prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.* [NCERT]

SOLUTION The coefficients of a^{r-1}, a^r and a^{r+1} in the binomial expansion of $(1 + a)^n$ are $^nC_{r-1}$, ^nC_r and $^nC_{r+1}$ respectively. It is given that $^nC_{r-1}$, ^nC_r and $^nC_{r+1}$ are in A.P.

$$\therefore \quad 2\,{}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$$

$$\Rightarrow \quad 2 = \frac{^nC_{r-1}}{^nC_r} + \frac{^nC_{r+1}}{^nC_r}$$

$$\Rightarrow \quad 2 = \frac{r}{n-r+1} + \frac{n-r}{r+1} \qquad \left[\because \frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad 2 = \frac{r(r+1) + (n-r)(n-r+1)}{(r+1)(n-r+1)}$$

$$\Rightarrow \quad 2\left\{(n-r+1)(r+1)\right\} = r(r+1) + (n-r)(n-r+1)$$

$$\Rightarrow \quad 2nr - 2r^2 + 2n + 2 = r^2 + r + n^2 - 2nr + r^2 + n - r$$

$$\Rightarrow \quad n^2 - 4nr - n + 4r^2 - 2 = 0 \Rightarrow n^2 - n(4r + 1) + 4r^2 - 2 = 0$$

EXAMPLE 32 *The coefficients of $(r-1)^{th}$, rth and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are in the ratio 1:3:5. Find n and r.* **[NCERT]**

SOLUTION We know that the coefficient of rth term in the expansion of $(x+1)^n$ is ${}^nC_{r-1}$. Therefore, coefficients of $(r-1)^{\text{th}}$, r^{th} and $(r+1)^{\text{th}}$ terms are ${}^nC_{r-2}$, ${}^nC_{r-1}$ and ${}^nC_r$ respectively.

It is given that

$${}^nC_{r-2} : {}^nC_{r-1} : {}^nC_r = 1:3:5$$

$$\Rightarrow \quad \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{5}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_{r-2}} = \frac{3}{1}$$

$$\Rightarrow \quad \frac{n-r+1}{r} = \frac{5}{3} \text{ and } \frac{n-r+2}{r-1} = \frac{3}{1} \qquad \left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad 3n - 8r + 3 = 0 \text{ and } n - 4r + 5 = 0 \Rightarrow n = 7 \text{ and } r = 3$$

LEVEL-2

Type VII **ON FINDING THE UNKNOWN WHEN THE VALUE OF A TERM IS GIVEN**

EXAMPLE 33 *If the third term in the expansion of $\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$ is 1000, then find x.*

SOLUTION We have,

$$T_3 = 1000$$

$$\Rightarrow \quad T_{2+1} = 1000$$

$$\Rightarrow \quad {}^5C_2 \left(\frac{1}{x}\right)^{5-2} (x^{\log x})^2 = 1000$$

$$\Rightarrow \quad 10\,(x^{\log_{10} x})^2 \times x^{-3} = 1000$$

$$\Rightarrow \quad x^{2\log_{10} x} \times x^{-3} = 100$$

$$\Rightarrow \quad x^{2\log_{10} x - 3} = 10^2$$

$$\Rightarrow \quad 2\log_{10} x - 3 = \log_x 10^2$$

$$\Rightarrow \quad 2\log_{10} x - 3 = \frac{2}{\log_{10} x}$$

$$\Rightarrow \quad 2y - 3 = \frac{2}{y}, \text{ where } y = \log_{10} x$$

$$\Rightarrow \quad 2y^2 - 3y - 2 = 0$$

$$\Rightarrow \quad (2y+1)(y-2) = 0$$

$$\Rightarrow \quad y = 2 \text{ or } y = -\frac{1}{2}$$

$$\Rightarrow \quad \log_{10} x = 2 \text{ or, } \log_{10} x = -\frac{1}{2} \Rightarrow x = 10^2 = 100 \text{ or, } x = 10^{-1/2} = \frac{1}{\sqrt{10}}.$$

EXAMPLE 34 *If the fourth term in the expansion of* $\left\{\sqrt{x^{\frac{1}{\log x+1}}}+x^{\frac{1}{12}}\right\}^6$ *is equal to 200 and* $x>1$, *then find x.*

SOLUTION It is given that

$$T_4 = 200$$

$$\Rightarrow \quad T_{3+1} = 200$$

$$\Rightarrow \quad {}^6C_3\left\{\sqrt{x^{\frac{1}{\log x+1}}}\right\}^{6-3}(x^{1/12})^3 = 200$$

$$\Rightarrow \quad 20\left(x^{\frac{1}{\log x+1}}\right)^{3/2} x^{1/4} = 200$$

$$\Rightarrow \quad x^{\frac{3}{2}\left(\frac{1}{\log x+1}\right)+\frac{1}{4}} = 10$$

$$\Rightarrow \quad \frac{3}{2}\left(\frac{1}{\log x+1}\right)+\frac{1}{4} = \log_x 10$$

$$\Rightarrow \quad \frac{3}{2}\left(\frac{1}{\log_{10} x+1}\right)+\frac{1}{4} = \frac{1}{\log_{10} x}$$

$$\Rightarrow \quad \frac{3}{2(y+1)}+\frac{1}{4} = \frac{1}{y}, \text{ where } y = \log_{10} x$$

$$\Rightarrow \quad \frac{6+y+1}{4(y+1)} = \frac{1}{y}$$

$$\Rightarrow \quad y^2+3y-4 = 0$$

$$\Rightarrow \quad (y+4)(y-1) = 0$$

$$\Rightarrow \quad y = 1, -4$$

$$\Rightarrow \quad \log_{10} x = 1, -4$$

$$\Rightarrow \quad x = 10 \text{ or, } x = 10^{-4} \Rightarrow x = 10 \qquad [\because x > 1]$$

EXAMPLE 35 *For what value of x is the ninth term in the expansion of*

$$\left\{3^{\log_3\sqrt{25^{x-1}+7}} + 3^{(-1/8)\log_3(5^{x-1}+1)}\right\}^{10}$$ *is equal to 180?*

SOLUTION We know that $a^{\log_a N} = N$.

$$\therefore \quad \left\{3^{\log_3\sqrt{25^{x-1}+7}} + 3^{(-1/8)\log_3(5^{x-1}+1)}\right\}^{10} = \left\{\sqrt{25^{x-1}+7}+(5^{x-1}+1)^{-1/8}\right\}^{10}$$

Let T_9 be the 9th term in the above expansion. Then,

$$T_9 = 180$$

$\Rightarrow \quad {}^{10}C_8 \left\{\sqrt{25^{x-1}+7}\right\}^{10-8} \left\{(5^{x-1}+1)^{-1/8}\right\}^8 = 180$

$\Rightarrow \quad {}^{10}C_8 \ (25^{x-1}+7) \ (5^{x-1}+1)^{-1} = 180$

$\Rightarrow \quad \dfrac{45\,(25^{x-1}+7)}{5^{x-1}+1} = 180$

$\Rightarrow \quad \dfrac{25^{x-1}+7}{5^{x-1}+1} = 4$

$\Rightarrow \quad \dfrac{y^2+7}{y+1} = 4$, where $y = 5^{x-1}$

$\Rightarrow \quad y^2 - 4y + 3 = 0$

$\Rightarrow \quad (y-3)(y-1) = 0$

$\Rightarrow \quad y = 3, -1$

$\Rightarrow \quad 5^{x-1} = 3$ or, $5^{x-1} = 1 \Rightarrow 5^x = 15$ or, $5^x = 5 \Rightarrow x = \log_5 15$ or, $x = 1$.

EXAMPLE 36 *If the fourth term in the expansion of* $\left(ax + \dfrac{1}{x}\right)^n$ *is* $\dfrac{5}{2}$, *then find the values of a and n.*

SOLUTION It is given that

$$T_4 = \frac{5}{2}$$

$\Rightarrow \quad T_{3+1} = \dfrac{5}{2}$

$\Rightarrow \quad {}^nC_3\,(ax)^{n-3}\left(\dfrac{1}{x}\right)^3 = \dfrac{5}{2} \Rightarrow {}^nC_3\,a^{n-3}\,x^{n-6} = \dfrac{5}{2}$ ...(i)

Clearly, RHS of the above equality is independent of x.

$\therefore \quad n - 6 = 0 \Rightarrow n = 6.$

Putting $n = 6$ in (i), we get

$${}^6C_3\,a^3 = \frac{5}{2} \Rightarrow \frac{6\times5\times4}{3\times2\times1}a^3 = \frac{5}{2} \Rightarrow a^3 = \frac{1}{8} \Rightarrow a = \frac{1}{2}$$

Hence, $a = \dfrac{1}{2}$ and $n = 6$.

Type VIII **ON MIDDLE TERM (S) IN A BINOMIAL EXPENSION**

EXAMPLE 37 *Find the value of* α *for which the coefficients of the middle terms in the expansions of* $(1+\alpha x)^4$ *and* $(1-\alpha x)^6$ *are equal, find* α.

SOLUTION In the expansion of $(1+\alpha x)^4$.

Middle term $= {}^4C_2\ (\alpha x)^2 = 6\alpha^2 x^2$

In the expansion of $(1-\alpha x)^6$.

Middle term $= {}^6C_3\,(-\alpha x)^3 = -20\,\alpha^3 x^3$

It is given that:

Coefficient of the middle term in $(1+\alpha x)^4$ = Coefficient of the middle term in $(1-\alpha x)^6$

$\Rightarrow 6\,\alpha^2 = -20\alpha^3 \Rightarrow \alpha = 0, \alpha = -\dfrac{3}{10}$

EXAMPLE 38 *If the middle term in the binomial expansion of* $\left(\frac{1}{x}+x\sin x\right)^{10}$ *is equal to* $\frac{63}{8}$, *find the value of x.* **[NCERT EXEMPLAR]**

SOLUTION In the binomial expansion of $\left(\frac{1}{x}+x\sin x\right)^{10}$, $\left(\frac{10}{2}+1\right)^{th}$ i.e. 6th term is the middle term.

It is given that

$$T_6=\frac{63}{8}$$

$$\Rightarrow \quad {}^{10}C_5\left(\frac{1}{x}\right)^{10-5}(x\sin x)^5=\frac{63}{8}$$

$$\Rightarrow \quad \frac{10!}{5!5!}(\sin x)^5=\frac{63}{8}$$

$$\Rightarrow \quad (\sin x)^5=\left(\frac{1}{2}\right)^5$$

$$\Rightarrow \quad \sin x=\frac{1}{2}=\sin\frac{\pi}{6}$$

$$\Rightarrow \quad x=n\pi+(-1)^n\frac{\pi}{6}, n\in Z$$

Type IX **ON COEFFICIENTS OF TERMS IN A BINOMIAL EXPANSION**

EXAMPLE 39 *The sum of the coefficients of first three terms in the expansion of* $\left(x-\frac{3}{x^2}\right)^m$, $x\neq 0$, *m being a natural number, is 559. Find the term of the expansion containing* x^3. **[NCERT]**

SOLUTION We have,

$$\left(x-\frac{3}{x^2}\right)^m={}^mC_0\,x^m+{}^mC_1\,x^{m-1}\left(-\frac{3}{x^2}\right)+{}^mC_2\,x^{m-2}\left(-\frac{3}{x^2}\right)^2+\ldots+{}^mC_m\,x^0\left(-\frac{3}{x^2}\right)^m$$

$$\Rightarrow \left(x-\frac{3}{x^2}\right)^m={}^mC_0\,x^m+(-3\times{}^mC_1)\,x^{m-3}+(9\times{}^mC_2)\,x^{m-6}+\ldots+{}^mC_m\,(-3)^m\times x^{-2m}$$

Clearly, the coefficients of first three terms are: ${}^mC_0$, $-3\times{}^mC_1$ and $9\times{}^mC_2$

It is given that the sum of these coefficients is 559.

$$\therefore \quad {}^mC_0-3\times{}^mC_1+9\times{}^mC_2=559$$

$$\Rightarrow \quad 1-3m+\frac{9m(m-1)}{2}=559$$

$$\Rightarrow \quad 2-6m+9m(m-1)=1118$$

$$\Rightarrow \quad 9m^2-15m-1116=0$$

$$\Rightarrow \quad 3m^2-5m-372=0$$

$$\Rightarrow \quad 3m^2-36m+31m-372=0$$

$$\Rightarrow \quad 3m(m-12)+31(m-12)=0$$

$$\Rightarrow \quad (m-12)(3m+31)=0$$

$$\Rightarrow \quad m=12 \qquad [\because m\in N \therefore 3m+31\neq 0]$$

Suppose $(r+1)^{th}$ term contains x^3.

Now,

$$T_{r+1} = {}^mC_r\,(x)^{m-r}\left(-\frac{3}{x^2}\right)^r = {}^mC_r\,(-3)^r\,x^{m-3r} = {}^{12}C_r\,(-3)^r\,x^{12-3r} \qquad [\because\ m=12]$$

This will contain x^3, if $12 - 3r = 3$ i.e. $r = 3$.

Putting $r = 3$ in T_{r+1}, we get

Required term $= T_5 = {}^{12}C_3\,(-3)^3\,x^{12-9} = -5940x^3$

EXAMPLE 40 *Find the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ and x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ and find the relation between a and b so that these coefficients are equal.*

SOLUTION Suppose x^7 occurs in $(r+1)^{th}$ term of the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$.

Now,

$$T_{r+1} = {}^{11}C_r\,(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r\,a^{11-r}\,b^{-r}\,x^{22-3r} \qquad \text{...(i)}$$

This will contain x^7, if

$22 - 3r = 7 \Rightarrow 3r = 15 \Rightarrow r = 5.$

Putting $r = 5$ in (i), we obtain that

Coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is ${}^{11}C_5\,a^6\,b^{-5}$.

Suppose x^{-7} occurs in $(r+1)^{th}$ term of the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$.

Now, $$T_{r+1} = {}^{11}C_r\,(ax)^{11-r}\left(-\frac{1}{bx^2}\right)^r = {}^{11}C_r\,a^{11-r}\,(-1)^r\,b^{-r}\,x^{11-3r} \qquad \text{... (ii)}$$

This will contain x^{-7}, if

$11 - 3r = -7 \Rightarrow 3r = 18 \Rightarrow r = 6.$

Putting $r = 6$ in (ii), we obtain that

Coefficient of x^{-7} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6\,a^5\,b^{-6}\,(-1)^6$.

If the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$, then

$${}^{11}C_5\,a^6\,b^{-5} = {}^{11}C_6\,a^5\,b^{-6}\,(-1)^6 \Rightarrow {}^{11}C_5\,ab = {}^{11}C_6 \Rightarrow ab = 1 \qquad \left[\because\ {}^{11}C_5 = {}^{11}C_6\right]$$

EXAMPLE 41 *If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, prove that its coefficient is*

$$\left\{\frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}\right\}.$$

SOLUTION Suppose x^p occurs in $(r+1)^{th}$ term in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$.

Now, $T_{r+1} = {}^{2n}C_r\,(x^2)^{2n-r}\left(\frac{1}{x}\right)^r = {}^{2n}C_r\,x^{4n-3r}$ ...(i)

For this term to contain x^p, we must have

$$4n-3r = p \Rightarrow r = \frac{4n-p}{3}$$

$\therefore$ Coefficient of $x^p = {}^{2n}C_r$ where $r=\frac{4n-p}{3}$

$\Rightarrow$ Coefficient of $x^p = \frac{(2n)!}{(2n-r)!\,r!}$, where $r=\frac{4n-p}{3}$

$\Rightarrow$ Coefficient of $x^p = \frac{(2n)!}{\left\{2n-\left(\frac{4n-p}{3}\right)\right\}!\left(\frac{4n-p}{3}\right)!} = \frac{(2n)!}{\left(\frac{2n+p}{3}\right)!\left(\frac{4n-p}{3}\right)!}$

EXAMPLE 42 *Find the coefficient of x^n in the expansion of $(1+x)(1-x)^n$.*

SOLUTION Coefficient of x^n in $(1+x)(1-x)^n$

$= $ Coefficient of x^n in $(1-x)^n$ + Coefficient of x^{n-1} in $(1-x)^n$

$= (-1)^n\;{}^nC_n + (-1)^{n-1}\,{}^nC_{n-1}$

$= (-1)^n\;(1-n)$

EXAMPLE 43 *Find the coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^{11}$.*

[NCERT EXEMPLAR]

SOLUTION $(1+x+x^2+x^3)^{11} = \{(1+x)+x^2(1+x)\}^{11} = \{(1+x)(1+x^2)^{11}\} = (1+x)^{11}(1+x^2)^{11}$

$$=\left({}^{11}C_0+{}^{11}C_1+{}^{11}C_2\,x^2+{}^{11}C_3\,x^3+{}^{11}C_4\,x^4+{}^{11}C_5\,x^5+\ldots\right)\times$$

$$\left({}^{11}C_0+{}^{11}C_1x^2+{}^{11}C_2(x^2)^2+{}^{11}C_3(x^2)^3+\ldots\right)$$

$\therefore$ Coefficient of x^4 in $(1+x+x^2+x^3)^4 = {}^{11}C_0\times{}^{11}C_2+{}^{11}C_2\times{}^{11}C_1+{}^{11}C_4\times{}^{11}C_0$

$= 55+55\times 11+330 = 990$

EXAMPLE 44 *If the coefficients of x and x^2 in the expansion of $(1+x)^m(1-x)^n$ are 3 and -6 respectively. Find the values of m and n.*

SOLUTION We have,

$(1+x)^m(1-x)^n$

$$=\left\{{}^mC_0+{}^mC_1\,x+{}^mC_2\,x^2+\ldots+{}^mC_m\,x^m\right\}\times\left\{{}^nC_0-{}^nC_1\,x+{}^nC_2\,x^2\,\ldots+(-1)^n\,{}^nC_n\,x^n\right\}$$

$$={}^mC_0\,{}^nC_0-\left({}^mC_0\,{}^nC_1-{}^nC_0\,{}^mC_1\right)x+\left({}^mC_0\,{}^nC_2+{}^nC_0\,{}^mC_2-{}^mC_1\,{}^nC_1\right)x^2+\ldots$$

It is given that the coefficients of x and x^2 in the expansion of $(1+x)^m(1-x)^n$ are 3 and -6 respectively.

$\therefore \quad -({}^mC_0\ {}^nC_1 - {}^nC_0\ {}^mC_1) = 3$ and, ${}^mC_0\ {}^nC_2 + {}^nC_0\ {}^mC_2 - {}^mC_1\ {}^nC_1 = -6$

$\Rightarrow \quad m - n = 3$ and $n(n-1) + m(m-1) - 2mn = -12$

$\Rightarrow \quad m - n = 3$ and $(m-n)^2 - (m+n) = -12$

$\Rightarrow \quad m - n = 3$ and $m + n = 21$

$\Rightarrow \quad m = 12, n = 9$

Type X **ON FINDING THE TERM INDEPENDENT OF THE VARIABLE**

EXAMPLE 45 *Find the coefficient of the term independent of x in the expansion of*

$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}.$$

SOLUTION We have,

$$\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}$$

$$=\frac{(x^{1/3})^3+1^3}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x^{1/2}\,(x^{1/2}-1)}$$

$$=\frac{(x^{1/3}+1)\,(x^{2/3}-x^{1/3}+1)}{x^{2/3}-x^{1/3}+1}-\frac{\left(x^{1/2}+1\right)\left(x^{1/2}-1\right)}{x^{1/2}\left(x^{1/2}-1\right)}$$

$$=\left(x^{1/3}+1\right)-\left(\frac{x^{1/2}+1}{x^{1/2}}\right)= x^{1/3}+1-1-x^{-1/2}= x^{1/3}-x^{-1/2}$$

$$\therefore \quad \left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10} = (x^{1/3}-x^{-1/2})^{10}$$

Let T_{r+1} be the general term in $\left(x^{1/3}-x^{-1/2}\right)^{10.}$ Then,

$$T_{r+1} = {}^{10}C_r\ (x^{1/3})^{10-r}\ (-1)^r\ (x^{-1/2})^r = (-1)^r\ {}^{10}C_r\ x^{\frac{10-r}{3}-\frac{r}{2}}$$

For this term to be independent of x, we must have

$$\frac{10-r}{3}-\frac{r}{2} = 0 \Rightarrow 20-2r-3r = 0 \Rightarrow r = 4$$

So, required coefficient $= {}^{10}C_4\,(-1)^4 = 210.$

EXAMPLE 46 *Find the greatest value of the term independent of x in the expansion of* $\left(x\sin\alpha+\frac{\cos\alpha}{x}\right)^{10}$*, where* $\alpha \in R$.

SOLUTION Let $(r+1)^{th}$ term be independent of x.

Now, $T_{r+1} = {}^{10}C_r\,(x\sin\alpha)^{10-r}\left(\frac{\cos\alpha}{x}\right)^r = {}^{10}C_r\,x^{10-2r}\,(\sin\alpha)^{10-r}\,(\cos\alpha)^r$

If it is independent of x, then $r = 5$.

$\therefore \quad$ Term independent of $x = T_6 = {}^{10}C_5\,(\sin\alpha\cos\alpha)^5 = {}^{10}C_5\times 2^{-5}\,(\sin 2\alpha)^5$

Clearly, it is greatest when $2\alpha = \pi/2$ and its greatest value is ${}^{10}C_5\times 2^{-5} = \frac{10!}{2^5\,(5!)^2}$

Type XI ON COEFFICIENTS OF TERMS IN A BINOMIAL EXPANSION

EXAMPLE 47 *Find the coefficient of x^5 in the expansion of $(1+x)^{21}+(1+x)^{22}+\ldots+(1+x)^{30}$.*

SOLUTION We have,

$$(1+x)^{21}+(1+x)^{22}+\ldots+(1+x)^{30}$$

$$=(1+x)^{21}\left\{\frac{(1+x)^{10}-1}{(1+x)-1}\right\}=\frac{1}{x}\left\{(1+x)^{31}-(1+x)^{21}\right\}$$

$\therefore$ Coefficient of x^5 in the given expression = Coefficient of x^5 in $\left[\frac{1}{x}\left\{(1+x)^{31}-(1+x)^{21}\right\}\right]$

$$= \text{Coefficient of } x^6 \text{ in } \left\{(1+x)^{31}-(1+x)^{21}\right\}$$

$$= {}^{31}C_6 - {}^{21}C_6$$

EXAMPLE 48 *Find the coefficient of x^{50} after simplifying and collecting the like terms in the expansion of $(1+x)^{1000}+x(1+x)^{999}+x^2(1+x)^{998}+\ldots+x^{1000}$.*

SOLUTION Let $S=(1+x)^{1000}+x(1+x)^{999}+x^2(1+x)^{998}+\ldots+x^{1000}$. Clearly, it is a G.P. consisting of 1001 terms with first term $(1+x)^{1000}$ and common ratio $\frac{x}{1+x}$.

$$\therefore \quad S=(1+x)^{1000}\left\{\frac{1-\left(\frac{x}{1+x}\right)^{1001}}{1-\left(\frac{x}{1+x}\right)}\right\}$$

$$\Rightarrow \quad S=(1+x)^{1000}\left\{\frac{(1+x)^{1001}-x^{1001}}{(1+x)^{1000}}\right\}=(1+x)^{1001}-x^{1001}$$

$\therefore$ Coefficient of x^{50} in S = Coefficient of x^{50} in $\left\{(1+x)^{1001}-x^{1001}\right\}$

$$= \text{Coefficient of } x^{50} \text{ in } (1+x)^{1001}$$

$$= {}^{1001}C_{50}.$$

EXAMPLE 49 *If n is a positive integer, find the coefficient of x^{-1} in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$.*

SOLUTION Clearly, **[NCERT EXEMPLAR]**

$$(1+x)^n\left(1+\frac{1}{x}\right)^n=\frac{(1+x)^n(1+x)^n}{x^n}=\frac{(1+x)^{2n}}{x^n}$$

$\therefore$ Coefficient of x^{-1} in $(1+x)^n\left(1+\frac{1}{x}\right)^n$ = Coefficient of x^{-1} in $\frac{(1+x)^{2n}}{x^n}$

$$= \text{Coefficient of } x^{n-1} \text{ in } (1+x)^{2n}$$

$$= {}^{2n}C_{n-1}$$

EXAMPLE 50 *If in the expansion of $(1-x)^{2n-1}$, the coefficient of x^r is denoted by a_r, then prove that $a_{r-1}+a_{2n-r}=0$.*

SOLUTION We have,

$$a_{r-1} = \text{Coefficient of } x^{r-1} \text{ in } (1-x)^{2n-1} = (-1)^{r-1}\ {}^{2n-1}C_{r-1}$$

$$a_{2n-r} = \text{Coefficient of } x^{2n-r} \text{ in } (1-x)^{2n-1} = (-1)^{2n-r}\ {}^{2n-1}C_{2n-r}$$

$$\therefore\ a_{r-1} + a_{2n-r} = (-1)^{r-1}\ {}^{2n-1}C_{r-1} + (-1)^{2n-r}\ {}^{2n-1}C_{2n-r}$$

$$= (-1)^{r-1}\ {}^{2n-1}C_{(2n-1)-(r-1)} + (-1)^{2n}(-1)^{-r}\ {}^{2n-1}C_{2n-r} \qquad [\because\ {}^nC_r = {}^nC_{n-r}]$$

$$= (-1)^{r-1}\ {}^{2n-1}C_{2n-r} + (-1)^{-r}\ {}^{2n-1}C_{2n-r} \qquad [\because\ (-1)^{2n} = 1]$$

$$= [(-1)^{r-1} + (-1)^{-r}]\ {}^{2n-1}C_{2n-r}$$

$$= \left\{(-1)^{r-1} + \frac{1}{(-1)^r}\right\}\ {}^{2n-1}C_{2n-r}$$

$$= \left\{\frac{(-1)^{2r-1}+1}{(-1)^r}\right\}\ {}^{2n-1}C_{2n-r} = \left\{\frac{-1+1}{(-1)^r}\right\}\ {}^{2n-1}C_{2n-r} = 0 \qquad [\because\ (-1)^{2r-1} = -1]$$

Type XII **ON CONSECUTIVE TERMS AND THEIR COEFFICIENTS**

EXAMPLE 51 *If a_1, a_2, a_3, a_4 be the coefficients of four consecutive terms in the expansion of $(1+x)^n$, then prove that:* $\dfrac{a_1}{a_1+a_2} + \dfrac{a_3}{a_3+a_4} = \dfrac{2a_2}{a_2+a_3}$. **[NCERT EXEMPLAR]**

SOLUTION Let a_1, a_2, a_3, a_4 be the coefficients of 4 consecutive terms viz. the rth, the $(r+1)$th, the $(r+2)$th and the $(r+3)$th terms. Then,

$$a_1 = {}^nC_{r-1},\ a_2 = {}^nC_r,\ a_3 = {}^nC_{r+1} \text{ and } a_4 = {}^nC_{r+2}$$

Now, $a_1 + a_2 = {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r,\ a_2 + a_3 = {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

and, $a_3 + a_4 = {}^nC_{r+1} + {}^nC_{r+2} = {}^{n+1}C_{r+2}$

$$\therefore\ \frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{{}^nC_{r-1}}{{}^{n+1}C_r} + \frac{{}^nC_{r+1}}{{}^{n+1}C_{r+2}}$$

$$= \frac{{}^nC_{r-1}}{\left(\frac{n+1}{r}\right){}^nC_{r-1}} + \frac{{}^nC_{r+1}}{\left(\frac{n+1}{r+2}\right){}^nC_{r+1}} \qquad \left[\because\ {}^nC_r = \frac{n}{r}\cdot{}^{n-1}C_{r-1}\right]$$

$$= \frac{r}{n+1} + \frac{r+2}{n+1} = 2\left(\frac{r+1}{n+1}\right) \qquad \text{...(i)}$$

and, $$2\frac{a_2}{a_2+a_3} = 2\frac{{}^nC_r}{{}^{n+1}C_{r+1}} = 2\left(\frac{{}^nC_r}{\frac{n+1}{r+1}\cdot{}^nC_r}\right) = 2\left(\frac{r+1}{n+1}\right) \qquad \text{...(ii)}$$

From (i) and (ii), we obtain

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}.$$

EXAMPLE 52 *The* 3^{rd}, 4^{th} *and* 5^{th} *terms in the expansion of* $(x+a)^n$ *are respectively* 84, 280 *and* 560, *find the values of* x, a *and* n.

SOLUTION It is given that: $T_3 = 84$, $T_4 = 280$ and $T_5 = 560$

We have,

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r\, x^{n-r}\, a^r}{{}^nC_{r-1}\, x^{n-r+1}\, a^{r-1}} = \frac{n-r+1}{r}\cdot\frac{a}{x}$$

$$\therefore \quad \frac{T_4}{T_3} = \frac{n-2}{3}\cdot\frac{a}{x} \text{ and } \frac{T_5}{T_4} = \frac{n-3}{4}\cdot\frac{a}{x}$$

$$\Rightarrow \quad \frac{280}{84} = \frac{n-2}{3}\cdot\frac{a}{x} \text{ and } \frac{560}{280} = \frac{n-3}{4}\cdot\frac{a}{x} \qquad [\because T_3 = 84, T_4 = 280 \text{ and } T_5 = 560]$$

$$\Rightarrow \quad \frac{10}{3} = \frac{n-2}{3}\cdot\frac{a}{x} \text{ and } \frac{2}{1} = \frac{n-3}{4}\cdot\frac{a}{x}$$

$$\Rightarrow \quad \frac{a}{x} = \frac{10}{n-2} \text{ and } \frac{a}{x} = \frac{8}{n-3}$$

$$\Rightarrow \quad \frac{10}{n-2} = \frac{8}{n-3} \Rightarrow 5n-15 = 4n-8 \Rightarrow n = 7$$

Putting $n = 7$ in $\frac{a}{x} = \frac{10}{n-2}$, we get

$$\frac{a}{x} = \frac{10}{5} \Rightarrow 2x = a$$

Now, $T_3 = 84$

$$\Rightarrow \quad {}^nC_2\, x^{n-2}\, a^2 = 84$$

$$\Rightarrow \quad {}^7C_2\, x^5\, (2x)^2 = 84 \qquad [\because a = 2x \text{ and } n = 7]$$

$$\Rightarrow \quad 21 \times 2^4 \times x^7 = 84 \Rightarrow x^7 = 1 \Rightarrow x = 1$$

$$\therefore \quad a = 2x = 2 \times 1 = 2$$

Hence, $n = 7$, $a = 2$ and $x = 1$.

Type XIII **ON APPLICATIONS OF BINOMIAL THEOREM**

EXAMPLE 53 *How many terms are free from radical signs in the expansion of* $(x^{1/5} + y^{1/10})^{55}$.

SOLUTION The general term in the expansion of $(x^{1/5} + y^{1/10})^{55}$ is given by

$$T_{r+1} = {}^{55}C_r \left(x^{1/5}\right)^{55-r} \left(y^{1/10}\right)^r \Rightarrow T_{r+1} = {}^{55}C_r\, x^{11-r/5}\, y^{r/10}$$

Clearly, T_{r+1} will be free from radical signs, if $\frac{r}{5}$ and $\frac{r}{10}$ are integers for $0 \le r \le 55$

$\therefore \quad r = 0, 10, 20, 30, 40, 50.$

Hence, there are 6 terms in the expansion of $(x^{1/5} + y^{1/10})^{55}$ which are independent of radical signs.

EXAMPLE 54 *Find the number of integral terms in the expansion of* $\left(5^{1/2} + 7^{1/8}\right)^{1024}$.

SOLUTION The general term T_{r+1} in the expansion of $\left(5^{1/2} + 7^{1/8}\right)^{1024}$ is given by

$$T_{r+1} = {}^{1024}C_r \left(5^{1/2}\right)^{1024-r} \left(7^{1/8}\right)^r$$

$\Rightarrow$ $$T_{r+1} = {}^{1024}C_r \; 5^{512-\frac{r}{2}} \; 7^{r/8}$$

$\Rightarrow$ $$T_{r+1} = \left\{{}^{1024}C_r \; 5^{512-r}\right\} \times 5^{r/2} \times 7^{r/8}$$

$\Rightarrow$ $$T_{r+1} = \left\{{}^{1024}C_r \; 5^{512-r}\right\} \times \left(5^4 \times 7\right)^{r/8}$$

Clearly, T_{r+1} will be an integer, iff

$\frac{r}{8}$ is an integer such that $0 \le r \le 1024$

$\Rightarrow$ r is a multiple of 8 satisfying $0 \le r \le 1024$

$\Rightarrow$ $r = 0, 8, 16, 24, \ldots, 1024$

$\Rightarrow$ r can assume 129 values.

Hence, there are 129 integral terms in the expansion of $\left(5^{1/2} + 7^{1/8}\right)^{1024}$.

EXERCISE 18.2

LEVEL-1

1. Find the 11th term from the beginning and the 11th term from the end in the expansion of $\left(2x - \frac{1}{x^2}\right)^{25}$.

2. Find the 7th term in the expansion of $\left(3x^2 - \frac{1}{x^3}\right)^{10}$.

3. Find the 5th term from the end in the expansion of $\left(3x - \frac{1}{x^2}\right)^{10}$.

4. Find the 8th term in the expansion of $(x^{3/2} y^{1/2} - x^{1/2} y^{3/2})^{10}$.

5. Find the 7th term in the expansion of $\left(\frac{4x}{5} + \frac{5}{2x}\right)^8$.

6. Find the 4th term from the beginning and 4th term from the end in the expansion of $\left(x + \frac{2}{x}\right)^9$.

7. Find the 4th term from the end in the expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$.

8. Find the 7th term from the end in the expansion of $\left(2x^2 - \frac{3}{2x}\right)^8$.

9. Find the coefficient of:

   (i) x^{10} in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$.

(ii) x^7 in the expansion of $\left(x-\frac{1}{x^2}\right)^{40}$.

(iii) x^{-15} in the expansion of $\left(3x^2-\frac{a}{3x^3}\right)^{10}$.

(iv) x^9 in the expansion of $\left(x^2-\frac{1}{3x}\right)^9$.

(v) x^m in the expansion of $\left(x+\frac{1}{x}\right)^n$.

(vi) x in the expansion of $(1-2x^3+3x^5)\left(1+\frac{1}{x}\right)^8$.

(vii) $a^5 b^7$ in the expansion of $(a-2b)^{12}$. [NCERT]

(viii) x in the expansion of $(1-3x+7x^2)(1-x)^{16}$. [NCERT EXEMPLAR]

10. Which term in the expansion of $\left\{\left(\frac{x}{\sqrt{y}}\right)^{1/3}+\left(\frac{y}{x^{1/3}}\right)^{1/2}\right\}^{21}$ contains x and y to one and the same power?

11. Does the expansion of $\left(2x^2-\frac{1}{x}\right)^{20}$ contain any term involving x^9?

12. Show that the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$ does not contain any term involving x^{-1}.

13. Find the middle term in the expansion of:

(i) $\left(\frac{2}{3}x-\frac{3}{2x}\right)^{20}$

(ii) $\left(\frac{a}{x}+bx\right)^{12}$

(iii) $\left(x^2-\frac{2}{x}\right)^{10}$

(iv) $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$

14. Find the middle terms in the expansion of:

(i) $\left(3x-\frac{x^3}{6}\right)^9$ [NCERT EXEMPLAR]

(ii) $\left(2x^2-\frac{1}{x}\right)^7$

(iii) $\left(3x-\frac{2}{x^2}\right)^{15}$

(iv) $\left(x^4-\frac{1}{x^3}\right)^{11}$

15. Find the middle term(s) in the expansion of:

(i) $\left(x-\frac{1}{x}\right)^{10}$

(ii) $(1-2x+x^2)^n$

(iii) $(1+3x+3x^2+x^3)^{2n}$

(iv) $\left(2x-\frac{x^2}{4}\right)^9$

(v) $\left(x-\frac{1}{x}\right)^{2n+1}$ (vi) $\left(\frac{x}{3}+9y\right)^{10}$ [NCERT]

(vii) $\left(3-\frac{x^3}{6}\right)^7$ (viii) $\left(2ax-\frac{b}{x^2}\right)^{12}$ [NCERT EXEMPLAR]

(ix) $\left(\frac{p}{x}+\frac{x}{p}\right)^9$ [NCERT EXEMPLAR] (x) $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$ [NCERT EXEMPLAR]

16. Find the term independent of x in the expansion of the following expressions:

(i) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ (ii) $\left(2x+\frac{1}{3x^2}\right)^9$

(iii) $\left(2x^2-\frac{3}{x^3}\right)^{25}$ (iv) $\left(3x-\frac{2}{x^2}\right)^{15}$ [NCERT EXEMPLAR]

(v) $\left(\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{2x^2}\right)^{10}$ [NCERT EXEMPLAR] (vi) $\left(x-\frac{1}{x^2}\right)^{3n}$

(vii) $\left(\frac{1}{2}x^{1/3}+x^{-1/5}\right)^8$ (viii) $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

[NCERT EXEMPLAR]

(ix) $\left(\sqrt[3]{x}+\frac{1}{2\sqrt[3]{x}}\right)^{18}$, $x>2$ (x) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^6$ [NCERT]

17. If the coefficients of $(2r+4)$th and $(r-2)$th terms in the expansion of $(1+x)^{18}$ are equal, find r. [NCERT EXEMPLAR]

18. If the coefficients of $(2r+1)$th term and $(r+2)$th term in the expansion of $(1+x)^{43}$ are equal, find r.

19. Prove that the coefficient of $(r+1)$th term in the expansion of $(1+x)^{n+1}$ is equal to the sum of the coefficients of rth and $(r+1)$th terms in the expansion of $(1+x)^n$.

20. Prove that the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\cdot 2^n$.

21. The coefficients of 5th, 6th and 7th terms in the expansion of $(1+x)^n$ are in A.P., find n.

22. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^{2n}$ are in A.P., show that $2n^2-9n+7=0$. [NCERT EXEMPLAR]

23. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^n$ are in A.P., then find the value of n.

24. If in the expansion of $(1+x)^n$, the coefficients of pth and qth terms are equal, prove that $p+q=n+2$, where $p\neq q$.

25. Find a, if the coefficients of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal. [NCERT]

26. Find the coefficient of a^4 in the product $(1+2a)^4(2-a)^5$ using binomial theorem.

[NCERT]

LEVEL-2

27. In the expansion of $(1+x)^n$ the binomial coefficients of three consecutive terms are respectively 220, 495 and 792, find the value of n.
28. If in the expansion of $(1+x)^n$, the coefficients of three consecutive terms are 56, 70 and 56, then find n and the position of the terms of these coefficients.
29. If 3rd, 4th, 5th and 6th terms in the expansion of $(x+\alpha)^n$ be respectively a, b, c and d, prove that $\dfrac{b^2-ac}{c^2-bd}=\dfrac{5a}{3c}$.
30. If a, b, c and d in any binomial expansion be the 6th, 7th, 8th and 9th terms respectively, then prove that $\dfrac{b^2-ac}{c^2-bd}=\dfrac{4a}{3c}$.
31. If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ be 76, 95 and 76, find n.
32. If the 6th, 7th and 8th terms in the expansion of $(x+a)^n$ are respectively 112, 7 and 1/4, find x, a, n.
33. If the 2nd, 3rd and 4th terms in the expansion of $(x+a)^n$ are 240, 720 and 1080 respectively, find x, a, n. [NCERT]
34. Find a, b and n in the expansion of $(a+b)^n$, if the first three terms in the expansion are 729, 7290 and 30375 respectively. [NCERT]
35. If the term free from x in the expansion of $\left(\sqrt{x}-\dfrac{k}{x^2}\right)^{10}$ is 405, find the value of k. [NCERT EXEMPLAR]
36. Find the sixth term in the expansion $\left(y^{1/2}+x^{1/3}\right)^n$, if the binomial coefficient of the third term from the end is 45. [NCERT EXEMPLAR]
37. If p is a real number and if the middle term in the expansion of $\left(\dfrac{p}{2}+2\right)^8$ is 1120, find p. [NCERT EXEMPLAR]
38. Find n in the binomial $\left(\sqrt[3]{2}+\dfrac{1}{\sqrt[3]{3}}\right)^n$, if the ratio of 7th term from the beginning to the 7th term from the end is $\dfrac{1}{6}$. [NCERT EXEMPLAR]
39. If the seventh term from the beginning and end in the binomial expansion of $\left(\sqrt[3]{2}+\dfrac{1}{\sqrt[3]{3}}\right)^n$ are equal, find n. [NCERT EXEMPLAR]

ANSWERS

1. ${}^{25}C_{10}\left(\dfrac{2^{15}}{x^5}\right), -{}^{25}C_{15}\left(\dfrac{2^{10}}{x^{20}}\right)$
2. $\dfrac{17010}{x^{10}}$
3. $\dfrac{17010}{x^8}$

4. $-120\,x^8y^{12}$ 5. $\frac{4375}{x^4}$ 6. $672\,x^3, \frac{5376}{x^3}$ 7. $\frac{10500}{x^3}$

8. $4032\,x^{10}$ 9. (i) ${}^{20}C_{10}\cdot 2^{10}$ (ii) $-{}^{40}C_{11}$ (iii) $-\frac{40}{27}a^7$

(iv) $-\frac{28}{9}$ (v) $\frac{n!}{\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}$ (vi) 154 (vii) -101376 (viii) -19

10. 10th 11. No

13. (i) ${}^{20}C_{10}$ (ii) $924\,a^6b^6$ (iii) $-8064\,x^5$ (iv) -252

14. (i) $\frac{189}{8}x^{17}, -\frac{21}{16}x^{19}$ (ii) $-560\,x^5, 280\,x^2$

(iii) $\frac{-6435\times 3^8\times 2^7}{x^6}, \frac{6437\times 3^7\times 2^8}{x^9}$ (iv) $-462\,x^9,\ 462\,x^2$

15. (i) -252 (ii) $\frac{(2n)!}{(n!)^2}(-1)^n x^n$ (iii) $\frac{(6n)!}{[(3n)!]^2}\ x^{3n}$

(iv) $\frac{63}{4}x^{13}, -\frac{63}{32}x^{14}$ (v) $(-1)^n\,.\,{}^{2n+1}C_n\,x,\ (-1)^{n+1}\,.\,{}^{2n+1}C_n\frac{1}{x}$

(vi) $61236\,x^5y^5$ (vii) $-\frac{105}{8}x^9, \frac{35}{48}x^{12}$ (viii) $\frac{59\,136\,a^6b^6}{x^6}$

(ix) $\frac{126x}{p}$ (x) -252

16. (i) $\frac{7}{18}$ (ii) $\frac{64}{27}\times{}^9C_3$ (iii) ${}^{25}C_{10}\,(2^{15}\times 3^{10})$

(iv) $-3003\times 3^{10}\times 2^5$ (v) $\frac{5}{12}$ (vi) $(-1)^n\ {}^{3n}C_n$

(vii) 7 (viii) $\frac{17}{54}$ (ix) $\frac{{}^{18}C_9}{2^9}$ (x) $\frac{5}{12}$

17. 6 18. 14 21. 7 or 14 23. 7

25. $\frac{8}{7}$ 26. -438 27. 12 28. $n=8$, 4th, 5th, 6th

31. 8 32. $n=8,\ x=4, a=\frac{1}{2}$ 33. $n=5, x=2, a=3$

34. $a=3, b=5, n=6$ 35. $k=\pm 3$ 36. $252\,y^{5/2}x^{5/3}$ 37. $p=\pm 2$

38. $n=9$ 39. $n=12$

HINTS TO NCERT & SELECTED PROBLEMS

9. (vii) Let T_{r+1} be the $(r+1)^{th}$ term in the expansion of $(a-2b)^{12}$. Then,

$$T_{r+1} = {}^{12}C_r\, a^{12-r}\,(-2b)^r = {}^{12}C_r\,(-1)^r\,2^r\,a^{12-r}b^r$$

If a^5b^7 appears in $(r+1)^{th}$ term, then

$$12-r=5 \text{ and } r=7 \Rightarrow r=7$$

Thus, a^5b^7 appears in 8^{th} term given by $T_8 = {}^{12}C_7\,(-1)^7\,2^7\,a^5b^7 = -101376a^5b^7$

Hence, Coefficient of $a^5b^7 = -101376$

(viii) $(1-3x+7x^2)(1-x)^{16} = (1-3x+7x^2)({}^{16}C_0 - {}^{16}C_1\, x\, {}^{16}C_2\, x^2 - {}^{16}C_3\, x^3 + \ldots)$

$\therefore$ Coefficient of x in $(1-3x+7x^2)(1-x)^{16} = 1 \times -{}^{16}C_1 - 3 \times {}^{16}C_0 = -16-3 = -19$

15. (vi) In the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ there are 11 terms. So, $\left(\frac{10}{2}+1\right)^{th}$ i.e. 6th term is the middle term.

$$\text{Now, } T_6 = T_{5+1} = {}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5 = 61236x^5y^5$$

16. (x) Let $(r+1)^{th}$ term in the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^6$ be independent of x. Then the exponent of x in $(r+1)^{th}$ term must be zero.

$$\text{Now, } T_{r+1} = {}^6C_r\left(\frac{3}{2}x^2\right)^{6-r}\left(-\frac{1}{3x}\right)^r$$

$$\Rightarrow \quad T_{r+1} = {}^6C_r\left(\frac{3}{2}\right)^{6-r}\left(-\frac{1}{3}\right)^r x^{12-3r} \qquad \ldots(i)$$

For T_{r+1} to be independent of x, we must have

$12-3r=0 \Rightarrow r=4$

Hence, 5^{th} term is independent of x.

Puting $r=4$ in (i), we get

$$T_5 = {}^6C_4\left(\frac{3}{2}\right)^2\left(-\frac{1}{3}\right)^4 = 15 \times \frac{1}{4\times 9} = \frac{5}{12}$$

18. ${}^{18}C_{2r+3} = {}^{18}C_{r-3} \Rightarrow (2r+3)+(r-3)=18 \Rightarrow r=6$

25. We have,

$$(3+ax)^9 = {}^9C_0 \times 3^9 + {}^9C_1 \times 3^8 \times (ax)^1 + {}^9C_2 \times 3^7 \times (ax)^2 + {}^9C_3 \times 3^6 \times (ax)^3 + \ldots + {}^9C_9(ax)^9$$

$\therefore$ Coefficient of $x^2 = {}^9C_2 \times 3^7 \times a^2$ and, Coefficient of $x^3 = {}^9C_3 \times 3^6 \times a^3$

Now, Coefficient of x^2 = Coefficient of x^3

$$\Rightarrow \quad {}^9C_2 \times 3^7 \times a^2 = {}^9C_3 \times 3^6 \times a^3 \Rightarrow 36 \times 3^7 \times a^2 = 84 \times 3^6 \times a^3 \Rightarrow a = \frac{36\times 3^7}{84 \times 3^6} = \frac{9}{7}$$

26. $$(1+2a)^4(2-a)^5 = \left\{{}^4C_0 + {}^4C_1(2a) + {}^4C_2(2a)^2 + {}^4C_3(2a)^3 + {}^4C_4(2a)^4\right\}$$
$$\times\left\{{}^5C_0\,2^5 - {}^5C_1\,2^4a + {}^5C_2\,2^3a^2 - {}^5C_3\,2^2a^3 + {}^5C_4(2)\,a^4 - {}^5C_5\,a^5\right\}$$

$\therefore$ $$\text{Coefficient of } a^4 = {}^4C_0 \times \left({}^5C_4 \times 2\right) + \left({}^4C_1 \times 2\right)\times\left(-{}^5C_3 \times 2^2\right) + \left({}^4C_2 \times 2^2\right)\times\left({}^5C_2 \times 2^3\right)$$
$$+\left({}^4C_3 \times 2^3\right)\times\left(-{}^5C_1 \times 2^4\right) + \left({}^4C_4 \times 2^4\right)\times\left({}^5C_0 \times 2^5\right)$$

$= 10 + 8 \times (-40) + 24 \times 80 + (4 \times 8)(-80) + (16 \times 32)$

$= 10 - 320 + 1920 - 2560 + 512 = -438$

33. It is given that in the expansion of $(x + a)^n$

$T_2 = 240, T_3 = 720$ and $T_4 = 1080$

$\Rightarrow \dfrac{T_3}{T_2} = 3$ and $\dfrac{T_4}{T_3} = \dfrac{3}{2}$

$\Rightarrow \dfrac{n-2+1}{2}\dfrac{a}{x} = 3$ and $\dfrac{n-3+1}{3}\dfrac{a}{x} = \dfrac{3}{2}$ $\left[\because \dfrac{T_{r+1}}{T_r} = \dfrac{n-r+1}{r}\dfrac{a}{x}\right]$

$\Rightarrow (n-1)\dfrac{a}{x} = 6$ and $(n-2)\dfrac{a}{x} = \dfrac{9}{2}$

$\therefore \dfrac{n-1}{n-2} = \dfrac{6}{9} \times 2 \Rightarrow \dfrac{n-1}{n-2} = \dfrac{4}{3} \Rightarrow 4n - 8 = 3n - 3 \Rightarrow n = 5$

Putting $n = 5$ in $(n-1)\dfrac{a}{x} = 6$, we get $2a = 3x$.

Now, $T_2 = 240$

$\Rightarrow {}^nC_1 x^{n-1} a = 240$

$\Rightarrow n x^{n-1} a = 240$

$\Rightarrow 5x^4 a = 240$ $[\because n = 5]$

$\Rightarrow x^4 \times \dfrac{3x}{2} = 48$ $\left[\because a = \dfrac{3x}{2}\right]$

$\Rightarrow x^5 = 32 \Rightarrow x^5 = (2)^5 \Rightarrow x = 2$

$\therefore 2a = 3x \Rightarrow a = 3$

Hence, $x = 2, a = 3$ and $n = 5$.

34. We have,

${}^nC_0 a^n b^0 = 729, {}^nC_1 a^{n-1} b = 7290$ and ${}^nC_2 a^{n-2} b^2 = 30375$

$\Rightarrow a^n = 729, n a^{n-1} b = 7290$ and $n(n-1) a^{n-2} b^2 = 60750$

$\therefore \dfrac{n a^{n-1} b}{a^n} = \dfrac{7290}{729}$ and $\dfrac{n(n-1) a^{n-2} b^2}{n a^{n-1} b} = \dfrac{60750}{7290}$

$\Rightarrow \dfrac{nb}{a} = 10$ and $\dfrac{(n-1)b}{a} = \dfrac{25}{3}$

$\Rightarrow \dfrac{(n-1)\frac{b}{a}}{n\frac{b}{a}} = \dfrac{25}{30} \Rightarrow \dfrac{n-1}{n} = \dfrac{5}{6} \Rightarrow n = 6$

Now, $a^n = 729 \Rightarrow a^6 = 3^6 \Rightarrow a = 3$

$\therefore \dfrac{nb}{a} = 10 \Rightarrow \dfrac{6 \times b}{3} = 10 \Rightarrow b = 5$

35. Let $(r+1)^{th}$ term, in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$, be free from x and be equal to T_{r+1}. Then,

$$T_{r+1} = {}^{10}C_r\,(\sqrt{x})^{10-r}\left(\frac{-k}{x^2}\right)^r = {}^{10}C_r\, x^{5-\frac{5r}{2}}\,(-k)^r \qquad \text{...(i)}$$

If T_{r+1} is independent of x, then

$$5-\frac{5r}{2}=0 \Rightarrow r=2$$

Putting $r=2$ in (i), we obtain

$$T_3 = {}^{10}C_r\,(-k)^2 = 45k^2$$

But, it is given that the value of term free from x is 405.

$\therefore \quad 45k^2 = 405 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$

36. In the binomial expansion of $(y^{1/2}+x^{1/3})^n$, there are $(n+1)$ terms. The third term from the end is $((n+1)-3+1)^{th}$ i.e. $(n-1)^{th}$ term from the beginning.

$\therefore$ The binomial coefficient of 3rd term from the end

= The binomial coefficient of $(n-1)^{th}$ term from the beginning $= {}^nC_{n-2} = {}^nC_2$

It is given that the binomial coefficient of the third term from the end is 45.

$\therefore \quad {}^nC_2 = 45 \Rightarrow \frac{n(n-1)}{2} = 45 \Rightarrow n^2-n-90=0 \Rightarrow (n-10)(n+9)=0 \Rightarrow n=10.$

Let T_6 be the sixth term in the binomial expansion of $(y^{1/2}+x^{1/3})^n$. Then,

$$T_6 = {}^nC_5\,(y^{1/2})^{n-5}\,(x^{1/3})^5 = {}^{10}C_5\, y^{5/2}\, x^{5/3} = 252\, y^{5/2} x^{5/3} \qquad [\because n=10]$$

37. In the expansion of $\left(\frac{p}{2}+2\right)^8$, we observe that $\left(\frac{8}{2}+1\right)^{th}$ i.e. 5th term is the middle term. It is given that the middle term is 1120.

$\therefore \quad T_5 = 1120$

$\Rightarrow \quad {}^8C_4\left(\frac{p}{2}\right)^{8-4}(2)^4 = 1120 \Rightarrow p^4 = 16 \Rightarrow p = \pm 2$

38. In the binomial expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, $\left((n+1)-7+1\right)^{th}$ i.e. $(n-5)^{th}$ term from the beginning is the 7th term from the end.

Now,

$$T_7 = {}^nC_6\,(\sqrt[3]{2})^{n-6}\left(\frac{1}{\sqrt[3]{3}}\right)^6 = {}^nC_6 \times 2^{\frac{n}{3}-2} \times \frac{1}{3^2}$$

and, $T_{n-5} = {}^nC_{n-6}\,(\sqrt[3]{2})^6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6} = {}^nC_6 \times 2^2 \times \frac{1}{3^{n/3-2}}$

It is given that

$$\frac{T_7}{T_{n-5}} = \frac{1}{6}$$

$$\Rightarrow \quad \frac{{}^nC_6 \times 2^{(n/3)-2} \times \frac{1}{3^2}}{{}^nC_6 \times 2^2 \times \frac{1}{3^{(n/3)-2}}} = \frac{1}{6}$$

$$\Rightarrow \quad 2^{(n/3)-4} \times 3^{(n/3)-4} = 6^{-1}$$

$$\Rightarrow \quad 6^{(n/3)-4} = 6^{-1} \Rightarrow \frac{n}{3} - 4 = -1 \Rightarrow n = 9$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the number of terms in the expansion of $(2 + \sqrt{3}x)^{10} + (2 - \sqrt{3}x)^{10}$.
2. Write the sum of the coefficients in the expansion of $(1 - 3x + x^2)^{111}$.
3. Write the number of terms in the expansion of $(1 - 3x + 3x^2 - x^3)^8$.
4. Write the middle term in the expansion of $\left(\frac{2x^2}{3} + \frac{3}{2x^2}\right)^{10}$.
5. Which term is independent of x, in the expansion of $\left(x - \frac{1}{3x^2}\right)^9$?
6. If a and b denote respectively the coefficients of x^m and x^n in the expansion of $(1 + x)^{m+n}$, then write the relation between a and b.
7. If a and b are coefficients of x^n in the expansions of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then write the relation between a and b.
8. Write the middle term in the expansion of $\left(x + \frac{1}{x}\right)^{10}$.
9. If a and b denote the sum of the coefficients in the expansions of $(1 - 3x + 10x^2)^n$ and $(1 + x^2)^n$ respectively, then write the relation between a and b.
10. Write the coefficient of the middle term in the expansion of $(1 + x)^{2n}$.
11. Write the number of terms in the expansion of $\{(2x+y^3)^4\}^7$.
12. Find the sum of the coefficients of two middle terms in the binomial expansion of $(1+x)^{2n-1}$.
13. Find the ratio of the coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$.
14. Write last two digits of the number 3^{400}.
15. Find the number of terms in the expansion of $(a+b+c)^n$.
16. If a and b are the coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, find $\frac{a}{b}$.

17. Write the total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$.

18. If $(1-x+x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}$, find the value of $a_0 + a_2 + a_4 + \ldots + a_{2n}$.

ANSWERS

1. 6 2. -1 3. 25 4. 252 5. 4^{th} term 6. $a = b$ 7. $a = 2b$
8. ${}^{10}C_5$ 9. $a = b^3$ 10. ${}^{2n}C_n$ 11. 29 12. ${}^{2n}C_n$ 13. 1 14. 01
15. $\frac{n(n+1)}{2}$ 16. 2 17. 51 18. $\frac{3^n+1}{2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If in the expansion of $(1+x)^{20}$, the coefficients of rth and $(r+4)$th terms are equal, then r is equal to
(a) 7 (b) 8 (c) 9 (d) 10

2. The term without x in the expansion of $\left(2x - \frac{1}{2x^2}\right)^{12}$ is
(a) 495 (b) -495 (c) -7920 (d) 7920

3. If rth term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$ is without x, then r is equal to
(a) 8 (b) 7 (c) 9 (d) 10

4. If in the expansion of $(a+b)^n$ and $(a+b)^{n+3}$, the ratio of the coefficients of second and third terms, and third and fourth terms respectively are equal, then n is
(a) 3 (b) 4 (c) 5 (d) 6

5. If A and B are the sums of odd and even terms respectively in the expansion of $(x+a)^n$, then $(x+a)^{2n} - (x-a)^{2n}$ is equal to
(a) $4(A+B)$ (b) $4(A-B)$ (c) AB (d) $4AB$

6. The number of irrational terms in the expansion of $\left(4^{1/5} + 7^{1/10}\right)^{45}$ is
(a) 40 (b) 5 (c) 41 (d) none of these

7. The coefficient of x^{-17} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ is
(a) 1365 (b) -1365 (c) 3003 (d) -3003

8. In the expansion of $\left(x^2 - \frac{1}{3x}\right)^9$, the term without x is equal to
(a) $\frac{28}{81}$ (b) $\frac{-28}{243}$ (c) $\frac{28}{243}$ (d) none of these

9. If in the expansion of $(1+x)^{15}$, the coefficients of $(2r+3)^{\text{th}}$ and $(r-1)^{\text{th}}$ terms are equal, then the value of r is
(a) 5 (b) 6 (c) 4 (d) 3

10. The middle term in the expansion of $\left(\frac{2x^2}{3}+\frac{3}{2x^2}\right)^{10}$ is

(a) 251 (b) 252 (c) 250 (d) none of these

11. If in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$, x^{-17} occurs in rth term, then

(a) $r=10$ (b) $r=11$ (c) $r=12$ (d) $r=13$

12. In the expansion of $\left(x-\frac{1}{3x^2}\right)^9$, the term independent of x is

(a) T_3 (b) T_4 (c) T_5 (d) none of these

13. If in the expansion of $(1+y)^n$, the coefficients of 5th, 6th and 7th terms are in A.P., then n is equal to

(a) 7, 11 (b) 7, 14 (c) 8, 16 (d) none of these

14. In the expansion of $\left(\frac{1}{2}x^{1/3}+x^{-1/5}\right)^8$, the term independent of x is

(a) T_5 (b) T_6 (c) T_7 (d) T_8

15. If the sum of odd numbered terms and the sum of even numbered terms in the expansion of $(x+a)^n$ are A and B respectively, then the value of $(x^2-a^2)^n$ is

(a) A^2-B^2 (b) A^2+B^2 (c) $4AB$ (d) none of these

16. If the coefficient of x in $\left(x^2+\frac{\lambda}{x}\right)^5$ is 270, then $\lambda=$

(a) 3 (b) 4 (c) 5 (d) none of these

17. The coefficient of x^4 in $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$ is

(a) $\frac{405}{256}$ (b) $\frac{504}{259}$ (c) $\frac{450}{263}$ (d) none of these

18. The total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$ after simplification is

(a) 202 (b) 51 (c) 50 (d) none of these

19. If T_2/T_3 in the expansion of $(a+b)^n$ and T_3/T_4 in the expansion of $(a+b)^{n+3}$ are equal, then $n=$

(a) 3 (b) 4 (c) 5 (d) 6

20. The coefficient of $\frac{1}{x}$ in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$ is

(a) $\frac{n!}{\{(n-1)!(n+1)!\}}$ (b) $\frac{(2n)!}{[(n-1)!(n+1)!]}$

(c) $\frac{(2n)!}{(2n-1)!(2n+1)!}$ (d) none of these

21. If the sum of the binomial coefficients of the expansion $\left(2x+\frac{1}{x}\right)^n$ is equal to 256, then the term independent of x is

(a) 1120 (b) 1020 (c) 512 (d) none of these

22. If the fifth term of the expansion $(a^{2/3} + a^{-1})^n$ does not contain 'a'. Then n is equal to
(a) 2 (b) 5 (c) 10 (d) none of these

23. The coefficient of x^{-3} in the expansion of $\left(x - \frac{m}{x}\right)^{11}$ is
(a) $-924\, m^7$ (b) $-792\, m^5$ (c) $-792\, m^6$ (d) $-330\, m^7$

24. The coefficient of the term independent of x in the expansion of $\left(ax + \frac{b}{x}\right)^{14}$ is
(a) $14!\, a^7 b^7$ (b) $\frac{14!}{7!} a^7 b^7$ (c) $\frac{14!}{(7!)^2} a^7 b^7$ (d) $\frac{14!}{(7!)^3} a^7 b^7$

25. The coefficient of x^5 in the expansion of $(1 + x)^{21} + (1 + x)^{22} + \ldots + (1 + x)^{30}$ is
(a) ${}^{51}C_5$ (b) ${}^{9}C_5$ (c) ${}^{31}C_6 - {}^{21}C_6$ (d) ${}^{30}C_5 + {}^{20}C_5$

26. The coefficient of $x^8\, y^{10}$ in the expansion of $(x + y)^{18}$ is
(a) ${}^{18}C_8$ (b) ${}^{18}P_{10}$ (c) 2^{18} (d) none of these

27. If the coefficients of the $(n + 1)^{th}$ term and the $(n + 3)^{th}$ term in the expansion of $(1 + x)^{20}$ are equal, then the value of n is
(a) 10 (b) 8 (c) 9 (d) none of these

28. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^n$, $n \in N$ are in A.P., then $n =$
(a) 7 (b) 14 (c) 2 (d) none of these

29. The middle term in the expansion of $\left(\frac{2x}{3} - \frac{3}{2x^2}\right)^{2n}$ is
(a) ${}^{2n}C_n$ (b) $(-1)^n\ {}^{2n}C_n\, x^{-n}$ (c) ${}^{2n}C_n\, x^{-n}$ (d) none of these

30. If r^{th} term is the middle term in the expansion of $\left(x^2 - \frac{1}{2x}\right)^{20}$, then $(r + 3)^{th}$ term is
(a) ${}^{20}C_{14}\left(\frac{x}{2^{14}}\right)$ (b) ${}^{20}C_{12}\, x^2\, 2^{-12}$ (c) $-{}^{20}C_7\, x \,.\, 2^{-13}$ (d) none of these

31. The number of terms with integral coefficients in the expansion of $\left(17^{1/3} + 35^{1/2} x\right)^{600}$ is
(a) 100 (b) 50 (c) 150 (d) 101

32. Constant term in the expansion of $\left(x - \frac{1}{x}\right)^{10}$ is
(a) 152 (b) -152 (c) -252 (d) 252

33. If the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are the same, then the value of a is
(a) $-\frac{7}{9}$ (b) $-\frac{9}{7}$ (c) $\frac{7}{9}$ (d) $\frac{9}{7}$

ANSWERS

1. (c)	2. (d)	3. (c)	4. (c)	5. (d)	6. (c)	7. (b)	8. (c)
9. (a)	10. (b)	11. (c)	12. (b)	13. (b)	14. (b)	15. (a)	16. (a)
17. (a)	18. (b)	19. (c)	20. (b)	21. (a)	22. (c)	23. (d)	24. (c)
25. (b)	26. (a)	27. (c)	28. (a)	29. (b)	30. (c)	31. (d)	32. (c)
33. (d)							

SUMMARY

1. (*Binomial theorem*) If x and a are real numbers, then for all $n \in N$, we have

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots\ldots\ldots + {}^nC_r\, x^{n-r} a^r + \ldots + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x^0 a^n$$

i.e., $$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

This expansion has the following properties:

(i) It has $(n+1)$ terms.

(ii) The sum of the indices of x and a in each term is n.

(iii) The coefficients of terms equidistant from the beginning and the end are equal.

(vi) General term is given by $T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

(v) $(x+a)^n = \sum\limits_{r=0}^{n} {}^nC_r\ x^{n-r}\ a^r$ can also be expressed as $(x+a)^n = \sum\limits_{r+s=n} \frac{n!}{r!\,s!}\, x^r\ a^s$

(vi) Replacing a by $-a$ in the expansion of $(x+a)^n$, we get

$$(x-a)^n = {}^nC_0\ x^n a^0 - {}^nC_1\ x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 - {}^nC_3\, x^{n-3} a^3 + \ldots + (-1)^r\ {}^nC_r\ x^{n-r}\ a^r + \ldots + (-1)^n\ {}^nC_n\ x^0\ a^n$$

The general term in the expansion of $(x-a)^n$ is $T_{r+1} = (-1)^r\ {}^nC_r\ x^{n-r}\ a^r$

(vii) Putting $x = 1$ and replacing a by x in the expansion of $(x+a)^n$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_n\, x^n = \sum_{r=0}^{n} {}^nC_r\ x^r$$

This is expansion of $(1+x)^n$ is ascending powers of x. In this case, $T_{r+1} = {}^nC_r\, x^r$

(viii) Putting $a = 1$ in the expansion of $(x+a)^n$, we get

$$(1+x)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} + {}^nC_2\, x^{n-2} + \ldots + {}^nC_n\, x^0 = \sum_{r=0}^{n} {}^nC_r\ x^{n-r}$$

This is the expansion of $(1+x)^n$ in descending powers of x. In this case, $T_{r+1} = {}^nC_r\ x^{n-r}$

(ix) $$(x+a)^n + (x-a)^n = 2\left\{{}^nC_0\, x^n a^0 + {}^nC_2\, x^{n-2} a^2 + \ldots\right\}$$
$$= 2\,\{\text{Sum of the odd terms in the expansion of } (x+a)^n\}$$

$$(x+a)^n - (x-a)^n = 2\left\{{}^nC_1\, x^{n-1} a^1 + {}^nC_3\, x^{n-3} a^3 + \ldots\right\}$$
$$= 2\,\{\text{Sum of the even terms in the expansion of } (x+a)^n\}$$

If n is odd, then $\left\{(x+a)^n + (x-a)^n\right\}$ and $\left\{(x+a)^n - (x-a)^n\right\}$ both have $\left(\dfrac{n+1}{2}\right)$ terms.

If n is even, then $\left\{(x+a)^n + (x-a)^n\right\}$ has $\left(\dfrac{n}{2}+1\right)$ terms whereas $\left\{(x+a)^n - (x-a)^n\right\}$ has $\left(\dfrac{n}{2}\right)$ terms.

(x) If O and E denote respectively the sums of odd terms and even terms in the expansion of $(x+a)^n$, then

(a) $(x+a)^n = O+E$ and $(x-a)^n = O-E$ (b) $(x^2-a^2)^n = O^2-E^2$

(c) $4\,OE = (x-a)^{2n} - (n-a)^{2n}$ (d) $(x+a)^{2n} + (x-a)^{2n} = 2\,(O^2+E^2)$

(xi) If n is even, then $\left(\dfrac{n}{2}+1\right)^{\text{th}}$ term is the middle term.

If n is odd, then $\left(\dfrac{n+1}{2}\right)$ and $\left(\dfrac{n+3}{2}\right)$ are middle terms.

CHAPTER 19

ARITHMETIC PROGRESSIONS

19.1 SEQUENCE

A sequence is a function whose domain is the set N of natural numbers.

It is customary to denote a sequence by a letter 'a' and the image $a(n)$ of $n \in N$ under a by a_n. Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range. The images of $1, 2, 3, ..., n, ...$ under a sequence 'a' are generally denoted by $a_1, a_2, a_3, ... a_n, ...$ respectively. $a_1, a_2, a_3, ... a_n, ...$ are known as first term, second term ..., nth term,... respectively of the sequence. If a_n is the nth term of a sequence, 'a' then we write $a = < a_n >$.

REAL SEQUENCE *A sequence whose range is a subset of R is called a real sequence.*

In other words, a real sequence is a function with domain N and the range a subset of the set R of real numbers.

REPRESENTATION OF A SEQUENCE *There are several ways of representing a real sequence.*

One way to represent a real sequence is to list its first few terms till the rule for writing down other terms becomes clear. For example, 1, 3, 5, ... is a sequence whose nth term is $(2n - 1)$.

Another way to represent a real sequence is to give a rule of writing the nth term of the sequence. For example, the sequence 1, 3, 5, 7, ... can be written as $a_n = 2n - 1$.

Sometimes we represent a real sequence by using a recursive relation. For example, the Fibonacci sequence is given by

$$a_1 = 1, a_2 = 1 \text{ and } a_{n+1} = a_n + a_{n-1}, n \geq 2$$

The terms of this sequence are 1, 1, 2, 3, 5, 8, ...

ILLUSTRATION 1 *Give first 3 terms of the sequence defined by* $a_n = \dfrac{n}{n^2 + 1}$.

SOLUTION Putting $n = 1, 2, 3$ in $a_n = \dfrac{n}{n^2+1}$, we get

$$a_1 = \frac{1}{1^2 + 1} = \frac{1}{2},\ a_2 = \frac{2}{2^2 + 1} = \frac{2}{5} \text{ and } a_3 = \frac{3}{3^2 + 1} = \frac{3}{10}.$$

ILLUSTRATION 2 *Find the first four terms of the sequence whose first term is 1 and whose $(n + 1)$th term is obtained by subtracting n from its nth term.*

SOLUTION We are given that $a_1 = 1$ and $a_{n+1} = a_n - n$.

Putting $n = 1$, we obtain

$$a_2 = a_1 - 1 \Rightarrow a_2 = 1 - 1 = 0 \qquad [\because a_1 = 1]$$

Putting $n = 2$, we obtain

$$a_3 = a_2 - 2 \Rightarrow a_3 = 0 - 2 = -2$$

Similarly, by putting $n = 3$, we obtain

$$a_4 = a_3 - 3 = -2 - 3 = -5$$

SERIES *If $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ is a sequence, then the expression $a_1 + a_2 + a_3 + a_4 + \ldots + a_n + \ldots$ is a series.*

A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

PROGRESSIONS *It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the nth term. Those sequences whose terms follow certain patterns are called progressions.*

In this chapter, we shall study arithmetical progressions as defined below.

19.2 ARITHMETIC PROGRESSION (A.P.)

A sequence is called an arithmetic progression if the difference of a term and the previous term is always same.

i.e. $a_{n+1} - a_n = \text{constant } (= d) \text{ for all } n \in N$

The constant difference, generally denoted by d is called the common difference.

ILLUSTRATION 1 *1, 4, 7, 10, ... is an A.P. whose first term is 1 and the common difference is equal to* $4 - 1 = 3$.

ILLUSTRATION 2 *11, 7, 3, –1, ... is an A.P. whose first term is 11 and the common difference is equal to* $7 - 11 = -4$.

In order to determine whether a sequence is an A.P. or not when its nth term is given, we may use the following algorithm.

ALGORITHM

STEP I *Obtain a_n.*

STEP II *Replace n by $n+1$ in a_n to get a_{n+1}.*

STEP III *Calculate $a_{n+1} - a_n$.*

STEP IV *If $a_{n+1} - a_n$ is independent of n, the given sequence is an A.P. Otherwise it is not an A.P.*

Following examples illustrate the procedure:

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Show that the sequence defined by $a_n = 4n + 5$ is an A.P. Also, find its common difference.*

SOLUTION We have, $a_n = 4n + 5$

Replacing n by $(n+1)$, we get

$$a_{n+1} = 4(n+1) + 5 = 4n + 9$$

$$\therefore \quad a_{n+1} - a_n = (4n+9) - (4n+5) = 4$$

Clearly, $a_{n+1} - a_n$ is independent of n and is equal to 4.

So, the given sequence is an A.P. with common difference 4.

EXAMPLE 2 *Show that the sequence defined by $a_n = 2n^2 + 1$ is not an A.P.*

SOLUTION We have, $a_n = 2n^2 + 1$

Replacing n by $(n+1)$ in a_n, we obtain

$$a_{n+1} = 2(n+1)^2 + 1 = 2n^2 + 4n + 3$$

$$\therefore \quad a_{n+1} - a_n = (2n^2 + 4n + 3) - (2n^2 + 1) = 4n + 2$$

Clearly, $a_{n+1} - a_n$ is not independent of n and is therefore not constant. So, the given sequence is not an A.P.

EXAMPLE 3 *Show that the sequence* $\log a, \log\left(\frac{a^2}{b}\right), \log\left(\frac{a^3}{b^2}\right), \log\left(\frac{a^4}{b^3}\right), \ldots\ldots$ *forms an A.P.*

SOLUTION We have,

$$\log\left(\frac{a^2}{b}\right) - \log a = \log\left(\frac{a^2}{b} \times \frac{1}{a}\right) = \log\left(\frac{a}{b}\right)$$

$$\log\left(\frac{a^3}{b^2}\right) - \log\left(\frac{a^2}{b}\right) = \log\left(\frac{a^3}{b^2} \times \frac{b}{a^2}\right) = \log\left(\frac{a}{b}\right)$$

$$\log\left(\frac{a^4}{b^3}\right) - \log\left(\frac{a^3}{b^2}\right) = \log\left(\frac{a^4}{b^3} \times \frac{b^2}{a^3}\right) = \log\left(\frac{a}{b}\right)$$

This shows that the difference of a term and the preceding term is always same.
Hence, the given sequence forms an A.P.

ALITER From the symmetry, we obtain

$$a_n = \log\left(\frac{a^n}{b^{n-1}}\right)$$

$$\Rightarrow \quad a_{n+1} = \log\left(\frac{a^{n+1}}{b^n}\right)$$

$$\therefore \quad a_{n+1} - a_n = \log\left(\frac{a^{n+1}}{b^n}\right) - \log\left(\frac{a^n}{b^{n-1}}\right) = \log\left(\frac{a^{n+1}}{b^n} \times \frac{b^{n-1}}{a^n}\right) = \log\left(\frac{a}{b}\right)$$

Clearly, $a_{n+1} - a_n$ is constant for all values of n.

So, the given sequence is an A.P. with common difference $\log\left(\frac{a}{b}\right)$.

LEVEL-2

EXAMPLE 4 *Show that a sequence is an A.P. if its nth term is a linear expression in n and in such a case the common difference is equal to the coefficient of n.*

SOLUTION Let a_n be the n^{th} term of a sequence. Let a_n be a linear expression in n.

i.e. $\quad a_n = An + B$, where A, B are constants.

$$\Rightarrow \quad a_{n+1} = A(n+1) + B$$

$$\therefore \quad a_{n+1} - a_n = \{A(n+1) + B\} - \{An + B\} = A$$

Clearly, $a_{n+1} - a_n$ is independent of n and is therefore a constant.
Hence, the sequence is an A.P. with common difference A.

NOTE *Students are advised to use the statement of the above example as a standard result.*

EXAMPLE 5 *The n^{th} term of a sequence is $3n - 2$. Is the sequence an A.P. ? If so, find its 10th term.*

SOLUTION Here, $a_n = 3n - 2$.

Clearly, a_n is a linear expression in n. So, the given sequence is an A.P. with common difference 3.

Putting $n = 10$, we get: $a_{10} = 3 \times 10 - 2 = 28$

REMARK *It is evident from the above examples that a sequence is not an A.P. if its nth term is not a linear expression in n.*

EXERCISE 19.1

1. If the n^{th} term a_n of a sequence is given by $a_n = n^2 - n + 1$, write down its first five terms.
2. A sequence is defined by $a_n = n^3 - 6n^2 + 11n - 6, n \in N$. Show that the first three terms of the sequence are zero and all other terms are positive.
3. Find the first four terms of the sequence defined by $a_1 = 3$ and, $a_n = 3a_{n-1} + 2$, for all $n > 1$. [NCERT]
4. Write the first five terms in each of the following sequences:
   (i) $a_1 = 1, a_n = a_{n-1} + 2, n > 1$
   (ii) $a_1 = 1 = a_2, a_n = a_{n-1} + a_{n-2}, n > 2$
   (iii) $a_1 = a_2 = 2, a_n = a_{n-1} - 1, n > 2$ [NCERT]
5. The Fibonacci sequence is defined by $a_1 = 1 = a_2, a_n = a_{n-1} + a_{n-2}$ for $n > 2$.
   Find $\frac{a_{n+1}}{a_n}$ for $n = 1, 2, 3, 4, 5$. [NCERT]
6. Show that each of the following sequences is an A.P. Also, find the common difference and write 3 more terms in each case.
   (i) $3, -1, -5, -9 \ldots$ (ii) $-1, 1/4, 3/2, 11/4, \ldots$
   (iii) $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, \ldots$ (iv) $9, 7, 5, 3, \ldots$
7. The n^{th} term of a sequence is given by $a_n = 2n + 7$. Show that it is an A.P. Also, find its 7th term.
8. The n^{th} term of a sequence is given by $a_n = 2n^2 + n + 1$. Show that it is not an A.P.

ANSWERS

1. $a_1 = 1,\ a_2 = 3,\ a_3 = 7,\ a_4 = 13,\ a_5 = 21$ 3. $a_1 = 3, a_2 = 11, a_3 = 35, a_4 = 107$
4. (i) $a_1 = 1, a_2 = 3, a_3 = 5, a_4 = 7, a_5 = 9$ (ii) $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5$
   (iii) $a_1 = 2, a_2 = 2, a_3 = 1, a_4 = 0, a_5 = -1$
6. (i) -4 (ii) $\frac{5}{4}$ (iii) $2\sqrt{2}$ (iv) -2 7. 21

HINTS TO NCERT & SELECTED PROBLEMS

3. We have, $a_1 = 3$ and $a_n = 3a_{n-1} + 2$ for $n > 1$
   $\therefore\ a_2 = 3a_1 + 2 = 3 \times 3 + 2 = 11,\ a_3 = 3a_2 + 2 = 3 \times 11 + 2 = 35$
   and, $a_4 = 3a_3 + 2 = 3 \times 35 + 2 = 107$.
4. (iii) We have, $a_1 = a_2 = 2$ and $a_n = a_{n-1} - 1$ for $n > 2$.
   $\therefore\ a_3 = a_2 - 1 = 2 - 1 = 1,\ a_4 = a_3 - 1 = 1 - 1 = 0$ and, $a_5 = a_4 - 1 = 0 - 1 = -1$
5. We have, $a_1 = 1 = a_2$ and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$.
   $\therefore\ a_3 = a_2 + a_1 = 1 + 1 = 2,\ a_4 = a_3 + a_2 = 2 + 1 = 3$
   $a_5 = a_4 + a_3 = 3 + 2 = 5,\ a_6 = a_5 + a_4 = 5 + 3 = 8$
   We have to find $\frac{a_{n+1}}{a_n}$ for $n = 1, 2, 3, 4, 5$. i.e. $\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \frac{a_5}{a_4}$, and $\frac{a_6}{a_5}$

Clearly, $\frac{a_2}{a_1} = \frac{1}{1} = 1, \frac{a_3}{a_2} = \frac{2}{1} = 2, \frac{a_4}{a_3} = \frac{3}{2}, \frac{a_5}{a_4} = \frac{5}{3}$ and $\frac{a_6}{a_5} = \frac{8}{5}$

7. We have,

$a_n = 2n + 7 \Rightarrow a_{n+1} = 2(n+1) + 7 = 2n + 9.$

$\therefore \quad a_{n+1} - a_n = (2n+9) - (2n+7) = 2$ (a constant). So, the sequence is an A.P.

8. Show that $a_{n+1} - a_n$ is not independent of n.

19.3 GENERAL TERM OF AN A.P.

THEOREM *Let a be the first term and d be the common difference of an A.P. Then, its nth* term is $a + (n-1)\,d$ *i.e.* $a_n = a + (n-1)\,d$.

PROOF *Let $a_1, a_2, a_3, a_4, \dots, a_n, \dots$ be the given A.P. Then,*

$a_1 =$ First term $= a$

$\Rightarrow \quad a_1 = a + (1-1)\,d.$

By the definition, we have

$$a_2 - a_1 = d \Rightarrow a_2 = a_1 + d \Rightarrow a_2 = a + d \Rightarrow a_2 = a + (2-1)\,d$$
$$a_3 - a_2 = d \Rightarrow a_3 = a_2 + d \Rightarrow a_3 = (a+d) + d \Rightarrow a_3 = a + 2d \Rightarrow a_3 = a + (3-1)\,d$$
$$a_4 - a_3 = d \Rightarrow a_4 = a_3 + d \Rightarrow a_4 = (a+2d) + d \Rightarrow a_4 = a + 3d \Rightarrow a_4 = a + (4-1)\,d$$

Similarly, $a_5 = a + (5-1)\,d,\ a_6 = a + (6-1)\,d, \dots,\ a_n = a + (n-1)\,d.$

Hence, nth term of an A.P. with first term a and common difference d is $a_n = a + (n-1)\,d.$

Q.E.D.

19.3.1 nth TERM OF AN A.P. FROM THE END

Let a be the first term and d be the common difference of an A.P. having m terms. Then, n^{th} term from the end is $(m-n+1)^{th}$ term from the beginning.

$\therefore \quad n^{th}$ term from the end $= a_{m-n+1} = a + (m-n+1-1)\,d = a + (m-n)\,d$

For finding the n^{th} term from the end, we may take a_m as the first term and $-d$ as the common difference.

Taking a_m as the first term and common difference equal to '$-d$', we find that

n^{th} term from the end $= a_m + (n-1)(-d)$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE INDICATED TERM OF AN A.P.**

EXAMPLE 1 *Show that the sequence 9, 12, 15, 18, ... is an A.P. Find its 16th term and the general term.*

SOLUTION Clearly, $(12-9) = (15-12) = (18-15) = 3$, so the given sequence is an A.P. with common difference $d = 3$ and first term $a = 9$.

$\therefore \quad$ 16th term $= a_{16} = a + (16-1)\,d = a + 15d = 9 + 15 \times 3 = 54 \quad [\because a_n = a + (n-1)\,d]$

and, General term $= n$th term $= a_n = a + (n-1)\,d = 9 + (n-1) \times 3 = 3n + 6$

EXAMPLE 2 *Show that the sequence $\log a, \log(ab), \log(ab^2), \log(ab^3), \dots$ is an A.P. Find its nth term.*

SOLUTION We have,

$$\log(ab) - \log a = \log\left(\frac{ab}{a}\right) = \log b,\ \log(ab^2) - \log(ab) = \log\left(\frac{ab^2}{ab}\right) = \log b$$

$$\log(ab^3) - \log(ab^2) = \log\left(\frac{ab^3}{ab^2}\right) = \log b$$

It follows from the above results that the difference of a term and the preceding term is always same. So, the given sequence is an A.P. with common difference log b.

$\therefore \quad a_n = a + (n-1)\, d$

$\Rightarrow \quad a_n = \log a + (n-1) \log b = \log a + \log b^{n-1} = \log (ab^{n-1})$

EXAMPLE 3 *Which term of the sequence 72, 70, 68, 66, ... is 40?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 72$ and common difference $d = -2$. Let its nth term be 40.

i.e. $\quad a_n = 40$

$\Rightarrow \quad a + (n-1) d = 40$

$\Rightarrow \quad 72 + (n-1)(-2) = 40 \qquad [\because a_n = a + (n-1)\, d]$

$\Rightarrow \quad 72 - 2n + 2 = 40 \Rightarrow 2n = 34 \Rightarrow n = 17$

Hence, 17th term of the given sequence is 40.

EXAMPLE 4 *Which term of the sequence 4, 9, 14, 19, ... is 124?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 4$ and common difference $d = 5$. Let 124 be the nth term of the given sequence. Then,

$$a_n = 124 \Rightarrow a + (n-1)\, d = 124 \Rightarrow 4 + (n-1) \times 5 = 124 \Rightarrow n = 25$$

Hence, 25th term of the given sequence is 124.

EXAMPLE 5 *How many terms are there in the sequence 3, 6, 9, 12, ..., 111?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 3$ and common difference $d = 3$. Let there be n terms in the given sequence. Then,

$$n^{\text{th}} \text{ term} = 111 \Rightarrow a + (n-1)\, d = 111 \Rightarrow 3 + (n-1) \times 3 = 111 \Rightarrow n = 37$$

Thus, the given sequence contains 37 terms.

EXAMPLE 6 *Is 184 a term of the sequence 3, 7, 11, ... ?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 3$ and common difference $d = 4$. Let the nth term of the given sequence be 184. Then,

$$a_n = 184 \Rightarrow a + (n-1)\, d = 184 \Rightarrow 3 + (n-1) \times 4 = 184 \Rightarrow 4n = 185 \Rightarrow n = 46\frac{1}{4}.$$

Since n is not a natural number. So, 184 is not a term of the given sequence.

EXAMPLE 7 *Which term of the sequence* $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$ *is the first negative term?*

SOLUTION The given sequence is an A.P. in which first term $a = 20$ and common difference $d = -\frac{3}{4}$. Let the nth term of the given A.P. be the first negative term. Then,

$\quad a_n < 0$

$\Rightarrow \quad a + (n-1)\, d < 0$

$\Rightarrow \quad 20 + (n-1) \times (-3/4) < 0 \Rightarrow \frac{83}{4} - \frac{3n}{4} < 0 \Rightarrow 83 - 3n < 0 \Rightarrow 3n > 83 \Rightarrow n > 27\frac{2}{3}$

Since 28 is the natural number just greater than $27\frac{2}{3}$. So, $n = 28$. Thus, 28th term of the given sequence is the first negative term.

EXAMPLE 8 *Which term of the sequence* $8 - 6i, 7 - 4i, 6 - 2i, \ldots$ *is (i) purely real (ii) purely imaginary?*

SOLUTION The given sequence is clearly an A.P. with first term $a = 8 - 6i$ and common difference $d = -1 + 2i$. The nth term of the given A.P. is given by

$$a_n = a + (n-1)\,d = 8 - 6i + (n-1)(-1+2i) = (9-n) + i\,(2n-8)$$

(i) Let the nth term of the given sequence be purely real. Then, a_n is purely real.

$\therefore$ $(9-n) + i(2n-8)$ is purely real $\Rightarrow 2n - 8 = 0 \Rightarrow n = 4$

So, 4th term of the given sequence is purely real.

(ii) Let the nth term of the given sequence be purely imaginary. Then, a_n is purely imaginary

$\therefore$ $(9-n) + i(2n-8)$ is purely imaginary

$\Rightarrow$ $9 - n = 0 \Rightarrow n = 9$

Thus, 9th term of the given sequence is purely imaginary.

Type II **PROBLEMS BASED UPON** $a_n = a + (n-1)\,d$

EXAMPLE 9 *If pth, qth and rth terms of an A.P. are a, b, c respectively, then show that:*

(i) $a(q-r) + b(r-p) + c(p-q) = 0$ *(ii)* $(a-b)\,r + (b-c)\,p + (c-a)\,q = 0$ **[NCERT]**

SOLUTION Let A be the first term and D be the common difference of the given A.P. Then,

$$a = p\text{th term} \Rightarrow a = A + (p-1)\,D \quad \text{...(i)}$$
$$b = q\text{th term} \Rightarrow b = A + (q-1)\,D \quad \text{...(ii)}$$
$$c = r\text{th term} \Rightarrow c = A + (r-1)\,D \quad \text{...(iii)}$$

(i) Substituting these values of a, b, c, in $a\,(q-r) + b\,(r-p) + c\,(p-q)$, we obtain

$$\begin{aligned}
&a\,(q-r) + b\,(r-p) + c\,(p-q)\\
&= \{A + (p-1)\,D\}\,(q-r) + \{A + (q-1)\,D\}\,(r-p) + \{A + (r-1)\,D\}\,(p-q)\\
&= A\{(q-r) + (r-p) + (p-q)\} + D\{(p-1)\,(q-r) + (q-1)\,(r-p) + (r-1)\,(p-q)\}\\
&= A\,.\,0 + D\,\{p\,(q-r) + q\,(r-p) + r\,(p-q) - (q-r) - (r-p) - (p-q)\}\\
&= A\,.\,0 + D\,.\,0 = 0
\end{aligned}$$

(ii) On subtracting (ii) from (i); (iii) from (ii) and (i) from (iii), we get

$$a - b = (p-q)\,D \;\text{...(iv)}\quad (b-c) = (q-r)\,D \;\text{...(v)}\quad c - a = (r-p)\,D \quad \text{...(vi)}$$

$$\begin{aligned}
\therefore\quad (a-b)\,r + (b-c)\,p + (c-a)\,q &= (p-q)\,D\,r + (q-r)\,D\,p + (r-p)\,D\,q\\
&= D[(p-q)\,r + (q-r)\,p + (r-p)\,q] = D \times 0 = 0
\end{aligned}$$

EXAMPLE 10 *Show that the sum of* $(m+n)^{th}$ *and* $(m-n)^{th}$ *term of an A.P. is equal to twice the* m^{th} *term.* **[NCERT]**

SOLUTION Let a be the first term and d be the common different of the AP. Then,

$$a_{m+n} = (m+n)^{th}\text{ term} = a + (m+n-1)\,d \text{ and, } a_{m-n} = (m-n)^{th}\text{ term} = a + (m-n-1)\,d$$

$$\begin{aligned}
\therefore\quad a_{m+n} + a_{m-n} &= \{a + (m+n-1)\,d\} + \{a + (m-n-1)\,d\}\\
&= 2a + (m+n-1+m-n-1)\,d\\
&= 2a + 2\,(m-1)\,d\\
&= 2\,\{a + (m-1)\,d\}\\
&= 2\,a_m.
\end{aligned}$$

EXAMPLE 11 *If m times the mth term of an A.P. is equal to n times its nth term, show that the (m + n)th term of the A.P. is zero.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

m times m^{th} term = n times nth term

$\Rightarrow$ $m\,a_m = n\,a_n$

$\Rightarrow$ $m\{a + (m-1)\,d\} = n\{a + (n-1)\,d\}$

$\Rightarrow \quad m\{a+(m-1)d\} - n\{a+(n-1)d\} = 0$

$\Rightarrow \quad a(m-n) + \{m(m-1) - n(n-1)\}d = 0$

$\Rightarrow \quad a(m-n) + \{(m^2-n^2) - (m-n)\}d = 0$

$\Rightarrow \quad a(m-n) + (m-n)(m+n-1)d = 0$

$\Rightarrow \quad (m-n)\{a+(m+n-1)d\} = 0$

$\Rightarrow \quad a+(m+n-1)d = 0 \qquad [\because m \neq n]$

$\Rightarrow \quad a_{m+n} = 0$

Hence, the $(m+n)^{\text{th}}$ term of the given A.P. is zero.

EXAMPLE 12 *If the pth term of an A.P. is q and the q^{th} term is p, prove that its nth term is $(p+q-n)$.* [NCERT]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

pth term $= q \Rightarrow a+(p-1)d = q \qquad$ ...(i)

qth term $= p \Rightarrow a+(q-1)d = p \qquad$ ...(ii)

Subtracting (ii) from (i), we get

$$(p-q)d = (q-p) \Rightarrow d = -1$$

Putting $d=-1$ in (i), we get: $a = (p+q-1)$

$\therefore \quad n$th term $= a+(n-1)d = (p+q-1)+(n-1)\times(-1) = (p+q-n)$

EXAMPLE 13 *If the mth term of an A.P. be 1/n, and nth term be 1/m then show that its (mn)th term is 1.*

SOLUTION Let a and d be the first term and common difference respectively of the given A.P. Then,

$\frac{1}{n} = m$th term $\Rightarrow \frac{1}{n} = a+(m-1)d \qquad$ ...(i)

$\frac{1}{m} = n$th term $\Rightarrow \frac{1}{m} = a+(n-1)d \qquad$ ...(ii)

On subtracting (ii) from (i), we get

$$\frac{1}{n}-\frac{1}{m} = (m-n)d \Rightarrow \frac{m-n}{mn} = (m-n)d \Rightarrow d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in (i), we get

$$\frac{1}{n} = a+\frac{(m-1)}{mn} \Rightarrow \frac{1}{n} = a+\frac{1}{n}-\frac{1}{mn} \Rightarrow a = \frac{1}{mn}$$

$\therefore \quad (mn)$th term $= a+(mn-1)d = \frac{1}{mn}+(mn-1)\frac{1}{mn} = 1$

EXAMPLE 14 *Determine the number of terms in the A.P. 3, 7, 11, ... 407. Also, find its 20th term from the end.*

SOLUTION Clearly, the given sequence is an A.P. with first term 3 and the common difference 4. Let there be n terms in the given A.P. Then,

$$407 = n\text{th term} \Rightarrow 407 = 3+(n-1)\times 4 \Rightarrow 4n = 408 \Rightarrow n = 102$$

Now,

20th term from the end $=$ [102 − 20 + 1]th term from the beginning

$=$ 83rd term from the beginning $= 3+(83-1)\times 4 = 331$

ALITER To find 20th term from the end, we consider the given sequence as an A.P. with first term = 407 and common difference − 4.

$\therefore \quad$ 20th term from the end $= 407+(20-1)\times(-4) = 331$.

EXAMPLE 15 *How many numbers of two digits are divisible by 7 ?*

SOLUTION First two digit number divisible by 7 is 14 and last two digit number divisible by 7 is 98. So, we have to determine the number of terms in the sequence 14, 21, 28, ..., 98. Let there be n terms in this sequence. Then,

$$98 = n\text{th term} \Rightarrow 98 = 14 + (n-1) \times 7 \Rightarrow 7n = 91 \Rightarrow n = 13$$

LEVEL-2

EXAMPLE 16 *Show that there is no A.P. which consists of only distinct prime numbers.*

SOLUTION Let $a_1, a_2, a_3, ..., a_n, ...$ be an A.P. consisting only of prime numbers. Let d be the common difference of the A.P. Since the difference of two consecutive prime numbers is greater than or equal to 1. Therefore, $d > 1$.

Now,

$$(a_1 + 1)^{\text{th}} \text{ term of this A.P.} = a_1 + (a_1 + 1 - 1)\, d = a_1(1 + d)$$

$$\Rightarrow \quad (a_1 + 1)^{\text{th}} \text{ term is not a prime number}$$

This is a contradiction that the A.P. consists of only prime numbers as its terms.
Hence, there cannot be an A.P. which consists only of distinct prime numbers.

EXAMPLE 17 *Show that in an A.P. the sum of the terms equidistant from the beginning and end is always same and equal to the sum of first and last terms.* **[NCERT EXEMPLAR]**

SOLUTION Let $a_1, a_2, a_3, ..., a_n$ be an A.P. with common difference 'd'. Then,

kth term from the beginning $= a_k = a_1 + (k-1)\, d$

and, kth term from the end $= (n - k + 1)$th term from the beginning

$$= a_{n-k+1}$$
$$= a_1 + (n - k + 1 - 1)\, d = a_1 + (n-k)\, d$$

$\therefore$ (kth term from the beginning) + (kth term from the end)

$$= a_k + a_{n-k+1}$$
$$= \{a_1 + (k-1)\, d\} + \{a_1 + (n-k)\, d\} = 2a_1 + (n-1)\, d = a_1 + \{a_1 + (n-1)\, d\} = a_1 + a_n$$

Thus, $a_k + a_{n-k+1} = a_1 + a_n$ for all $k = 1, 2, ... n$

$$\Rightarrow \quad a_2 + a_{n-1} = a_3 + a_{n-2} = a_4 + a_{n-3} = ... = a_1 + a_n$$

Hence, the sum of the terms equidistant from the beginning and end is always same and equal to the sum of first and last terms.

NOTE *The statement of the above example may be treated as a standard result.*

EXAMPLE 18 *In the arithmetic progressions 2, 5, 8, ... upto 50 terms, and 3, 5, 7, 9, ... upto 60 terms, find how many terms are identical.*

SOLUTION Let the mth term of the first A.P. be equal to the nth term of the second A.P. Then,

$$2 + (m-1) \times 3 = 3 + (n-1) \times 2$$
$$\Rightarrow \quad 3m - 1 = 2n + 1$$
$$\Rightarrow \quad 3m = 2n + 2$$
$$\Rightarrow \quad \frac{m}{2} = \frac{n+1}{3} = k \text{ (say)}$$
$$\Rightarrow \quad m = 2k \text{ and } n = 3k - 1$$
$$\Rightarrow \quad 2k \le 50 \text{ and } 3k - 1 \le 60 \qquad [\because m \le 50 \text{ and } n \le 60]$$
$$\Rightarrow \quad k \le 25 \text{ and } k \le 20\frac{1}{3}$$

$\Rightarrow \quad k \leq 20$ [$\because$ k is a natural number]

$\Rightarrow \quad k = 1, 2, 3, \ldots 20$

Corresponding to each value of k, we get a pair of identical terms.

Hence, there are 20 identical terms in the two A.P.'s.

EXAMPLE 19 *Find the number of terms common to the two A.P.'s: 3, 7, 11, ... 407 and 2, 9, 16, ..., 709.*

SOLUTION Let the number of terms in two A.P.'s be m and n respectively. Then,

407 = mth term of first A.P. and, 709 = nth term of second A.P.

$$\Rightarrow \quad 407 = 3 + (m-1) \times 4 \quad \text{and} \quad 709 = 2 + (n-1) \times 7$$

$$\Rightarrow \quad m = 102 \quad \text{and} \quad n = 102$$

So, each A.P. consists of 102 terms.

Let pth term of first A.P. be identical to qth term of the second A.P. Then,

$$3 + (p-1) \times 4 = 2 + (q-1) \times 7$$

$$\Rightarrow \quad 4p - 1 = 7q - 5$$

$$\Rightarrow \quad 4p + 4 = 7q$$

$$\Rightarrow \quad 4(p+1) = 7q$$

$$\Rightarrow \quad \frac{p+1}{7} = \frac{q}{4} = k \text{ (say)}$$

$$\Rightarrow \quad p = 7k - 1 \quad \text{and} \quad q = 4k$$

Since each A.P. consists of 102 terms.

$$\therefore \quad p \leq 102 \quad \text{and} \quad q \leq 102$$

$$\Rightarrow \quad 7k - 1 \leq 102 \quad \text{and} \quad 4k \leq 102$$

$$\Rightarrow \quad k \leq 14\frac{5}{7} \quad \text{and} \quad k \leq 25\frac{1}{2}$$

$$\Rightarrow \quad k \leq 14 \Rightarrow k = 1, 2, 3, \ldots, 14$$

Corresponding to each value of k, we get a pair of identical terms.

Hence, there are 14 identical terms in two A.P.'s.

EXAMPLE 20 *If $a_1, a_2, a_3, \ldots, a_n$ are in A.P., where $a_i > 0$ for all i, show that*

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}.$$

[NCERT EXEMPLAR]

SOLUTION Let d be the common difference of the given A.P. Then

$$a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d \quad \ldots \text{(i)}$$

Now, $\dfrac{1}{\sqrt{a_1} + \sqrt{a_2}} + \dfrac{1}{\sqrt{a_2} + \sqrt{a_3}} + \dfrac{1}{\sqrt{a_3} + \sqrt{a_4}} + \ldots + \dfrac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}}$

$$= \frac{\sqrt{a_2} - \sqrt{a_1}}{(\sqrt{a_2} + \sqrt{a_1})(\sqrt{a_2} - \sqrt{a_1})} + \frac{\sqrt{a_3} - \sqrt{a_2}}{(\sqrt{a_3} + \sqrt{a_2})(\sqrt{a_3} - \sqrt{a_2})} + \ldots + \frac{\sqrt{a_n} - \sqrt{a_{n-1}}}{(\sqrt{a_n} + \sqrt{a_{n-1}})(\sqrt{a_n} - \sqrt{a_{n-1}})}$$

$$= \frac{\sqrt{a_2} - \sqrt{a_1}}{a_2 - a_1} + \frac{\sqrt{a_3} - \sqrt{a_2}}{a_3 - a_2} + \ldots + \frac{\sqrt{a_n} - \sqrt{a_{n-1}}}{a_n - a_{n-1}}$$

$$= \frac{\sqrt{a_2} - \sqrt{a_1}}{d} + \frac{\sqrt{a_3} - \sqrt{a_2}}{d} + \ldots + \frac{\sqrt{a_n} - \sqrt{a_{n-1}}}{d} \quad \text{[Using (i)]}$$

$$= \frac{1}{d}\left\{(\sqrt{a_2} - \sqrt{a_1}) + (\sqrt{a_3} - \sqrt{a_2}) + \ldots + (\sqrt{a_n} - \sqrt{a_{n-1}})\right\}$$

$$= \frac{1}{d}\left\{\sqrt{a_n} - \sqrt{a_1}\right\}$$

$$= \frac{(\sqrt{a_n} - \sqrt{a_1})(\sqrt{a_n} + \sqrt{a_1})}{d(\sqrt{a_n} + \sqrt{a_1})} = \frac{a_n - a_1}{d(\sqrt{a_n} + \sqrt{a_1})} = \frac{a_1 + (n-1)d - a_1}{d(\sqrt{a_n} + \sqrt{a_1})} = \frac{n-1}{\sqrt{a_n} + \sqrt{a_1}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$$

EXAMPLE 21 *If $a_1, a_2, a_3, \dots, a_n$ be an A.P. of non-zero terms, prove that*

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}.$$

SOLUTION Let 'd' be the common difference of the given A.P. Then,

$$a_2 - a_1 = a_3 - a_2 = \dots = a_n - a_{n-1} = d \quad \text{(say).} \qquad \dots(i)$$

Now,

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} + \dots + \frac{1}{a_{n-1} a_n}$$

$$= \frac{1}{d}\left\{\frac{d}{a_1 a_2} + \frac{d}{a_2 a_3} + \frac{d}{a_3 a_4} + \dots + \frac{d}{a_{n-1} a_n}\right\}$$

$$= \frac{1}{d}\left\{\frac{(a_2 - a_1)}{a_1 a_2} + \frac{(a_3 - a_2)}{a_2 a_3} + \frac{(a_4 - a_3)}{a_3 a_4} + \dots + \frac{(a_n - a_{n-1})}{a_{n-1} a_n}\right\}$$

$$= \frac{1}{d}\left\{\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + \left(\frac{1}{a_3} - \frac{1}{a_4}\right) + \dots + \left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right)\right\}$$

$$= \frac{1}{d}\left\{\frac{1}{a_1} - \frac{1}{a_n}\right\} = \frac{1}{d}\left\{\frac{a_n - a_1}{a_1 a_n}\right\} = \frac{1}{d}\left\{\frac{a_1 + (n-1)d - a_1}{a_1 a_n}\right\} = \frac{n-1}{a_1 a_n}$$

EXAMPLE 22 *If $a_1, a_2, a_3, \dots, a_n$ are in AP with common difference d (where $d \neq 0$), then the sum of series.*

$\sin d(\text{cosec}\, a_1 \text{cosec}\, a_2 + \text{cosec}\, a_2 \text{cosec}\, a_3 + \dots + \text{cosec}\, a_{n-1} \text{cosec}\, a_n)$ *is equal to* $\cot a_1 - \cot a_n$.

SOLUTION We have, **[NCERT EXEMPALR]**

$\sin d(\text{cosec}\, a_1 \text{cosec}\, a_2 + \text{cosec}\, a_2 \text{cosec}\, a_3 + \dots + \text{cosec}\, a_{n-1} \text{cosec}\, a_n)$

$$= \frac{\sin d}{\sin a_1 \sin a_2} + \frac{\sin d}{\sin a_2 \sin a_3} + \frac{\sin d}{\sin a_3 \sin a_4} + \dots + \frac{\sin d}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin(a_2 - a_1)}{\sin a_1 \sin a_2} + \frac{\sin(a_3 - a_2)}{\sin a_2 \sin a_3} + \frac{\sin(a_4 - a_3)}{\sin a_3 \sin a_4} + \dots + \frac{\sin(a_n - a_{n-1}}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin a_2 \cos a_2 - \cos a_1 \sin a_2}{\sin a_1 \sin a_2} + \frac{\sin a_3 \cos a_2 - \cos a_3 \sin a_2}{\sin a_2 \sin a_3} + \dots + \frac{\sin a_n \cos a_{n-1} - \cos a_n \sin a_{n-1}}{\sin a_{n-1} \sin a_n}$$

$$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + \dots + (\cot a_{n-1} - \cot a_n)$$

$$= \cot a_1 - \cot a_n$$

EXERCISE 19.2

LEVEL-1

1. Find:
   (i) 10th term of the A.P. 1, 4, 7, 10, ... (ii) 18th term of the A.P. $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, \dots$
   (iii) *nth* term of the A.P. 13, 8, 3, –2, ...

2. If the sequence $< a_n >$ is an A.P., show that $a_{m+n} + a_{m-n} = 2a_m$.
3. (i) Which term of the A.P. 3, 8, 13, ... is 248 ?
   (ii) Which term of the A.P. 84, 80, 76, ... is 0 ?
   (iii) Which term of the A.P. 4, 9, 14, ... is 254 ?
4. (i) Is 68 a term of the A.P. 7, 10, 13, ... ?
   (ii) Is 302 a term of the A.P. 3, 8, 13, ... ?
5. (i) Which term of the sequence 24, $23\frac{1}{4}, 22\frac{1}{2}, 21\frac{3}{4}$... is the first negative term ?
   (ii) Which term of the sequence $12 + 8i, 11 + 6i, 10 + 4i, \ldots$ is (a) purely real (b) purely imaginary ?
6. (i) How many terms are there in the A.P. 7, 10, 13, ... 43 ?
   (ii) How many terms are there in the A.P. $-1, -\frac{5}{6}, -\frac{2}{3}, -\frac{1}{2}, \ldots, \frac{10}{3}$?
7. The first term of an A.P. is 5, the common difference is 3 and the last term is 80; find the number of terms.
8. The 6th and 17th terms of an A.P. are 19 and 41 respectively, find the 40th term.
9. If 9th term of an A.P. is zero, prove that its 29th term is double the 19th term.
10. If 10 times the 10th term of an A.P. is equal to 15 times the 15th term, show that 25th term of the A.P. is zero.
11. The 10th and 18th terms of an A.P. are 41 and 73 respectively. Find 26th term.
12. In a certain A.P. the 24th term is twice the 10th term. Prove that the 72nd term is twice the 34th term.
13. If $(m + 1)$th term of an A.P. is twice the $(n + 1)$th term, prove that $(3m + 1)$th term is twice the $(m + n + 1)$th term.
14. If the nth term of the A.P. 9, 7, 5, ... is same as the nth term of the A.P. 15, 12, 9, ... find n.
15. Find the 12th term from the end of the following arithmetic progressions:
   (i) 3, 5, 7, 9, ... 201 (ii) 3, 8, 13, ... , 253 (iii) 1, 4, 7, 10, ..., 88
16. The 4th term of an A.P. is three times the first and the 7th term exceeds twice the third term by 1. Find the first term and the common difference.
17. Find the second term and *nth* term of an A.P. whose 6th term is 12 and the 8th term is 22.
18. How many numbers of two digit are divisible by 3 ?
19. An A.P. consists of 60 terms. If the first and the last terms be 7 and 125 respectively, find 32nd term.
20. The sum of 4th and 8th terms of an A.P. is 24 and the sum of the 6th and 10th terms is 34. Find the first term and the common difference of the A.P.

LEVEL-2

21. The first and the last terms of an A.P. are a and l respectively. Show that the sum of *nth* term from the beginning and *nth* term from the end is $a + l$.
22. If an A.P. is such that $\frac{a_4}{a_7} = \frac{2}{3}$, find $\frac{a_6}{a_8}$.
23. If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP, whose common difference is d, show that
   $\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n = \frac{\tan\theta_n - \tan\theta_1}{\sin d}$ [NCERT EXEMPALR]

ANSWERS

1. (i) 28 (ii) $35\sqrt{2}$ (iii) $-5n+18$ 3. (i) 50 (ii) 22 (iii) 51
4. (i) No (ii) No 5. (i) 34th (ii) (a) 5 (b) 13
6. (i) 13 (ii) 27 7. 26 8. 87 11. 105 14. 7
15. (i) 179 (ii) 198 (iii) 55 16. First term = 3, common difference = 2
17. $a_2 = -8$, $a_n = 5n - 18$ 18. 30 19. 69 20. $-\frac{1}{2}, \frac{5}{2}$ 22. $\frac{4}{5}$

19.4 SELECTION OF TERMS IN AN A.P.

Sometimes we require certain number of terms in A.P. The following ways of selecting terms are generally very convenient.

Number of terms	Terms	Common difference
3	$a-d, a, a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d
6	$a-5d, a-3d, a-d, a+d, a+3d, a+5d$	$2d$

It should be noted that in case of an odd number of terms, the middle term is a and the common difference is d while in case of an even number of terms the middle terms are $a-d$, $a+d$ and the common differences is $2d$.

The following examples will illustrate the use of such representations.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The sum of three numbers in A.P. is – 3, and their product is 8. Find the numbers.*

SOLUTION Let the numbers be $(a-d)$, a, $(a+d)$. Then,

$$\text{Sum} = -3 \Rightarrow (a-d)+a+(a+d) = -3 \Rightarrow 3a = -3 \Rightarrow a = -1$$

and, $\text{Product} = 8$

$$\Rightarrow (a-d)(a)(a+d) = 8$$

$$\Rightarrow a(a^2-d^2) = 8$$

$$\Rightarrow (-1)(1-d^2) = 8 \qquad [\because a=-1]$$

$$\Rightarrow d^2 = 9 \Rightarrow d = \pm 3$$

When $a=-1$ and $d=3$, the numbers are $-4, -1, 2$. When $a=-1$ and $d=-3$, the numbers are $2, -1, -4$. So, the numbers are $-4, -1, 2$, or $2, -1, -4$.

EXAMPLE 2 *Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.*

SOLUTION Let the numbers be $(a-3d)$, $(a-d)$, $(a+d)$, $(a+3d)$. Then,

$$\text{Sum} = 20 \Rightarrow (a-3d)+(a-d)+(a+d)+(a+3d) = 20 \Rightarrow 4a = 20 \Rightarrow a = 5$$

and, Sum of the squares $=120$

$$\Rightarrow (a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2 = 120$$

$$\Rightarrow \quad 4a^2 + 20d^2 = 120$$

$$\Rightarrow \quad a^2 + 5d^2 = 30$$

$$\Rightarrow \quad 25 + 5d^2 = 30 \qquad [\because a = 5]$$

$$\Rightarrow \quad 5d^2 = 5 \Rightarrow d = \pm 1$$

If $d = 1$, and $a = 5$, then the numbers are $2, 4, 6, 8$. If $d = -1$, and $a = 5$, then the numbers are $8, 6, 4, 2$.

Thus, the numbers are $2, 4, 6, 8$ or $8, 6, 4, 2$.

EXAMPLE 3 *Divide 32 into four parts which are in A.P. such that the product of extremes is to the product of means is 7 : 15.*

SOLUTION Let the four parts be $(a - 3d)$, $(a - d)$, $(a + d)$ and $(a + 3d)$. Then,

$$\text{Sum} = 32 \Rightarrow (a - 3d) + (a - d) + (a + d) + (a + 3d) = 32 \Rightarrow 4a = 32 \Rightarrow a = 8$$

It is given that the product of extremes is to the product of means is 7 : 15.

$$\therefore \quad \frac{(a - 3d)(a + 3d)}{(a - d)(a + d)} = \frac{7}{15}$$

$$\Rightarrow \quad \frac{a^2 - 9d^2}{a^2 - d^2} = \frac{7}{15} \Rightarrow \frac{64 - 9d^2}{64 - d^2} = \frac{7}{15} \Rightarrow 128d^2 = 512 \Rightarrow d^2 = 4 \Rightarrow d = \pm 2$$

Thus, the four of 32 parts are 2, 6, 10, 14.

When $a = 8$ and $d = 2$ four parts are: 2, 6, 10, and 14. When $a = 8$ and $d = -2$ four parts are 14, 10, 6 and 2.

LEVEL-2

EXAMPLE 4 *The product of three numbers in A.P. is 224, and the largest number is 7 times the smallest. Find the numbers.* [NCERT EXEMPLAR]

SOLUTION Let the three numbers in A.P. be $a - d$, a, $a + d$, where $d > 0$. Clearly, $a + d$ is the largest number and $a - d$ is the smallest number.

It is given that :

Product of numbers = 224 and, The largest number = 7 (The smallest numbers)

$$\Rightarrow \quad (a - d)\, a\, (a + d) = 224 \text{ and, } a + d = 7(a - d)$$

$$\Rightarrow \quad a(a^2 - d^2) = 224 \quad \text{and,} \quad 6a = 8d$$

$$\Rightarrow \quad a(a^2 - d^2) = 224 \quad \text{and,} \quad d = \frac{3a}{4}$$

$$\Rightarrow \quad a\left(a^2 - \frac{9}{16}a^2\right) = 224 \qquad \text{[On eliminating } d\text{]}$$

$$\Rightarrow \quad \frac{7a^3}{16} = 224$$

$$\Rightarrow \quad a^3 = 512 = 8^3$$

$$\Rightarrow \quad a = 8.$$

Putting $a = 8$ in $d = \frac{3a}{4}$, we obtain $d = 6$.

Hence, three numbers are 2, 8, 14.

EXAMPLE 5 *If the fourth power of the common difference of an A.P. with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer.*

SOLUTION Let $a-3d, a-d, a+d, a+3d$ be four consecutive terms of an A.P. with integer terms. Clearly, the common difference is $2d$. Since the terms are integers, therefore a and d are also integers.

Now, Given sum $= (a-3d)(a-d)(a+d)(a+3d)+(2d)^4$

$$= (a^2-9d^2)(a^2-d^2)+16d^4$$

$$= a^4-10a^2d^2+9d^4+16d^4$$

$$= a^4-10a^2d^2+25d^4$$

$= (a^2-5d^2)^2$, which is square of an integer as a and d are integers.

EXERCISE 19.3

LEVEL-1

1. The sum of three terms of an A.P. is 21 and the product of the first and the third terms exceeds the second term by 6, find three terms.
2. Three numbers are in A.P. If the sum of these numbers be 27 and the product 648, find the numbers.
3. Find the four numbers in A.P., whose sum is 50 and in which the greatest number is 4 times the least.

LEVEL-2

4. The sum of three numbers in A.P. is 12 and the sum of their cubes is 288. Find the numbers.
5. If the sum of three numbers in A.P. is 24 and their product is 440, find the numbers. [NCERT]
6. The angles of a quadrilateral are in A.P. whose common difference is 10°. Find the angles.

ANSWERS

1. 1, 7, 13	2. 6, 9, 12	3. 5, 10, 15, 20	4. 2, 4, 6 or 6, 4, 2
5. 5, 8, 11	6. 75°, 85°, 95°, 105°		

HINTS TO NCERT & SELECTED PROBLEMS

5. Let the three numbers be $a-d, a, a+d$. It is given that the sum and product of these numbers are 24 and 440 respectively. Therefore,

$$a-d+a+a+d=24 \text{ and } (a-d)\,a\,(a+d)=440$$

$$\Rightarrow \quad 3a=24 \text{ and } a(a^2-d^2)=440$$

$$\Rightarrow \quad a=8 \text{ and } a(a^2-d^2)=440$$

Now,

$$a(a^2-d^2)=440 \Rightarrow 8(64-d^2)=440 \Rightarrow 64-d^2=55 \Rightarrow d^2=9 \Rightarrow d=\pm 3$$

Hence, three numbers are 5, 8, 11 or 11, 8, 5.

19.5 SUM TO n TERMS OF AN A.P.

THEOREM *Show that the sum S_n of n terms of an A.P. with first term 'a' and common difference 'd' is*

$$S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$$

or, $$S_n = \frac{n}{2}(a + l), \text{ where } l = \text{last term} = a + (n-1)\,d$$

PROOF Let $a_1, a_2, a_3, \ldots$ be an A.P. with first term a and common difference d. Then

$$a_1 = a, a_2 = a + d, a_3 = a + 2d, a_4 = a + 3d, \ldots, a_n = a + (n-1)\,d$$

Now,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

$$\Rightarrow \quad S_n = a + (a+d) + (a+2d) + \ldots + (a + (n-2)\,d) + \{a + (n-1)\,d\} \quad \ldots(i)$$

Writing the above series in a reverse order, we get

$$S_n = \{a + (n-1)\,d\} + \{a + (n-2)\,d\} + \ldots + (a+d) + a \quad \ldots(ii)$$

Adding the corresponding terms of (i) and (ii), we get

$$2S_n = \{2a + (n-1)\,d\} + \{2a + (n-1)\,d\} + \ldots + \{2a + (n-1)\,d\}$$

$$\Rightarrow \quad 2S_n = n\{2a + (n-1)\,d\} \quad [\because 2a + (n-1)\,d \text{ repeats } n \text{ times}]$$

$$\Rightarrow \quad S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$$

Now, l = last term $= a + (n-1)d$

$$\therefore \quad S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\} = \frac{n}{2}\left[a + \left\{a + (n-1)\,d\right\}\right] = \frac{n}{2}\left(a + l\right)$$

ALITER We have,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-2} + a_{n-1} + a_n \quad \ldots(i)$$

or, $$S_n = a_n + a_{n-1} + a_{n-2} + \ldots + a_3 + a_2 + a_1 \quad \ldots(ii)$$

Adding corresponding terms in (i) and (ii), we get

$$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \ldots + (a_{n-1} + a_2) + (a_n + a_1)$$

$$\Rightarrow \quad 2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n) + (a_1 + a_n)$$

$$\Rightarrow \quad 2S_n = n\,(a_1 + a_n) \quad [\because a_1 + a_n = a_k + a_{n-k+1} \text{ for } k = 2, 3, \ldots, n]$$

$$\Rightarrow \quad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\Rightarrow \quad S_n = \frac{n}{2}\left\{a_1 + a_1 + (n-1)\,d\right\} \quad [\because a_n = a_1 + (n-1)\,d]$$

$$\Rightarrow \quad S_n = \frac{n}{2}\left\{2a_1 + n - 1)\,d\right\}$$

NOTE 1 *In the formula $S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$, there are four quantities viz. S_n, a, n and d. If any three of these are known, the fourth can be determined. Sometimes two of these quantities are given, in such cases remaining two quantities are provided by some other relations.*

NOTE 2 *If the sum S_n of n terms of a sequence is given, then nth term a_n of the sequence can be determined by the using formula: $a_n = S_n - S_{n-1}$*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE SUM OF GIVEN NUMBER OF TERMS OF AN A.P.**

Formula: $S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$ or, $S_n = \frac{n}{2}\left\{a + l\right\}$.

EXAMPLE 1 *Find the sum of 20 terms of the A.P. 1, 4, 7, 10, ...*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Clearly, $a = 1$, $d = 3$. We have to find the sum of 20 terms of the given A.P. Putting $a = 1$, $d = 3$, $n = 20$ in $S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$, we get

$$S_{20} = \frac{20}{2}\left\{2 \times 1 + (20-1) \times 3\right\} = 10 \times 59 = 590$$

EXAMPLE 2 *Find the sum of the series : 5 + 13 + 21 + ... + 181.*

SOLUTION The terms of the given series form an A.P. with first term $a = 5$ and common difference $d = 8$. Let there be n terms in the given series. Clearly,

$$a_n = 181 \Rightarrow a + (n-1)\,d = 181 \Rightarrow 5 + (n-1) \times 8 = 181 \Rightarrow 8n = 184 \Rightarrow n = 23$$

$$\therefore \quad \text{Required sum} = \frac{n}{2}(a + l) = \frac{23}{2}(5 + 181) = 2139.$$

EXAMPLE 3 *Find the sum of all three digit natural numbers, which are divisible by 7.*

SOLUTION The smallest and the largest numbers of three digits, which are divisible by 7 are 105 and 994 respectively. So, the sequence of three digit numbers which are divisible by 7 are 105, 112, 119, ..., 994. Clearly, these numbers are in A.P. with first term $a = 105$ and common difference $d = 7$.

Let there be n terms in this sequence. Then,

$$a_n = 994 \Rightarrow a + (n-1)\,d = 994 \Rightarrow 105 + (n-1) \times 7 = 994 \Rightarrow n = 128$$

$$\therefore \quad \text{Required sum} = \frac{n}{2}\left\{2a + (n-1)\,d\right\} = \frac{128}{2}\left\{2 \times 105 + (128-1) \times 7\right\} = 70336$$

EXAMPLE 4 *Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3.*

SOLUTION Clearly, the numbers between 250 and 1000 which are divisible by 3 are 252, 255, 258, ..., 999. These numbers are in A.P. with first term $a = 252$, common difference = 3 and last term = 999. Let there be n terms in this A.P. Then,

$$a_n = 999 \Rightarrow a + (n-1)\,d = 999 \Rightarrow 252 + (n-1) \times 3 = 999 \Rightarrow n = 250$$

$$\therefore \quad \text{Required sum} = S_n = \frac{n}{2}(a + l) = \frac{250}{2}(252 + 999) = 156375$$

EXAMPLE 5 *Find the sum of all odd integers between 2 and 100 divisible by 3.*

SOLUTION The odd integers between 2 and 100 which are divisible by 3 are 3, 9, 15, 21, ..., 99. Clearly, these numbers are in A.P. with first term $a = 3$ and common difference $d = 6$. Let there be n terms in this sequence. Then,

$$a_n = 99 \Rightarrow a + (n-1)\,d \Rightarrow 3 + (n-1) \times 6 = 99 \Rightarrow n = 17$$

$$\therefore \quad \text{Required sum} = \frac{n}{2}(a + l) = \frac{17}{2}(3 + 99) = 867.$$

EXAMPLE 6 *Find the sum of first 20 terms of an A.P., in which 3rd term is 7 and 7th term is two more than thrice of its 3rd term.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. It is given that

$$a_3 = 7 \quad \text{and} \quad a_7 = 3a_3 + 2$$

$$\Rightarrow \quad a + 2d = 7 \quad \text{and} \quad a + 6d = 3(a + 2d) + 2$$

$\Rightarrow \quad a + 2d = 7$ and $a = -1$

$\Rightarrow \quad a = -1, d = 4$

$\therefore \quad S_{20} = \frac{20}{2}\left\{2 \times -1 + (20-1) \times 4\right\}$ $\quad \left[\text{Using: } S_n = \frac{n}{2}\left\{2a + (n-1)d\right\}\right]$

$\Rightarrow \quad S_{20} = \frac{20}{2}(-2 + 76) = 740$

EXAMPLE 7 *The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.* [NCERT]

SOLUTION Let there be n terms in the A.P. with first term $a = 11$ and common difference d. It is given that

Sum of first four terms $= 56$

$\Rightarrow \quad \frac{4}{2}\left\{2 \times 11 + (4-1)d\right\} = 56$

$\Rightarrow \quad 22 + 3d = 28 \Rightarrow 3d = 6 \Rightarrow d = 2$

It is also given that

Sum of last four terms $= 112$

$\Rightarrow \quad a_n + a_{n-1} + a_{n-2} + a_{n-3} = 112$

$\Rightarrow \quad \frac{4}{2}(a_n + a_{n-3}) = 112$ $\quad \left[\text{Using: } S_n = \frac{n}{2}(a + l)\right]$

$\Rightarrow \quad a_n + a_{n-3} = 56$

$\Rightarrow \quad \{11 + (n-1) \times 2\} + \{11 + (n-4) \times 2\} = 56$

$\Rightarrow \quad 22 + 2(2n-5) = 56$

$\Rightarrow \quad 4n = 44 \Rightarrow n = 11.$

Hence, there are 11 terms in the A.P.

EXAMPLE 8 *If the sum of n terms of an A.P. is $pn + qn^2$, where p and q are constants, find the common difference.* [NCERT]

SOLUTION Let S_n denote the sum of n terms and a_n denote the nth term of the A.P. Then,

$S_n = pn + qn^2$

$\Rightarrow \quad S_{n-1} = p(n-1) + q(n-1)^2$ $\quad$ [On replacing n by $(n-1)$ in S_n]

Now, $\quad a_n = S_n - S_{n-1}$

$\Rightarrow \quad a_n = \{pn + qn^2\} - \{p(n-1) + q(n-1)^2\}$

$\Rightarrow \quad a_n = pn - p(n-1) + qn^2 - q(n-1)^2$

$\Rightarrow \quad a_n = p\{n - (n-1)\} + q\{n^2 - (n-1)^2\}$

$\Rightarrow \quad a_n = p + q(2n-1)$

$\therefore \quad a_{n-1} = p + q\{2(n-1) - 1\}$ $\quad$ [Replacing n by $(n-1)$ in a_n]

Let d be the common difference of the A.P. Then,

$d = a_n - a_{n-1}$

$\Rightarrow \quad d = \{p + q(2n-1)\} - [p + q\{2(n-1) - 1\}]$

$\Rightarrow \quad d = \{p + q(2n-1)\} - \{p + q(2n-3)\}$

$\Rightarrow \quad d = q(2n - 1 - 2n + 3) = 2q$

Hence, the common difference $= 2q$.

EXAMPLE 9 *If the sum of n terms of an A.P. is $3n^2+5n$ and its mth term is 164, find the value of m.* **[NCERT]**

SOLUTION Let S_n denote the sum of n terms and a_n be the nth term of the given A.P. Then,

$$S_n = 3n^2 + 5n$$

$\Rightarrow \quad S_{n-1} = 3(n-1)^2 + 5(n-1) = 3n^2 - n - 2$ [On replacing n by $(n-1)$ in S_n]

Now, $\quad a_n = S_n - S_{n-1}$

$\Rightarrow \quad a_n = (3n^2+5n) - (3n^2 - n - 2)$

$\Rightarrow \quad a_n = 6n + 2$

Now, $\quad a_m = 164$ [Given]

$\Rightarrow \quad 6m + 2 = 164 \Rightarrow 6m = 162 \Rightarrow m = 27$

EXAMPLE 10 *Find the sum to n terms of the sequence given by $a_n = 5 - 6n, n \in N$.*

SOLUTION We have, $a_n = 5 - 6n$

$\therefore \quad a_{n+1} = 5 - 6(n+1) = -1 - 6n$

$\therefore \quad a_{n+1} - a_n = (-1-6n) - (5-6n) = -6$, for all $n \in N$

Since $a_{n+1} - a_n$ is constant for all $n \in N$. So, the given sequence is an A.P. with common difference -6.

Putting $n = 1$, in $a_n = 5 - 6n$, we get: $a_1 = -1$.

So, the sum S_n to n terms is given by

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(-1 + 5 - 6n) = n(2 - 3n)$$

EXAMPLE 11 *If the m^{th} term of an A.P. is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, show that the sum of mn terms is $\frac{1}{2}(mn+1)$, where $m \neq n$.* **[NCERT]**

SOLUTION Let a be the first term and d be the common difference of the given A.P. It is given that

$$a_m = \frac{1}{n} \text{ and } a_n = \frac{1}{m}$$

Now, $\quad a_m = \frac{1}{n} \Rightarrow a + (m-1)d = \frac{1}{n}$ ...(i)

and, $\quad a_n = \frac{1}{m} \Rightarrow a + (n-1)d = \frac{1}{m}$ ...(ii)

Subtracting (ii) from (i), we get

$$(m-n)d = \frac{1}{n} - \frac{1}{m} \Rightarrow (m-n)d = \frac{m-n}{mn} \Rightarrow d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in (i), we get

$$a + (m-1)\frac{1}{mn} = \frac{1}{n} \Rightarrow a + \frac{1}{n} - \frac{1}{mn} = \frac{1}{n} \Rightarrow a = \frac{1}{mn}$$

$$\therefore \quad S_{mn} = \frac{mn}{2}\left\{2a + (mn-1)d\right\} = \frac{mn}{2}\left\{\frac{2}{mn} + (mn-1) \times \frac{1}{mn}\right\} = \frac{1}{2}(mn+1)$$

Type II **FINDING THE NUMBER OF TERMS IN AN A.P. WHEN THE SUM OF ITS n TERMS IS GIVEN**

EXAMPLE 12 *How many terms of the series 54, 51, 48, ... be taken so that their sum is 513 ? Explain the double answer.*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 54$ and common difference $d = -3$. Let the sum of n terms be 513. Then,

$$S_n = 513$$

$$\Rightarrow \frac{n}{2}\left\{2a + (n-1)\,d\right\} = 513$$

$$\Rightarrow \frac{n}{2}\left\{108 + (n-1) \times -3\right\} = 513 \Rightarrow n^2 - 37n + 342 = 0 \Rightarrow (n-18)(n-19) = 0 \Rightarrow n = 18 \text{ or, } 19$$

Here, the common difference is negative. So, the terms are in decreasing order and the value of 19^{th} term is $54 + (19-1) \times -3 = 0$. Thus, the sum of 18 terms as well as that of 19 terms is 513.

EXAMPLE 13 *Find the number of terms in the series* $20, 19\frac{1}{3}, 18\frac{2}{3}, \ldots$ *of which the sum is 300, explain the double answer.*

SOLUTION The given sequence is an A.P. with first term $a = 20$ and the common difference $d = -\frac{2}{3}$. Let the sum of n terms be 300. Then,

$$S_n = 300$$

$$\Rightarrow \quad \frac{n}{2}\left\{2a + (n-1)\,d\right\} = 300$$

$$\Rightarrow \quad \frac{n}{2}\left\{2 \times 20 + (n-1)\left(-\frac{2}{3}\right)\right\} = 300$$

$$\Rightarrow \quad n^2 - 61n + 900 = 0 \Rightarrow (n-25)(n-36) = 0 \Rightarrow n = 25 \text{ or, } 36$$

$\therefore$ Sum of 25 terms = Sum of 36 terms = 300.

Here, the common difference is negative therefore terms go on diminishing and 31*st* term becomes zero. All terms following 31*st* term are negative. These negative terms when added to positive terms from 26*th* term to 30*th term,* they cancel out each other and the sum remains same. Hence, the sum of 25 terms as well as that of 36 terms is 300.

EXAMPLE 14 *Solve* $1 + 6 + 11 + 16 + \ldots + x = 148$.

SOLUTION Clearly, terms of the given series form an A.P. with first term $a = 1$ and common difference $d = 5$. Let there be n terms in this series. Then,

$$1 + 6 + 11 + 16 + \ldots + x = 148$$

$$\Rightarrow \quad \text{Sum of } n \text{ terms} = 148$$

$$\Rightarrow \quad \frac{n}{2}\left\{2a + (n-1)\,d\right\} = 148$$

$$\Rightarrow \quad \frac{n}{2}\left\{2 + (n-1) \times 5\right\} = 148 \Rightarrow 5n^2 - 3n - 296 = 0 \Rightarrow (n-8)(5n+37) = 0 \Rightarrow n = 8$$

Clearly, $x = n^{th}$ term

$$\Rightarrow \quad x = a + (n-1)\,d = 1 + (8-1) \times 5 = 36 \qquad [\because a = 1, d = 5, n = 8]$$

Type III **PROVING RESULTS RELATED TO THE SUM OF n TERMS OF AN A.P.**

EXAMPLE 15 *The sum of the first p, q, r terms of an A.P. are a, b, c respectively. Show that*

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$ [NCERT]

SOLUTION Let A be the first term and D be the common difference of the given A.P. Then,

$$a = \text{Sum of } p \text{ terms} \Rightarrow a = \frac{p}{2}\left\{2A+(q-1)D\right\} \Rightarrow \frac{2a}{p}=\left\{2A+(p-1)D\right\} \quad ...(i)$$

$$b = \text{Sum of } q \text{ terms} \Rightarrow b = \frac{q}{2}\left\{2A+(q-1)D\right\} \Rightarrow \frac{2b}{q}=\left\{2A+(q-1)D\right\} \quad ...(ii)$$

and, $$c = \text{Sum of } r \text{ terms} \Rightarrow c = \frac{r}{2}\left\{2A+(r-1)D\right\} \Rightarrow \frac{2c}{r}=\left\{2A+(r-1)D\right\} \quad ...(iii)$$

Multiplying (i), (ii) and (iii) by $(q-r)$, $(r-p)$ and $(p-q)$ respectively and adding, we get

$$\frac{2a}{p}(q-r)+\frac{2b}{q}(r-p)+\frac{2c}{r}(p-q)$$

$$=\{2A+(p-1)D\}(q-r)+\{2A+(q-1)D\}(r-p)+\{2A+(r-1)D\}(p-q)$$

$$=2A(q-r+r-p+p-q)+D\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\}$$

$$=2A\times 0+D\times 0=0$$

EXAMPLE 16 *The sum of n, 2n, 3n terms of an A.P. are S_1, S_2, S_3 respectively. Prove that:* $S_3=3(S_2-S_1)$. [NCERT]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_1 = \text{Sum of } n \text{ terms} \Rightarrow S_1 = \frac{n}{2}\left\{2a+(n-1)d\right\} \quad ...(i)$$

$$S_2 = \text{Sum of } 2n \text{ terms} \Rightarrow S_2 = \frac{2n}{2}\left\{2a+(2n-1)d\right\} \quad ...(ii)$$

and, $$S_3 = \text{Sum of } 3n \text{ terms} \Rightarrow S_3 = \frac{3n}{2}\left\{2a+(3n-1)d\right\} \quad ...(iii)$$

Now, $$S_2-S_1=\frac{2n}{2}\left\{2a+(2n-1)d\right\}-\frac{n}{2}\left\{2a+(n-1)d\right\}$$

$$\Rightarrow \quad S_2-S_1=\frac{n}{2}\left[2\left\{2a+(2n-1)d\right\}-\left\{2a+(n-1)d\right\}\right]=\frac{n}{2}\left\{2a+(3n-1)d\right\}$$

$$\therefore \quad 3(S_2-S_1)=\frac{3n}{2}\left\{2a+(3n-1)d\right\} \quad ...(iv)$$

From (iii) and (iv), we get

$$3(S_2-S_1)=S_3$$

EXAMPLE 17 *The sums of n terms of three arithmetical progressions are S_1, S_2 and S_3. The first term of each is unity and the common differences are 1, 2 and 3 respectively. Prove that $S_1+S_3=2S_2$.*

SOLUTION We have,

S_1 = Sum of n terms of an A.P. with first term 1 and common difference 1

$$\Rightarrow \quad S_1=\frac{n}{2}\left\{2\times 1+(n-1)\times 1\right\}=\frac{n}{2}(n+1)$$

S_2 = Sum of n terms of an A.P. with first term 1 and common difference 2

$\Rightarrow \quad S_2 = \frac{n}{2}\left\{2 \times 1 + (n-1) \times 2\right\} = n^2$

$S_3 =$ Sum of n terms of and A.P. with first term 1 and common difference 3

$\Rightarrow \quad S_2 = \frac{n}{2}\left\{2 \times 1 + (n-1) \times 3\right\} = \frac{n}{2}(3n-1)$

$\therefore \quad S_1 + S_3 = \frac{n}{2}(n+1) + \frac{n}{2}(3n-1) = 2n^2$

Hence, $S_1 + S_3 = 2S_2$ $\qquad [\because S_2 = n^2]$

EXAMPLE 18 *If in an A.P. the sum of m terms is equal to n and the sum of n terms is equal to m, then prove that the sum of (m + n) terms is – (m + n). Also, find the sum of first (m – n) terms (m > n).* **[NCERT EXEMPLAR]**

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_m = n \Rightarrow \frac{m}{2}\left\{2a + (m-1)d\right\} = n \Rightarrow 2am + m(m-1)d = 2n \qquad \text{...(i)}$$

and, $$S_n = m \Rightarrow \frac{n}{2}\left\{2a + (n-1)d\right\} = m \Rightarrow 2an + n(n-1)d = 2m \qquad \text{...(ii)}$$

Subtracting (ii) from (i), we get

$$2a(m-n) + \{m(m-1) - n(n-1)\}d = 2n - 2m$$

$$\Rightarrow \quad 2a(m-n) + \{(m^2 - n^2) - (m-n)\}d = -2(m-n)$$

$$\Rightarrow \quad 2a + (m+n-1)d = -2 \qquad \text{[On dividing both sides by } (m-n)] \quad \text{...(iii)}$$

Now, $$S_{m+n} = \frac{m+n}{2}\left\{2a + (m+n-1)d\right\}$$

$$\Rightarrow \quad S_{m+n} = \frac{(m+n)}{2}(-2) \qquad \text{[Using (iii)]}$$

$$\therefore \quad S_{m+n} = -(m+n)$$

From (iii), we obtain

$$2a = -2 - (m+n-1)d \qquad \text{...(iv)}$$

Substituting this value of $2a$ in (i), we obtain

$$-2m - m(m+n-1)d + m(m-1)d = 2n$$

$$\Rightarrow \quad d = -2\left(\frac{m+n}{mn}\right) \qquad \text{...(v)}$$

Putting $d = -2\left(\frac{m+n}{mn}\right)$ in (iv), we obtain

$$2a = -2 + \frac{2}{mn}(m+n-1)(m+n) \qquad \text{...(vi)}$$

Now,

$$S_{m-n} = \frac{m-n}{2}\left\{2a + (m-n-1)d\right\}$$

$$\Rightarrow \quad S_{m-n} = \frac{m-n}{2}\left\{-2 + \frac{2}{mn}(m+n-1)(m+n) - \frac{2}{mn}(m-n-1)(m+n)\right\} \qquad \text{[Using (v) and (vi)]}$$

$$\Rightarrow \quad S_{m-n} = \frac{m-n}{2}\left\{-2 + \frac{4n}{mn}(m+n)\right\} = \frac{1}{m}(m-n)(m+2n)$$

EXAMPLE 19 *If the sum of first m terms of an A.P. is the same as the sum of its first n terms, show that the sum of its (m + n) terms is zero.* **[NCERT]**

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_m = S_n$$

$$\Rightarrow \quad \frac{m}{2}\left\{2a + (m-1)\,d\right\} = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$$

$$\Rightarrow \quad 2a\,(m-n) + \{m\,(m-1) - n\,(n-1)\}\,d = 0$$

$$\Rightarrow \quad 2a\,(m-n) + \{(m^2 - n^2) - (m-n)\}\,d = 0$$

$$\Rightarrow \quad (m-n)\,\{2a + (m+n-1)\,d\} = 0$$

$$\Rightarrow \quad 2a + (m+n-1)\,d = 0 \qquad [\because m - n \neq 0] \quad \text{... (i)}$$

$$\therefore \quad S_{m+n} = \frac{m+n}{2}\left\{2a + (m+n-1)\,d\right\} = \frac{m+n}{2} \times 0 = 0 \qquad \text{[Using (i)]}$$

EXAMPLE 20 *The ratio of the sums of m and n terms of an A.P. is $m^2 : n^2$. Show that the ratio of the mth and nth terms is $(2m-1):(2n-1)$.* [NCERT]

SOLUTION Let a be the first term and d the common difference of the given A.P. Then, the sums of m and n terms are given by

$$S_m = \frac{m}{2}\left\{2a + (m-1)\,d\right\} \quad \text{and,} \quad S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\} \text{ respectively.}$$

It is given that

$$\frac{S_m}{S_n} = \frac{m^2}{n^2}$$

$$\Rightarrow \quad \frac{\frac{m}{2}\left\{2a + (m-1)\,d\right\}}{\frac{n}{2}\left\{2a + (n-1)\,d\right\}} = \frac{m^2}{n^2}$$

$$\Rightarrow \quad \frac{2a + (m-1)\,d}{2a + (n-1)\,d} = \frac{m}{n}$$

$$\Rightarrow \quad \{2a + (m-1)\,d\}\,n = \{2a + (n-1)\,d\}\,m$$

$$\Rightarrow \quad 2a\,(n-m) = d\,\{(n-1)\,m - (m-1)\,n\}$$

$$\Rightarrow \quad 2a\,(n-m) = d\,(n-m)$$

$$\Rightarrow \quad d = 2a$$

$$\therefore \quad \frac{T_m}{T_n} = \frac{a + (m-1)\,d}{a + (n-1)\,d} = \frac{a + (m-1)\,2a}{a + (n-1)\,2a} = \frac{2m-1}{2n-1}$$

EXAMPLE 21 *The interior angles of a polygon are in A.P. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon.* [NCERT]

SOLUTION Let there be n sides of the polygon. Then, the sum of its interior angles is given by

$$S_n = (2n-4) \text{ right angles} = (n-2) \times 180° \qquad \text{...(i)}$$

Thus, the interior angles form an A.P. with first term $a = 120°$ and common difference $d = 5°$.

$$\therefore \quad S_n = \frac{n}{2}\left\{2 \times 120° + (n-1) \times 5°\right\} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$(n-2) \times 180° = \frac{n}{2}\left\{2 \times 120° + (n-1) \times 5°\right\}$$

$$\Rightarrow \quad (n-2) \times 360 = n\,(5n + 235)$$

$$\Rightarrow \quad n^2 - 25n + 144 = 0 \Rightarrow (n-16)\,(n-9) = 0 \Rightarrow n = 16 \text{ or, } n = 9$$

For $n = 16$, we obtain

Last angle $= a_n = a + (n-1)\,d = 120° + (16-1) \times 5 = 195°$, which is not possible.

Hence, $n = 9$.

EXAMPLE 22 *The first, second and the last terms of an A.P. are a, b, c respectively. Prove that the sum is* $\dfrac{(a+c)(b+c-2a)}{2(b-a)}$. **[NCERT EXEMPLAR]**

SOLUTION Let d be the common difference of the given A.P. Then, $d = b - a$. Let there be n terms in the given A.P. Then,

$$c = n\text{th term}$$

$$\Rightarrow \quad c = a + (n-1)\,d$$

$$\Rightarrow \quad c = a + (n-1)(b-a) \qquad [\because d = b - a]$$

$$\Rightarrow \quad n - 1 = \frac{c-a}{b-a} \Rightarrow n = \frac{c-a}{b-a} + 1 \Rightarrow n = \frac{b+c-2a}{b-a}$$

$\therefore$ Sum of the A.P. = Sum of its n terms

$$= \frac{n}{2}(a+c) \qquad \left[\text{Using}: S_n = \frac{n}{2}(a+l)\right]$$

$$= \frac{(a+c)(b+c-2a)}{2(b-a)}.$$

EXAMPLE 23 *Let S_n denote the sum of the first n terms of an A.P. If $S_{2n} = 3\,S_n$, then prove that* $\dfrac{S_{3n}}{S_n} = 6$.

SOLUTION Let a be the first term and d the common difference of the given A.P. Then,

$$S_{2n} = 3\,S_n$$

$$\Rightarrow \quad \frac{2n}{2}\left\{a + (2n-1)\,d\right\} = \frac{3n}{2}\left\{a + (n-1)\,d\right\}$$

$$\Rightarrow \quad 2\{a + (2n-1)\,d\} = 3\{a + (n-1)\,d\}$$

$$\Rightarrow \quad 2a - (3n - 3 - 4n + 2)\,d = 0$$

$$\Rightarrow \quad 2a - (n+1)\,d = 0$$

$$\Rightarrow \quad 2a = (n+1)\,d \qquad \text{...(i)}$$

Now,
$$\frac{S_{3n}}{S_n} = \frac{\frac{3n}{2}\left\{2a + (3n-1)\,d\right\}}{\frac{n}{2}\left\{2a + (n-1)\,d\right\}}$$

$$\Rightarrow \quad \frac{S_{3n}}{S_n} = \frac{3\left\{(n+1)\,d + (3n-1)\,d\right\}}{\left\{(n+1)\,d + (n-1)\,d\right\}} \qquad [\text{Using (i)}]$$

$$\Rightarrow \quad \frac{S_{3n}}{S_n} = \frac{12\,n\,d}{2nd} = 6.$$

LEVEL-2

Type IV **ON SUM OF TERMS OF AN A.P.**

EXAMPLE 24 *Prove that a sequence is an A.P. iff the sum of its n terms is of the form $An^2 + Bn$, where A, B are constants.*

SOLUTION Let S_n be the sum of n terms of an A.P. with first term a and common difference d. Then,

$$S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\} = an + \frac{n^2}{2}d - \frac{n}{2}d = \left(\frac{d}{2}\right)n^2 + \left(a - \frac{d}{2}\right)n$$

$\Rightarrow \quad S_n = An^2 + Bn$, where $A = \dfrac{d}{2}$ and $B = a - \dfrac{d}{2}$

Thus, the sum of n terms of an A.P. is of the form $An^2 + Bn$.

Conversely, let the sum S_n of n terms of a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ be of the form $An^2 + Bn$. Then, we have to show that the sequence is an A.P.

We have, $S_n = An^2 + Bn$

$\Rightarrow \quad S_{n-1} = A(n-1)^2 + B(n-1)$ [On replacing n by $n-1$]

Now, $\quad a_n = S_n - S_{n-1}$

$\Rightarrow \quad a_n = \{An^2 + Bn\} - \{A(n-1)^2 + B(n-1)\} = 2An + (B - A)$

$\Rightarrow \quad a_{n+1} = 2A(n+1) + (B - A)$ [On replacing n by $n+1$]

$\therefore \quad a_{n+1} - a_n = \{2A(n+1) + B - A\} - \{2An + (B - A)\} = 2A$

Clearly, $a_{n+1} - a_n = 2A$ for all $n \in N$. So, the sequence is an A.P. with common difference $2A$.

REMARK *It follows from this example that a sequence is an A.P. iff the sum of its n terms is of the form $An^2 + Bn$ i.e. a quadratic expression in n and in such a case the common difference is twice the coefficient of n^2. For example, if $S_n = 3n^2 + 2n$, one can easily say that it is the sum of n terms of an A.P. with common difference 6. Similarly, $S_n = nP + \frac{1}{2} n(n-1)Q = \frac{Q}{2} n^2 + \left(P - \frac{Q}{2}\right) n$ is the sum of n terms of an A.P, with common difference Q.*

EXAMPLE 25 *Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots$, if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.* [NCERT EXEMPLAR]

SOLUTION We know that in an A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term i.e. $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$ So, if an A.P. consists of 24 terms, then $a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$.

Now, $\quad a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$

$\Rightarrow \quad (a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225$

$\Rightarrow \quad 3(a_1 + a_{24}) = 225$

$\Rightarrow \quad a_1 + a_{24} = \dfrac{225}{3} = 75$ ...(i)

$\therefore \quad S_{24} = \dfrac{24}{2}(a_1 + a_{24})$ $\left[\text{Using } S_n = \dfrac{n}{2}(a_1 + a_n)\right]$

$\Rightarrow \quad S_{24} = 12(75) = 900$ [Using (i)]

EXAMPLE 26 *The first term of an A.P. is a and the sum of first p terms is zero, show that the sum of its next q terms is $-\dfrac{a(p+q)q}{p-1}$.* [NCERT EXEMPLAR]

SOLUTION Let d be the common difference of the A.P. It is gtiven that the sum of first p terms is zero

i.e. $\quad S_p = 0 \Rightarrow \dfrac{p}{2}\left\{2a + (p-1)d\right\} = 0 \Rightarrow d = -\dfrac{2a}{p-1}$.

Let S be the required sum. Then,

$S = a_{p+1} + a_{p+2} + \ldots + a_{p+q}$

$\Rightarrow \quad S = (a_1 + a_2 + \ldots + a_p + a_{p+1} + \ldots + a_{p+q}) - (a_1 + a_2 + \ldots + a_p)$

$\Rightarrow \quad S = S_{p+q} - S_p$

$\Rightarrow \quad S = S_{p+q} - 0 \qquad [\because\ S_p = 0 \text{ (given)}]$

$$\Rightarrow \quad S = \frac{p+q}{2}\left\{2a + (p+q-1)d\right\}$$

$$\Rightarrow \quad S = \frac{p+q}{2}\left\{2a + (p+q-1)\left(-\frac{2a}{p-1}\right)\right\}$$

$$\Rightarrow \quad S = (p+q)\,a\left\{1 - \left(\frac{p+q-1}{p-1}\right)\right\} = (p+q)\,a\left(\frac{p-1-p-q+1}{p-1}\right) = -\frac{(p+q)\,a}{p-1}$$

EXAMPLE 27 *If the first term of an A.P. is 2 and the sum of first five terms is equal to one-fourth of the sum of the next five terms, find the sum of first 30 terms.*

SOLUTION Let $a_1, a_2, a_3, \dots$ be given A.P. with common difference d. It is given that $a_1 = 2$ and the sum of first five terms is equal to one fourth of the sum of next five terms.

i.e. $$a_1 + a_2 + a_3 + a_4 + a_5 = \frac{1}{4}(a_6 + a_7 + a_8 + a_9 + a_{10})$$

$$\Rightarrow \quad 4(a_1 + a_2 + a_3 + a_4 + a_5) = (a_6 + a_7 + a_8 + a_9 + a_{10})$$

$$\Rightarrow \quad 5(a_1 + a_2 + a_3 + a_4 + a_5) = (a_1 + a_2 + \dots + a_{10})$$

$$\Rightarrow \quad 5S_5 = S_{10}$$

$$\Rightarrow \quad 5\left[\frac{5}{2}\left\{2\times 2 + (5-1)d\right\}\right] = \frac{10}{2}\left\{2\times 2 + (10-1)d\right\}$$

$$\Rightarrow \quad 50(1+d) = 20 + 45d$$

$$\Rightarrow \quad d = -6$$

Thus, we have $a = 2$ and $d = -6$.

$$\therefore \quad \text{Required sum} = S_{30} = \frac{30}{2}\left\{2\times 2 + (30-1)\times -6\right\} = -2550.$$

EXAMPLE 28 *The p^{th} term of an A.P. is a and qth term is b. Prove that the sum of its (p + q) terms is* $\frac{p+q}{2}\left\{a + b + \frac{a-b}{p-q}\right\}$. **[NCERT EXEMPLAR]**

SOLUTION Let A and D be the first term and common difference respectively of the given A.P. Then,

$$a = pth \text{ term} \Rightarrow a = A + (p-1)D \qquad \dots\text{(i)}$$

$$b = qth \text{ term} \Rightarrow b = A + (q-1)D \qquad \dots\text{(ii)}$$

Subtracting (ii) from (i), we get: $D = \dfrac{a-b}{p-q}$

Adding (i) and (ii), we get $a + b = 2A + (p+q-2)D$

$$\Rightarrow \quad a + b = 2A + (p+q-1)D - D$$

$$\Rightarrow \quad (a+b) + D = 2A + (p+q-1)D$$

$$\Rightarrow \quad (a+b) + \frac{a-b}{p-q} = 2A + (p+q-1)D \qquad \dots\text{(iii)}$$

Now, S_{p+q} = Sum of $(p+q)$ terms

$$\Rightarrow \quad S_{p+q} = \frac{p+q}{2}\left\{2A + (p+q-1)D\right\} = \frac{p+q}{2}\left\{a+b+\frac{a-b}{p-q}\right\} \qquad \text{[Using (iii)]}$$

EXAMPLE 29 *The ratio of the sum of n terms of two A.P.'s is $(7n+1):(4n+27)$. Find the ratio of their mth terms.*

SOLUTION Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the two given A.P.'s. Then, the sums S_n and S_n' of their n terms are given by

$$S_n = \frac{n}{2}\left\{2a_1 + (n-1)d_1\right\}, \text{ and } S_n' = \frac{n}{2}\left\{2a_2 + (n-1)d_2\right\}$$

$$\therefore \quad \frac{S_n}{S_n'} = \frac{\frac{n}{2}\left\{2a_1 + (n-1)d_1\right\}}{\frac{n}{2}\left\{2a_2 + (n-1)d_2\right\}} = \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2}$$

It is given that

$$\frac{S_n}{S_n'} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \quad \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \quad \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27} \qquad \ldots \text{(i)}$$

We have to find the ratio to m^{th} terms of two A.P.'s i.e., $\frac{a_1+(m-1)d_1}{a_1+(m-1)d_2}$. Clearly, this can be obtained by replacing $\frac{n-1}{2}$ by $(m-1)$ on the LHS of (i). Replacing $\frac{n-1}{2}$ by $m-1$ i.e. n by $(2m-1)$ on both side of (i), we get

$$\frac{a_1+(m-1)d_1}{a_2+(m-1)d_2} = \frac{7(2m-1)+1}{4(2m-1)+27} = \frac{14m-6}{8m+23}$$

Hence, the ratio of the mth terms of the two A.P.'s is $(14m-6):(8m+23)$.

REMARK *It is evident from the above example that if we are given the ratio of the sums of n terms of two A.P.'s then the ratio of their m^{th} terms is obtained by replacing n by $(2m-1)$.*

EXAMPLE 30 *The sum of n terms of two arithmetic progressions are in the ratio $(3n+8):(7n+15)$. Find the ratio of their 12th terms.* [NCERT]

SOLUTION Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the two given A.P.'s. Then, the sums of their n terms are given by

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d_1\} \text{ and, } S_n' = \frac{n}{2}\{2a_2 + (n-1)d_2\}$$

It is given that

$$\frac{S_n}{S_n'} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{\frac{n}{2}\left\{2a_1 + (n-1)d_1\right\}}{\frac{n}{2}\left\{2a_2 + (n-1)d_2\right\}} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{a_1 + \left(\frac{n-1}{2}\right) d_1}{a_2 + \left(\frac{n-1}{2}\right) d_2} = \frac{3n+8}{7n+15}$$

Replacing $\frac{n-1}{2}$ by 11 i.e. n by 23 on both sides, we get

$$\frac{a_1 + 11\, d_1}{a_2 + 11\, d_2} = \frac{3 \times 23 + 8}{7 \times 23 + 15} = \frac{77}{176} = \frac{7}{16}$$

Hence, the required ratio is 7 : 16.

ALITER If the ratio of the sums of n terms of two A.P.s is given, then the ratio of their m^{th} terms is obtained by replacing n by $(2m-1)$ in the given ratio. So, required ratio is obtained by replacing n by $2 \times 12 - 1 = 23$ in $(3n+8) : (7n+15)$.

Hence, required ratio $= (69+8) : (161+15) = 7 : 16$.

EXAMPLE 31 *If there are $(2n+1)$ terms in A.P., then prove that the ratio of the sum of odd terms and the sum of even terms is $(n+1) : n$.* [NCERT EXEMPLAR]

SOLUTION Let a and d be the first term and common difference respectively of the given A.P. Let a_k denote the k^{th} terms of the given A.P. Then, $a_k = a + (k-1)\, d$.

Now, S_1 = Sum of odd terms $= a_1 + a_3 + a_5 + \dots + a_{2n+1}$

$$\Rightarrow \quad S_1 = \frac{n+1}{2}\left(a_1 + a_{2n+1}\right) = \frac{n+1}{2}\left\{a + a + (2n+1-1)\, d\right\} = (n+1)\,(a + nd)$$

and, S_2 = Sum of even terms $= a_2 + a_4 + a_6 + \dots + a_{2n}$

$$\Rightarrow \quad S_2 = \frac{n}{2}\left(a_2 + a_{2n}\right) = \frac{n}{2}\left[(a+d) + \left\{a + (2n-1)\, d\right\}\right] = n\,(a + nd)$$

$$\therefore \quad S_1 : S_2 = (n+1)\,(a+nd) : n\,(a+nd) = (n+1) : n$$

EXAMPLE 32 *Let S_k be the sum of first k terms of an A.P. What must this progression be for the ratio $\frac{S_{kx}}{S_x}$ to be independent of x ?*

SOLUTION Let a be the first term and d common difference of the given progression. Then,

$$\frac{S_{kx}}{S_x} = \frac{\frac{kx}{2}\left\{2a + (kx-1)\, d\right\}}{\frac{x}{2}\left\{2a + (x-1)\, d\right\}} = \frac{k\left\{kxd + (2a-d)\right\}}{\left\{xd + (2a-d)\right\}}$$

Clearly, the RHS of the above relation will be independent of x iff $2a - d = 0$ i.e. $d = 2a$.

Hence, the progression is $a, 3a, 5a, 7a, \dots,$ where a is any non-zero real number.

EXAMPLE 33 *Let S_n be the sum of first n terms of an A.P. with non-zero common difference. Find the ratio of first term and common difference if $\frac{S_{n_1 n_2}}{S_{n_1}}$ is independent of n_1.*

SOLUTION Let the first term and common difference of the A.P. be a and d respectively. Then,

$$S_{n_1 n_2} = \frac{n_1 n_2}{2}\left\{2a + (n_1 n_2 - 1)\, d\right\} \text{ and, } S_{n_1} = \frac{n_1}{2}\left\{2a + (n_1 - 1)\, d\right\}$$

$$\therefore \quad \frac{S_{n_1 n_2}}{S_{n_1}} = \frac{n_2\left\{2a + (n_1 n_2 - 1)\, d\right\}}{\left\{2a + (n_1 - 1)\, d\right\}} = \frac{n_2\left\{(2a-d) + n_1 n_2\, d\right\}}{\left\{(2a-d) + n_1\, d\right\}}$$

Clearly, RHS will be independent of n_1 iff $2a - d = 0$ i.e. $d = 2a$.

Hence, $\frac{a}{d} = \frac{1}{2}$.

EXAMPLE 34 *If $S_1, S_2, S_3, \dots S_m$ are the sums of n terms of m A.P.'s whose first terms are 1, 2, 3, ..., m and common differences are 1, 3, 5, ..., (2 m − 1) respectively. Show that*

$$S_1 + S_2 + \dots + S_m = \frac{mn}{2}(mn+1)$$

SOLUTION The first terms, common difference and the sums of their n terms are as under:

First terms	*Common differences*	*Sums of n terms*
1	1	$S_1 = \frac{n}{2}\left\{2\times 1 + (n-1)\times 1\right\}$
2	3	$S_2 = \frac{n}{2}\left\{2\times 2 + (n-1)\times 3\right\}$
3	5	$S_3 = \frac{n}{2}\left\{2\times 3 + (n-1)\times 5\right\}$
$\vdots$	$\vdots$	
m	$2m-1$	$S_m = \frac{n}{2}\left\{2m + (n-1)(2m-1)\right\}$

$$\begin{aligned}
\therefore\quad & S_1 + S_2 + \dots + S_m \\
&= \frac{n}{2}\left[2\times 1 + (n-1)\times 1\right] + \frac{n}{2}\left[2\times 2 + (n-1)\times 3\right] + \dots + \frac{n}{2}\left[2m + (n-1)(2m-1)\right] \\
&= \frac{n}{2}\left[2\times(1+2+3+\dots+m) + (n-1)(1+3+5+\dots+(2m-1))\right] \\
&= \frac{n}{2}\left[2\times\frac{m}{2}(1+m) + (n-1)\frac{m}{2}\{1+(2m-1)\}\right] \\
&= \frac{n}{2}\left[m(m+1) + m^2(n-1)\right] = \frac{mn}{2}(mn+1)
\end{aligned}$$

EXAMPLE 35 *If the sum of m terms of an A.P. is equal to the sum of either the next n terms or the next p terms, then prove that*

$$(m+n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m+p)\left(\frac{1}{m}-\frac{1}{n}\right).$$

[NCERT EXEMPLAR]

SOLUTION Let a denote the first term and d the common difference of the A.P. Further, let a_k denote the k^{th} term of the A.P. Then,

Sum of m terms = Sum of next n terms

$$\begin{aligned}
\Rightarrow\quad & a_1 + a_2 + a_3 + \dots + a_m = a_{m+1} + a_{m+2} + \dots + a_{m+n} \\
\Rightarrow\quad & 2(a_1 + a_2 + \dots + a_m) = a_1 + a_2 + \dots + a_m + a_{m+1} + a_{m+2} + \dots + a_{m+n} \\
\Rightarrow\quad & 2S_m = S_{m+n} \\
\Rightarrow\quad & 2\,\frac{m}{2}\left\{2a + (m-1)d\right\} = \frac{m+n}{2}\left\{2a + (m+n-1)d\right\} \\
\Rightarrow\quad & \frac{2m}{m+n} = \frac{2a+(m+n-1)d}{2a+(m-1)d} \\
\Rightarrow\quad & \frac{2m}{m+n} - 1 = \frac{2a+(m+n-1)d}{2a+(m-1)d} - 1 \\
\Rightarrow\quad & \frac{m-n}{m+n} = \frac{nd}{2a+(m-1)d} \qquad \text{...(i)}
\end{aligned}$$

Similarly,

Sum of m terms = Sum of next p terms

$$\Rightarrow \quad \frac{m-p}{m+p} = \frac{pd}{2a+(m-1)d} \quad \text{...(ii)}$$

Dividing (i) by (ii), we get

$$\frac{m-n}{m+n} \cdot \frac{m+p}{m-p} = \frac{n}{p}$$

$$\Rightarrow \quad \frac{(m-n)(m+p)}{n} = \frac{(m+n)(m-p)}{p}$$

$$\Rightarrow \quad \frac{(m-n)(m+p)}{nm} = \frac{(m+n)(m-p)}{mp}$$

$$\Rightarrow \quad (m+p)\left(\frac{m-n}{mn}\right) = (m+n)\left(\frac{m-p}{mp}\right)$$

$$\Rightarrow \quad (m+n)\left(\frac{1}{p}-\frac{1}{m}\right) = (m+p)\left(\frac{1}{n}-\frac{1}{m}\right)$$

$$\Rightarrow \quad (m+n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$$

EXERCISE 19.4

LEVEL-1

1. Find the sum of the following arithmetic progressions:
   (i) 50, 46, 42, ... to 10 terms
   (ii) 1, 3, 5, 7, ... to 12 terms
   (iii) 3, 9/2, 6, 15/2, ... to 25 terms
   (iv) 41, 36, 31, ... to 12 terms
   (v) $a+b, a-b, a-3b, \ldots$ to 22 terms
   (vi) $(x-y)^2, (x^2+y^2), (x+y)^2, \ldots$ to n terms
   (vii) $\frac{x-y}{x+y}, \frac{3x-2y}{x+y}, \frac{5x-3y}{x+y}, \ldots$ to n terms
2. Find the sum of the following series:
   (i) $2+5+8+\ldots+182$
   (ii) $101+99+97+\ldots+47$
   (iii) $(a-b)^2+(a^2+b^2)+(a+b)^2+\ldots+[(a+b)^2+6ab]$
3. Find the sum of first n natural numbers.
4. Find the sum of all natural numbers between 1 and 100, which are divisible by 2 or 5. [NCERT]
5. Find the sum of first n odd natural numbers.
6. Find the sum of all odd numbers between 100 and 200.
7. Show that the sum of all odd integers between 1 and 1000 which are divisible by 3 is 83667.
8. Find the sum of all integers between 84 and 719, which are multiples of 5.

9. Find the sum of all integers between 50 and 500 which are divisible by 7.
10. Find the sum of all even integers between 101 and 999.
11. Find the sum of all integers between 100 and 550, which are divisible by 9.
12. Find the sum of the series: $3 + 5 + 7 + 6 + 9 + 12 + 9 + 13 + 17 + \dots$ to $3n$ terms.
13. Find the sum of all those integers between 100 and 800 each of which on division by 16 leaves the remainder 7.
14. Solve: (i) $25 + 22 + 19 + 16 + \dots + x = 115$ (ii) $1 + 4 + 7 + 10 + \dots + x = 590$.
15. Find the rth term of an A.P., the sum of whose first n terms is $3n^2 + 2n$. **[NCERT EXEMPLAR]**
16. How many terms are there in the A.P. whose first and fifth terms are -14 and 2 respectively and the sum of the terms is 40 ?
17. The sum of first 7 terms of an A.P. is 10 and that of next 7 terms is 17. Find the progression.
18. The third term of an A.P. is 7 and the seventh term exceeds three times the third term by 2. Find the first term, the common difference and the sum of first 20 terms.
19. The first term of an A.P. is 2 and the last term is 50. The sum of all these terms is 442. Find the common difference.
20. The number of terms of an A.P. is even; the sum of odd terms is 24, of the even terms is 30, and the last term exceeds the first by 10 1/2, find the number of terms and the series.
21. If $S_n = n^2 p$ and $S_m = m^2 p, m \neq n$, in an A.P., prove that $S_p = p^3$.
22. If 12th term of an A.P. is -13 and the sum of the first four terms is 24, what is the sum of first 10 terms ?
23. If the 5*th* and 12*th* terms of an A.P. are 30 and 65 respectively, what is the sum of first 20 terms ?
24. Find the sum of n terms of the A.P. whose kth terms is $5k + 1$. **[NCERT]**
25. Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder. **[NCERT]**
26. If the sum of a certain number of terms of the AP 25, 22, 19, ... is 116. Find the last term. **[NCERT]**
27. Find the sum of odd integers from 1 to 2001. **[NCERT]**
28. How many terms of the A.P. $-6, -\frac{11}{2}, -5, \dots$ are needed to give the sum -25?
29. In an A.P. the first term is 2 and the sum of the first five terms is one fourth of the next five terms. Show that 20th term is -112. **[NCERT]**

LEVEL-2

30. If S_1 be the sum of $(2n + 1)$ terms of an A.P. and S_2 be the sum of its odd terms, then prove that: $S_1 : S_2 = (2n + 1) : (n + 1)$.
31. Find an A.P. in which the sum of any number of terms is always three times the squared number of these terms.
32. If the sum of n terms of an A.P. is $nP + \frac{1}{2} n(n-1) Q$, where P and Q are constants, find the common difference. **[NCERT]**
33. The sums of n terms of two arithmetic progressions are in the ratio $5n + 4 : 9n + 6$. Find the ratio of their 18th terms. **[NCERT]**
34. The sums of first n terms of two A.P.'s are in the ratio $(7n + 2) : (n + 4)$. Find the ratio of their 5*th* terms.

ANSWERS

1. (i) 320 (ii) 144 (iii) 525 (iv) 162 (v) $22a-440b$ (vi) $n\{(x-y)^2+(n-1)xy\}$
(vii) $\frac{n}{2(x+y)}\left\{n(2x-y)-y\right\}$ 2. (i) 5612 (ii) 2072 (iii) $6(a^2+b^2+3ab)$
3. $\frac{n(n+1)}{2}$ 4. 3050 5. n^2 6. 7500 8. 50800 9. 17696 10. 246950
11. 16425 12. $3n(2n+3)$ 13. 19668 14. (i) -2 (ii) 58 15. $6r-1$
16. 10 17. $a=1,\ d=1/7$ 18. $-1, 4, 740$ 19. 3
20. 8 terms, $1\frac{1}{2}, 3, 4\frac{1}{2}, \ldots$ 22. 0 23. 1150 24. $\frac{n}{2}(5n+7)$ 25. 1210
26. 4 27. 1002001 28. 5 or 20 31. 3, 9, 15, 21 32. Q
33. 179 : 321 34. 5 : 1

HINTS TO NCERT & SELECTED PROBLEMS

3. Required sum $=1+2+3+\ldots+n=\frac{n}{2}(1+n)$

4. Required sum = Sum of natural numbers between1 and 100 which are divisible by 2
+ Sum of natural numbers between 1 and 100 which are divisible by 5
− Sum of natural numbers between 1 and 100 which are divisible by 2 and 5 both i.e. by 10

$$= (2+4+\ldots+100)+(5+10+15+\ldots+100)-(10+20+\ldots+100)$$
$$= \frac{50}{2}(2+100)+\frac{20}{2}(5+100)-\frac{10}{2}(10+100)$$
$$= 2550+1050-550 = 3050$$

5. Required sum $=1+3+5+\ldots+(2n-1)=\frac{n}{2}\left\{1+(2n-1)\right\}=n^2$

6. Required sum $=101+103+\ldots+199=\frac{50}{2}(101+199)=7500$

8. Required sum $=85+90+\ldots+715=\frac{127}{2}(85+715)=50800$

9. Required sum $=56+63+\ldots+497$

10. Required sum $=102+104+\ldots+998$

11. Required sum $=108+117+\ldots+549$

12. Required sum $=(3+6+9+\ldots$ to n terms$)+(5+9+13+\ldots$ to n terms$)+(7+12+17+\ldots$ to n terms$)$

13. Required sum $=103+119+135+\ldots+791$

15. $a_r=S_r-S_{r-1}=(3r^2+2r)-\left\{3(r-1)^2+2(r-1)\right\}=6r-1$

17. We have, $S_7=10$ and $S_{14}=10+17=27$

24. We have,

$$a_k=5k+1 \Rightarrow a_1=6 \text{ and } a_n=5n+1$$
$$\therefore\quad S_n=\frac{n}{2}\left(a_1+a_n\right)$$
$$\Rightarrow\quad S_n=\frac{n}{2}\left(6+5n+1\right)=\frac{n}{2}\left(5n+7\right)$$

25. We have to find the sum of all two digit numbers of the form $4k+1$, $k \in N$. Clearly, such numbers are 13, 17, 21, 25, ..., 97 and are forming an A.P. with common difference 4. Let such numbers be n in number. Then,

$$97 = n^{\text{th}} \text{ term of AP with first term 13 and common difference 4}$$
$$\Rightarrow \quad 97 = 13 + (n-1) \times 4$$
$$\Rightarrow \quad n - 1 = 21$$
$$\Rightarrow \quad n = 22$$

Let S be the sum of such numbers. Then,

$$S = \frac{n}{2}\left(a_1 + a_n\right)$$
$$\Rightarrow \quad S = \frac{22}{2}\left(13 + 97\right) = 1210$$

26. Let the sum of n terms of the A.P. 25, 22, 19, ... be 116. Then,

$$116 = \frac{n}{2}\left\{2 \times 25 + (n-1) \times (-3)\right\}$$
$$\Rightarrow \quad 232 = n(-3n + 53)$$
$$\Rightarrow \quad 3n^2 - 53n + 232 = 0$$
$$\Rightarrow \quad 3n^2 - 24n - 29n + 232 = 0$$
$$\Rightarrow \quad (n-8)(3n-29) = 0$$
$$\Rightarrow \quad n = 8 \qquad \left[\because n \neq \frac{29}{3}\right]$$
$$\therefore \quad a_8 = 25 + 7 \times (-3) = 4$$

27. The odd integers from 1 to 2001 are $1, 3, 5, 7, \ldots, 2001$. Let the number of such integers be n. Then,

$$2001 = 1 + (n-1) \times 2 \Rightarrow n = 1001$$
$$\therefore \quad \text{Required sum} = \frac{1001}{2}(1 + 2001) = 1001 \times 1001 = 1002001.$$

<u>ALITER</u> The sum of first n odd integers is n^2. So, the sum of odd integers $1, 3, 5, 7, \ldots, 2001$ is $(1001)^2 = 1002001$.

29. We have,

$$a_1 = 2 \text{ and } a_1 + a_2 + \ldots + a_5 = \frac{1}{4}(a_6 + a_7 + \ldots + a_{10})$$

Now,

$$a_1 + a_2 + \ldots + a_5 = \frac{1}{4}(a_6 + a_7 + \ldots + a_{10})$$
$$\Rightarrow \quad 4(a_1 + a_2 + \ldots + a_5) = a_6 + a_7 + \ldots + a_{10}$$
$$\Rightarrow \quad 4S_5 = S_{10} - S_5$$
$$\Rightarrow \quad 5S_5 = S_{10}$$
$$\Rightarrow \quad 5\left[\frac{5}{2}\{2 \times 2 + (5-1)d\}\right] = \frac{10}{2}\left[2 \times 2 + (10-1)d\right]$$
$$\Rightarrow \quad \frac{25}{2}(4 + 4d) = \frac{10}{2}(9d + 4)$$
$$\Rightarrow \quad 20(1 + d) = 2(9d + 4) \Rightarrow 10 + 10d = 9d + 4 \Rightarrow d = -6$$

$\therefore \quad a_{20} = a_1 + 19d = 2 + 19 \times (-6) = -112$

31. Use $S_n = 3n^2$ and $a_n = S_n - S_{n-1}$.

32. We have,

$$S_n = nP + \frac{1}{2}n(n-1)Q \Rightarrow S_{n-1} = (n-1)P + \frac{1}{2}(n-1)(n-2)Q$$

Let a_n be the n^{th} term. Then,

$$a_n = S_n - S_{n-1}$$

$$\Rightarrow \quad a_n = \left\{nP + \frac{1}{2}n(n-1)Q\right\} - \left\{(n-1)P + \frac{1}{2}(n-1)(n-2)Q\right\}$$

$$\Rightarrow \quad a_n = P + \frac{1}{2}(n-1)\{n-(n-2)\}Q$$

$$\Rightarrow \quad a_n = P + (n-1)Q$$

$$\Rightarrow \quad a_{n-1} = P + (n-2)Q$$

Let d be the common difference. Then,

$$d = a_n - a_{n-1} = \{P + (n-1)Q\} - \{P + (n-2)Q\} = Q$$

ALITER We have,

$$S_n = nP + \frac{1}{2}n(n-1)Q \Rightarrow S_n = \frac{1}{2}n^2Q + \left(P - \frac{Q}{2}\right)n$$

Clearly, S_n is of the form $An^2 + Bn$. Hence, the sequence is an A.P. with common difference $2A = Q$.

33. Let S_n and S'_n be the sums of n terms of two arithmetic progressions. Then,

$$\frac{S_n}{S_n'} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{\frac{n}{2}\{2a_1 + (n-1)d_1\}}{\frac{n}{2}\{2a_2 + (n-1)d_2\}} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{5n+4}{9n+6}$$

Replacing $\frac{n-1}{2}$ by 17 i.e. n by 35, we get $\frac{a_1 + 17d_1}{a_2 + 17d_2} = \frac{179}{321}$

ALITER If the ratio of the sums of n terms is given, then to find the ratio of their n^{th} terms, we replace n by $(2n-1)$. So, to find the ratio of 18th terms, we replace n by $2 \times 18 - 1 = 35$ in the ratio $5n + 4 : 9n + 6$

Hence, required ratio is $(5 \times 35 + 4) : (9 \times 35 + 6)$ i.e. $179 : 321$.

19.6 PROPERTIES OF ARITHMETIC PROGRESSIONS

In this section, we shall discuss some properties of arithmetical progressions which will be frequently used in this chapter and in the subsequent chapters.

PROPERTY 1 *If a constant is added to or subtracted from each term of an A.P., then the resulting sequence is also an A.P. with the same common difference.*

PROOF Let $a_1, a_2, a_3, \ldots$ be an A.P. with common difference d, and let k be a fixed constant which is added to each term of this A.P. Then, the resulting sequence is $a_1 + k, a_2 + k, a_3 + k_1 \ldots$

Let $b_n = a_n + k, n = 1, 2, \ldots$ Then, the new sequence is $b_1, b_2, b_3, \ldots$

Now, $b_{n+1} - b_n = (a_{n+1} + k) - (a_n + k) = a_{n+1} - a_n = d$ for all $n \in N$

Thus, the new sequence is also an A.P. with common difference d.

PROPERTY 2 *If each term of a given A.P. is multiplied or divided by a non-zero constant k, then the resulting sequence is also an A.P. with common difference kd or d/k, where d is the common difference of the given A.P.*

PROOF Let $a_1, a_2, a_3, \ldots$ be an A.P. with common difference d and let k be a non-zero constant. Let $b_1, b_2, b_3, \ldots$ be sequence obtained by multiplying each term of the given A.P. by k. Then,

$$b_1 = a_1 k, b_2 = a_2 k, \ldots, b_n = a_n k, \ldots$$

Now, $b_{n+1} - b_n = a_{n+1} k - a_n k = (a_{n+1} - a_n) k = dk$ for all $n \in N$ $[\because a_{n+1} - a_n = d$ for all $n \in N]$

This shows that the new sequence is an A.P. with common difference dk.

Similarly, it can be proved that on dividing each term of a given A.P. by a non-zero constant, we obtain a sequence which is also an A.P.

PROPERTY 3 *In a finite A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.*

i.e. $a_k + a_{n-(k-1)} = a_1 + a_n$ for all $k = 1, 2, 3, \ldots, n-1$.

PROOF Let $a_1, a_2, a_3, \ldots, a_n$ be an A.P. with common difference d. We have to show that

$$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = a_4 + a_{n-3} = \ldots$$

i.e. $a_1 + a_n = a_k + a_{n-(k-1)}$ for all $k = 1, 2, 3, \ldots, n-1$

For any $k = 1, 2, \ldots, n-1$

$$\begin{aligned} a_k + a_{n-(k-1)} &= a_k + a_{n+1-k} \\ &= [a_1 + (k-1) d] + [a_1 + (n+1-k-1) d] \\ &= 2a_1 + (k-1+n+1-k-1) d \\ &= 2a_1 + (n-1) d = a_1 + [a_1 + (n-1) d] = a_1 + a_n. \end{aligned}$$

PROPERTY 4 *Three numbers a, b, c are in A.P. iff $2b = a + c$.*

PROOF First, let a, b, c be in A.P. Then,

$b - a$ = Common difference and, $c - b$ = Common difference

$\Rightarrow \quad b - a = c - b$

$\Rightarrow \quad 2b = a + c$

Conversely, let a, b, c be three numbers such that $2b = a + c$. Then, we have to show that a, b, c are in A.P.

We have, $2b = a + c \Rightarrow b - a = c - b \Rightarrow a, b, c$ are in A.P.

ILLUSTRATION *If $\frac{2}{3}, k, \frac{5}{8}$ are in A.P., find the value of k.*

SOLUTION It is given that,

$$\frac{2}{3}, k, \frac{5}{8} \text{ are in A.P.} \Rightarrow 2k = \frac{2}{3} + \frac{5}{8} \Rightarrow 2k = \frac{31}{24} \Rightarrow k = \frac{31}{48}.$$

PROPERTY 5 *A sequence is an A.P. iff its nth term is a linear expression in n i.e. $a_n = An + B$, where A, B are constants. In such a case the coefficient of n in a_n is the common difference of the A.P.*

PROOF See example 3 on page 19.3.

PROPERTY 6 *A sequence is an A.P. iff the sum of its first n terms is of the form $A n^2 + Bn$, where A, B are constants independent of n. In such a case the common difference is 2A i.e. 2 times the coefficient of n^2.*

PROOF See example 6 on page 19.17.

PROPERTY 7 *If the terms of an A.P. are chosen at regular intervals, then they form an A.P.*

PROPERTY 8 *If a_n, a_{n+1} and a_{n+2} are three consecutive terms of an A.P., then $2a_{n+1} = a_n + a_{n+2}$.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **TO PROVE THAT THREE NUMBERS ARE IN A.P. WHEN THREE GIVEN NUMBERS ARE IN A.P.**

EXAMPLE 1 *If a, b, c are in A.P., prove that the following are also in A.P.*

(i) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ (ii) $b+c, c+a, a+b$

(iii) $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ [NCERT] (iv) $a^2(b+c), b^2(c+a), c^2(a+b)$

(v) $\left\{(b+c)^2-a^2\right\}, \left\{(c+a)^2-b^2\right\}, \left\{(a+b)^2-c^2\right\}$ (vi) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$

SOLUTION (i) a, b, c are in A.P.

$\Rightarrow \quad \frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P. [On dividing each term by abc and using Property 2]

$\Rightarrow \quad \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

Thus, a, b, c are in A.P. $\Rightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

(ii) It is given that

a, b, c are in A.P.

$\Rightarrow a-(a+b+c), b-(a+b+c), c-(a+b+c)$ are in A.P. [Subtracting $a+b+c$ from each term]

$\Rightarrow -(b+c), -(c+a), -(a+b)$ are in A.P. [Multiplying each term by -1]

$\Rightarrow b+c, c+a, a+b$ are in A.P.

(iii) a, b, c are in A.P.

$\Rightarrow \quad \frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P. [On dividing each term by abc and using Property 2]

$\Rightarrow \quad \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

$\Rightarrow \quad \frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ca}, \frac{ab+bc+ca}{ab}$ are in A.P.

[On multiplying each term by $ab+bc+ca$ and using Property 2]

$\Rightarrow \quad \frac{ab+bc+ca}{bc}-1, \frac{ab+bc+ca}{ca}-1, \frac{ab+bc+ca}{ab}-1$ are in A.P.

[On adding -1 to each term and using Property 1]

$\Rightarrow \quad \frac{ab+ac}{bc}, \frac{ab+bc}{ca}, \frac{bc+ca}{ab}$ are in A.P.

$\Rightarrow \quad a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

(iv) $a^2(b+c), b^2(c+a), c^2(a+b)$ will be in A.P.

if $b^2(c+a) - a^2(b+c) = c^2(a+b) - b^2(c+a)$

i.e. if $c(b^2 - a^2) + ab(b-a) = a(c^2 - b^2) + bc(c-b)$

i.e. if $(b-a)(ab+bc+ca) = (c-b)(ab+bc+ca)$

i.e. if $b - a = c - b$

i.e. if $2b = a + c$

i.e. if a, b, c are in A.P.

Thus, a, b, c are in A.P. $\Rightarrow a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

ALITER It is given that

a, b, c are in A.P.

$\Rightarrow \frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P.

$\Rightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

$\Rightarrow \frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ca}, \frac{ab+bc+ca}{ab}$ are in A.P.

$\Rightarrow 1+\frac{ab+ca}{bc}, 1+\frac{ab+bc}{ca}, 1+\frac{bc+ca}{ab}$ are in A.P.

$\Rightarrow \frac{a(b+c)}{bc}, \frac{b(a+c)}{ca}, \frac{c(a+b)}{ab}$ are in A.P. [Subtracting 1 from each term]

$\Rightarrow \frac{a^2(b+c)}{abc}, \frac{b^2(a+c)}{abc}, \frac{c^2(a+b)}{abc}$ are in A.P.

$\Rightarrow a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P. [Multiplying each term by abc]

(v) It is given that

a, b, c are in A.P.

$\Rightarrow -2a, -2b, -2c$ are in A.P. [Multiplying each term by -2]

$\Rightarrow a+b+c-2a, a+b+c-2b, a+b+c-2c$ are in A.P. [Adding $a+b+c$ to each term]

$\Rightarrow b+c-a, c+a-b, a+b-c$ are in A.P.

$\Rightarrow (a+b+c)(b+c-a), (a+b+c)(c+a-b), (a+b+c)(a+b-c)$ are in A.P.

[Multiplying each terming $a+b+c$]

$\Rightarrow (b+c)^2 - a^2, (c+a^2) - b^2, (a+b)^2 - c^2$ are in A.P.

(vi) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ will be in A.P.

if $\frac{1}{\sqrt{c}+\sqrt{a}} - \frac{1}{\sqrt{b}+\sqrt{c}} = \frac{1}{\sqrt{a}+\sqrt{b}} - \frac{1}{\sqrt{c}+\sqrt{a}}$

i.e. if $\frac{\sqrt{b}-\sqrt{a}}{(\sqrt{c}+\sqrt{a})(\sqrt{b}+\sqrt{c})} = \frac{(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

i.e. if $\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{c}} = \frac{\sqrt{c}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

i.e. if $b - a = c - b$

i.e. if $2b = a + c$

i.e. if a, b, c are in A.P.

Thus, a, b, c are in A.P. $\Rightarrow \dfrac{1}{\sqrt{b}+\sqrt{c}}, \dfrac{1}{\sqrt{c}+\sqrt{a}}, \dfrac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P.

ALITER It is given that

a, b, c are in A.P.

$$\Rightarrow \quad b-a=c-b$$

$$\Rightarrow \quad \left(\sqrt{b}+\sqrt{a}\right)\left(\sqrt{b}-\sqrt{a}\right)=\left(\sqrt{c}-\sqrt{b}\right)\left(\sqrt{c}+\sqrt{b}\right)$$

$$\Rightarrow \quad \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{c}}=\frac{\sqrt{c}-\sqrt{b}}{\sqrt{b}+\sqrt{a}}$$

$$\Rightarrow \quad \frac{\left(\sqrt{b}+\sqrt{c}\right)-\left(\sqrt{a}+\sqrt{c}\right)}{\sqrt{b}+\sqrt{c}}=\frac{\left(\sqrt{c}+\sqrt{a}\right)-\left(\sqrt{b}+\sqrt{a}\right)}{\sqrt{b}+\sqrt{a}}$$

$$\Rightarrow \quad \frac{\left(\sqrt{b}+\sqrt{c}\right)-\left(\sqrt{c}+\sqrt{a}\right)}{\left(\sqrt{b}+\sqrt{c}\right)\left(\sqrt{c}+\sqrt{a}\right)}=\frac{\left(\sqrt{c}+\sqrt{a}\right)-\left(\sqrt{a}+\sqrt{b}\right)}{\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{c}+\sqrt{a}\right)}$$

$$\Rightarrow \quad \frac{1}{\sqrt{c}+\sqrt{a}}-\frac{1}{\sqrt{b}+\sqrt{c}}=\frac{1}{\sqrt{a}+\sqrt{b}}-\frac{1}{\sqrt{c}+\sqrt{a}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}} \text{ are in A.P.}$$

EXAMPLE 2 *If a^2, b^2, c^2 are in A.P, then prove that the following are also in A.P.*

(i) $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ (ii) $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$

SOLUTION (i) $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ will be in A.P.

if $$\frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a}$$

i.e. if $$\frac{b-a}{(c+a)(b+c)}=\frac{c-b}{(a+b)(c+a)}$$

i.e. if $$\frac{b-a}{b+c}=\frac{c-b}{a+b}$$

i.e. if $$b^2-a^2=c^2-b^2$$

i.e. if $$2b^2=a^2+c^2$$

i.e. if a^2, b^2, c^2 are in A.P.

Thus, a^2, b^2, c^2 are in A.P. $\Rightarrow \dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P.

ALITER (i) It is given that

a^2, b^2, c^2 are in A.P.

$$\Rightarrow \quad b^2-a^2=c^2-b^2$$

$$\Rightarrow \quad (b-a)(b+a)=(c-b)(c+b)$$

$\Rightarrow \quad \dfrac{b-a}{b+c}=\dfrac{c-b}{a+b}$

$\Rightarrow \quad \dfrac{(b+c)-(a+c)}{b+c}=\dfrac{(c+a)-(b+a)}{a+b}$

$\Rightarrow \quad \dfrac{(b+c)-(a+c)}{(a+c)(b+c)}=\dfrac{(c+a)-(b+a)}{(a+b)(a+c)}$ [Multiplying both side by $\dfrac{1}{a+b}$]

$\Rightarrow \quad \dfrac{1}{a+c}-\dfrac{1}{b+c}=\dfrac{1}{a+b}-\dfrac{1}{a+c}$

$\Rightarrow \quad \dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$, are in A.P.

(ii) $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ will be in A.P.

if $\dfrac{a}{b+c}+1, \dfrac{b}{c+a}+1, \dfrac{c}{a+b}+1$ are in A.P. [On adding 1 to each term]

i.e. if $\dfrac{a+b+c}{b+c}, \dfrac{a+b+c}{c+a}, \dfrac{a+b+c}{c+a}$ are in A.P.

i.e. if $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P. [On dividing each term by $a+b+c$]

i.e. if $\dfrac{1}{c+a}-\dfrac{1}{b+c}=\dfrac{1}{a+b}-\dfrac{1}{c+a}$

i.e. if $\dfrac{b-a}{(c+a)(b+c)}=\dfrac{c-b}{(a+b)(c+a)}$

i.e. if $\dfrac{b-a}{b+c}=\dfrac{c-b}{a+b}$

i.e. if $b^2-a^2=c^2-b^2$

i.e. if $2b^2=a^2+c^2$

i.e. if a^2, b^2, c^2 are in A.P.

Thus, a^2, b^2, c^2 are in A.P. $\Rightarrow \dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in A.P.

ALITER It is given that

a^2, b^2, c^2 are in A.P.

$\Rightarrow \quad b^2-a^2=c^2-b^2.$

$\Rightarrow \quad (b+a)(b-a)=(c+b)(c-b)$

$\Rightarrow \quad \dfrac{b-a}{b+c}=\dfrac{c-b}{a+b}$

$\Rightarrow \quad \dfrac{(b+c)-(a+c)}{b+c}=\dfrac{(c+a)-(b+a)}{a+b}$

$\Rightarrow \quad \dfrac{(b+c)-(a+c)}{(a+c)(b+c)}=\dfrac{(c+a)-(b+a)}{(a+b)(a+c)}$

$\Rightarrow \quad \dfrac{1}{a+c}-\dfrac{1}{b+c}=\dfrac{1}{a+b}-\dfrac{1}{a+c}$

$\Rightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$\Rightarrow \frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b}$ are in A.P.

$\Rightarrow 1+\frac{a}{b+c}, 1+\frac{b}{c+a}, 1+\frac{c}{a+b}$ are in A.P.

$\Rightarrow \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

EXAMPLE 3 *If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.*

SOLUTION $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

$\Rightarrow \left\{\frac{b+c-a}{a}+2\right\}, \left\{\frac{c+a-b}{b}+2\right\}, \left\{\frac{a+b-c}{c}+2\right\}$ are in A.P. [Adding 2 to each term]

$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in A.P.

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P $\left[\text{Dividing each term by } a+b+c\right]$

EXAMPLE 4 *If $a^2+2bc, b^2+2ac, c^2+2ab$ are in A.P., show that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.*

SOLUTION $a^2+2bc, b^2+2ac, c^2+2ab$ are in A.P.

$\Rightarrow (a^2+2bc)-(ab+bc+ca), (b^2+2ac)-(ab+bc+ca), (c^2+2ab)-(ab+bc+ca)$ are in A.P.

[On subtracting $(ab+bc+ca)$ from each term]

$\Rightarrow a^2+bc-ab-ca, b^2+ca-ab-bc, c^2+ab-bc-ca$ are in A.P.

$\Rightarrow (a-b)(a-c), (b-c)(b-a), (c-a)(c-b)$ are in A.P.

$\Rightarrow \frac{-1}{b-c}, \frac{-1}{c-a}, \frac{-1}{a-b}$ are in A.P. $\left[\text{On dividing each term by } (a-b)(b-c)(c-a)\right]$

$\Rightarrow \frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P. [On multiplying each term by -1]

EXAMPLE 5 *If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P., prove that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.*

SOLUTION $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P.

$\Rightarrow (c-a)^2-(b-c)^2 = (a-b)^2-(c-a)^2$

$\Rightarrow (b-a)(2c-a-b) = (c-b)(2a-b-c)$...(i)

$\Rightarrow (b-a)\{(c-a)+(c-b)\} = (c-b)\{(a-b)+(a-c)\}$

$\Rightarrow (b-a)(c-a)+(b-a)(c-b) = (c-b)(a-b)+(a-c)(c-b)$

$\Rightarrow -(a-b)(c-a)+(a-b)(b-c) = -(a-b)(b-c)+(b-c)(c-a)$

$\Rightarrow -\frac{1}{b-c}+\frac{1}{c-a} = -\frac{1}{c-a}+\frac{1}{a-b}$ [Dividing throughout by $(a-b)(b-c)(c-a)$]

$\Rightarrow \frac{2}{c-a} = \frac{1}{a-b}+\frac{1}{b-c}$

$\Rightarrow \quad \dfrac{1}{b-c}, \dfrac{1}{c-a}, \dfrac{1}{a-b}$ are in AP.

Thus, if $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P., then $\dfrac{1}{b-c}, \dfrac{1}{c-a}, \dfrac{1}{a-b}$ are in A.P.

EXAMPLE 6 *If a, b, c are in A.P., then prove that:*

(i) $(a-c)^2 = 4(b^2-ac)$ (ii) $a^3+4b^3+c^3 = 3b(a^2+c^2)$

SOLUTION (i) It is given that a, b, c are in A.P.

$\therefore \quad 2b = a+c \Rightarrow b = \dfrac{a+c}{2}$

Putting $b = \dfrac{a+c}{2}$ on RHS, we obtain

$$\text{RHS} = 4(b^2-ac) = 4\left\{\left(\frac{a+c}{2}\right)^2 - ac\right\} = 4\left\{\frac{(a+c)^2-4ac}{4}\right\} = (a+c)^2 - 4ac = (a-c)^2 = \text{LHS}$$

(ii) It is given that a, b, c are in A.P.

$\therefore \quad 2b = a+c \Rightarrow b = \dfrac{a+c}{2}$

$$\text{LHS} = a^3+4b^3+c^3$$
$$= a^3 + 4\left(\frac{a+c}{2}\right)^3 + c^3 = (a^3+c^3) + \frac{1}{2}(a+c)^3$$
$$= \frac{1}{2}\left\{2(a^3+c^3)+(a+c)^3\right\} = \frac{1}{2}\left\{2(a+c)(a^2-ac+c^2)+(a+c)^3\right\}$$
$$= \frac{1}{2}(a+c)\left\{2(a^2-ac+c^2)+(a+c)^2\right\}$$
$$= \frac{1}{2}(a+c)\,3(a^2+c^2) = 3\left(\frac{a+c}{2}\right)(a^2+c^2) = 3b(a^2+c^2) = \text{RHS}$$

<u>ALITER</u> $\text{LHS} = a^3+4b^3+c^3$

$$= (a^3+c^3)+4b^3$$
$$= (a+c)^3 - 3ac(a+c) + 4b^3$$
$$= (2b)^3 - 3ac(2b) + 4b^3$$
$$= 12b^3 - 6abc$$
$$= 3b(4b^2-2ac) = 3b\left\{(2b)^2 - 2ac\right\} = 3b\left\{(a+c)^2-2ac\right\} = 3b(a^2+c^2) = \text{RHS}$$

EXAMPLE 7 *If $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P., show that either a, b, c are in A.P. or $ab+bc+ca=0$.*

SOLUTION $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

$\Rightarrow \quad b^2(c+a) - a^2(b+c) = c^2(a+b) - b^2(c+a)$

$\Rightarrow \quad (b^2a-a^2b)+(b^2c-a^2c) = (c^2b-b^2c)+(c^2a-b^2a)$

$\Rightarrow \quad (b-a)(ab+bc+ca) = (c-b)(ab+bc+ca)$

$\Rightarrow \quad (ab+bc+ca)(2b-a-c) = 0$

$\Rightarrow \quad ab+bc+ca = 0$ or $2b-a-c = 0$

$\Rightarrow \quad ab+bc+ca = 0$ or a, b, c are in A.P.

EXERCISE 19.5

LEVEL-1

1. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., prove that:
   (i) $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P. (ii) $a(b+c), b(c+a), c(a+b)$ are in A.P.
2. If a^2, b^2, c^2 are in A.P., prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.
3. If a, b, c are in A.P., then show that:
   (i) $a^2(b+c), b^2(c+a), c^2(a+b)$ are also in A.P.
   (ii) $b+c-a, c+a-b, a+b-c$ are in A.P.
   (iii) $bc-a^2, ca-b^2, ab-c^2$ are in A.P.
4. If $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P., prove that:
   (i) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. (ii) bc, ca, ab are in A.P.
5. If a, b, c are in A.P., prove that:
   (i) $(a-c)^2 = 4(a-b)(b-c)$ (ii) $a^2 + c^2 + 4ac = 2(ab + bc + ca)$
   (iii) $a^3 + c^3 + 6abc = 8b^3$
6. If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.
7. Show that x^2+xy+y^2, z^2+zx+x^2 and y^2+yz+z^2 are consecutive terms of an A.P., if x, y and z are in A.P. [NCERT EXEMPLAR]

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5. (i) Put $b = \frac{a+c}{2}$ on RHS (ii) Put $b = \frac{a+c}{2}$ on RHS (iii) Put $b = \frac{a+c}{2}$ on RHS and LHS

6. $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

$\Rightarrow$ $a\left(\frac{1}{b}+\frac{1}{c}\right)+1, b\left(\frac{1}{c}+\frac{1}{a}\right)+1, c\left(\frac{1}{a}+\frac{1}{b}\right)+1$ are A.P. [$\because$ Adding 1 throughout]

$\Rightarrow$ $a\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{a}{a}, b\left(\frac{1}{c}+\frac{1}{a}\right)+\frac{b}{b}, c\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{c}{c}$ are in A.P.

$\Rightarrow$ $a\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right), b\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right), c\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ are in A.P.

$\Rightarrow$ a, b, c are in A.P. $\left[\text{Dividing each term by } \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right]$

19.7 INSERTION OF ARITHMETIC MEANS

If between two given quantities a and b we have to insert n quantities $A_1, A_2, \dots, A_n$ such that $a, A_1, A_2, \dots A_n, b$ form an A.P., then we say that $A_1, A_2, \dots, A_n$ are arithmetic means between a and b.

ILLUSTRATION *Since 15, 11, 7, 3, –1, –5 are in A.P., it follows that 11, 7, 3, –1 are four arithmetic means between 15 and –5.*

If a, A, b are in A.P., we say that A is the arithmetic mean of a and b.

19.7.1 INSERTION OF ARITHMETIC MEANS

Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between two quantities a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an A.P. Let d be the common difference of this A.P. Clearly, it contains $(n+2)$ terms.

$$\therefore \quad b = (n+2)^{\text{th}} \text{ term} \Rightarrow b = a + (n+1)\,d \Rightarrow d = \frac{b-a}{n+1}$$

Now,
$$A_1 = a + d = a + \frac{b-a}{n+1}$$
$$A_2 = a + 2d = a + \frac{2(b-a)}{n+1}$$
$$A_3 = a + 3d = a + \frac{3(b-a)}{n+1}$$
$$\vdots \qquad \vdots \quad \vdots$$
$$A_n = a + nd = a + \frac{n(b-a)}{n+1}$$

These are the required arithmetic means between a and b.

19.7.2 INSERTION OF A SINGLE ARITHMETIC MEAN BETWEEN TWO NUMBERS

Let a and b be two numbers and A be the single arithmetic mean between them. Then,

a, A, b are in A.P.

$$\Rightarrow \quad A - a = b - A$$
$$\Rightarrow \quad 2A = a + b \Rightarrow A = \frac{a+b}{2}$$

Thus, the arithmetic mean of a and b is $\frac{a+b}{2}$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Insert three arithmetic means between 3 and 19.*

SOLUTION Let A_1, A_2, A_3 be 3 A.M.'s between 3 and 19. Then $3, A_1, A_2, A_3, 19$ are in A.P. whose common difference d is given by $d = \frac{19-3}{3+1} = 4$.

$$\therefore \quad A_1 = 3 + d = 3 + 4 = 7,\ A_2 = 3 + 2d = 3 + 2 \times 4 = 11,\ A_3 = 3 + 3d = 3 + 3 \times 4 = 15.$$

Hence, the required A.M.'s are 7, 11, 15.

EXAMPLE 2 *For what value of n, $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the arithmetic mean of a and b?* [NCERT]

SOLUTION The A.M. of a and b is $\frac{a+b}{2}$. Therefore, $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ will be the A.M. of a and b, if

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \frac{a+b}{2}$$
$$\Rightarrow \quad 2(a^{n+1} + b^{n+1}) = (a^n + b^n)(a+b)$$
$$\Rightarrow \quad 2a^{n+1} + 2b^{n+1} = a^{n+1} + a^n b + b^n a + b^{n+1}$$
$$\Rightarrow \quad a^{n+1} + b^{n+1} = a^n b + b^n a$$
$$\Rightarrow \quad a^n (a-b) = b^n (a-b)$$
$$\Rightarrow \quad a^n = b^n \Rightarrow \frac{a^n}{b^n} = 1 \Rightarrow \left(\frac{a}{b}\right)^n = 1 \Rightarrow \left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0 \Rightarrow n = 0.$$

EXAMPLE 3 *If n arithmetic means are inserted between 20 and 80 such that the ratio of first mean to the last mean is 1 : 3, then find the value of n.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between 20 and 80 and let d be the common difference of the A.P. 20, $A_1, A_2, \ldots A_n$, 80. Then,

$$d = \frac{80-20}{n+1} = \frac{60}{n+1} \qquad \left[\text{Using: } d = \frac{b-a}{n+1}\right]$$

Now, $A_1 = 20 + d \Rightarrow A_1 = 20 + \dfrac{60}{n+1} = 20\left(\dfrac{n+4}{n+1}\right)$

And, $A_n = 20 + nd \Rightarrow A_n = 20 + \dfrac{60n}{n+1} = 20\left(\dfrac{4n+1}{n+1}\right)$

It is given that

$$\frac{A_1}{A_n} = \frac{1}{3} \Rightarrow \frac{\frac{20(n+4)}{n+1}}{\frac{20(4n+1)}{n+1}} = \frac{1}{3} \Rightarrow \frac{n+4}{4n+1} = \frac{1}{3} \Rightarrow 4n+1 = 3n+12 \Rightarrow n = 11$$

EXAMPLE 4 *Between 1 and 31 are inserted m arithmetic means so that the ratio of the 7th and $(m-1)$th means is 5 : 9. Find the value of m.* [NCERT]

SOLUTION Let $A_1, A_2, \ldots, A_m$ be m arithmetic means between 1 and 31. Then 1, $A_1, A_2, \ldots, A_m$, 31 is an A.P. with common difference d given by

$$d = \frac{31-1}{m+1} = \frac{30}{m+1} \qquad \left[\text{Using: } d = \frac{b-a}{n+1}\right]$$

Now, $A_7 = 1 + 7d = 1 + \dfrac{7 \times 30}{m+1} = \dfrac{m+211}{m+1}$

and, $A_{m-1} = 1 + (m-1)\,d = 1 + \dfrac{30(m-1)}{m+1} = \dfrac{31m-29}{m+1}$

It is given that

$$\frac{A_7}{A_{m-1}} = \frac{5}{9} \Rightarrow \frac{m+211}{31m-29} = \frac{5}{9} \Rightarrow 9m + 1899 = 155m - 145 \Rightarrow 146m = 2044 \Rightarrow m = 14$$

LEVEL-2

EXAMPLE 5 *Prove that the sum of n arithmetic means between two numbers is n times the single A.M. between them.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an A.P. with common difference d given by $d = \dfrac{b-a}{n+1}$.

Now, $A_1 + A_2 + \ldots + A_n = \dfrac{n}{2}\left(A_1 + A_n\right)$ $\qquad \left[\because S_n = \dfrac{n}{2}(a+l)\right]$

$= \dfrac{n}{2}(a+b)$ $\qquad [\because a, A_1, A_2, \ldots, A_n, b$ is an A.P. $\therefore a + b = A_1 + A_n]$

$= n\left(\dfrac{a+b}{2}\right) = n \times$ (A.M. between a and b)

EXAMPLE 6 *The sum of two numbers is $\frac{13}{6}$. An even number of arithmetic means are being inserted between them and their sum exceeds their number by 1. Find the number of means inserted.*

SOLUTION Let a and b be two numbers such that $a+b=\frac{13}{6}$ ...(i)

Let $A_1, A_2, \ldots, A_{2n}$ be $2n$ arithmetic means between a and b. Then,

$$A_1 + A_2 + \ldots + A_{2n} = 2n\left(\frac{a+b}{2}\right) \quad \text{[Using result of Example 5]}$$

$$\Rightarrow \quad A_1 + A_2 + \ldots + A_{2n} = n(a+b) = \frac{13}{6}n \quad \text{[Using (i)]}$$

$$\Rightarrow \quad 2n+1 = \frac{13}{6}n \quad [\because A_1 + A_2 + \ldots + A_{2n} = 2n+1 \text{ (given)}]$$

$$\Rightarrow \quad 12n+6 = 13$$

$$\Rightarrow \quad n = 6$$

EXAMPLE 7 *If the A.M. between pth and qth terms of an A.P. be equal to the A.M. between rth and sth terms of the A.P., then show that $p+q=r+s$.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then

$$a_p = p\text{th term} = a + (p-1)d; \; a_q = q\text{th term} = a+(q-1)d$$

$$a_r = r\text{th term} = a+(r-1)d \text{ and, } a_s = s\text{th term} = a+(s-1)d$$

It is given that

A.M. between a_p and a_q = A.M. between a_r and a_s

$$\Rightarrow \quad \frac{1}{2}(a_p + a_q) = \frac{1}{2}(a_r + a_s)$$

$$\Rightarrow \quad a_p + a_q = a_r + a_s$$

$$\Rightarrow \quad \{a+(p-1)d\} + \{a+(q-1)d\} = \{a+(r-1)d\} + \{a+(s-1)d\}$$

$$\Rightarrow \quad (p+q-2)d = (r+s-2)d$$

$$\Rightarrow \quad p+q = r+s$$

EXAMPLE 8 *Suppose x and y are two real numbers such that the rth mean between x and $2y$ is equal to the rth mean between $2x$ and y when n arithmetic means are inserted between them in both the cases. Show that $\frac{n+1}{r} - \frac{y}{x} = 1$.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between x and $2y$. Then, $x, A_1, A_2, \ldots, A_n, 2y$ are in AP with common difference d_1 given by $d_1 = \frac{2y-x}{n+1}$.

$$\therefore \quad r^{\text{th}} \text{ mean} = A_r = x + r\,d_1 = x + r\left(\frac{2y-x}{n+1}\right)$$

Let $A_1', A_2', \ldots, A_n'$ be n arithmetic means between $2x$ and y. Then, $2x, A_1', A_2', \ldots, A_n', y$ are in A.P. with common difference d_2 given by $d_2 = \frac{y-2x}{n+1}$.

$$\therefore \quad r^{\text{th}} \text{ mean} = A_r' = 2x + r\,d_2 = 2x + r\left(\frac{y-2x}{n+1}\right)$$

It is given that :

$$A_r = A_r'$$

$$\Rightarrow \quad x + r\left(\frac{2y-x}{n+1}\right) = 2x + r\left(\frac{y-2x}{n+1}\right)$$

$$\Rightarrow \quad (n+1)\,x + r\,(2y - x) = (n+1)\,2x + r\,(y - 2x)$$
$$\Rightarrow \quad (n+1)\,x - ry = rx$$
$$\Rightarrow \quad \frac{n+1}{r} - \frac{y}{x} = 1$$

EXERCISE 19.6

LEVEL-1

1. Find the A.M. between:
   (i) 7 and 13 (ii) 12 and -8 (iii) $(x - y)$ and $(x + y)$.
2. Insert 4 A.M.s between 4 and 19.
3. Insert 7 A.M.s between 2 and 17.
4. Insert six A.M.s between 15 and -13.
5. There are n A.M.s between 3 and 17. The ratio of the last mean to the first mean is 3 : 1. Find the value of n.
6. Insert A.M.s between 7 and 71 in such a way that the 5th A.M. is 27. Find the number of A.M.s.
7. If n A.M.s are inserted between two numbers, prove that the sum of the means equidistant from the beginning and the end is constant.
8. If x, y, z are in A.P. and A_1 is the A.M. of x and y and A_2 is the A.M. of y and z, then prove that the A.M. of A_1 and A_2 is y.
9. Insert five numbers between 8 and 26 such that the resulting sequence is an A.P. [NCERT]

ANSWERS

1. (i) 10 (ii) 2 (iii) x 2. 7, 10, 13, 16
3. $\frac{31}{8}, \frac{23}{4}, \frac{61}{8}, \frac{19}{2}, \frac{91}{8}, \frac{53}{4}, \frac{121}{8}$
4. 11, 7, 3, -1, -5, -9 5. 6 6. 15 9. 11, 14, 17, 20, 23

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9. Let a_1, a_2, a_3, a_4, a_5 be five natural numbers between 8 and 26 such that $8, a_1, a_2, a_3, a_4, a_5, 26$ is an A.P. Let d be the common difference. Then,

$$d = \frac{26-8}{5+1} = 3 \qquad \left[\because\ d = \frac{b-a}{n+1}\right]$$

$\therefore$ $a_1 = 8 + 3 = 11$, $a_2 = a_1 + 3 = 14$, $a_3 = 17$, $a_4 = 20$ and $a_5 = 23$

Hence, five numbers are 11, 14, 17, 20 and 23.

19.8 APPLICATIONS OF A.P.

In this section, we shall discuss some problems based upon the applications of arithmetic progressions.

LEVEL-1

EXAMPLE 1 *The digits of a positive integer, having three digits, are in A.P. and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.*

SOLUTION Let the digits at ones, tens and hundreds place be $(a - d)$, a and $(a + d)$ respectively. Then the number is

$$(a+d) \times 100 + a \times 10 + (a-d) = 111a + 99d$$

The number obtained by reversing the digits is

$$(a-d)\times 100 + a\times 10 + (a+d) = 111a - 99d$$

It is given that $(a-d) + a + (a+d) = 15$

and, $111a - 99d = 111a + 99d - 594$

$\Rightarrow$ $3a = 15$ and $198d = 594 \Rightarrow a = 5$ and $d = 3$

So, the number is $111a + 99d = 111\times 5 + 99\times 3 = 852$.

EXAMPLE 2 *Two cars start together in the same direction from the same place. The first goes with uniform speed of 10 km/h. The second goes at a speed of 8 km/h in the first hour and increases the speed by 1/2 km each succeeding hour. After how many hours will the second car overtake the first car if both cars go non-stop ?*

SOLUTION Suppose the second car overtakes the first car after t hours. Then the two cars travel the same distance in t hours.

Distance travelled by the first car in t hours = $10\,t$ km.

Distance travelled by the second car in t hours

= Sum of t terms of an A.P. with first term 8 and common difference 1/2.

$$= \frac{t}{2}\left\{2\times 8 + (t-1)\times\frac{1}{2}\right\} = \frac{t\,(t+31)}{4}$$

When the second car overtakes the first car. The distance travelled by both cars is same.

$$\therefore \quad 10t = \frac{t\,(t+31)}{4} \Rightarrow t\,(t-9) = 0 \Rightarrow t = 9 \qquad [\because t \neq 0]$$

Thus, the second car will overtake the first car in 9 hours.

EXAMPLE 3 *A man repays a loan of ₹ 3250 by paying ₹ 20 in the first month and then increases the payment by ₹ 15 every month. How long will it take him to clear the loan ?*

SOLUTION Suppose the loan is cleared in n months. Clearly, the amounts form an A.P. with first term 20 and the common difference 15.

$\therefore$ Sum of the amounts = 3250

$$\Rightarrow \quad \frac{n}{2}\left\{2\times 20 + (n-1)\times 15\right\} = 3250$$

$$\Rightarrow \quad 3n^2 + 5n - 1300 = 0 \Rightarrow (n-20)(3n+65) = 0 \Rightarrow n = 20 \qquad \left[\because 3n + 65 \neq 0\right]$$

Thus, the loan is cleared in 20 months.

LEVEL-2

EXAMPLE 4 *150 workers were engaged to finish a piece of work in a certain number of days. Four workers dropped the second day, four more workers dropped the third day and so on. It takes 8 more days to finish the work now. Find the number of days in which the work was completed.*

SOLUTION Suppose the work is completed in n days when the workers started dropping. Since 4 workers are dropped on every day except the first day. Therefore, the total number of workers who worked all the n days is the sum of n terms of an A.P. with first term 150 and common difference – 4.

$$\text{i.e.} \quad \frac{n}{2}\left\{2\times 150 + (n-1)\times -4\right\} = n\,(152 - 2n)$$

Had the workers not dropped then the work would have finished in $(n-8)$ days with 150 workers working on each day. Therefore, the total number of workers who would have worked all the n days is $150\,(n-8)$.

$$\therefore \quad n\,(152-2n) = 150\,(n-8) \Rightarrow n^2 - n - 600 = 0 \Rightarrow (n-25)(n+24) = 0 \Rightarrow n = 25.$$

Thus, the work is completed in 25 days.

EXAMPLE 5 *Along a road lie an odd number of stones placed at intervals of 10 metres. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried the job with one of the end stones by carrying them in succession. In carrying all the stones he covered a distance of 3 km. Find the number of stones.*

SOLUTION Let there be $(2n + 1)$ stones. Clearly, one stone lies in the middle and n stones on each side of it in a row. Let P be the mid-stone and let A and B be the end stones on the left and right of P respectively. Clearly, there are n intervals each of length 10 metres on both the sides of P. Now, suppose the man starts from A. He picks up the end stone on the left of mid-stone and goes to the mid-stone, drops it and goes to $(n - 1)$th stone on left, picks it up, goes to the mid-stone and drops it. This process is repeated till he collects all stones on the left of the mid-stone at the mid-stone. So, distance covered in collecting stones on the left of the mid-stones

$$= 10 \times n + 2\,[10 \times (n-1) + 10 \times (n-2) + \ldots + 10 \times 2 + 10 \times 1].$$

After collecting all stones on left of the mid-stone the man goes to the stone B on the right side of the mid-stone, picks it up, goes to the mid-stone and drops it. Then he goes to $(n-1)^{\text{th}}$ stone on the right and the process is repeated till he collects all stones at the mid-stone.

Distance covered in collecting the stones on the right side of the mid-stone

$$= 2\,[10 \times n + 10 \times (n-1) + 10 \times (n-2) + \ldots + 10 \times 2 + 10 \times 1]$$

$\therefore$ Total distance covered

$$= 10 \times n + 2\,[10 \times (n-1) + 10 \times (n-2) + \ldots + 10 \times 2 + 10 \times 1] + 2\,[10 \times n + 10 \times (n-1) + \ldots + 10 \times 2 + 10 \times 1]$$

$$= 4\,[10 \times n + 10 \times (n-1) + \ldots + 10 \times 2 + 10 \times 1] - 10 \times n$$

$$= 40\left[1 + 2 + 3 + \ldots + n\right] - 10n = 40\left\{\frac{n}{2}(1+n)\right\} - 10n = 20n^2 + 10n.$$

But, the total distance covered is 3 km = 3000 m.

$$\therefore \quad 20n^2 + 10n = 3000 \Rightarrow 2n^2 + n - 300 = 0 \Rightarrow (n-12)(2n+25) = 0 \Rightarrow n = 12$$

Hence, the number of stones $= 2n + 1 = 25$.

EXERCISE 19.7

LEVEL-1

1. A man saved ₹ 16500 in ten years. In each year after the first he saved ₹ 100 more than he did in the receding year. How much did he save in the first year ?
2. A man saves ₹ 32 during the first year, ₹ 36 in the second year and in this way he increases his savings by ₹ 4 every year. Find in what time his saving will be ₹ 200.
3. A man arranges to pay off a debt of ₹ 3600 by 40 annual instalments which form an arithmetic series. When 30 of the instalments are paid, he dies leaving one-third of the debt unpaid, find the value of the first instalment.
4. A manufacturer of radio sets produced 600 units in the third year and 700 units in the seventh year. Assuming that the product increases uniformly by a fixed number every year, find (i) the production in the first year (ii) the total product in 7 years and (iii) the product in the 10th year.
5. There are 25 trees at equal distances of 5 metres in a line with a well, the distance of the well from the nearest tree being 10 metres. A gardener waters all the trees separately starting from the well and he returns to the well after watering each tree to get water for the next. Find the total distance the gardener will cover in order to water all the trees.
6. A man is employed to count ₹ 10710. He counts at the rate of ₹ 180 per minute for half an hour. After this he counts at the rate of ₹ 3 less every minute than the preceding minute. Find the time taken by him to count the entire amount.

7. A piece of equipment cost a certain factory ₹ 600,000. If it depreciates in value, 15% the first, 13.5% the next year, 12% the third year, and so on. What will be its value at the end of 10 years, all percentages applying to the original cost ?
8. A farmer buys a used tractor for ₹ 12000. He pays ₹ 6000 cash and agrees to pay the balance in annual instalments of ₹ 500 plus 12% interest on the unpaid amount. How much the tractor cost him?
9. Shamshad Ali buys a scooter for ₹ 22000. He pays ₹ 4000 cash and agrees to pay the balance in annual instalments of ₹ 1000 plus 10% interest on the unpaid amount. How much the scooter will cost him.
10. The income of a person is ₹ 300,000 in the first year and he receives an increase of ₹ 10000 to his income per year for the next 19 years. Find the total amount, he received in 20 years. [NCERT]
11. A man starts repaying a loan as first instalment of ₹ 100. If he increases the instalments by ₹ 5 every month, what amount he will pay in the 30th instalment? [NCERT]
12. A carpenter was hired to build 192 window frames. The first day he made five frames and each day thereafter he made two more frames than he made the day before. How many days did it take him to finish the job? [NCERT EXEMPLAR]
13. We know that the sum of the interior angles of a triangle is 180°. Show that the sums of the interior angles of polygons with 3, 4, 5, 6,... sides form an arithmetic progression. Find the sum of the interior angles for a 21 sided polygon. [NCERT EXEMPLAR]
14. In a potato race 20 potatoes are placed in a line at intervals of 4 meters with the first potato 24 metres from the starting point. A contestant is required to bring the potatoes back to the starting place one at a time. How far would he run in bringing back all the potatoes? [NCERT EXEMPLAR]
15. A man accepts a position with an initial salary of ₹ 5200 per month. It is understood that he will receive an automatic increase of ₹320 in the very next month and each month thereafter.
    (i) Find his salary for the tenth month.
    (ii) What is his total earnings during the first year?
16. A man saved ₹66000 in 20 years. In each succeeding year after the first year he saved ₹ 200 more than what he saved in the previous year. How much did he save in the first year?
17. In a cricket team tournament 16 teams participated. A sum of ₹ 8000 is to be awarded among themselves as prize money. If the last place team is awarded ₹ 275 in prize money and the award increases by the same amount for successive finishing places, how much amount will the first place team receive?

ANSWERS

1. ₹ 1200 2. 5 yrs 3. ₹ 51 4. (i) 550 (ii) 4375 (iii) 775
5. 3500 m 6. 89 minutes 7. ₹ 105000 8. ₹ 16680
9. ₹ 39100 10. ₹7900,000 11. ₹ 245 12. 12 days
13. 3420° 14. 2480 m 15. (i) ₹ 8080 (ii) ₹ 83520
16. ₹1400 17. ₹ 725

HINTS TO NCERT & SELECTED PROBLEMS

10. Here, $a = 300{,}000$, $d = 10{,}000$ and $n = 20$. Let S be the total amount received in 20 years. Then,

$$S = ₹\ \frac{20}{2}\{2 \times 300{,}000 + (20-1) \times 10{,}000\} = ₹10\,(600{,}000 + 190{,}000) = ₹\ 7900{,}000$$

11. Here, $a = 100$, $d = 5$ and $n = 30$.

∴ Amount to be paid in 30th instalment $= a_{30} = a + 29d = 100 + 29 \times 5 = 245$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the common difference of an A.P. whose nth term is $xn + y$.
2. Write the common difference of an A.P. the sum of whose first n terms is $\frac{P}{2}n^2 + Qn$.
3. If the sum of n terms of an AP is $2n^2 + 3n$, then write its nth term.
4. If $\log 2$, $\log (2^x - 1)$ and $\log (2^x + 3)$ are in A.P., write the value of x.
5. If the sums of n terms of two arithmetic progressions are in the ratio $2n + 5 : 3n + 4$, then write the ratio of their mth terms.
6. Write the sum of first n odd natural numbers.
7. Write the sum of first n even natural numbers.
8. Write the value of n for which nth terms of the A.P.s 3, 10, 17, ... and 63, 65, 67,.... are equal.
9. If $\frac{3 + 5 + 7 + \ldots + \text{upto } n \text{ terms}}{5 + 8 + 11 + \ldots \text{ upto 10 terms}} = 7$, then find the value of n.
10. If mth term of an A.P. is n and nth term is m, then write its pth term.
11. If the sums of n terms of two A.P.'s are in the ratio $(3n + 2) : (2n + 3)$, find the ratio of their 12^{th} terms.

ANSWERS

1. x	2. P	3. $4n + 1$	4. $\log_2 5$
5. $(4m + 3) : (6m + 1)$		6. n^2	7. $n(n + 1)$
8. 13	9. 35	10. $m + n - p$	11. 71 : 49

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If 7th and 13th terms of an A.P. be 34 and 64 respectively, then its 18th term is
   (a) 87 (b) 88 (c) 89 (d) 90
2. If the sum of p terms of an A.P. is q and the sum of q terms is p, then the sum of $p + q$ terms will be
   (a) 0 (b) $p - q$ (c) $p + q$ (d) $-(p + q)$
3. If the sum of n terms of an A.P. be $3n^2 - n$ and its common difference is 6, then its first term is
   (a) 2 (b) 3 (c) 1 (d) 4
4. Sum of all two digit numbers which when divided by 4 yield unity as remainder is
   (a) 1200 (b) 1210 (c) 1250 (d) none of these.
5. In A.M.'s are introduced between 3 and 17 such that the ratio of the last mean to the first mean is 3 : 1, then the value of n is
   (a) 6 (b) 8 (c) 4 (d) none of these.
6. If S_n denotes the sum of first n terms of an A.P. $< a_n >$ such that $\frac{S_m}{S_n} = \frac{m^2}{n^2}$, then $\frac{a_m}{a_n} =$
   (a) $\frac{2m+1}{2n+1}$ (b) $\frac{2m-1}{2n-1}$ (c) $\frac{m-1}{n-1}$ (d) $\frac{m+1}{n+1}$.

7. The first and last terms of an A.P. are 1 and 11. If the sum of its terms is 36, then the number of terms will be
(a) 5 (b) 6 (c) 7 (d) 8

8. If the sum of n terms of an A.P., is $3n^2+5n$ then which of its terms is 164?
(a) 26th (b) 27th (c) 28th (d) none of these.

9. If the sum of n terms of an A.P. is $2n^2+5n$, then its nth term is
(a) $4n-3$ (b) $3n-4$ (c) $4n+3$ (d) $3n+4$

10. If $a_1, a_2, a_3, \ldots, a_n$ are in A.P. with common difference d, then the sum of the series $\sin d$ $[\text{cosec } a_1 \text{ cosec } a_2 + \text{cosec } a_1 \text{ cosec } a_3 + \ldots + \text{cosec } a_{n-1} \text{ cosec } a_n]$ is
(a) $\sec a_1 - \sec a_n$ (b) $\text{cosec } a_1 - \text{cosec } a_n$
(c) $\cot a_1 - \cot a_n$ (d) $\tan a_1 - \tan a_n$

11. In the arithmetic progression whose common difference is non-zero, the sum of first $3n$ terms is equal to the sum of next n terms. Then the ratio of the sum of the first $2n$ terms to the next $2n$ terms is
(a) 1/5 (b) 2/3 (c) 3/4 (d) none of these

12. If $a_1, a_2, a_3, \ldots, a_n$ are in A.P. with common difference d, then the sum of the series
$\sin d\, [\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \ldots + \sec a_{n-1} \sec a_n]$, is
(a) $\sec a_1 - \sec a_n$ (b) $\text{cosec } a_1 - \text{cosec } a_n$
(c) $\cot a_1 - \cot a_n$ (d) $\tan a_n - \tan a_1$

13. If four numbers in A.P. are such that their sum is 50 and the greatest number is 4 times the least, then the numbers are
(a) 5, 10, 15, 20 (b) 4, 10, 16, 22 (c) 3, 7, 11, 15 (d) none of these

14. If n arithmetic means are inserted between 1 and 31 such that the ratio of the first mean and nth mean is 3 : 29, then the value of n is
(a) 10 (b) 12 (c) 13 (d) 14

15. Let S_n denote the sum of n terms of an A.P. whose first term is a. If the common difference d is given by $d = S_n - k S_{n-1} + S_{n-2}$, then $k =$
(a) 1 (b) 2 (c) 3 (d) none of these

16. The first and last term of an A. P. are a and l respectively. If S is the sum of all the terms of the A.P. and the common difference is given by $\dfrac{l^2-a^2}{k-(l+a)}$, then $k =$
(a) S (b) $2S$ (c) $3S$ (d) none of these

17. If the sum of first n even natural numbers is equal to k times the sum of first n odd natural numbers, then $k =$
(a) $\dfrac{1}{n}$ (b) $\dfrac{n-1}{n}$ (c) $\dfrac{n+1}{2n}$ (d) $\dfrac{n+1}{n}$

18. If the first, second and last term of an A.P are a, b and $2a$ respectively, then its sum is
(a) $\dfrac{ab}{2(b-a)}$ (b) $\dfrac{ab}{b-a}$ (c) $\dfrac{3ab}{2(b-a)}$ (d) none of these

19. If, S_1 is the sum of an arithmetic progression of 'n' odd number of terms and S_2 the sum of the terms of the series in odd places, then $\dfrac{S_1}{S_2} =$
(a) $\dfrac{2n}{n+1}$ (b) $\dfrac{n}{n+1}$ (c) $\dfrac{n+1}{2n}$ (d) $\dfrac{n+1}{n}$

20. If in an A.P., $S_n = n^2 p$ and $S_m = m^2 p$, where S_r denotes the sum of r terms of the A.P., then S_p is equal to

(a) $\frac{1}{2} p^3$ (b) mnp (c) p^3 (d) $(m+n)p^2$

21. If in an A.P., the pth term is q and $(p+q)^{th}$ term is zero, then the q^{th} term is

(a) $-p$ (b) p (c) $p+q$ (d) $p-q$

22. The 10th common term between the A.P.s 3, 7, 11, 15, ... and 1, 6, 11, 16, ... is

(a) 191 (b) 193 (c) 211 (d) none of these

23. If in an A.P. $S_n = n^2 q$ and $S_m = m^2 q$, where S_r denotes the sum of r terms of the A.P., then S_q equals

(a) $\frac{q^3}{2}$ (b) mnq (c) q^3 (d) $(m^2+n^2)q$

24. Let S_n denote the sum of first n terms of an A.P. If $S_{2n} = 3S_n$, then $S_{3n} : S_n$ is equal to

(a) 4 (b) 6 (c) 8 (d) 10

ANSWERS

1. (c)	2. (d)	3. (a)	4. (b)	5. (a)	6. (b)	7. (b)	8. (b)
9. (c)	10. (c)	11. (a)	12. (d)	13. (a)	14. (d)	15. (b)	16. (b)
17. (d)	18. (c)	19. (a)	20. (c)	21. (b)	22. (a)	23. (c)	24. (b)

SUMMARY

1. A sequence is a function whose domain is the set N of all natural numbers or some subsets of the type $\{1, 2, 3, ..., n\}$.
   A sequence containing a finite number of terms is called a finite sequence.
   A sequence is called an infinite sequence if it is not a finite sequence.
2. If $a_1, a_2, a_3, ..., a_n, ...$ is a sequence, then the expression $a_1 + a_2 + a_3 + ... + a_n + ...$ is called a series.
   A series is called a finite series if it has got finite number of terms, otherwise, it is called an infinite series.
3. Those sequences whose terms follow certain patterns are called progressions.
4. A sequence is called an arithmetic progression if the difference of a term and the previous term is always same, i.e. $a_{n+1} - a_n$ = constant $(= d)$ for all $n \in N$
   The constant difference 'd' is called the common difference.
5. A sequence is an arithmetic progression if and only if its nth terms is a linear expression in n and in such a case the common difference is equal to the coefficient of n.
6. If a is the first term and d is the common difference of an A.P., then its nth term is given by
$$a_n = a + (n-1)d$$
7. If an A.P. consists of m terms, then nth term from the end is equal to $(m-n+1)^{th}$ term from the beginning.
8. The following ways of selecting terms of an A.P. are generally very convenient:

Number of terms	*Terms*	*Common difference*
3	$a-d, a, a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d
6	$a-5d, a-3d, a-d, a+d, a+3d, a+5d$	$2d$

9. The sum S_n of n terms of an A.P. with first term 'a' and common difference 'd' is given by
$$S_n = \frac{n}{2}\{2a + (n-1)d\} \text{ or, } S_n = \frac{n}{2}(a + l), \text{ where } l = \text{last term} = a + (n-1)d.$$

10. If the sum S_n of n terms of a sequence is given, then n^{th} term a_n of the sequence can be determined by using the formula $a_n = S_n - S_{n-1}$

11. A sequence is an A.P. iff the sum of its n terms is of the form $An^2 + Bn$ i.e. a quadratic expression in n and in such a case the common difference is twice the coefficient of n^2.

12. If the ratio of the sums of n terms of two A.P.'s is given, then the ratio of their n^{th} terms is obtained by replacing n by $(2n-1)$ in the given ratio

13. Three numbers a, b, c are in A.P. iff $2b = a + c$. In such a case b is called the arithmetic mean of a and c.

14. The arithmetic mean of a and b is $\frac{a+b}{2}$.

15. If n numbers $A_1, A_2, \ldots, A_n$ are inserted between two given numbers a and b such that $a, A_1, A_2, \ldots, A_n, b$ is an arithmetic progression, then $A_1, A_2, \ldots, A_n$ are known as n arithmetic means between a and b and the common difference of the A.P. is $d = \frac{b-a}{n+1}$.

Also, $A_1 + A_2 + \ldots + A_n = n\left(\frac{a+b}{2}\right)$.

16. In an A.P. the sum of the terms equidistant from the beginning and the end is always same and is equal to the sum of first and last term.

CHAPTER 20

GEOMETRIC PROGRESSIONS

20.1 GEOMETRIC PROGRESSION

A sequence of non-zero numbers is called a geometric progression (abbreviated as G.P.) if the ratio of a term and the term preceding to it is always a constant quantity.

The constant ratio is called the common ratio of the G.P.

In other words, a sequence, $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called a geometric progression if $\frac{a_{n+1}}{a_n}$ = constant for all $n \in N$.

ILLUSTRATION 1 *The sequence 4, 12, 36, 108, ... is a G.P., because* $\frac{12}{4} = \frac{36}{12} = \frac{108}{36} = \ldots = 3$, *which is constant.*

Clearly, this sequence is a G.P. with first term 4 and common ratio 3.

ILLUSTRATION 2 *The sequence* $\frac{1}{3}, -\frac{1}{2}, \frac{3}{4}, -\frac{9}{8}, \ldots$ *is a G.P. with first term* $\frac{1}{3}$ *and common ratio equal to* $\left(-\frac{1}{2}\right) \div \left(\frac{1}{3}\right) = -\frac{3}{2}$.

ILLUSTRATION 3 *Show that the sequence given by $a_n = 3(2^n)$, for all $n \in N$, is a G.P. Also, find its common ratio.*

SOLUTION We have, $a_n = 3(2^n)$

$\therefore \quad a_{n+1} = 3(2^{n+1})$

So, $\frac{a_{n+1}}{a_n} = \frac{3(2^{n+1})}{3(2^n)} = 2$, which is constant for all $n \in N$.

So, the given sequence is a G.P. with common ratio 2.

GEOMETRIC SERIES If $a_1, a_2, a_3, \ldots a_n, \ldots$ is a G.P., then the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is *called a geometric series.*

Note that the geometric series is finite or infinite according as the corresponding G.P. consists of finite or infinite number of terms.

20.2 GENERAL TERM OF A G.P.

THEOREM *Prove that the nth term of a G.P. with first term a and common ratio r is given by $a_n = ar^{n-1}$.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be the given G.P. Then,

$$a_1 = a \Rightarrow a_1 = ar^{1-1}$$

Since $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P. with common ratio r. Therefore,

$$\frac{a_2}{a_1} = r \Rightarrow a_2 = a_1 r \Rightarrow a_2 = ar \Rightarrow a_2 = ar^{2-1}$$

$$\frac{a_3}{a_2} = r \Rightarrow a_3 = a_2 r \Rightarrow a_3 = (ar) r \Rightarrow a_3 = ar^2 \Rightarrow a_3 = ar^{3-1}$$

$$\frac{a_4}{a_3} = r \Rightarrow a_4 = a_3 r \Rightarrow a_4 = (ar^2)\, r \Rightarrow a_4 = ar^3 \Rightarrow a_4 = ar^{4-1}$$

Continuing in this manner, we get $a_n = ar^{n-1}$ Q.E.D.

NOTE *It follows from the above discussion that if a is the first term and r is the common ratio of a G.P., then the G.P. can be written as* $a, ar, ar^2, ..., ar^{n-1}$ *or*, $a, ar, ar^2, ar^3, ar^4, ..., ar^{n-1}, ...$ *according as it is finite or infinite.*

20.2.1 *n*th TERM FROM THE END OF A FINITE G.P.

THEOREM 1 *Prove that the nth term from the end of a finite G.P. consisting of m terms is* ar^{m-n}, *where a is the first term and r is the common ratio of the G.P.*

PROOF Since the G.P. consists of m terms.

$\therefore$ nth term from the end $=$ $(m-n+1)$th term from the beginning $= ar^{m-n}$

THEOREM 2 *Prove that the nth term from the end of a G.P. with last term l and common ratio r is given by* $a_n = l\left(\frac{1}{r}\right)^{n-1}$.

PROOF Clearly, when we look at the terms of a G.P. from the last term and move towards the beginning we find that the progression is a G.P. with common ratio $1/r$.

So, nth term from the end $= l\left(\frac{1}{r}\right)^{n-1}$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **FINDING THE INDICATED TERM OF A G.P. WHEN ITS FIRST TERM AND THE COMMON RATIO ARE GIVEN**

EXAMPLE 1 *Find the 9th term and the general term of the progression:* $\frac{1}{4}, -\frac{1}{2}, 1, -2, ...$

SOLUTION The given progression is clearly a G.P. with first term $a = 1/4$ and common ratio $r = -2$.

$\therefore$ 9th term $= a_9 = ar^{(9-1)} = ar^8 = \frac{1}{4}(-2)^8 = 64$

and, General term $= a_n = ar^{(n-1)} = \frac{1}{4}(-2)^{n-1} = (-1)^{n-1}\, 2^{n-3}$

EXAMPLE 2 *Find the 5th term of the progression*

$$1, \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \left(\frac{3-2\sqrt{2}}{12}\right), \left(\frac{5\sqrt{2}-7}{24\sqrt{3}}\right), ...$$

SOLUTION Clearly, the given progression is a G.P. with first term $a = 1$ and common ratio $\frac{\sqrt{2}-1}{2\sqrt{3}}$. So, its 5th term is given by

$$a_5 = ar^{(5-1)} = 1 \times \left(\frac{\sqrt{2}-1}{2\sqrt{3}}\right)^4 = \frac{(\sqrt{2}-1)^4}{144}$$

EXAMPLE 3 *Find 4th term from the end of the G.P. 3, 6, 12, 24, ..., 3072.*

SOLUTION Clearly, the given progression is a G.P. with common ratio $r = 2$.

$\therefore$ 4th term from the end $= l\left(\frac{1}{r}\right)^{4-1} = (3072)\left(\frac{1}{2}\right)^{4-1} = 384$

Type II **ON FINDING THE POSITION OF A GIVEN TERM IN A GIVEN G.P.**

EXAMPLE 4 *Which term of the G.P.* $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$ *is* $\frac{1}{128}$?

SOLUTION Clearly, the given progression is a G.P. with first term $a=2$ and common ratio $r=1/2$. Let the nth term be $\frac{1}{128}$. Then,

$$a_n=\frac{1}{128} \Rightarrow ar^{n-1}=\frac{1}{128} \Rightarrow 2\left(\frac{1}{2}\right)^{n-1}=\frac{1}{128} \Rightarrow \left(\frac{1}{2}\right)^{n-2}=\left(\frac{1}{2}\right)^7 \Rightarrow n-2=7 \Rightarrow n=9$$

Thus, 9th term of the given G.P. is $\frac{1}{128}$.

EXAMPLE 5 *Which term of the G.P. 5, 10, 20, 40, ... is 5120 ?*

SOLUTION Clearly, the given G.P. has first term $a=5$ and the common ratio $r=2$. Let the nth term be 5120. Then,

$$a_n=5120$$

$$\Rightarrow ar^{n-1}=5120 \Rightarrow 5(2^{n-1})=5120 \Rightarrow 2^{n-1}=1024 \Rightarrow 2^{n-1}=2^{10} \Rightarrow n-1=10 \Rightarrow n=11$$

Thus, 11th term of the given G.P. is 5120.

EXAMPLE 6 *Which term of the G.P. 2, 8, 32, ... is 131072 ?* [NCERT]

SOLUTION Here, $a=2$ and $r=4$. Let the nth term be 131072. Then,

$$a_n=131072$$

$$\Rightarrow ar^{n-1}=131072 \Rightarrow 2\times 4^{n-1}=131072 \Rightarrow 4^{n-1}=65536 \Rightarrow 4^{n-1}=4^8 \Rightarrow n-1=8 \Rightarrow n=9$$

Hence, 131072 is the 9th term of the given G.P.

Type III **PROBLEMS BASED ON THE DEFINITION OF A G.P. AND THE FORMULA** $a_n = ar^{n-1}$

EXAMPLE 7 *The fourth, seventh and the last term of a G.P. are 10, 80 and 2560 respectively. Find the first term and the number of terms in the G.P.*

SOLUTION Let a be the first term and r be the common ratio of the given G.P. Then,

$$a_4=10,\ a_7=80 \Rightarrow ar^3=10 \text{ and } ar^6=80 \Rightarrow \frac{ar^6}{ar^3}=\frac{80}{10} \Rightarrow r^3=8 \Rightarrow r=2.$$

Putting $r=2$ in $ar^3=10$, we get:

$$a(2)^3=10 \Rightarrow a=\frac{10}{8}=\frac{5}{4}.$$

Let there be n terms in the given G.P. Then,

$$a_n=2560 \Rightarrow ar^{n-1}=2560$$

$$\Rightarrow \quad \frac{5}{4}(2^{n-1})=2560 \Rightarrow 2^{n-4}=256 \Rightarrow 2^{n-4}=2^8 \Rightarrow n-4=8 \Rightarrow n=12.$$

EXAMPLE 8 *The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio of the G.P.* [NCERT]

SOLUTION Let r be the common ratio of the G.P. It is given that the first term $a=1$.

Now, $a_3+a_5=90$

$$\Rightarrow \quad ar^2+ar^4=90$$

$$\Rightarrow \quad r^2+r^4=90$$

$\Rightarrow \quad r^4 + r^2 - 90 = 0$

$\Rightarrow \quad r^4 + 10r^2 - 9r^2 - 90 = 0 \Rightarrow (r^2+10)(r^2-9) = 0 \Rightarrow r^2 - 9 = 0 \Rightarrow r = \pm 3.$

Hence, the common ratio of the given G.P. is 3 or -3.

EXAMPLE 9 *If the 4th and 9th terms of a G.P. be 54 and 13122 respectively, find the G.P.*

SOLUTION Let a be the first term and r the common ratio of the given G.P. Then,

$a_4 = 54$ and $a_9 = 13122$

$\Rightarrow \quad ar^3 = 54$ and $ar^8 = 13122$

$$\Rightarrow \quad \frac{ar^8}{ar^3} = \frac{13122}{54} \Rightarrow r^5 = 243 \Rightarrow r^5 = 3^5 \Rightarrow r = 3$$

Putting $r = 3$ in $ar^3 = 54$, we get: $a(3)^3 = 54 \Rightarrow a = 2$

Thus, the given G.P. is $a, ar, ar^2, ar,^3 \dots$ i.e. 2, 6, 18, 54, ...

EXAMPLE 10 *Find a G.P. for which the sum of first two terms is –4 and the fifth term is 4 times the third term.* [NCERT]

SOLUTION Let a be the first term and r be the common ratio of the given G.P. It is given that
The sum of first two terms= –4. $\Rightarrow a_1 + a_2 = -4 \Rightarrow a + ar = -4$ ...(i)
It is also given that

$a_5 = 4a_3 \Rightarrow ar^4 = 4ar^2 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$

Putting $r = 2$ and -2 respectively in (i), we get $a = -\frac{4}{3}$ and $a = 4$ respectively.

Thus, the required G.P. is $-\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, \dots$ or $4, -8, 16, -32, \dots$

EXAMPLE 11 *The third term of a G.P. is 4. Find the product of its first five terms.*

SOLUTION Let a be the first term and r the common ratio. Then,

$a_3 = 4 \Rightarrow ar^2 = 4$ ...(i)

$\therefore$ Product of first five terms $= a_1 a_2 a_3 a_4 a_5 = a(ar)(ar^2)(ar^3)(ar^4)$

$= a^5 r^{10} = (ar^2)^5 = 4^5$ [Using (i)]

EXAMPLE 12 *If the pth, qth and rth terms of a G.P. are a, b, c respectively, prove that:*
$a^{(q-r)} \cdot b^{(r-p)} \cdot c^{(p-q)} = 1.$ [NCERT]

SOLUTION Let A be the first term and R be the common ratio of the given G.P. Then,
$a = p$th term $= AR^{(p-1)}$, $b = q$th term $= AR^{(q-1)}$, and $c = r$th term $= AR^{(r-1)}$

Substituting the values of a, b and c, we get

$$a^{(q-r)} \cdot b^{(r-p)} \cdot c^{(p-q)}$$
$$= \left\{AR^{(p-1)}\right\}^{(q-r)} \cdot \left\{AR^{(q-1)}\right\}^{(r-p)} \cdot \left\{AR^{(r-1)}\right\}^{(p-q)}$$
$$= A^{(q-r)} R^{(p-1)(q-r)} \cdot A^{(r-p)} R^{(q-1)(r-p)} \cdot A^{(p-q)} R^{(r-1)(p-q)}$$
$$= A^{(q-r+r-p+p-q)} R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$$
$$= A^0 R^{p(q-r)+q(r-p)+r(p-q)-(q-r)-(r-p)-(p-q)} = A^0 R^0 = 1.$$

EXAMPLE 13 *If a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of a G.P., show that*
$$(q-r)\log a+(r-p)\log b+(p-q)\log c=0.$$

SOLUTION Let A be the first term and R the common ratio of the given G.P. Then,

$$a=p\text{th term} \Rightarrow a=AR^{p-1} \Rightarrow \log a=\log A+(p-1)\log R \quad \text{...(i)}$$
$$b=q\text{th term} \Rightarrow b=AR^{q-1} \Rightarrow \log b=\log A+(q-1)\log R \quad \text{...(ii)}$$
$$c=r\text{th term} \Rightarrow c=AR^{r-1} \Rightarrow \log c=\log A+(r-1)\log R \quad \text{...(iii)}$$

Substituting the values of $\log a$, $\log b$ and $\log c$, we get

$$(q-r)\log a+(r-p)\log b+(p-q)\log c$$
$$=(q-r)\left\{\log A+(p-1)\log R\right\}+(r-p)\left\{\log A+(q-1)\log R\right\}+(p-q)\left\{\log A+(r-1)\log R\right\}$$
$$=\log A\left\{(q-r)+(r-p)+(p-q)\right\}+\log R\left\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\right\}$$
$$=(\log A)\,0+\left\{p(q-r)+q(r-p)+r(p-q)-(q-r)-(r-p)-(p-q)\right\}\log R$$
$$=(\log A)\,0+(\log R)\,0=0.$$

EXAMPLE 14 *Find four numbers forming a geometric progression in which the third term is greater than the first terms by 9, and second term is greater than the 4th by 18.* **[NCERT]**

SOLUTION Let the four numbers in G.P. be a, ar, ar^2 and ar^3. It is given that

$$ar^2=a+9 \text{ and } ar=ar^3+18$$
$$\Rightarrow a(r^2-1)=9 \text{ and } ar(1-r^2)=18$$
$$\therefore \frac{ar(1-r^2)}{a(r^2-1)}=\frac{18}{9} \Rightarrow -r=2 \Rightarrow r=-2$$

Putting $r=-2$ in $a(r^2-1)=9$, we get

$$a(4-1)=9 \Rightarrow a=3$$

Hence, the numbers are: $3, 3(-2), 3(-2)^2, 3(-2)^3$ or, $3, -6, 12, -24$.

EXAMPLE 15 *The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and nth hour?* **[NCERT]**

SOLUTION Clearly, number of bacteria at the end of different hours forms a G.P. with first term $a=30$ and common ratio $r=2$.

Number of bacteria present at the end of 2nd hour

= Third term of the G.P. with first term $a=30$ and common ratio $r=2$

$=ar^2=30\times 2^2=120$

Number of bacteria present at the end of 4th hour

= 5th term of the G.P. with first term $a=30$ and common ratio $r=2$

$=ar^4=30\times 2^4=480$

Number of bacteria present at the end of nth hour

$=(n+1)^{th}$ term of the G.P. with first term $a=30$ and common ratio $r=2$

$=ar^n=30\times 2^n$

EXAMPLE 16 *What will ₹ 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?* [NCERT]

SOLUTION We have,

P = Principal = ₹ 500, R = Rate of interest = 10%

$\therefore$ Amount at the end of one year $= ₹\left(P+\frac{PR}{100}\right) = ₹\, P\left(1+\frac{R}{100}\right)$

Amount at the end of second year $= ₹\left\{P\left(1+\frac{R}{100}\right)+P\left(1+\frac{R}{100}\right)\frac{R}{100}\right\}$

$$= ₹\, P\left(1+\frac{R}{100}\right)\left(1+\frac{R}{100}\right) = \text{Rs } P\left(1+\frac{R}{100}\right)^2$$

Amount at the end of the third year $= ₹\left\{P\left(1+\frac{R}{100}\right)^2+P\left(1+\frac{R}{100}\right)^2\cdot\frac{R}{100}\right\}$

$$= ₹\, P\left(1+\frac{R}{100}\right)^3$$

and so on.

Clearly, amounts at the end of various year form a G.P. with first term and common ratio $\left(1+\frac{R}{100}\right)$.

$\therefore$ Amount at the end of 10th year = 11^{th} term of the G.P.

$$= ₹\, P\left(1+\frac{R}{100}\right)^{10}$$

$$= ₹\, 500\left(1+\frac{10}{100}\right)^{10} = ₹\, 500\times\left(\frac{11}{10}\right)^{10} = ₹\, 500\times(1.1)^{10}$$

EXAMPLE 17 *A manufacturer reckons that the value of a machine, which costs him ₹ 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.* [NCERT]

SOLUTION We have,

Initial value of the machine $= V_0 =$ ₹ 15625 and, R = Rate of depreciation = 20%

$\therefore$ Depreciated value at the end of first year $= V_0 - \frac{V_0 R}{100} = V_0\left(1-\frac{R}{100}\right)$

Depreciated value at the end of second year $= V_1 - \frac{V_1 R}{100} = V_1\left(1-\frac{R}{100}\right) = V_0\left(1-\frac{R}{100}\right)^2$

ans so on.

Clearly, depreciated values at the end of different years form a G.P. with first term V_0 and common ratio $\left(1-\frac{R}{100}\right)$.

$\therefore$ Depriciated value at the end of 5 years

= 6th term of the G.P. with first term V_0 (= Rs 15625) and common ratio $r = \left(1-\frac{R}{100}\right)$

$$= V_0\left(1-\frac{R}{100}\right)^5 = ₹\left\{15625\left(1-\frac{20}{100}\right)^5\right\} = ₹\left\{15625\times\left(\frac{4}{5}\right)^5\right\} = ₹\, 5120$$

LEVEL-2

EXAMPLE 18 *In a G.P. of positive terms, if any terms is equal to the sum of next two terms, find the common ratio of the G.P.*

SOLUTION Let a be the first term and r be the common ratio of the G.P. By hypothesis

$$a_n = a_{n+1} + a_{n+2}$$

$$\Rightarrow \quad ar^{n-1} = ar^n + ar^{n+1}$$

$$\Rightarrow \quad 1 = r + r^2 \Rightarrow r^2 + r - 1 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

But, $r > 0$. Therefore, $r = \frac{-1+\sqrt{5}}{2} = 2\left(\frac{\sqrt{5}-1}{4}\right) = 2 \sin 18°$

EXAMPLE 19 *If the first and the nth terms of a G.P. are a and b respectively and if P is the product of the first n terms, prove that* $P^2 = (ab)^n$. **[NCERT]**

SOLUTION Let r be the common ratio of the given G.P. Then,

$$b = nth \text{ term} = ar^{n-1} \Rightarrow r^{n-1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n-1}}$$

Now,

$$P = \text{Product of the first } n \text{ terms}$$

$$\Rightarrow \quad P = a \,.\, ar \,.\, ar^2 \ldots ar^{n-1}$$

$$\Rightarrow \quad P = a^n \, r^{1+2+3+\ldots+(n-1)}$$

$$\Rightarrow \quad P = a^n \, r^{\frac{n(n-1)}{2}} \qquad \left[\because 1+2+3+\ldots+(n-1) = \left(\frac{n-1}{2}\right)\left(1+(n-1)\right) = \frac{n(n-1)}{2}\right]$$

$$\Rightarrow \quad P = a^n \left\{\left(\frac{b}{a}\right)^{\frac{1}{n-1}}\right\}^{\frac{n(n-1)}{2}} = a^n \left(\frac{b}{a}\right)^{n/2} = a^{n/2} b^{n/2} = (ab)^{n/2}$$

$$\therefore \quad P^2 = \left\{(ab)^{n/2}\right\}^2 = (ab)^n$$

EXAMPLE 20 *The $(m+n)$th and $(m-n)$th terms of a G.P. are p and q respectively. Show that the mth and nth terms are* $\sqrt{pq}$ *and* $p\left(\frac{q}{p}\right)^{m/2n}$ *respectively.*

SOLUTION Let a be the first term and r be the common ratio. Then,

$$a_{m+n} = p \text{ and } a_{m-n} = q$$

$$\Rightarrow \quad ar^{m+n-1} = p \text{ and } ar^{m-n-1} = q$$

$$\Rightarrow \quad \frac{ar^{m+n-1}}{ar^{m-n-1}} = \frac{p}{q} \Rightarrow r^{2n} = \frac{p}{q} \Rightarrow r = \left(\frac{p}{q}\right)^{1/2n} \Rightarrow \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}$$

Now, $a_m = ar^{m-1}$

$$\Rightarrow \quad a_m = ar^{(m+n-1)}\left(\frac{1}{r}\right)^n$$

$$\Rightarrow \quad a_m = a_{m+n}\left(\frac{1}{r}\right)^n \qquad [\because a_{m+n} = ar^{m+n-1}]$$

$$\Rightarrow \quad a_m = p\left(\frac{q}{p}\right)^{n/2n} \qquad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

$$\Rightarrow \quad a_m = p\left(\frac{q}{p}\right)^{1/2} = \sqrt{pq}$$

and, $\quad a_n = ar^{n-1}$

$$\Rightarrow \quad a_n = ar^{m+n-1}\left(\frac{1}{r}\right)^m = a_{m+n}\left(\frac{1}{r}\right)^m \qquad [\because a_{m+n} = ar^{m+n-1}]$$

$$\Rightarrow \quad a_n = p\left(\frac{q}{p}\right)^{m/2n} \qquad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

EXAMPLE 21 *If pth, qth and rth terms of an A.P. as well as a G.P. are a, b and c respectively. Prove that*

$$a^{b-c}\, b^{c-a}\, c^{a-b} = 1.$$

SOLUTION Let A be the first term and d be the common difference of the A.P. It is given that a, b and c are p^{th}, q^{th} and r^{th} terms of the A.P. Therefore,

$$a = A+(p-1)\,d,\; b = A+(q-1)\,d,\; c = A+(r-1)\,d$$

$$\therefore \quad b-c = (q-r)\,d,\; c-a = (r-p)\,d \text{ and } a-b = (p-q)\,d.$$

Let α be the first term and R be the common ratio of the G.P. Then,

$$a = \alpha R^{p-1},\; b = \alpha R^{q-1}, \text{ and } c = \alpha R^{r-1}$$

$$a^{b-c}\, b^{c-a}\, c^{a-b} = \left\{\alpha R^{p-1}\right\}^{(q-r)d} \times \left\{\alpha R^{q-1}\right\}^{(r-p)d} \times \left\{\alpha R^{r-1}\right\}^{(p-q)d}$$

$$= \alpha^{(q-r)d+(r-p)d+(p-q)d}\, R^{(p-1)(q-r)d+(q-1)(r-p)d+(r-1)(p-q)d}$$

$$= \alpha^{(q-r+r-p+p-q)d}\, R^{\{p(q-r)+q(r-p)+r(p-q)-(q-r)-(r-p)-(p-q)\}d}$$

$$= \alpha^0 R^0 = 1.$$

EXAMPLE 22 *Find all sequences which are simultaneously A.P. and G.P.*

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n \ldots$ be a sequence which is both an A.P. as well as a G.P.
Let a_n, a_{n+1}, a_{n+2} be three consecutive terms of the A.P. Then,

$$2a_{n+1} = a_n + a_{n+2},\; n \in N \qquad \ldots(i)$$

Let r be the common ratio of the sequence when it is considered a G.P. Then,

$$a_n = a_1 r^{n-1},\; a_{n+1} = a_1 r^n \text{ and } a_{n+2} = a_1 r^{n+1}$$

Putting these values in (i), we get

$$2a_1 r^n = a_1 r^{n-1} + a_1 r^{n+1} \Rightarrow 2r = 1 + r^2 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$$

Putting $r = 1$ in $a_1, a_2 = a_1 r, a_3 = a_1 r^2, a_4 = a_1 r^3, \ldots$, we obtain

$a_1, a_2 = a_1, a_3 = a_1, a_4 = a_1 \ldots$, which is a constant squance.

Hence, the constant sequence is the only sequence which is both an A.P. as well as G.P.

EXAMPLE 23 *In a finite G.P. the product of the terms equidistant from the beginning and the end is always same and equal to the product of first and last term.*

SOLUTION Let $a_1, a_2, a_3, \dots, a_{n-1}, a_n$ be a finite G.P. with common ratio r.
Now,

$$a_k = k\text{th term from the beginning} = a_1 r^{k-1}$$

and, $$a_{n-k+1} = k\text{th term from the end} = a_n \left(\frac{1}{r}\right)^{k-1}, \text{ where } 1 < k < n$$

$$\therefore \quad a_k\, a_{n-k+1} = \left(a_1 r^{k-1}\right) a_n \left(\frac{1}{r}\right)^{k-1} = a_1\, a_n \quad \text{for all } k \text{ satisfying } 1 < k < n.$$

Hence, the product of terms equidistant from the beginning and the end is always equal to the product of first and last term.

EXAMPLE 24 *Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \dots, ar^{n-1}$ and $A, AR, AR^2, \dots, AR^{n-1}$ form a G.P., and find the common ratio.* [NCERT]

SOLUTION The sequence formed by the products of the corresponding terms of the given sequences is

$$aA,\ aA\, rR,\ aA\, r^2 R^2, \dots, aA\, r^{n-1} R^{n-1}$$

or, $$aA,\ aA\,(rR),\ aA\,(rR)^2,\ aA\,(rR)^3, \dots, aA\,(rR)^{n-1}$$

Clearly, the ratio of any term and preceding term in the above squance is same equal to rR.
So, it is a G.P. with common ratio rR.

EXAMPLE 25 *If p^{th}, q^{th}, r^{th} and s^{th} terms of an A.P. are in G.P., then show that $(p-q), (q-r), (r-s)$ are also in G.P.* [NCERT]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Further, let a_p, a_q, a_r and a_s be its p^{th}, q^{th}, r^{th} and s^{th} terms respectively. Then,

$$a_p = a + (p-1)\,d,\ a_q = a + (q-1)\,d,\ a_r = a + (r-1)\,d \text{ and } a_s = a + (s-1)\,d.$$

$$\Rightarrow \quad a_p - a_q = (p-q)\,d,\ a_q - a_r = (q-r)\,d \text{ and } a_r - a_s = (r-s)\,d \qquad \dots \text{(i)}$$

It is given that a_p, a_q, a_r and a_s are in G.P. Let A be the first term and R be the common ratio of the G.P. Then,

$$A = a_p,\ AR = a_q,\ AR^2 = a_r \text{ and } AR^3 = a_s.$$

$$\therefore \quad A - AR = a_p - a_q,\ AR - AR^2 = a_q - a_r \text{ and } AR^2 - AR^3 = a_r - a_s$$

$$\Rightarrow \quad A(1-R) = a_p - a_q,\ AR(1-R) = a_q - a_r \text{ and } AR^2(1-R) = a_r - a_s$$

$$\Rightarrow \quad (a_q - a_r)^2 = \left\{AR\ (1-R)\right\}^2 = \left\{A\ (1-R)\right\}\left\{AR^2(1-R)\right\} = (a_p - a_q)(a_r - a_s)$$

$$\Rightarrow \quad (q-r)^2 d^2 = \left\{(p-q)\,d\right\}\left\{(r-s)\,d\right\} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad (q-r)^2 = (p-q)(r-s)$$

$$\Rightarrow \quad p-q,\ q-r,\ r-s \text{ are in G.P.}$$

EXERCISE 20.1

LEVEL-1

1. Show that each one of the following progressions is a G.P. Also, find the common ratio in each case:
   (i) $4, -2, 1, -1/2, \dots$ (ii) $-2/3, -6, -54, \dots$
   (iii) $a, \frac{3a^2}{4}, \frac{9a^3}{16}, \dots$ (iv) $1/2, 1/3, 2/9, 4/27, \dots$

2. Show that the sequence defined by $a_n = \frac{2}{3^n}, n \in N$ is a G.P.

3. Find:
   (i) the ninth term of the G.P. 1, 4, 16, 64, ...
   (ii) the 10th term of the G.P. $-\frac{3}{4}, \frac{1}{2}, -\frac{1}{3}, \frac{2}{9}, \ldots$
   (iii) the 8th term of the G.P. 0.3, 0.06, 0.012, ...
   (iv) the 12th term of the G.P. $\frac{1}{a^3 x^3}, ax, a^5 x^5, \ldots$
   (v) nth term of the G.P. $\sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}}, \ldots$
   (vi) the 10th term of the G.P. $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \ldots$

4. Find the 4th term from the end of the G.P. $\frac{2}{27}, \frac{2}{9}, \frac{2}{3}, \ldots, 162$.

5. Which term of the progression 0.004, 0.02, 0.1, ... is 12.5 ?

6. Which term of the G.P. :
   (i) $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}, \ldots$ is $\frac{1}{512\sqrt{2}}$?
   (ii) $2, 2\sqrt{2}, 4, \ldots$ is 128 ? [NCERT]
   (iii) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729? [NCERT]
   (iv) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots$ is $\frac{1}{19683}$? [NCERT]

7. Which term of the progression $18, -12, 8, \ldots$ is $\frac{512}{729}$?

8. Find the 4th term from the end of the G.P. $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots, \frac{1}{4374}$.

9. The fourth term of a G.P. is 27 and the 7th term is 729, find the G.P.

10. The seventh term of a G.P. is 8 times the fourth term and 5th term is 48. Find the G.P.

11. If the G.P.'s 5, 10, 20, ... and 1280, 640, 320, ... have their nth terms equal, find the value of n.

12. If 5^{th}, 8^{th} and 11^{th} terms of a G.P. are p, q and s respectively, prove that $q^2 = ps$. [NCERT]

13. The 4th term of a G.P. is square of its second term, and the first term is -3. Find its 7^{th} term. [NCERT]

14. In a GP the 3^{rd} term is 24 and the 6^{th} term is 192. Find the 10^{th} term. [NCERT]

LEVEL-2

15. If a, b, c, d and p are different real numbers such that:
$(a^2 + b^2 + c^2)\, p^2 - 2(ab + bc + cd)\, p + (b^2 + c^2 + d^2) \leq 0$, then show that a, b, c and d are in G.P. [NCERT]

16. If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$ $(x \neq 0)$, then show that a, b, c and d are in G.P. [NCERT]

17. If the p^{th} and q^{th} terms of a G.P. are q and p respectively, show that $(p+q)^{th}$ term is $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$.

ANSWERS

1. (i) $-\frac{1}{2}$ (ii) 9 (iii) $\frac{3a}{4}$ (iv) $\frac{2}{3}$
3. (i) 4^8 (ii) $\frac{1}{2}\left(\frac{2}{3}\right)^8$ (iii) $(0.3)(0.2)^7$ (iv) $(ax)^{41}$ (v) $\sqrt{3}\left(\frac{1}{3}\right)^{n-1}$ (vi) $\frac{1}{\sqrt{2}} \times \frac{1}{2^8}$
4. 6
5. 6
6. (i) 11^{th} (ii) 13^{th} (iii) 12^{th} (iv) 9^{th}
7. 9
8. $\frac{1}{162}$
9. 1, 3, 9, ...
10. 3, 6, 12, ..
11. 5
13. -2187
14. 3072

HINTS TO NCERT & SELECTED PROBLEMS

6. (ii) Let n^{th} term of the G.P. $2, 2\sqrt{2}, 4, \ldots$ be 128. Then,

$$2(\sqrt{2})^{n-1} = 128 \Rightarrow 2^{\frac{n+1}{2}} = 2^7 \Rightarrow \frac{n+1}{2} = 7 \Rightarrow n = 13$$

Thus, 13th term of the G.P. $2, 2\sqrt{2}, 4, \ldots$ is 128.

(iii) Let n^{th} term of the G.P. $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ be 729. Then,

$$\sqrt{3} \times (\sqrt{3})^{n-1} = 729 \Rightarrow 3^{\frac{n}{2}} = 3^6 \Rightarrow \frac{n}{2} = 6 \Rightarrow n = 12$$

Hence, 12^{th} term of the given G.P. is 729.

(iv) Let nth term of the G.P. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ be $\frac{1}{19683}$. Then,

$$\frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{1}{19683} \Rightarrow n^3 = 3^9 \Rightarrow n = 9$$

12. Let a be the first term and r the common ratio of the given G.P. It is given that

$p = 5^{th}$ term, $q = 8^{th}$ term, $s = 11^{th}$ term

$\Rightarrow \quad p = ar^4, \; q = ar^7, \; s = ar^{10}$

$\therefore \quad q^2 = a^2 r^{14}$ and $ps = a^2 r^{14}$

$\Rightarrow \quad q^2 = ps.$

13. Let the common ratio of the given G.P. be r. It is given that the fourth term is square of its second term.

$\therefore \quad (-3) r^3 = (-3r)^2 \Rightarrow -3r^3 = 9r^2 \Rightarrow r = -3$

Hence, 7^{th} term $= (-3) r^6 = -3(-3)^6 = -2187.$

14. Let the first term and common ratio of the given G.P. be a and r respectively.

It is given that 3^{rd} term $= 24$ and 6^{th} term $= 192$

$\Rightarrow \quad ar^2 = 24$ and $ar^5 = 192$

$$\Rightarrow \quad \frac{ar^5}{ar^2} = \frac{192}{24}$$

$\Rightarrow \quad r^3 = 8 \Rightarrow r = 2$

Putting $r = 2$ in $ar^2 = 24$, we get $a = 6$.

$\therefore \quad 10^{th}$ term $= ar^9 = 6 \times 2^9 = 3072$

15. It is given that

$$(a^2+b^2+c^2)\,p^2-2\,(ab+bc+cd)\,p+(b^2+c^2+d^2)\le 0$$

$\Rightarrow\ (a^2p^2-2\,abp+b^2)+(b^2p^2-2\,bcp+c^2)+(c^2p^2-2\,cdp+d^2)\le 0$

$\Rightarrow\ (ap-b)^2+(bp-c)^2+(cp-d)^2\le 0$

$\Rightarrow\ (ap-b)^2+(bp-c)^2+(cp-d)^2=0$ $[\because (ap-b)^2+(bp-c)^2+(cp-d)^2$ cannot be negative$]$

$\Rightarrow\ ap-b=0,\ bp-c=0,\ cp-d=0$

$\Rightarrow\ \dfrac{b}{a}=\dfrac{c}{b}=\dfrac{d}{c}=p$

$\Rightarrow\ a, b, c, d$ are in G.P. with common ratio p.

16. We have,

$$\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$$

Now,

$$\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}$$

$\Rightarrow\ \dfrac{(a+bx)+(a-bx)}{(a+bx)-(a-bx)}=\dfrac{(b+cx)+(b-cx)}{(b+cx)-(b-cx)}$ [Applying componendo-dividendo]

$\Rightarrow\ \dfrac{a}{bx}=\dfrac{b}{cx}\Rightarrow\dfrac{b}{a}=\dfrac{c}{b}$

Similarly, $\dfrac{b+cx}{b-cx}=\dfrac{c+dx}{c-dx}\ \Rightarrow\ \dfrac{c}{b}=\dfrac{d}{c}$

$\therefore\ \dfrac{b}{a}=\dfrac{c}{b}=\dfrac{d}{c}\Rightarrow a, b, c, d$ are in G.P.

20.3 SELECTION OF TERMS IN G.P.

Sometimes it is required to select a finite number of terms in G.P. It is always convenient if we select the terms in the following manner:

No. of terms	*Terms*	*Common ratio*
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

If the product of the numbers is not given, then the numbers are taken as $a, ar, ar^2, ar^3, \ldots$

The following examples illustrate the application of the above selections.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If the sum of three numbers in G.P. is 38 and their product is 1728, find them.*

SOLUTION Let the numbers be $\frac{a}{r}$, a, ar. It is given that the product and sum of these numbers are 38 and 1728 respectively.

Now, Product $=1728\ \Rightarrow\ \frac{a}{r}(a)(ar)=1728\Rightarrow a^3=1728\Rightarrow a=12$

and, Sum = 38

$$\Rightarrow \quad \frac{a}{r} + a + ar = 38$$

$$\Rightarrow \quad a\left(\frac{1}{r} + 1 + r\right) = 38$$

$$\Rightarrow \quad 12\left(\frac{1 + r + r^2}{r}\right) = 38$$

$$\Rightarrow \quad 6 + 6r + 6r^2 = 19r \Rightarrow 6r^2 - 13r + 6 = 0 \Rightarrow (3r - 2)(2r - 3) = 0 \Rightarrow r = 3/2 \text{ or, } r = 2/3$$

Putting the values of a and r in $\frac{a}{r}, a, ar$, we find that the required numbers are 8, 12, 18 or 18, 12, 8.

EXAMPLE 2 *If the continued product of three numbers in G.P. is 216 and the sum of their products in pairs is 156, find the numbers.*

SOLUTION Let the three numbers be a/r, a, ar. Then,

Product = 216 $\Rightarrow (a/r).(a).(ar) = 216 \Rightarrow a^3 = 6^3 \Rightarrow a = 6.$

Sum of the products in pairs = 156

$$\Rightarrow \quad \frac{a}{r}.a + a.ar + \frac{a}{r}.ar = 156$$

$$\Rightarrow \quad a^2\left(\frac{1}{r} + r + 1\right) = 156$$

$$\Rightarrow \quad 36\left(\frac{1 + r^2 + r}{r}\right) = 156$$

$$\Rightarrow \quad 3(r^2 + r + 1) = 13r \Rightarrow 3r^2 - 10r + 3 = 0 \Rightarrow (3r - 1)(r - 3) = 0 \Rightarrow r = \frac{1}{3} \text{ or, } r = 3$$

Putting the values of a and r, the required numbers are 18, 6, 2 or 2, 6, 18.

EXAMPLE 3 *Three numbers are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the means by 5, they will be in A.P. Find the numbers.*

SOLUTION Let the numbers be a, ar, ar^2. It is given that the sum of these numbers is 70.

$$\therefore \quad a(1 + r + r^2) = 70 \qquad \text{...(i)}$$

It is also given that $4a, 5ar, 4ar^2$ are in A.P.

$$\therefore \quad 2(5ar) = 4a + 4ar^2$$

$$\Rightarrow \quad 5r = 2 + 2r^2 \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (2r - 1)(r - 2) = 0 \Rightarrow r = 2 \text{ or, } r = 1/2$$

Putting $r = 2$ in (i), we obtain $a = 10$. So, the numbers are 10, 20, 40

Putting $r = 1/2$ in (i), we get $a = 40$. So, the numbers are 40, 20, 10.

EXAMPLE 4 *Find three numbers in G.P. whose sum is 52 and the sum of whose products in pairs is 624.*

SOLUTION Let the required numbers be a, ar, ar^2. Then,

$$\text{Sum} = 52 \Rightarrow a + ar + ar^2 = 52 \Rightarrow a(1 + r + r^2) = 52 \qquad \text{...(i)}$$

Sum of the products in pairs = 624

$$\Rightarrow \quad a.ar + ar.ar^2 + a.ar^2 = 624$$

$$\Rightarrow \quad a^2 r(1 + r + r^2) = 624 \qquad \text{...(ii)}$$

Dividing (ii) by (i), we get

$$ar = 12 \Rightarrow a = \frac{12}{r} \qquad \text{....(iii)}$$

Putting $a = \frac{12}{r}$ in (i), we get

$$\frac{12}{r}(1 + r + r^2) = 52 \Rightarrow 3r^2 - 10r + 3 = 0 \Rightarrow (3r - 1)(r - 3) = 0 \Rightarrow r = 1/3 \text{ or, } r = 3$$

Putting $r = 3$ in (iii), we obtain $a = 4$. So, the numbers are 4, 12, 36.

Putting $r = \frac{1}{3}$ in (iii), we get $a = 36$. So, the numbers are 36, 12, 4.

EXAMPLE 5 *The product of first three terms of a G.P. is 1000. If 6 is added to its second term and 7 added to its third term, the terms become in A.P. Find the G.P.*

SOLUTION Let first three terms of the given G.P. be $\frac{a}{r}$, a, ar. Then,

$$\text{Product} = 1000 \Rightarrow a^3 = 1000 \Rightarrow a = 10.$$

It is given that $\frac{a}{r}$, $a + 6$, $ar + 7$ are in A.P.

$$\therefore \quad 2(a + 6) = \frac{a}{r} + ar + 7$$

$$\Rightarrow \quad 32 = \frac{10}{r} + 10r + 7$$

$$\Rightarrow \quad 25 = \frac{10}{r} + 10r$$

$$\Rightarrow \quad 5 = \frac{2}{r} + 2r \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (2r - 1)(r - 2) = 0 \Rightarrow r = 2, 1/2$$

Hence, the G.P. is 5, 10, 20, ... or 20, 10, 5, ...

EXAMPLE 6 *The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.* [NCERT]

SOLUTION Let the numbers in G.P. be a, ar, ar^2. It is given that the sum of these numbers is 56.

$$\therefore \quad a + ar + ar^2 = 56 \qquad \text{...(i)}$$

It is also given that

$a - 1$, $ar - 7$ and $ar^2 - 21$ are in A.P.

$$2(ar - 7) = (a - 1) + (ar^2 - 21) \Rightarrow 2ar = a + ar^2 - 8 \Rightarrow a + ar^2 = 2ar + 8 \qquad \text{...(ii)}$$

From (i), we obtain

$$a + ar^2 = 56 - ar \qquad \text{...(iii)}$$

Substituting $a + ar^2 = 56 - ar$ on the LHS of (ii), we get

$$2ar + 8 = 56 - ar \Rightarrow 3ar = 48 \Rightarrow ar = 16 \Rightarrow r = \frac{16}{a}$$

Putting $r = \frac{16}{a}$ in (i), we get

$$a + 16 + \frac{256}{a} = 56$$

$$\Rightarrow \quad a^2 + 16a + 256 = 56a \Rightarrow a^2 - 40a + 256 = 0 \Rightarrow (a - 32)(a - 8) = 0 \Rightarrow a = 8, 32$$

Putting $a=8$, in $r=\frac{16}{a}$ we get: $r=\frac{16}{8}=2$

Putting $a=32$, in $r=\frac{16}{a}$ we get: $r=\frac{16}{32}=\frac{1}{2}$

When $a=8$ and $r=2$, we obtain 8, 16 and 32 as the numbers in G.P.

When $a=32$ and $r=\frac{1}{2}$, we obtain 32, 16, 8 as the numbers in G.P.

Hence, the numbers, in order, are 8, 16 and 32 or 32, 16 and 8.

LEVEL-2

EXAMPLE 7 *Find three numbers in G.P. whose sum is 13 and the sum of whose squares is 91.*

SOLUTION Let the numbers be a, ar, ar^2. Then,

$$\text{Sum} = 13 \Rightarrow a+ar+ar^2 = 13 \Rightarrow a(1+r+r^2) = 13 \quad \text{...(i)}$$

Sum of the squares = 91

$$\Rightarrow \quad a^2+a^2r^2+a^2r^4 = 91 \Rightarrow a^2(1+r^2+r^4) = 91 \quad \text{...(ii)}$$

Now, $a(1+r+r^2) = 13$

$$\Rightarrow \quad a^2(1+r+r^2)^2 = 169 \quad \text{[From (i)]}$$

$$\Rightarrow \quad a^2(1+r^2+r^4)+2a^2r(1+r+r^2) = 169$$

$$\Rightarrow \quad 91+2ar\left\{a(1+r+r^2)\right\} = 169$$

$$\Rightarrow \quad 91+2ar\times 13 = 169 \quad \text{[Using (ii)]}$$

$$\Rightarrow \quad ar = 3 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad a = \frac{3}{r} \quad \text{...(iii)}$$

Putting $a = \frac{3}{r}$ in (i), we get

$$\frac{3}{r}(1+r+r^2) = 13$$

$$\Rightarrow \quad \frac{3}{r}+3+3r = 13 \Rightarrow 3r^2-10r+3=0 \Rightarrow (3r-1)(r-3)=0 \Rightarrow r=3 \text{ or, } r=\frac{1}{3}$$

Putting $r=3$ in (iii), we get $a=1$. So, the numbers are 1, 3, 9.

Putting $r=\frac{1}{3}$ in (iii), we get $a=9$. So, the nmbers are 9, 3, 1.

Hence, the numbers are 1, 3, 9 or 9, 3, 1.

EXAMPLE 8 *Find four numbers in G.P. whose sum is 85 and product is 4096.*

SOLUTION Let the four numbers in G.P. be $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

It is given that

$$\text{Product} = 4096 \Rightarrow a^4 = 4096 \Rightarrow a^4 = 8^4 \Rightarrow a = 8$$

and, Sum = 85

$$\Rightarrow \quad a\left(\frac{1}{r^3}+\frac{1}{r}+r+r^3\right)=85$$

$$\Rightarrow \quad 8\left(r^3+\frac{1}{r^3}\right)+8\left(r+\frac{1}{r}\right) = 85$$

$$\Rightarrow \quad 8\left\{\left(r+\frac{1}{r}\right)^3-3\left(r+\frac{1}{r}\right)\right\}+8\left(r+\frac{1}{r}\right)=85$$

$$\Rightarrow \quad 8\left(r+\frac{1}{r}\right)^3-16\left(r+\frac{1}{r}\right)-85 = 0$$

$$\Rightarrow \quad 8x^3-16x-85 = 0, \text{ where } r+\frac{1}{r} = x$$

$$\Rightarrow \quad (2x-5)(4x^2+10x+17) = 0$$

$$\Rightarrow \quad 2x-5 = 0 \qquad [\because\ 4x^2+10x+17=0 \text{ has imaginary roots}]$$

$$\Rightarrow \quad x = \frac{5}{2}$$

$$\Rightarrow \quad r+\frac{1}{r} = \frac{5}{2} \Rightarrow 2r^2-5r+2=0 \Rightarrow (r-2)(2r-1)=0 \quad \Rightarrow r=2 \quad \text{or, } r=\frac{1}{2}$$

Putting $a=8$ and $r=2$ or $r=\frac{1}{2}$, we obtain that the four numbers are either 1, 4, 16, 64 or, 64, 16, 4, 1.

EXERCISE 20.2

LEVEL-1

1. Find three numbers in G.P. whose sum is 65 and whose product is 3375.
2. Find three numbers in G.P. whose sum is 38 and their product is 1728.
3. The sum of first three terms of a G.P. is 13/12 and their product is – 1. Find the G.P.
4. The product of three numbers in G.P. is 125 and the sum of their products taken in pairs is 87½. Find them.
5. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms. [NCERT]
6. The sum of three numbers in G.P. is 14. If the first two terms are each increased by 1 and the third term decreased by 1, the resulting numbers are in A.P. Find the numbers.
7. The product of three numbers in G.P. is 216. If 2, 8, 6 be added to them, the results are in A.P. Find the numbers.
8. Find three numbers in G.P. whose product is 729 and the sum of their products in pairs is 819.

LEVEL-2

9. The sum of three numbers in G.P. is 21 and the sum of their squares is 189. Find the numbers. [NCERT]

ANSWERS

1. 45, 15, 5 or 5, 15, 45
2. 8, 12, 18
3. $\frac{4}{3}, -1, \frac{3}{4}, \ldots$ or $\frac{3}{4}, -1, \frac{4}{3}, \ldots$
4. 10, 5, $\frac{5}{2}$ or $\frac{5}{2}$, 5, 10

5. $\frac{2}{5}, 1, \frac{5}{2}$. 6. 2, 4, 8 or 8, 4, 2 7. 18, 6, 2 or 2, 6, 18

8. 1, 9, 81 or 81, 9, 9. 3, 6, 12

HINTS TO NCERT & SELECTED PROBLEMS

3. Let the terms of the G.P. be $\frac{a}{r}$, a, ar. It is given that

$$\frac{a}{r} + a + ar = \frac{13}{12} \text{ and } \frac{a}{r} \times a \times ar = -1$$

$$\Rightarrow \quad a\left(\frac{r^2+r+1}{r}\right) = \frac{13}{12} \text{ and } a^3 = -1$$

$$\Rightarrow \quad a = -1 \text{ and } a(r^2+r+1) = \frac{13r}{12}$$

$$\Rightarrow \quad r^2 + r + 1 = -\frac{13r}{12}$$

$$\Rightarrow \quad 12r^2 + 25r + 12 = 0$$

$$\Rightarrow \quad 12r^2 + 16r + 9r + 12 = 0 \Rightarrow (3r+4)(4r+3) = 0 \Rightarrow r = -\frac{4}{3} \text{ or } -\frac{3}{4}$$

Hence, three numbers are $\frac{3}{4}, -1, \frac{4}{3}$ or, $\frac{4}{3}, -1, \frac{3}{4}$.

5. Let the terms of the G.P. be $\frac{a}{r}$, a, ar. It is given that

$$\frac{a}{r} + a + ar = \frac{39}{10} \text{ and } \frac{a}{r} \times a \times ar = 1$$

$$\Rightarrow \quad a\left(\frac{r^2+r+1}{r}\right) = \frac{39}{10} \text{ and } a^3 = 1$$

$$\Rightarrow \quad a = 1 \text{ and } a(r^2+r+1) = \frac{39r}{10}$$

$$\Rightarrow \quad 10(r^2+r+1) = 39r$$

$$\Rightarrow \quad 10r^2 - 29r + 10 = 0$$

$$\Rightarrow \quad (2r-5)(5r-2) = 0$$

$$\Rightarrow \quad r = \frac{5}{2} \text{ or } r = \frac{2}{5}$$

Hence, the numbers are $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$

20.4 SUM OF THE TERMS OF A G.P.

THEOREM *Prove that the sum of n terms of a G.P. with first term 'a' and common ratio 'r' is given by*

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right) \quad \text{or,} \quad S_n = a\left(\frac{1 - r^n}{1 - r}\right), r \neq 1$$

<u>PROOF</u> Let S_n denote the sum of n terms of the G.P. with first term 'a' and common ratio r. Then,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \quad \ldots\text{(i)}$$

Multiplying both sides by r, we get

$$r\,S_n = ar + ar^2 + \ldots + ar^{n-1} + a\,r^n \qquad \ldots\text{(ii)}$$

On subtracting (ii) from (i), we get

$$S_n - r\,S_n = a - a\,r^n$$

$$\Rightarrow \quad S_n(1-r) = a(1-r^n)$$

$$\Rightarrow \quad S_n = a\left(\frac{1-r^n}{1-r}\right), \text{ provided that } r \neq 1$$

$$\text{or,} \quad S_n = a\left(\frac{r^n-1}{r-1}\right)$$

$$\text{Hence,}\quad S_n = a\left(\frac{1-r^n}{1-r}\right) \quad \text{or,} \quad S_n = a\left(\frac{r^n-1}{r-1}\right), r \neq 1$$

Q.E.D

NOTE *Some authors state two different formulas for S_n viz.,*

$$S_n = a\left(\frac{1-r^n}{1-r}\right) \text{ for } r<1 \quad \text{and} \quad S_n = a\left(\frac{r^n-1}{r-1}\right) \text{ for } r>1.$$

In fact these two are exactly identical. The only thing which must be noted is that the above formulae do not hold for $r = 1$. For $r = 1$, the sum of n terms of the G.P. is

$$S_n = a + a + a + \ldots + a \text{ (n times)} = na$$

REMARK 1 *If l is the last term of the G.P., then $l = a\,r^{n-1}$.*

$$\therefore \quad S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a-ar^n}{1-r} = \frac{a-(a\,r^{n-1})\,r}{1-r} = \frac{a-lr}{1-r}$$

$$\text{Thus,}\quad S_n = \frac{a-lr}{1-r} \quad \text{or,} \quad S_n = \frac{lr-a}{r-1}, \; r \neq 1$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **FINDING THE SUM OF GIVEN NUMBER OF TERMS OF A GIVEN G.P.**

EXAMPLE 1 *Find the sum of 7 terms of the G.P. 3, 6, 12, ...*

SOLUTION Here, $a = 3, r = 2$ and $n = 7$.

$$\therefore \quad S_7 = a\left(\frac{r^7-1}{r-1}\right) = 3\left(\frac{2^7-1}{2-1}\right) = 3(128-1) = 381$$

EXAMPLE 2 *Find the sum of 10 terms of the G.P.* $1, 1/2, 1/4, 1/8 \ldots$

SOLUTION Here, $a = 1, r = 1/2$ and $n = 10$.

$$\therefore \quad S_{10} = a\left(\frac{r^{10}-1}{r-1}\right)$$

$$\Rightarrow \quad S_{10} = 1\left\{\frac{(1/2)^{10}-1}{(1/2)-1}\right\} = 2\left(1-\frac{1}{2^{10}}\right) = 2\left(\frac{2^{10}-1}{2^{10}}\right) = \frac{(1024-1)}{512} = \frac{1023}{512}$$

EXAMPLE 3 *Find the sum to 7 terms of the sequence*

$$\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right), \left(\frac{1}{5^4}+\frac{2}{5^5}+\frac{3}{5^6}\right), \left(\frac{1}{5^7}+\frac{2}{5^8}+\frac{3}{5^9}\right), \ldots$$

SOLUTION The given sequence is

$$\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right), \frac{1}{5^3}\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right), \frac{1}{5^6}\left(\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}\right), \ldots$$

Clearly, this is a G.P. with first term $a = \frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3} = \frac{38}{125}$ and common ratio $r = \frac{1}{5^3}$.

$$\therefore \quad S_7 = a\left(\frac{1-r^7}{1-r}\right) \Rightarrow S_7 = \frac{38}{125}\left\{\frac{1-(1/5^3)^7}{1-(1/5^3)}\right\} = \frac{38}{125}\left\{\frac{1-1/5^{21}}{1-\frac{1}{125}}\right\} = \frac{19}{62}\left(1-\frac{1}{5^{21}}\right)$$

EXAMPLE 4 *Sum the series:* $x(x+y) + x^2(x^2+y^2) + x^3(x^3+y^3) + \ldots$ *to n terms*

SOLUTION Let S_n denote the sum to n terms of the given series. Then,

$$S_n = x(x+y) + x^2(x^2+y^2) + x^3(x^3+y^3) + \ldots + x^n(x^n+y^n)$$

$$\Rightarrow \quad S_n = (x^2+x^4+x^6+\ldots+x^{2n}) + (xy + x^2y^2 + x^3y^3 + \ldots + x^ny^n)$$

$$\Rightarrow \quad S_n = x^2\left\{\frac{(x^2)^n-1}{x^2-1}\right\} + xy\left\{\frac{(xy)^n-1}{xy-1}\right\}$$

$$\Rightarrow \quad S_n = x^2\left\{\frac{x^{2n}-1}{x^2-1}\right\} + xy\left\{\frac{(xy)^n-1}{xy-1}\right\}$$

EXAMPLE 5 *Find the sum of the series* $2+6+18+\ldots+4374$.

SOLUTION The given series is a geometric series in which $a = 2$, $r = 3$ and $l = 4374$.

$$\therefore \quad \text{Required sum} = \frac{(lr-a)}{(r-1)} = \frac{4374\times 3-2}{3-1} = 6560.$$

EXAMPLE 6 *Find the sum of the following series:*

(i) $5+55+555+\ldots$ to n terms (ii) $0.7+0.77+0.777+\ldots$ to n terms

SOLUTION (i) Let S be the sum of the series $5+55+555+\ldots$ to n terms. Then,

$$S = 5\left\{1+11+111+\ldots+\text{to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S = \frac{5}{9}\left\{9+99+999+\ldots+\text{to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S = \frac{5}{9}\left\{(10-1)+(10^2-1)+(10^3-1)+\ldots+(10^n-1)\right\}$$

$$\Rightarrow \quad S = \frac{5}{9}\left\{(10+10^2+10^3+\ldots+10^n) - \underbrace{(1+1+1+\ldots+1)}_{n \text{ times}}\right\}$$

$$\Rightarrow \quad S = \frac{5}{9}\left\{10\times\frac{(10^n-1)}{10-1} - n\right\} = \frac{5}{9}\left\{\frac{10}{9}(10^n-1)-n\right\} = \frac{5}{81}\left\{10^{n+1}-10-9n\right\}$$

(ii) Let S be the sum $0.7 + 0.77 + 0.777 + \ldots$ to n terms. Then,

$$S = 7 \times 0.1 + 7 \times 0.11 + 7 \times 0.111 + \ldots \text{ to } n \text{ terms}$$

$$\Rightarrow \quad S = 7\left\{0.1 + 0.11 + 0.111 + \ldots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{0.9 + 0.99 + 0.999 + \ldots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{100}\right) + \left(1 - \frac{1}{1000}\right) + \ldots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \ldots + \left(1 - \frac{1}{10^n}\right)\right\}$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{n - \left(\frac{1}{10} + \frac{1}{10^2} + \ldots + \frac{1}{10^n}\right)\right\} = \frac{7}{9}\left[n - \frac{1}{10}\frac{\left\{1 - \left(\frac{1}{10}\right)^n\right\}}{\left(1 - \frac{1}{10}\right)}\right]$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right\} = \frac{7}{81}\left\{9n - 1 + \frac{1}{10^n}\right\}$$

EXAMPLE 7 *The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Find the sum of n terms of the G.P.* [NCERT]

SOLUTION Let a be the first term and r the common ratio of the G.P. It is given that

$$a + ar + ar^2 = 16 \quad \ldots\text{(i)} \quad \text{and,} \quad ar^3 + ar^4 + ar^5 = 128 \quad \ldots\text{(ii)}$$

$$\Rightarrow \quad a(1 + r + r^2) = 16 \quad \text{and,} \quad ar^3(1 + r + r^2) = 128$$

$$\Rightarrow \quad \frac{ar^3(1 + r + r^2)}{a(1 + r + r^2)} = \frac{128}{16} \Rightarrow r^3 = 8 \Rightarrow r = 2$$

Putting $r = 2$ in (i), we get: $a = \frac{16}{7}$.

$$\therefore \quad S_n = a\left(\frac{r^n - 1}{r - 1}\right) = \frac{16}{7}\left(\frac{2^n - 1}{2 - 1}\right) = \frac{16}{7}(2^n - 1)$$

EXAMPLE 8 *Find a G.P. for which the sum of the first two terms is -4 and the fifth term is 4 times the third term.* [NCERT]

SOLUTION Let a be the first term and r be the common ratio of the G.P.

We have,

$$a_1 + a_2 = -4 \quad \text{and} \quad a_5 = 4a_3$$

$$\Rightarrow \quad a + ar = -4 \quad \text{and} \quad ar^4 = 4ar^2$$

$$\Rightarrow \quad a(1 + r) = -4 \quad \text{and} \quad r^2 = 4 \Rightarrow a(1 + r) = -4 \quad \text{and} \quad r = \pm 2$$

When $r = 2$,

$$a(1 + r) = -4 \Rightarrow a = -\frac{4}{3}$$

When $r = -2$,

$$a(1+r) = -4 \Rightarrow a = 4$$

Hence, required G.P. is $-\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, \ldots$ or, $4, -8, 16, \ldots$

Type II **FINDING VALUE(S) OF n, r AND a WHEN THE SUM OF n TERMS OF A G.P. IS GIVEN**

EXAMPLE 9 *Determine the number of terms in G.P. $< a_n >$, if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.*

SOLUTION Let r be the common ratio of the given G.P. Then,

$$a_n = 96 \Rightarrow a_1 r^{n-1} = 96 \Rightarrow 3r^{n-1} = 96 \Rightarrow r^{n-1} = 32 \quad \ldots\text{(i)}$$

Now, $S_n = 189$

$$\Rightarrow \quad a_1\left(\frac{r^n - 1}{r-1}\right) = 189$$

$$\Rightarrow \quad 3\left\{\frac{(r^{n-1})\, r - 1}{r-1}\right\} = 189$$

$$\Rightarrow \quad 3\left(\frac{32r-1}{r-1}\right) = 189 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad 32r - 1 = 63r - 63 \Rightarrow 31r = 62 \Rightarrow r = 2$$

Putting $r = 2$ in (i), we get

$$2^{n-1} = 32 \Rightarrow 2^{n-1} = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6.$$

EXAMPLE 10 *How many terms of the geometric series $1 + 4 + 16 + 64 + \ldots$ will make the sum 5461 ?*

SOLUTION Let the sum of n terms of the given series 5461.

Here, $a = 1, r = 4$ and $S_n = 5461$.

$\therefore \quad S_n = 5461$

$$\Rightarrow \quad a\left(\frac{r^n - 1}{r-1}\right) = 5461$$

$$\Rightarrow \quad \frac{4^n - 1}{4-1} = 5461 \quad [\because a = 1 \text{ and } r = 4]$$

$$\Rightarrow \quad 4^n - 1 = 16383 \Rightarrow 4^n = 16384 \Rightarrow 4^n = 4^7 \Rightarrow n = 7$$

EXAMPLE 11 *The sum of some terms of a G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.* [NCERT]

SOLUTION Let there be n terms in the G.P. with first term $a = 5$ and common ratio $r = 2$. Then,

Sum of n terms $= 315$

$$\Rightarrow \quad a\left(\frac{r^n - 1}{r-1}\right) = 315$$

$$\Rightarrow \quad 5\left(\frac{2^n - 1}{2-1}\right) = 315$$

$$\Rightarrow \quad 2^n - 1 = 63 \Rightarrow 2^n = 64 = 2 \Rightarrow n = 6 \quad [\because a = 5 \text{ and } r = 2]$$

$$\therefore \quad \text{Last term} = ar^{n-1} = 5 \times 2^{6-1} = 160$$

EXAMPLE 12 *In an increasing G.P., the sum of the first and the last term is 66, the product of the second and the last but one is 128 and the sum of the terms is 126. How many terms are there in the progression?*

SOLUTION Let a be the first term and r the common ratio of the given G.P. Further, let there be n terms in the given G.P. It is given that the sum of the first and last term is 66.

i.e. $a_1 + a_n = 66$

$\Rightarrow \quad a + ar^{n-1} = 66$ ...(i)

It is also given that the product of second and the second last term is 128.

i.e. $a_2 \cdot a_{n-1} = 128 \Rightarrow ar \cdot ar^{n-2} = 128 \Rightarrow a^2 r^{n-1} = 128 \Rightarrow a \cdot (ar^{n-1}) = 128 \Rightarrow ar^{n-1} = \frac{128}{a}$

Putting this value of ar^{n-1} in (i), we get

$$a + \frac{128}{a} = 66 \Rightarrow a^2 - 66a + 128 = 0 \Rightarrow (a-2)(a-64) = 0 \Rightarrow a = 2, 64$$

Putting $a = 2$ in (i), we get

$$2 + 2 \cdot r^{n-1} = 66 \Rightarrow r^{n-1} = 32$$

Putting $a = 64$ in (i), we get

$$64 + 64r^{n-1} = 66 \Rightarrow r^{n-1} = \frac{1}{32}$$

We reject the second value as the G.P. is an increasing G.P. and therefore $r > 1$. Thus, we obtain $a = 2$ and $r^{n-1} = 32$.

Now, $S_n = 126$

$$\Rightarrow \quad 2\left(\frac{r^n - 1}{r - 1}\right) = 126$$

$$\Rightarrow \quad \frac{r^n - 1}{r - 1} = 63 \Rightarrow \frac{r^{n-1} r - 1}{r - 1} = 63 \Rightarrow \frac{32r - 1}{r - 1} = 63 \Rightarrow r = 2$$

$$\therefore \quad r^{n-1} = 32 \Rightarrow 2^{n-1} = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6$$

Hence, there are 6 terms in the progression.

EXAMPLE 13 *Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.* [NCERT]

SOLUTION If $a, ar, ar^2, \ldots$ and $A, AR, AR^2, \ldots$ are two geometric sequences, then the sequence having terms as the product of corresponding terms of the two sequences is also a geometric sequence with first term aA and common ratio rR.

Given sequences are geometric sequences with first terms 2 and 128 respectively and common ratios 2 and $\frac{1}{4}$ respectively. Therefore, the sequence formed by multiplying the corresponding terms of the given sequences is a G.P. with first term $a = 2 \times 128 = 256$ and common ratio $r = 2 \times \frac{1}{4} = \frac{1}{2}$.

Since each sequence contains 5 terms. Therefore, the sequence formed by the products of the corresponding terms has 5 terms.

$$\text{Hence, required sum} = 256\left\{\frac{1 - \left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}}\right\} = 256\left\{\frac{\left(1 - \frac{1}{32}\right)}{\frac{1}{2}}\right\} = 512\left(1 - \frac{1}{32}\right) = 512 \times \frac{31}{32} = 496$$

ALITER Required sum $= 2 \times 128 + 4 \times 32 + 8 \times 8 + 16 \times 2 + 32 \times \frac{1}{2}$

$$= 256 + 128 + 64 + 32 + 16$$

$$= 256 \left\{ \frac{1 - \left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}} \right\} = 512\left(1 - \frac{1}{32}\right) = 512 \times \frac{31}{32} = 496$$

Type III **ON PROVING RESULTS BASED UPON THE FORMULA FOR THE SUM OF n TERMS OF A G.P.**

EXAMPLE 14 *If S_1, S_2 and S_3 be respectively the sum of n, $2n$ and $3n$ terms of a G.P., prove that*

$$S_1 (S_3 - S_2) = (S_2 - S_1)^2$$

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$S_1 = a\left(\frac{r^n - 1}{r - 1}\right), S_2 = a\left(\frac{r^{2n} - 1}{r - 1}\right) \text{ and } S_3 = a\left(\frac{r^{3n} - 1}{r - 1}\right)$$

Now,

$$S_1 (S_3 - S_2) = a\left(\frac{r^n - 1}{r - 1}\right)\left\{a\left(\frac{r^{3n} - 1}{r - 1}\right) - a\left(\frac{r^{2n} - 1}{r - 1}\right)\right\}$$

$$\Rightarrow \quad S_1 (S_3 - S_2) = \frac{a^2}{(r-1)^2} (r^n - 1)\left\{(r^{3n} - 1) - (r^{2n} - 1)\right\}$$

$$\Rightarrow \quad S_1 (S_3 - S_2) = \frac{a^2}{(r-1)^2} (r^n - 1)(r^{3n} - r^{2n})$$

$$\Rightarrow \quad S_1 (S_3 - S_2) = \frac{a^2}{(r-1)^2} (r^n - 1)\, r^{2n} (r^n - 1)$$

$$\Rightarrow \quad S_1 (S_3 - S_2) = \left\{ar^n\left(\frac{r^n - 1}{r - 1}\right)\right\}^2$$

and, $$(S_2 - S_1)^2 = \left\{a\left(\frac{r^{2n} - 1}{r - 1}\right) - a\left(\frac{r^n - 1}{r - 1}\right)\right\}^2$$

$$\Rightarrow \quad (S_2 - S_1)^2 = \frac{a^2}{(r-1)^2}\left\{(r^{2n} - 1) - (r^n - 1)\right\}^2$$

$$\Rightarrow \quad (S_2 - S_1)^2 = \frac{a^2}{(r-1)^2}\left\{r^n (r^n - 1)\right\}^2 = \left\{a\, r^n\left(\frac{r^n - 1}{r - 1}\right)\right\}^2$$

Hence, $S_1 (S_3 - S_1) = (S_2 - S_1)^2$

EXAMPLE 15 *If S be the sum, P the product and R the sum of the reciprocals of n terms of a G.P., prove that* $\left(\frac{S}{R}\right)^n = P^2$. **[NCERT]**

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$S = a + ar + ar^2 + \dots + ar^{n-1} = a\left(\frac{r^n - 1}{r - 1}\right) \quad \text{...(i)}$$

$$P = a \cdot ar\, ar^2 \dots ar^{n-1} = a^n r^{1+2+3+\dots+(n-1)} = a^n r^{\frac{n(n-1)}{2}} \quad \text{...(ii)}$$

and,
$$R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \dots + \frac{1}{ar^{n-1}}$$

$$\Rightarrow \quad R = \frac{1}{a}\frac{\left\{(1/r)^n - 1\right\}}{\left\{(1/r) - 1\right\}} = \frac{1}{a}\left(\frac{1 - r^n}{1 - r}\right)\frac{1}{r^{n-1}}$$

$$\Rightarrow \quad R = \frac{1}{a}\left(\frac{r^n - 1}{r - 1}\right)\frac{1}{r^{n-1}} \quad \text{...(iii)}$$

$$\therefore \quad \frac{S}{R} = a\left(\frac{r^n - 1}{r - 1}\right) \cdot a\left(\frac{r - 1}{r^n - 1}\right) r^{n-1} = a^2 r^{n-1}$$

$$\Rightarrow \quad \left(\frac{S}{R}\right)^n = a^{2n} r^{n(n-1)} = \left\{a^n r^{\frac{n(n-1)}{2}}\right\}^2 = P^2 \quad \text{[Using (ii)]}$$

Hence, $\left(\frac{S}{R}\right)^n = P^2$

EXAMPLE 16 *A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8th set of letter is mailed* [NCERT]

SOLUTION Amount spent on mailing one letter = ₹ $\frac{1}{2}$

$\therefore$ Amount spent when first set of 4 letters is mailed = ₹ 2

Amount spent when second set of $4 \times 4 = 16$ letters is mailed = ₹ $(2 \times 4) = 8$

Amount spent when third set of $4 \times 4 \times 4 = 64$ letters is mailed = ₹ $(8 \times 4) = 32$

Clearly, 2, 8, 32, ... is a G.P. with first term 2 and common ratio 4.

$\therefore$ Total amount spent when 8th set of letters is mailed = Sum of 8 terms of the G.P.

$$= a\left(\frac{r^8 - 1}{r - 1}\right)$$

$$= ₹\left\{2\left(\frac{4^8 - 1}{4 - 1}\right)\right\} \quad [\because a = ₹\ 2 \text{ and } r = 4]$$

$$= ₹\left\{2 \times \left(\frac{65536 - 1}{3}\right)\right\} = ₹\ (2 \times 21845)$$

$$= ₹\ 43690$$

LEVEL-2

EXAMPLE 17 *Find the sum to n terms of the sequence*

$$\left(x+\frac{1}{x}\right)^2, \left(x^2+\frac{1}{x^2}\right)^2, \left(x^3+\frac{1}{x^3}\right)^2, \ldots$$

SOLUTION Let S_n denote the sum to n terms of the given sequence. Then,

$$S_n = \left(x+\frac{1}{x}\right)^2 + \left(x^2+\frac{1}{x^2}\right)^2 + \left(x^3+\frac{1}{x^3}\right)^2 + \ldots + \left(x^n+\frac{1}{x^n}\right)^2$$

$$\Rightarrow \quad S_n = \left(x^2+\frac{1}{x^2}+2\right) + \left(x^4+\frac{1}{x^4}+2\right) + \left(x^6+\frac{1}{x^6}+2\right) + \ldots + \left(x^{2n}+\frac{1}{x^{2n}}+2\right)$$

$$\Rightarrow \quad S_n = (x^2+x^4+x^6+\ldots+x^{2n}) + \left(\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+\ldots+\frac{1}{x^{2n}}\right) + \underbrace{(2+2+\ldots)}_{n \text{ times}}$$

$$\Rightarrow \quad S_n = x^2\left\{\frac{(x^2)^n-1}{x^2-1}\right\} + \frac{1}{x^2}\left\{\frac{(1/x^2)^n-1}{(1/x^2)-1}\right\} + 2n$$

$$\Rightarrow \quad S_n = x^2\left(\frac{x^{2n}-1}{x^2-1}\right) + \frac{1}{x^{2n}}\left(\frac{1-x^{2n}}{1-x^2}\right) + 2n$$

$$\Rightarrow \quad S_n = x^2\left(\frac{x^{2n}-1}{x^2-1}\right) + \frac{1}{x^{2n}}\left(\frac{x^{2n}-1}{x^2-1}\right) + 2n$$

$$\Rightarrow \quad S_n = \left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right) + 2n$$

EXAMPLE 18 *Find the sum to n terms of the sequence given by* $a_n = 2^n + 3n, n \in N$.

SOLUTION Let S_n denote the sum to terms of the given sequence. Then,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$

$$\Rightarrow \quad S_n = (2^1 + 3\times 1) + (2^2 + 3\times 2) + (2^3 + 3\times 3) + \ldots + (2^n + 3\times n)$$

$$\Rightarrow \quad S_n = (2^1 + 2^2 + 2^3 + \ldots + 2^n) + (3\times 1 + 3\times 2 + 3\times 3 + \ldots + 3\times n)$$

$$\Rightarrow \quad S_n = (2^1 + 2^2 + 2^3 + \ldots + 2^n) + 3(1+2+3+\ldots+n)$$

$$\Rightarrow \quad S_n = 2\left(\frac{2^n-1}{2-1}\right) + 3\left\{\frac{n}{2}(1+n)\right\} = 2(2^n-1) + \frac{3n}{2}(n+1)$$

EXAMPLE 19 *Prove that the sum to n terms of the series:* $11 + 103 + 1005 + \ldots$ *is* $\frac{10}{9}(10^n-1)+n^2$.

SOLUTION Let S_n denote the sum to n terms of the given series. Then,

$$S_n = 11 + 103 + 1005 + \ldots \text{ to } n \text{ terms}$$

$$\Rightarrow \quad S_n = (10+1) + (10^2+3) + (10^3+5) + \ldots + \{10^n + (2n-1)\}$$

$$\Rightarrow \quad S_n = (10+10^2+\ldots+10^n) + \{1+3+5+\ldots+(2n-1)\}$$

$$\Rightarrow \quad S_n = \frac{10(10^n-1)}{(10-1)} + \frac{n}{2}(1+2n-1) = \frac{10}{9}(10^n-1)+n^2$$

EXAMPLE 20 *Find the least value of n for which the sum $1+3+3^2+\ldots$ to n terms is greater than 7000.*

SOLUTION We have,

$$S_n = 1+3+3^2+\ldots \text{ to } n \text{ terms}$$

$$\Rightarrow \quad S_n = 1\times\left(\frac{3^n-1}{3-1}\right) = \frac{3^n-1}{2}$$

Now, $S_n > 7000$

$$\Rightarrow \quad \frac{3^n-1}{2} > 7000$$

$$\Rightarrow \quad 3^n - 1 > 14000$$

$$\Rightarrow \quad 3^n > 14001 \Rightarrow n\log 3 > \log 14001 \Rightarrow n > \frac{\log 14001}{\log 3} \Rightarrow n > \frac{4.1461}{0.4771} = 8.69$$

Hence, the least value of n is 9.

EXAMPLE 21 *If f is a function satisfying $f(x+y) = f(x)\,f(y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum\limits_{x=1}^{n} f(x) = 120$, find the value of n.* [NCERT]

SOLUTION We have,

$$f(x+y) = f(x)\,f(y) \text{ for all } x, y \in N$$

$$\therefore \quad f(x) = f(\underbrace{1+1+1+\ldots+1}_{x-\text{times}}) = \underbrace{f(1)\,f(1)\,f(1)\ldots f(1)}_{x\text{-times}} = [f(1)]^x \text{ for all } x \in N$$

$$\Rightarrow \quad f(x) = 3^x \text{ for all } x \in N \qquad [\because f(1)=3]$$

Now,

$$\sum_{x=1}^{n} f(x) = 120$$

$$\Rightarrow \quad \sum_{x=1}^{n} 3^x = 120$$

$$\Rightarrow \quad 3+3^2+\ldots+3^n = 120$$

$$\Rightarrow \quad 3\left(\frac{3^n-1}{3-1}\right) = 120 \Rightarrow 3^n - 1 = 80 \Rightarrow 3^n = 81 \Rightarrow 3^n = 3^4 \Rightarrow n = 4.$$

EXAMPLE 22 *Find the natural number a for which $\sum\limits_{k=1}^{n} f(a+k) = 16\,(2^n-1)$, where the function f statisfies $f(x+y) = f(x)\cdot f(y)$ for all natural numbers x, y and further $f(1) = 2$.* [NCERT]

SOLUTION Proceding as in Example 20, we obtain $f(x) = (f(1))^x = 2^x$ for all $x \in N$.

$$\therefore \quad \sum_{k=1}^{n} f(a+k) = 16\,(2^n-1)$$

$$\Rightarrow \quad \sum_{k=1}^{n} 2^{a+k} = 16\,(2^n-1)$$

$$\Rightarrow \quad 2^a\left(\sum_{k=1}^{n} 2^k\right) = 16\,(2^n-1)$$

$$\Rightarrow \quad 2^a(2+2^2+2^3+\ldots+2^n) = 16\,(2^n-1)$$

$$\Rightarrow \quad 2^a\left\{2\left(\frac{2^n-1}{2-1}\right)\right\} = 16\,(2^n-1)$$

$$\Rightarrow \quad 2^{a+1}(2^n-1) = 16\,(2^n-1)$$

$$\Rightarrow \quad 2^{a+1} = 2^4$$

$$\Rightarrow \quad a+1 = 4 \Rightarrow a = 3.$$

EXERCISE 20.3

LEVEL-1

1. Find the sum of the following geometric progressions:
   (i) 2, 6, 18, ... to 7 terms (ii) 1, 3, 9, 27, ... to 8 terms
   (iii) $1, -1/2, 1/4, -1/8, \ldots$ (iv) $(a^2-b^2), (a-b), \left(\frac{a-b}{a+b}\right)$, ... to n terms
   (v) 4, 2, 1, 1/2 ... to 10 terms.
2. Find the sum of the following geometric series:
   (i) 0.15 + 0.015 + 0.0015 + ... to 8 terms;
   (ii) $\sqrt{2} + \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} + \ldots$ to 8 terms;
   (iii) $\frac{2}{9} - \frac{1}{3} + \frac{1}{2} - \frac{3}{4} + \ldots$ to 5 terms;
   (iv) $(x+y) + (x^2+xy+y^2) + (x^3+x^2y+xy^2+y^3) + \ldots$ to n terms;
   (v) $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \ldots$ to $2n$ terms;
   (vi) $\frac{a}{1+i} + \frac{a}{(1+i)^2} + \frac{a}{(1+i)^3} + \ldots + \frac{a}{(1+i)^n}$.
   (vii) $1, -a, a^2, -a^3, \ldots$ to n terms $(a \neq 1)$ [NCERT]
   (viii) $x^3, x^5, x^7, \ldots$ to n terms [NCERT]
   (ix) $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$ to n terms [NCERT]
3. Evaluate the following:
   (i) $\sum_{n=1}^{11} (2+3^n)$ [NCERT] (ii) $\sum_{k=1}^{n} (2^k + 3^{k-1})$ (iii) $\sum_{n=2}^{10} 4^n$
4. Find the sum of the following series:
   (i) 5 + 55 + 555 + ... to n terms. [NCERT]
   (ii) 7 + 77 + 777 + ... to n terms. [NCERT]
   (iii) 9 + 99 + 999 + ... to n terms.
   (iv) 0.5 + 0.55 + 0.555 + ... to n terms.
   (v) 0.6 + 0.66 + 0.666 + ... to n terms. [NCERT]
5. How many terms of the G.P. 3, 3/2, 3/4, ... be taken together to make $\frac{3069}{512}$?
6. How many terms of the series 2 + 6 + 18 + ... must be taken to make the sum equal to 728?
7. How many terms of the sequence $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ must be taken to make the sum $39 + 13\sqrt{3}$?
8. The sum of n terms of the G.P. 3, 6, 12, ... is 381. Find the value of n.
9. The common ratio of a G.P. is 3 and the last term is 486. If the sum of these terms be 728, find the first term.

10. The ratio of the sum of first three terms is to that of first 6 terms of a G.P. is 125 : 152. Find the common ratio.

11. The 4th and 7th terms of a G.P. are $\frac{1}{27}$ and $\frac{1}{729}$ respectively. Find the sum of n terms of the G.P.

12. Find the sum : $\sum_{n=1}^{10}\left\{\left(\frac{1}{2}\right)^{n-1}+\left(\frac{1}{5}\right)^{n+1}\right\}$.

13. The fifth term of a G.P. is 81 whereas its second term is 24. Find the series and sum of its first eight terms.

14. If S_1, S_2, S_3 be respectively the sums of n, $2n$, $3n$ terms of a G.P., then prove that $S_1^2+S_2^2 = S_1(S_2+S_3)$.

15. Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$. [NCERT]

16. If a and b are the roots of $x^2-3x+p=0$ and c, d are the roots $x^2-12x+q=0$, where a, b, c, d form a G.P. Prove that $(q+p):(q-p) = 17:15$. [NCERT]

17. How many terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}$ .....are needed to give the sum $\frac{3069}{512}$? [NCERT]

18. A person has 2 parents, 4 grandparents, 8 great grand parents, and so on. Find the number of his ancestors during the ten generations preceding his own. [NCERT]

LEVEL-2

19. If $S_1, S_2, \ldots, S_n$ are the sums of n terms of n G.P.'s whose first term is 1 in each and common ratios are 1, 2, 3, ... , n respectively, then prove that

$$S_1+S_2+2S_3+3S_4+\ldots(n-1)S_n=1^n+2^n+3^n+\ldots+n^n.$$

20. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of the terms occupying the odd places. Find the common ratio of the G.P. [NCERT]

21. Let a_n be the nth term of the G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n}=\alpha$ and $\sum_{n=1}^{100} a_{2n-1}=\beta$, such that $\alpha \neq \beta$. Prove that the common ratio of the G.P. is α/β.

22. Find the sum of $2n$ terms of the series whose every even term is 'a' times the term before it and every odd term is 'c' times the term before it, the first term being unity.

ANSWERS

1. (i) 2186 (ii) 3280 (iii) $\frac{171}{256}$ (iv) $\frac{a-b}{(a+b)^{n-2}}\left\{\frac{(a+b)^n-1}{(a+b)-1}\right\}$ (v) $8\left(1-\frac{1}{1024}\right)$

2. (i) $\frac{1}{6}\left(1-\frac{1}{10^8}\right)$ (ii) $\frac{255\sqrt{2}}{128}$ (iii) $\frac{55}{72}$ (iv) $\frac{1}{x-y}\left\{x^2\left(\frac{x^n-1}{x-1}\right)-y^2\left(\frac{y^n-1}{y-1}\right)\right\}$

(v) $\frac{19}{24}\left(1-\frac{1}{5^{2n}}\right)$ (vi) $-ai\{1-(1+i)^{-n}\}$ (vii) $\frac{1-(-a)^n}{1+a}$ (viii) $x^3\frac{(x^{2n}-1)}{x^2-1}$

(xi) $\sqrt{7}\left(\dfrac{3^{n/2}-1}{\sqrt{3}-1}\right)$ 3. (i) 265741 (ii) $\frac{1}{2}(2^{n+2}+3^n-5)$ (iii) $\frac{16}{3}(4^9-1)$

4. (i) $\frac{5}{81}[10^{n+1}-9n-10]$ (ii) $\frac{7}{81}[10^{n+1}-9n-10]$ (iii) $\frac{1}{9}[10^{n+1}-9n-10]$

(iv) $\frac{5}{9}\left\{n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right\}$ (v) $\frac{6}{9}\left\{n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right\}$ 5. 10 6. 6

7. 6 8. 7 9. 2 10. $\frac{3}{5}$ 11. $\frac{3}{2}\left(1-\frac{1}{3^n}\right)$

12. $\dfrac{2^{10}-1}{2^9}+\dfrac{5^{10}-1}{4\times 5^{11}}$ 13. $a=16,\ r=\frac{3}{2},\ S_8=\frac{65\times 97}{8}$ 17. 10

18. 2046 20. 4 22. $(a+1)\left\{\dfrac{(ac)^n-1}{ac-1}\right\}$

HINTS TO NCERT & SELECTED PROBLEMS

2. (vii) Let S_n denote the sum of n terms of the G.P. $1-a, a^2, -a^3, \ldots$ Then,

$$S_n = 1\left\{\frac{(-a)^n-1}{-a-1}\right\} = \frac{1-(-1)^n a^n}{1+a}$$

(viii) Let S_n be the sum of n terms of the G.P. $x^3, x^5, x^7, \ldots$ Then,

$$S_n = x^3\left\{\frac{(x^2)^n-1}{x^2-1}\right\} = x^3\left(\frac{x^{2n}-1}{x^2-1}\right).$$

(ix) Let S_n denote the sum of n terms of the G.P. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$. Then,

$$S_n = \sqrt{7}\left\{\frac{(\sqrt{3})^n-1}{\sqrt{3}-1}\right\} = \sqrt{7}\left(\frac{3^{n/2}-1}{3^{1/2}-1}\right)$$

3. (i) $\displaystyle\sum_{n=1}^{11}(2+3^n)=\sum_{n=1}^{11}2+\sum_{n=1}^{11}3^n = 2\times 11+3\left(\frac{3^{11}-1}{3-1}\right)=22+\frac{3}{2}(3^{11}-1)=265741.$

4. (i) Let $S_n = 5+55+555+\ldots$ to n terms. Then,

$S_n = 5(1+11+111+\ldots$ to n terms)

$\Rightarrow S_n = \frac{5}{9}(9+99+999+\ldots$ to n terms)

$$\Rightarrow S_n = \frac{5}{9}\left\{(10-1)+(10^2-1)+(10^3-1)+\ldots+(10^n-1)\right\}$$

$$\Rightarrow S_n = \frac{5}{9}\left\{(10+10^2+10^3+\ldots+10^n)-n\right\}$$

$$\Rightarrow S_n = \frac{5}{9}\left\{10\left(\frac{10^n-1}{10-1}\right)-n\right\}$$

$$\Rightarrow S_n = \frac{5}{9}\left\{\frac{10}{9}(10^n-1)-n\right\} = \frac{5}{81}(10^{n+1}-9n-10).$$

(ii) Proceed as in (i)

(v) Let $S_n = 0.6 + 0.66 + 0.666 + \dots$ to n terms. Then,

$$S_n = 0.6 + 0.66 + 0.666 + \dots \text{ to } n \text{ terms}$$

$$\Rightarrow \quad S_n = \frac{6}{9}(0.1 + 0.11 + 0.111 + \dots \text{ to } n \text{ terms})$$

$$\Rightarrow \quad S_n = \frac{6}{9}\left\{0.9 + 0.99 + 0.999 + \dots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S_n = \frac{6}{9}\left\{\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\dots+\left(1-\frac{1}{10^n}\right)\right\}$$

$$\Rightarrow \quad S_n = \frac{6}{9}\left\{n-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\dots+\frac{1}{10^n}\right)\right\}$$

$$\Rightarrow \quad S_n = \frac{6}{9}\left\{n-\frac{1}{10}\frac{\left(1-\left(\frac{1}{10}\right)^n\right)}{1-\frac{1}{10}}\right\} = \frac{6}{9}\left\{n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right\}$$

15. Required ratio $= \dfrac{a_1 + a_2 + \dots + a_n}{a_{n+1} + a_{n+2} + \dots + a_{2n}} = \dfrac{a + ar + \dots + ar^{n-1}}{ar^n + ar^{n+1} + \dots + ar^{2n-1}}$

$$= \frac{a\left(\frac{1-r^n}{1-r}\right)}{ar^n\left(\frac{1-r^n}{1-r}\right)} = \frac{1}{r^n}$$

16. We have, $a+b=3$, $ab=p$, $c+d=12$ and $cd=q$ Let $b=ar$, $c=ar^2$ and $d=ar^3$. Then,

$$a+b=3 \text{ and } c+d=12$$

$$\Rightarrow \quad a(1+r)=3 \text{ and } ar^2(1+r)=12$$

$$\Rightarrow \quad \frac{ar^2(1+r)}{a(1+r)} = \frac{12}{3} \Rightarrow r=2$$

$$\therefore \quad a(1+r)=3 \Rightarrow a=1$$

Now, $p = ab = a.ar = 2$, $q = cd = ar^2 \times ar^3 = 2^5 = 32$

$$\therefore \quad \frac{q+p}{q-p} = \frac{32+2}{32-2} = \frac{34}{30} = \frac{17}{15}$$

17. Let the sum of n terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}, \dots$ be $\frac{3069}{512}$. Then,

$$3\left\{\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right\} = \frac{3069}{512} \Rightarrow 1-\frac{1}{2^n} = \frac{1023}{1024} \Rightarrow \frac{1}{2^n} = \frac{1}{2^{10}} \Rightarrow n=10$$

Hence, the sum of 10 terms of the given G.P. is $\frac{3069}{512}$.

18. Number of ancestors during the ten generations preceding his own generation

$= $ Sum of 10 terms of the G.P. 2, 4, 8, ...

$$= 2\left(\frac{2^{10}-1}{2-1}\right) = 2046.$$

20. Let there be $2n$ terms in the G.P. with first term a and common ratio r. Then,

Sum of all the terms $= 5$ (Sum of the terms occupying the odd places)

$$\Rightarrow \quad a_1 + a_2 + \ldots + a_{2n} = 5\,(a_1 + a_3 + a_5 + \ldots + a_{2n-1})$$

$$\Rightarrow \quad a + ar + \ldots + ar^{2n-1} = 5\,(a + ar^2 + \ldots + ar^{2n-2})$$

$$\Rightarrow \quad a\left\{\frac{1-r^{2n}}{1-r}\right\} = 5a\left\{\frac{1-(r^2)^n}{1-r^2}\right\} \Rightarrow 1 + r = 5 \Rightarrow r = 4$$

21. Let a be the first term and r be the common ratio of the G.P. Then,

$$\sum_{n=1}^{100} a_{2n} = \alpha \text{ and } \sum_{n=1}^{100} a_{2n-1} = \beta$$

$$\Rightarrow \quad a_2 + a_4 + \ldots + a_{200} = \alpha \text{ and } a_1 + a_3 + \ldots + a_{199} = \beta$$

$$\Rightarrow \quad ar + ar^3 + \ldots + ar^{199} = \alpha \text{ and } a + ar^2 + \ldots + ar^{198} = \beta$$

$$\Rightarrow \quad ar\left\{\frac{1-(r^2)^{100}}{1-r^2}\right\} = \alpha, \text{ and } a\left\{\frac{1-(r^2)^{100}}{1-r^2}\right\} = \beta$$

$$\Rightarrow \quad ar\left(\frac{1-r^{200}}{1-r^2}\right) = \alpha, \text{ and } a\left(\frac{1-r^{200}}{1-r^2}\right) = \beta$$

$$\Rightarrow \quad r = \frac{\alpha}{\beta}$$

22. Let $a_1 + a_2 + a_3, \ldots + a_{2n}$ be the given series. It is given that

$$a_1 = 1,\ a_2 = a\,a_1,\ a_3 = ca_2,\ a_4 = aa_3,\ a_5 = ca_4 \text{ and so on.}$$

$$\Rightarrow \quad a_1 = 1,\ a_2 = a,\ a_3 = ac,\ a_4 = a^2 c,\ a_5 = a^2 c^2,\ a_6 = a^3 c^2, \ldots$$

$\therefore$ Required sum $= a_1 + a_2 + a_3 + \ldots + a_{2n}$

$$= 1 + a + ac + a^2 c + a^2 c^2 + \ldots \text{ to } 2n \text{ term}$$

$$= (1 + a) + ac\,(1 + a) + a^2 c^2\,(1 + a) + \ldots \text{ to } n \text{ terms}$$

$$= (1 + a)\left\{\frac{1-(ac)^n}{1-ac}\right\} = (1 + a)\left\{\frac{(ac)^n - 1}{ac - 1}\right\}$$

20.5 SUM OF AN INFINITE G.P.

THEOREM *The sum of an infinite G.P. with first term a and common ratio* $r\ (-1 < r < 1$ i.e., $|r| < 1)$ is $S = \dfrac{a}{1-r}$.

PROOF Consider an infinite G.P. with first term a and common ratio r, where $-1 < r < 1$ i.e. $|r| < 1$. The sum of n terms of this G.P. is given by

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r} - \frac{ar^n}{1-r} \qquad \ldots\text{(i)}$$

Since $-1 < r < 1$, therefore r^n decreases as n increases and tends to zero as n tends to infinity

i.e. $r^n \to 0$ as $n \to \infty$.

$\therefore \quad \dfrac{a\,r^n}{1-r} \to 0$ as $n \to \infty$.

Hence, from (i), the sum of an infinite G.P. is given by

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right) = \frac{a}{1-r}, \text{ if } |r| < 1$$

NOTE *If $r \geq 1$, then the sum of an infinite G.P. tends to infinity.* Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **FINDING THE SUM TO INFINITY OF A G.P. OR A GEOMETRIC SERIES**

EXAMPLE 1 *Find the sum to infinity of the G.P.* $-\dfrac{5}{4}, \dfrac{5}{16}, -\dfrac{5}{64}, \ldots$

SOLUTION The given G.P. has first term $a = -5/4$ and the common ratio $r = -1/4$. Also, $|r| < 1$. Hence, the sum S to infinity is given by

$$S = \frac{a}{1-r} = \frac{-5/4}{1-(-1/4)} = -1$$

EXAMPLE 2 *Sum the following geometric series to infinity:*

(i) $(\sqrt{2}+1) + 1 + (\sqrt{2}-1) + \ldots \infty$ (ii) $\dfrac{1}{2} + \dfrac{1}{3^2} + \dfrac{1}{2^3} + \dfrac{1}{3^4} + \dfrac{1}{2^5} + \dfrac{1}{3^6} + \ldots \infty$

SOLUTION (i) The given series is a geometric series with first term $a = \sqrt{2}+1$ and the common ratio r given by

$$r = \frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \sqrt{2}-1$$

Hence, the sum S to infinity is given by

$$S = \frac{a}{1-r} = \frac{\sqrt{2}+1}{1-(\sqrt{2}-1)} = \frac{\sqrt{2}+1}{2-\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}(\sqrt{2}-1)}$$

$$\Rightarrow \quad S = \frac{(\sqrt{2}+1)^2}{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{3+2\sqrt{2}}{\sqrt{2}} = \frac{4+3\sqrt{2}}{2}$$

(ii) We have,

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \ldots \text{to } \infty$$

$$= \left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \ldots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \ldots\right)$$

$$= \left(\text{An infinite G.P. with } a = \frac{1}{2}, r = \frac{1}{2^2}\right) + \left(\text{An infinite G.P. with } a = \frac{1}{3^2}, r = \frac{1}{3^2}\right)$$

$$= \left\{\frac{(1/2)}{1-(1/2^2)}\right\} + \left\{\frac{(1/3^2)}{1-(1/3^2)}\right\} = \frac{2}{3} + \frac{1}{8} = \frac{19}{24}$$

EXAMPLE 3 *Prove that:* $6^{1/2} \times 6^{1/4} \times 6^{1/8} \ldots\ldots \infty = 6$.

SOLUTION Clearly,

$$6^{1/2} \times 6^{1/4} \times 6^{1/8} \ldots \infty = 6^{\{1/2 + 1/4 + 1/8 + \ldots \infty\}}$$

$$= 6^{\{(1/2)/(1-1/2)\}} \quad \left[\because \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \text{ to } \infty = \frac{1/2}{1-1/2} = 1\right]$$

$$= 6^1 = 6$$

Type II **ON PROVING RESULTS BASED UPON THE FORMULA FOR THE SUM TO INFINITY OF A G.P.**

EXAMPLE 4 *If* $b = a + a^2 + a^3 + \ldots \infty$, *prove that* $a = \dfrac{b}{1+b}$.

SOLUTION We have,

$$b = a + a^2 + a^3 + \ldots \infty$$

Clearly, RHS is a geometric series with first term 'a' and common ratio 'a'

$$\therefore \quad b = \frac{a}{1-a} \Rightarrow b - ab = a \Rightarrow a = \frac{b}{1+b}$$

EXAMPLE 5 *If* $x = a + \dfrac{a}{r} + \dfrac{a}{r^2} + \ldots \infty$, $y = b - \dfrac{b}{r} + \dfrac{b}{r^2} - \ldots \infty$ *and*, $z = c + \dfrac{c}{r^2} + \dfrac{c}{r^4} + \ldots \infty$, *prove that* $\dfrac{xy}{z} = \dfrac{ab}{c}$.

SOLUTION Clearly, x, y and z are the sums of infinite geometric progressions.

$$\therefore \quad x = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1}, \quad y = \frac{b}{1-\left(-\frac{1}{r}\right)} = \frac{br}{1+r} \text{ and, } z = \frac{c}{1-\frac{1}{r^2}} = \frac{cr^2}{r^2-1}$$

$$\Rightarrow \quad xy = \left(\frac{ar}{r-1}\right)\left(\frac{br}{r+1}\right) = \frac{abr^2}{r^2-1}$$

$$\Rightarrow \quad \frac{xy}{z} = \left\{\left(\frac{abr^2}{r^2-1}\right) \div \frac{cr^2}{(r^2-1)}\right\} = \frac{ab}{c}$$

EXAMPLE 6 *If* $x = 1 + a + a^2 + \ldots \infty$, *where* $|a| < 1$ *and* $y = 1 + b + b^2 + \ldots \infty$, *where* $|b| < 1$. *Prove that:*

$$1 + ab + a^2b^2 + \ldots \infty = \frac{xy}{x+y-1}$$

SOLUTION We have,

$$x = 1 + a + a^2 + \ldots \infty$$

$$\therefore \quad x = \frac{1}{1-a} \Rightarrow 1 - a = \frac{1}{x} \Rightarrow a = 1 - \frac{1}{x} \qquad \text{...(i)}$$

and, $y = 1 + b + b^2 + b^3 + \ldots\ldots \infty$

$$\Rightarrow \quad y = \frac{1}{1-b} \Rightarrow 1 - b = \frac{1}{y} \Rightarrow b = 1 - \frac{1}{y} \qquad \text{...(ii)}$$

$$\therefore \quad 1 + ab + (ab)^2 + (ab)^3 + \ldots \infty$$

$$= \frac{1}{1-ab} = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} \qquad \text{[Using (i) and (ii)]}$$

$$= \frac{xy}{x+y-1}$$

EXAMPLE 7 *If* $A = 1 + r^a + r^{2a} + \ldots$ *to* ∞ *and* $B = 1 + r^b + r^{2b} + \ldots \infty$, *prove that*

$$r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$$

SOLUTION We have,

$$A = 1 + r^a + r^{2a} + \ldots \infty \quad \text{and,} \quad B = 1 + r^b + r^{2b} + \ldots \infty$$

$$\Rightarrow \quad A = \frac{1}{1-r^a} \text{ and, } B = \frac{1}{1-r^b}$$

$$\Rightarrow \quad 1 - r^a = \frac{1}{A} \text{ and, } 1 - r^b = \frac{1}{B}$$

$$\Rightarrow \quad r^a = 1 - \frac{1}{A} \text{ and, } r^b = 1 - \frac{1}{B}$$

$$\Rightarrow \quad l = \left(\frac{A-1}{A}\right)^{1/a} \text{ and, } r = \left(\frac{B-1}{B}\right)^{1/b}$$

Hence, $$r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$$

Type III FINDING REQUIRED UNKNOWN WHEN THE SUM OF AN INFINITE G.P. IS GIVEN

EXAMPLE 8 *The first term of a G.P. is 2 and the sum to infinity is 6. Find the common ratio.*

SOLUTION Let r be the common ratio of the given G.P. It is given that, $a = 2$ and $S_\infty = 6$.

Now, $S_\infty = 6 \Rightarrow \dfrac{a}{1-r} = 6 \Rightarrow \dfrac{2}{1-r} = 6 \Rightarrow 6 - 6r = 2 \Rightarrow r = 2/3.$

EXAMPLE 9 *The sum of an infinite G.P. is 8, its second term is 2, find the first term.*

SOLUTION Let a be the first term and r the common ratio of the G.P. It is given that

$$S_\infty = 8 \text{ and } ar = 2$$

$$\Rightarrow \quad \frac{a}{1-r} = 8 \text{ and } r = \frac{2}{a}$$

$$\Rightarrow \quad \frac{a}{1-(2/a)} = 8 \qquad \text{[Eliminating } r\text{]}$$

$$\Rightarrow \quad a^2 - 8a + 16 = 0 \Rightarrow (a-4)^2 = 0 \Rightarrow a = 4.$$

EXAMPLE 10 *The sum of an infinite G.P. is 57 and the sum of their cubes is 9747, find the G.P.*

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$\text{Sum} = 57 \Rightarrow \frac{a}{1-r} = 57 \qquad \text{...(i)}$$

Sum of the cubes $= 9747$

$$\Rightarrow \quad a^3 + a^3 r^3 + a^3 r^6 + \ldots = 9747 \Rightarrow \frac{a^3}{1-r^3} = 9747 \qquad \text{...(ii)}$$

Dividing the cube of (i) by (ii), we get

$$\frac{a^3}{(1-r)^3} \times \frac{(1-r^3)}{a^3} = \frac{(57)^3}{9747}$$

$$\Rightarrow \quad \frac{1-r^3}{(1-r)^3} = 19$$

$$\Rightarrow \quad \frac{1+r+r^2}{(1-r)^2} = 19$$

$$\Rightarrow \quad 18r^2 - 39r + 18 = 0$$

$$\Rightarrow \quad (3r-2)(6r-9) = 0$$

$$\Rightarrow \quad r = 2/3 \text{ or, } r = 3/2$$

$$\Rightarrow \quad r = 2/3 \qquad [\because r \neq 3/2, \text{ because } -1 < r < 1 \text{ for an infinite G.P.}]$$

Putting $r = 2/3$ in (i), we get

$$\frac{a}{1-(2/3)} = 57 \Rightarrow a = 19$$

Hence, the G.P. is 19, 38/3, 76/9, ...

Type IV **FINDING A RATIONAL NUMBER WHOSE DECIMAL EXPANSION IS GIVEN**

EXAMPLE 11 *Which is the rational number having the decimal expansion* $0.3\overline{56}$?

SOLUTION We have,

$$0.3\overline{56} = 0.3 + 0.056 + 0.00056 + 0.0000056 + \dots \infty$$

$$= 0.3 + \left\{\frac{56}{10^3} + \frac{56}{10^5} + \frac{56}{10^7} + \dots \infty\right\}$$

$$= \frac{3}{10} + \frac{\frac{56}{10^3}}{1-\frac{1}{10^2}} = \frac{3}{10} + \frac{56}{990} = \frac{353}{990}$$

EXAMPLE 12 *Use geometric series to express* $0.555\ldots = 0.\overline{5}$ *as a rational number.*

SOLUTION We have,

$$0.\overline{5} = 0.5555\ldots$$

$$= 0.5 + 0.05 + 0.005 + \dots \infty$$

$$= \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \dots \infty$$

$$= \frac{(5/10)}{1-(1/10)} = \frac{5}{9}$$

Type V **ON APPLICATIONS OF INFINITE G.P.**

EXAMPLE 13 *A square is drawn by joining the mid-points of the sides of a square. A third square is drawn inside the second square in the same way and the process is continued indefinitely. If the side of the square is 10 cm, find the sum of the areas of all the squares so formed.*

SOLUTION Let $A_1A_2A_3A_4$ be the first square with each side equal to 10 cm. Let B_1, B_2, B_3, B_4 be the mid-points of its sides. Then,

$$B_1B_2 = \sqrt{A_2B_1^{\,2} + A_2B_2^{\,2}} = \sqrt{5^2+5^2} = 5\sqrt{2} \text{ cm.}$$

Let C_1, C_2, C_3, C_4 be the mid-points of the sides of the square $B_1B_2B_3B_4$. Then,

$$C_1C_2 = \sqrt{B_1C_2^{\,2} + B_1C_1^{\,2}} = \sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{2}}{2}\right)^2} = 5 \text{ cm}$$

Similarly, the side of fourth square is $\frac{5}{\sqrt{2}}$ cm and so on.

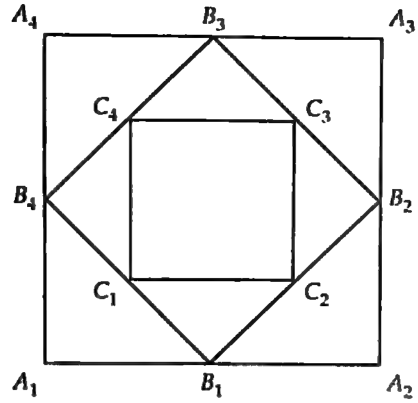


Fig. 20.1

$\therefore$ Sum of the areas of all the squares so formed

$$= \left\{10^2 + (5\sqrt{2})^2 + (5)^2 + \left(\frac{5}{\sqrt{2}}\right)^2 + \dots \infty\right\} \text{ sq. cm.} \qquad [\because \text{ Area} = (\text{Side})^2]$$

$$= \left\{100 + 50 + 25 + \frac{25}{2} + \dots \infty\right\} = \frac{100}{1-(1/2)} = 200 \text{ sq. cm.}$$

EXAMPLE 14 *After striking a floor a certain ball rebounds* $\left(\frac{4}{5}\right)^{th}$ *of the height from which it has fallen. Find the total distance that it travels before coming to rest, if it is gently dropped from a height of 120 metres.*

SOLUTION Initially the ball falls from a height of 120 metres. After striking the floor it rebounds and goes to a height of $\frac{4}{5}(120)$ metres. Now, it falls from a height of $\frac{4}{5}(120)$ metres and after rebounding again it goes to a height of $\frac{4}{5}\left(\frac{4}{5}(120)\right)$ metres. This process is continued till the ball comes to rest.

$\therefore$ The total distance traveled $= 120 + 2\left\{\frac{4}{5}(120) + \left(\frac{4}{5}\right)^2 (120) + \dots \infty\right\}$

$$= 120 + 2 \times \left\{\frac{\frac{4}{5}(120)}{1-\frac{4}{5}}\right\} = 120 + 960 = 1080 \text{ metres.}$$

EXAMPLE 15 *The inventor of the chess board suggested a reward of one grain of wheat for the first square, 2 grains for the second, 4 grains for the third and so on, doubling the number of the grains for subsequent squares. How many grains would have to be given to inventor ? (There are 64 squares in the chess board).*

SOLUTION Clearly, required number of grains is the sum of an infinite G.P. with first term 1 and common ratio 2.

$\therefore$ Number of grains $= 1 + 2 + 2^2 + 2^3 + \dots$ to 64 terms $= 1\left(\frac{2^{64}-1}{2-1}\right) = 2^{64} - 1.$

LEVEL-2

Type VI ON FINDING THE SUM OF AN INFINITE G.P.

EXAMPLE 16 *Find the sum of an infinitely decreasing G.P. whose first term is equal to b + 2 and the common ratio to 2/c, where b is the least value of the product of the roots of the equation* $(m^2+1)x^2 - 3x + (m^2+1)^2 = 0$, *and c is the greatest value of the sum of its roots.*

SOLUTION We have,

$$(m^2+1)x^2 - 3x + (m^2+1)^2 = 0$$

$\therefore$ Sum of the roots $= \dfrac{3}{m^2+1}$ and, Product of the roots $= (m^2+1)$

Now, b = Least value of the product of roots

b = Least value of (m^2+1)

$\Rightarrow$ $b = 1$ $[\because m^2+1 > 1 \text{ for all } m]$

c = Greatest value of the sum of the roots

$\Rightarrow$ c = Greatest value of $\dfrac{3}{m^2+1}$

Clearly, $\dfrac{3}{m^2+1}$ is greatest when m^2+1 is least and the least value of m^2+1 is 1.

$$\therefore \quad c = \frac{3}{1} = 3$$

So, first term of the infinite G.P. is $b+2 = 1+2 = 3$ and, the common ratio is $\dfrac{2}{c} = \dfrac{2}{3}$.

Hence, the sum S of the infinite G.P. is given by

$$S = \frac{3}{1-\frac{2}{3}} = 9 \qquad \left[\text{Using : } S = \frac{a}{1-r}\right]$$

Type VII ON PROVING RESULTS BASED UPON SUM OF AN INFINITE G.P.

EXAMPLE 17 *If* $x = \sum\limits_{n=0}^{\infty} \cos^{2n}\theta,\ y = \sum\limits_{n=0}^{\infty} \sin^{2n}\phi,\ z = \sum\limits_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi$, *where* $0 < \theta, \phi < \pi/2$ *then prove that* $xz + yz - z = xy$.

SOLUTION We have,

$$x = \sum_{n=0}^{\infty} \cos^{2n}\theta = 1 + \cos^2\theta + \cos^4\theta + \ldots \infty$$

$$\Rightarrow \quad x = \frac{1}{1-\cos^2\theta} \Rightarrow \sin^2\theta = \frac{1}{x}$$

$$\Rightarrow \quad y = \sum_{n=0}^{\infty} \sin^{2n}\phi = 1 + \sin^2\phi + \sin^4\phi + \ldots \infty$$

$$\Rightarrow \quad y = \frac{1}{1-\sin^2\phi} \Rightarrow \cos^2\phi = \frac{1}{y}$$

and, $$z = \sum_{n=0}^{\infty} \cos^{2n}\theta, \sin^{2n}\phi = 1 + \cos^2\theta\sin^2\phi + \cos^4\theta\sin^4\phi + \ldots \infty$$

$$\Rightarrow \quad z = \frac{1}{1-\cos^2\theta \sin^2\phi}$$

$$\Rightarrow \quad z = \frac{1}{1-(1-\sin^2\theta)(1-\cos^2\phi)}$$

$$\Rightarrow \quad z = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} \Rightarrow z = \frac{1}{\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}} \Rightarrow z = \frac{xy}{x+y-1} \Rightarrow xz + yz - z = xy$$

EXAMPLE 18 *If $|x|<1$ and $|y|<1$, find the sum to infinity of the following series:*

$$(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots$$

SOLUTION We have,

$$(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots\ \infty$$

$$= \frac{1}{x-y}\left\{(x^2-y^2)+(x^3-y^3)+(x^4-y^4)+\ldots \text{ to } \infty\right\}$$

$$\left[\because \frac{x^n-a^n}{x-a} = x^{n-1}+x^{n-2}a^1+x^{n-3}a^2+\ldots+a^{n-1}, n \in N\right]$$

$$= \frac{1}{x-y}\left\{(x^2+x^3+x^4+\ldots \text{ to } \infty)-(y^2+y^3+y^4+\ldots \text{ to } \infty)\right\}$$

$$= \frac{1}{x-y}\left\{\frac{x^2}{1-x}-\frac{y^2}{1-y}\right\}$$

$$= \frac{1}{x-y}\,\frac{\left\{x^2-x^2y-y^2+y^2x\right\}}{(1-x)(1-y)} = \frac{1}{(x-y)}\,\frac{\left\{(x^2-y^2)-xy(x-y)\right\}}{(1-x)(1-y)} = \frac{x+y-xy}{(1-x)(1-y)}$$

Type VIII **ON FINDING REQUIRED UNKNOWN WHEN SUM OF AN INFINITE G.P. IS GIVEN**

EXAMPLE 19 *The sum of an infinite geometric series is 15 and the sum of the squares of these terms is 45. Find the series.*

SOLUTION Let a be the first term and r be the common ratio of the infinite geometric series.

$$\text{Sum} = 15 \Rightarrow \frac{a}{1-r} = 15 \qquad \ldots\text{(i)}$$

Sum of the squares = 45

$$\Rightarrow \quad (a^2+a^2r^2+a^2r^4+\ldots\infty) = 45 \Rightarrow \frac{a^2}{1-r^2} = 45 \qquad \ldots\text{(ii)}$$

Dividing the square of (i), by (ii), we get

$$\frac{a^2}{(1-r)^2} \times \frac{1-r^2}{a^2} = \frac{(15)^2}{45} \Rightarrow \frac{1+r}{1-r} = 5 \Rightarrow 6r = 4 \Rightarrow r = \frac{2}{3}$$

Putting $r = \frac{2}{3}$ in (i), we get

$$\frac{a}{1-2/3} = 15 \Rightarrow a = 5$$

Hence, the required series is $5+\frac{10}{3}+\frac{20}{9}+\frac{40}{27}+\ldots\infty$.

EXAMPLE 20 *If each term of an infinite G.P. is twice the sum of the terms following it, then find the common ratio of the G.P.*

SOLUTION Let a be the first term and r the common ratio of the G.P. It is given that

$$a_n = 2[a_{n+1} + a_{n+2} + a_{n+3} + \dots \infty] \text{ for all } n \in N$$

$$\Rightarrow \quad ar^{n-1} = 2[ar^n + ar^{n+1} + \dots \infty]$$

$$\Rightarrow \quad ar^{n-1} = \frac{2\,ar^n}{1-r} \Rightarrow 1 = \frac{2r}{1-r} \Rightarrow r = \frac{1}{3}$$

EXERCISE 20.4

LEVEL-1

1. Find the sum of the following series to infinity:
   (i) $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} + \dots \infty$ (ii) $8 + 4\sqrt{2} + 4 + \dots \infty$
   (iii) $2/5 + 3/5^2 + 2/5^3 + 3/5^4 + \dots \infty$. (iv) $10 - 9 + 8.1 - 7.29 + \dots \infty$
   (v) $\frac{1}{3} + \frac{1}{5^2} + \frac{1}{3^3} + \frac{1}{5^4} + \frac{1}{3^5} + \frac{1}{5^6} + \dots \infty$ [NCERT]
2. Prove that: $(9^{1/3} . 9^{1/9} . 9^{1/27} \dots \infty) = 3$.
3. Prove that: $(2^{1/4} . 4^{1/8} . 8^{1/16} . 16^{1/32} \dots \infty) = 2$.
4. If S_p denotes the sum of the series $1 + r^p + r^{2p} + \dots$ to ∞ and s_p the sum of the series $1 - r^p + r^{2p} - \dots$ to ∞, prove that $S_p + s_p = 2\ S_{2p}$.
5. Find the sum of the terms of an infinite decreasing G.P. in which all the terms are positive, the first term is 4, and the difference between the third and fifth term is equal to 32/81.
6. Express the recurring decimal 0.125125125 ... as a rational number.
7. Find the rational number whose decimal expansion is $0.4\overline{23}$.
8. Find the rational numbers having the following decimal expansions:
   (i) $0.\overline{3}$ (ii) $0.\overline{231}$ (iii) $3.5\overline{2}$ (iv) $0.6\overline{8}$ [NCERT]
9. One side of an equilateral triangle is 18 cm. The mid-points of its sides are joined to form another triangle whose mid-points, in turn, are joined to form still another triangle. The process is continued indefinitely. Find the sum of the (i) perimeters of all the triangles. (ii) areas of all triangles.

LEVEL-2

10. Find an infinite G.P. whose first term is 1 and each term is the sum of all the terms which follow it.
11. The sum of first two terms of an infinite G.P. is 5 and each term is three times the sum of the succeeding terms. Find the G.P.
12. Show that in an infinite G.P. with common ratio r ($|r| < 1$), each term bears a constant ratio to the sum of all terms that follow it.
13. If S denotes the sum of an infinite G.P. and S_1 denotes the sum of the squares of its terms, then prove that the first term and common ratio are respectively $\frac{2SS_1}{S^2 + S_1}$ and $\frac{S^2 - S_1}{S^2 + S_1}$.

ANSWERS

1. (i) $\frac{3}{4}$ (ii) $8(2+\sqrt{2})$ (iii) $\frac{13}{24}$ (iv) 5.263 (v) $\frac{5}{12}$ 5. $6, \frac{12}{3-2\sqrt{2}}$

6. $\frac{125}{999}$ 7. $\frac{419}{990}$ 8. (i) $\frac{1}{3}$ (ii) $\frac{231}{999}$ (iii) $\frac{317}{90}$ (iv) $\frac{31}{45}$

9. (i) 108 cm (ii) $108\sqrt{3}$ square cm 10. $1, \frac{1}{2}, \frac{1}{4}, \dots$ 11. $4, 1, \frac{1}{4}, \frac{1}{16}, \dots$

HINTS TO NCERT & SELECTED PROBLEM

9. Sum of the perimeters $= 3\left\{18 + \frac{18}{2} + \frac{18}{4} + \dots \infty\right\}$

$$\text{Sum of the areas} = \frac{\sqrt{3}}{4}\left\{18^2 + \left(\frac{18}{2}\right)^2 + \left(\frac{18}{4}\right)^2 + \dots \infty\right\}$$

20.6 PROPERTIES OF GEOMETRIC PROGRESSIONS

In this section, we shall discuss some important properties of geometric progressions and geometric series.

PROPERTY I *If all the terms of a G.P. be multiplied or divided by the same non-zero constant, then it remains a G.P. with the same common ratio.*

PROOF Let $a_1, a_2, a_3, \dots, a_n, \dots$ be a G.P. with common ratio r. Then,

$$\frac{a_{n+1}}{a_n} = r, \text{ for all } n \in N \qquad \dots(i)$$

Let k be a non-zero constant. Multiplying all the terms of the given G.P. by k, we obtain the new sequence: $ka_1, ka_2, ka_3, \dots ka_n, \dots$

Clearly, $\frac{k a_{n+1}}{k a_n} = \frac{a_{n+1}}{a_n} = r$ for all $n \in N$ [Using (i)]

Hence, the new sequence also forms a G.P. with common ratio r.

PROPERTY II *The reciprocals of the terms of a given G.P. form a G.P.*

PROOF Let $a_1, a_2, a_3, \dots, a_n, \dots$ be a G.P. with common ratio r. Then,

$$\frac{a_{n+1}}{a_n} = r \text{ for all } n \in N \qquad \dots(i)$$

The sequence formed by the reciprocals of the terms of the given G.P. is

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots \frac{1}{a_n}, \dots$$

For this sequence the ratio of a term and the preceding term is given by

$$\frac{1/a_{n+1}}{1/a_n} = \frac{a_n}{a_{n+1}} = \frac{1}{r} \qquad \text{[Using (i)]}$$

So, the new sequence is a G.P. with common ratio $1/r$.

PROPERTY III *If each term of a G.P. be raised to the same power, the resulting sequence also forms a G.P.*

PROOF Let $a_1, a_2, a_3, \dots a_n, \dots$ be a G.P. with common ratio r. Then,

$$\frac{a_{n+1}}{a_n} = r \text{ for all } n \in N \qquad \dots(i)$$

Let k be a non-zero real number.

Consider the sequence whose terms are k^{th} powers of the terms of the given sequence

i.e. $a_1^k, a_2^k, a_3^k, \ldots, a_n^k, \ldots$

For this sequence, we have

$$\frac{a_{n+1}^k}{a_n^k} = \left(\frac{a_{n+1}}{a_n}\right)^k = r^k \text{ for all } n \in N \qquad \text{[Using (i)]}$$

Hence, $a_1^k, a_2^k, a_3^k, \ldots a_n^k, \ldots$ is a G.P. with common ratio r^k.

PROPERTY IV *In a finite G.P the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n$ be a finite G.P. with common ratio r. Then,

kth term from the beginning $= a_k = a_1 r^{k-1}$

kth term from the end $= (n-k+1)th$ term from the beginning $= a_{n-k+1} = a_1 r^{n-k}$

$\therefore$ (kth term from the beginning) (kth term from the end)

$$= a_k a_{n-k+1} = a_1 r^{k-1} a_1 r^{n-k} = a_1^2 r^{n-1} = a_1 \cdot a_1 r^{n-1} = a_1 a_n \quad \text{for all } k = 2, 3, \ldots, n-1$$

Hence, the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.

PROPERTY V *Three non-zero numbers a, b, c are in G.P. iff $b^2 = ac$*

PROOF Clearly,

$$a, b, c \text{ are in G.P.} \Leftrightarrow \frac{b}{a} = \frac{c}{b} = (\text{ common ratio}) \Leftrightarrow b^2 = ac$$

NOTE *When a, b, c are in G.P., then b is known as the Geometric mean of a and c.*

PROPERTY VI *If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.*

PROPERTY VII *If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P. of non-zero non-negative terms, then* $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ *is an A.P. and vice-versa.*

PROOF *Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a G.P. of non-zero non-negative terms with common ratio r. Then,*

$$a_n = a_1 r^{n-1}, \text{ for all } n \in N$$

$$\Rightarrow \quad \log a_n = \log a_1 + (n-1) \log r, \quad \text{for all } n \in N$$

Let $\quad b_n = \log a_n = \log a_1 + (n-1) \log r, \quad$ for all $n \in N$

Then, $\quad b_{n+1} - b_n = [\log a_1 + n \log r] - [\log a_1 + (n-1) \log r] = \log r \quad$ for all $n \in N$

Clearly, $b_{n+1} - b_n = \log r =$ Constant for all $n \in N$.

Hence, $b_1, b_2, \ldots, b_n, \ldots$ i.e. $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ is an A.P. with common difference $\log r$.

Conversely, let $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ be an A.P. with common difference d. Then,

$$\log a_{n+1} - \log a_n = d \quad \text{for all } n \in N.$$

$$\Rightarrow \quad \log\left(\frac{a_{n+1}}{a_n}\right) = d \quad \text{for all } n \in N.$$

$$\Rightarrow \quad \frac{a_{n+1}}{a_n} = e^d \text{ (a constant) for all } n \in N.$$

$$\Rightarrow \quad a_1, a_2, a_3, \ldots, a_n, \ldots \text{ is a G.P. with common ratio } e^d.$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS BASED UPON FOLLOWING RESULTS:**

(i) a, b, c are in G.P. iff $b^2 = ac$ (ii) a, b, c are in A.P. iff $2b = a + c$.

EXAMPLE 1 *If p, q, r are in A.P., show that the pth, qth and rth terms of any G.P. are in G.P.*

SOLUTION Let A be the first term and R the common ratio of a G.P. Then,

$$a_p = AR^{p-1}, \quad a_q = AR^{q-1} \text{ and } a_r = AR^{r-1}$$

We have to prove that a_p, a_q, a_r are in G.P. For this it is sufficient to show that

$$(a_q)^2 = a_p \cdot a_r$$

Now, $(a_q)^2 = (AR^{q-1})^2$

$\Rightarrow \quad (a_q)^2 = A^2 R^{2q-2}$

$\Rightarrow \quad (a_q)^2 = A^2 R^{p+r-2}$ $\quad [\because\ p, q, r \text{ are in A.P.} \therefore\ 2q = p + r]$

$\Rightarrow \quad (a_q)^2 = (AR^{p-1})(AR^{r-1}) = a_p \cdot a_r$

Hence, a_p, a_q, a_r are in G.P.

EXAMPLE 2 *If a, b, c are in G.P., then prove that* $\log a^n, \log b^n, \log c^n$ *are in A.P.*

SOLUTION It is given that a, b, c are in G.P.

$\therefore \quad b^2 = ac$

$\Rightarrow \quad (b^2)^n = (ac)^n$

$\Rightarrow \quad b^{2n} = a^n c^n$

$\Rightarrow \quad \log b^{2n} = \log (a^n c^n)$

$\Rightarrow \quad \log (b^n)^2 = \log a^n + \log c^n$

$\Rightarrow \quad 2 \log b^n = \log a^n + \log c^n$

$\Rightarrow \quad \log a^n, \log b^n, \log c^n$ are in A.P

EXAMPLE 3 *Three numbers whose sum is 15 are in A.P. If 1, 4, 19 be added to them respectively, then they are in G.P. Find the numbers.*

SOLUTION Let the three numbers be $a - d, a, a + d$. Then,

Sum $= 15 \Rightarrow (a - d) + a + (a + d) = 15 \Rightarrow a = 5$.

So, the numbers are $5 - d, 5, 5 + d$. Adding 1, 4, 19 respectively to these numbers, we get $6 - d,\ 9,\ 24 + d$. These numbers are in G.P.

$\therefore \quad 9^2 = (6 - d)(24 + d) \Rightarrow d^2 + 18d - 63 = 0 \Rightarrow (d + 21)(d - 3) = 0 \Rightarrow d = -21$ or, $d = 3$.

Hence, the numbers are 26, 5, –16 or 2, 5, 8.

Type II **PROBLEMS BASED UPON PROPERTIES OF G.P.**

EXAMPLE 4 *If a, b, c, d are in G.P., show that:*

(i) $(b - c)^2 + (c - a)^2 + (d - b)^2 = (a - d)^2$

(ii) $(ab + bc + cd)^2 = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$ **[NCERT]**

SOLUTION Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

(i) LHS $= (b - c)^2 + (c - a)^2 + (d - b)^2$

$$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$$
$$= a^2r^2 (1 - r)^2 + a^2 (r^2 - 1^2) + a^2r^2 (r^2 - 1)^2$$
$$= a^2 (r^6 - 2r^3 + 1) = a^2 (1 - r^3)^2 = (a - ar^3)^2 = (a - d)^2 = \text{RHS.}$$

(ii) $\text{LHS} = (ab + bc + cd)^2 = (a \times ar + ar \times ar^2 + ar^2 \times ar^3)^2 = a^4r^2 (1 + r^2 + r^4)^2$

$\text{RHS} = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$
$$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$$
$$= a^2 (1 + r^2 + r^4)\, a^2r^2 (1 + r^2 + r^4) = a^4r^2 (1 + r^2 + r^4)^2$$

$\therefore$ LHS = RHS.

EXAMPLE 5 *If a, b, c, d are in G.P., prove that $a + b, b + c, c + d$ are also in G.P.*

SOLUTION Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar$, $c = ar^2$ and $d = ar^3$

$\therefore$ $a + b = a + ar = a(1 + r)$, $b + c = ar + ar^2 = ar(1 + r)$ and $c + d = ar^2 + ar^3 = ar^2 (1 + r)$

Now, $(b + c)^2 = \left\{ar(1 + r)\right\}^2 = a^2r^2 (1 + r)^2 = \left\{a(1 + r)\right\}\left\{ar^2 (1 + r)\right\}$
$$= (a + b)(c + d) \qquad [\because a + b = a(1 + r), \text{ and } c + d = ar^2 (1 + r)]$$

Hence, $a + b, b + c, c + d$ are in G.P.

EXAMPLE 6 *If a, b, c, d are in G.P., prove that $a^n + b^n, b^n + c^n, c^n + d^n$ are also in G.P.* **[NCERT]**

SOLUTION Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

$\therefore$ $a^n + b^n = a^n + a^n r^n = a^n (1 + r^n)$

$b^n + c^n = a^n r^n + a^n r^{2n} = a^n r^n (1 + r^n)$, $c^n + d^n = a^n r^{2n} + a^n r^{3n} = a^n r^{2n} (1 + r^n)$

Clearly, $(b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$.

Hence, $a^n + b^n, b^n + c^n, c^n + d^n$ are in G.P.

LEVEL-2

EXAMPLE 7 *If a, b, c are in A.P. and x, y, z are in G.P., then show that $x^{b-c} . y^{c-a} . z^{a-b} = 1$.*

SOLUTION It is given that

a, b, c are in A.P. $\Rightarrow 2b = a + c$ ...(i)

x, y, z are in G.P. $\Rightarrow y^2 = xz$ ...(ii)

$\therefore$ $x^{b-c} y^{c-a} z^{a-b} = x^{b-c} (\sqrt{xz})^{c-a} z^{a-b}$ [Using (ii)]

$$= x^{b-c} x^{\frac{c-a}{2}} z^{\frac{c-a}{2}} z^{a-b}$$
$$= x^{b-c+\frac{c-a}{2}} z^{a-b+\frac{c-a}{2}}$$
$$= x^{\frac{2b-(a+c)}{2}} z^{\frac{(c+a)-2b}{2}} = x^0 z^0 = 1 \qquad \text{[Using (i)]}$$

EXAMPLE 8 *If mth, nth and pth terms of a G.P. form three consecutive terms of a G.P. Prove that m, n and p form three consecutive terms of an arithmetic sequence.*

SOLUTION Let a be the first term and r be the common ratio the G.P. Then,

$$a_m = ar^{m-1}, a_n = ar^{n-1} \text{ and } a_p = ar^{p-1}$$

It is given that a_m, a_n, a_p are in GP.

$$\therefore \quad (a_n)^2 = a_m a_p$$
$$\Rightarrow \quad (ar^{n-1})^2 = (ar^{m-1} \times ar^{p-1})$$
$$\Rightarrow \quad a^2 r^{2n-2} = a^2 r^{m+p-2}$$
$$\Rightarrow \quad r^{2n-2} = r^{m+p-2}$$
$$\Rightarrow \quad 2n-2 = m+p-2$$
$$\Rightarrow \quad 2n = m+p$$

$\Rightarrow$ m, n, p are in AP.

EXAMPLE 9 *If a, b, c are in G.P. and x, y are the arithmetic means of a, b and b, c respectively, then prove that:*

$$\frac{a}{x} + \frac{c}{y} = 2 \text{ and } \frac{1}{x} + \frac{1}{y} = \frac{2}{b}$$

SOLUTION It is given that

a, b, c are in G.P. $\Rightarrow$ $b^2 = ac$ ...(i)

x is the A.M. of a and $b \Rightarrow$ $x = \dfrac{a+b}{2}$ ...(ii)

and, y is the A.M. of b and $c \Rightarrow$ $y = \dfrac{b+c}{2}$ ...(iii)

Now, $\dfrac{a}{x} + \dfrac{c}{y} = \dfrac{2a}{a+b} + \dfrac{2c}{b+c} = \dfrac{2a(b+c) + 2c(a+b)}{(a+b)(b+c)}$ $\left[\because x = \dfrac{a+b}{2} \text{ and } y = \dfrac{b+c}{2}\right]$

$\Rightarrow$ $\dfrac{a}{x} + \dfrac{c}{y} = \dfrac{2(ab+2ac+bc)}{(ab+ac+b^2+bc)} = \dfrac{2(ab+2ac+bc)}{(ab+2ac+bc)} = 2$ [Using (i)]

And, $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{2}{a+b} + \dfrac{2}{b+c} = \dfrac{2(a+c+2b)}{(ab+b^2+ac+bc)} = \dfrac{2(a+c+2b)}{(ab+2b^2+bc)}$ [Using (i)]

$\Rightarrow$ $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{2(a+c+2b)}{b(a+2b+c)} = \dfrac{2}{b}$

EXAMPLE 10 *If a, b, c are in G.P. and $a^{1/x} = b^{1/y} = c^{1/z}$, prove that x, y, z are in A.P.* [NCERT]

SOLUTION We have,

$a^{1/x} = b^{1/y} = c^{1/z} = \lambda$ (say) $\Rightarrow$ $a = \lambda^x, b = \lambda^y$ and $c = \lambda^z$

Now, a, b, c are in G.P.

$\Rightarrow$ $b^2 = ac$

$\Rightarrow$ $(\lambda^y)^2 = \lambda^x \times \lambda^z \Rightarrow \lambda^{2y} = \lambda^{x+z} \Rightarrow 2y = x+z \Rightarrow$ x, y, z are in A.P.

EXAMPLE 11 *If $a^2+b^2, ab+bc$ and b^2+c^2 are in G.P., prove that a, b, c are also in G.P.*

SOLUTION It is given that

$a^2+b^2, ab+bc, b^2+c^2$ are in G.P.

$$\Rightarrow \quad (ab+bc)^2 = (a^2+b^2)(b^2+c^2)$$
$$\Rightarrow \quad a^2b^2 + b^2c^2 + 2ab^2c = a^2b^2 + a^2c^2 + b^2c^2 + b^4$$

$\Rightarrow$ $b^4 + a^2c^2 - 2ab^2c = 0 \Rightarrow (b^2 - ac)^2 = 0 \Rightarrow b^2 = ac \Rightarrow$ a, b, c are in G.P.

EXERCISE 20.5

LEVEL-1

1. If a, b, c are in G.P., prove that $\log a, \log b, \log c$ are in A.P.
2. If a, b, c are in G.P., prove that $\frac{1}{\log_a m}, \frac{1}{\log_b m}, \frac{1}{\log_c m}$ are in A.P.
3. Find k such that $k+9, k-6$ and 4 form three consecutive terms of a G.P.
4. Three numbers are in A.P. and their sum is 15. If 1, 3, 9 be added to them respectively, they form a G.P. Find the numbers.
5. The sum of three numbers which are consecutive terms of an A.P. is 21. If the second number is reduced by 1 and the third is increased by 1, we obtain three consecutive terms of a G.P. Find the numbers.
6. The sum of three numbers a, b, c in A.P. is 18. If a and b are each increased by 4 and c is increased by 36, the new numbers form a G.P. Find a, b, c.
7. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an A.P. Find the numbers.
8. If a, b, c are in G.P., prove that:
   (i) $a(b^2+c^2) = c(a^2+b^2)$
   (ii) $a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right) = a^3+b^3+c^3$
   (iii) $\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{a+b+c}{a-b+c}$
   (iv) $\frac{1}{a^2-b^2}+\frac{1}{b^2} = \frac{1}{b^2-c^2}$
   (v) $(a+2b+2c)(a-2b+2c) = a^2+4c^2$.
9. If a, b, c, d are in G.P., prove that:
   (i) $\frac{ab-cd}{b^2-c^2} = \frac{a+c}{b}$
   (ii) $(a+b+c+d)^2 = (a+b)^2+2(b+c)^2+(c+d)^2$
   (iii) $(b+c)(b+d) = (c+a)(c+d)$
10. If a, b, c are in G.P., prove that the following are also in G.P.:
    (i) a^2, b^2, c^2 (ii) a^3, b^3, c^3 (iii) $a^2+b^2, ab+bc, b^2+c^2$
11. If a, b, c, d are in G.P., prove that:
    (i) $(a^2+b^2), (b^2+c^2), (c^2+d^2)$ are in G.P.
    (ii) $(a^2-b^2), (b^2-c^2), (c^2-d^2)$ are in G.P.
    (iii) $\frac{1}{a^2+b^2}, \frac{1}{b^2+c^2}, \frac{1}{c^2+d^2}$ are in G.P.
    (iv) $(a^2+b^2+c^2), (ab+bc+cd), (b^2+c^2+d^2)$ are in G.P.
12. If $(a-b), (b-c), (c-a)$ are in G.P., then prove that $(a+b+c)^2 = 3(ab+bc+ca)$
13. If a, b, c are in G.P. then prove that: $\frac{a^2+ab+b^2}{bc+ca+ab} = \frac{b+a}{c+b}$
14. If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y and z respectively. Prove that x, y, z are in G.P. **[NCERT]**

LEVEL-2

15. If a, b, c are in A.P. and a, b, d are in G.P., then prove that $a, a-b, d-c$ are in G.P.
16. If pth, qth, rth and sth terms of an A.P. be in G.P., then prove that $p-q, q-r, r-s$ are in G.P. **[NCERT]**

17. If $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are three consecutive terms of an A.P., prove that a, b, c are the three consecutive terms of a G.P.

18. If $x^a = x^{b/2} z^{b/2} = z^c$, then prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

19. If a, b, c are in A.P. b, c, d are in G.P. and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P., prove that a, c, e are in G.P.

20. If a, b, c are in A.P. and a, x, b and b, y, c are in G.P., show that x^2, b^2, y^2 are in A.P.

21. If a, b, c are in A.P. and a, b, d are in G.P., show that $a, (a-b), (d-c)$ are in G.P.

22. If a, b, c are three distinct real numbers in G.P. and $a+b+c = xb$, then prove that either $x < -1$ or $x > 3$.

23. If p^{th}, q^{th} and r^{th} terms of an A.P. and G.P. are both a, b and c respectively, show that $a^{b-c} b^{c-a} c^{a-b} = 1$.

ANSWERS

3. 0 or 16 6. $a=-2,\ b=6,\ c=14$ or $a=46,\ b=6,\ c=-34$ 7. 8, 16, 32
8. 15, 5, –5 or 3, 5, 7 9. 12, 7, 2 or 3, 7, 11 14. 2046

HINTS TO NCERT & SELECTED PROBLEMS

1. a, b, c are in G.P. $\Rightarrow b^2 = ac \Rightarrow \log b^2 = \log ac \Rightarrow 2\log b = \log a + \log c$

2. a, b, c are in G.P.

$\therefore \quad b^2 = ac = \log_m b^2 = \log_m ac$

$\Rightarrow \quad 2\log_m b = \log_m a + \log_m c$

$\Rightarrow \quad \frac{2}{\log_b m} = \frac{1}{\log_a m} + \frac{1}{\log_c m} \Rightarrow \frac{1}{\log_a m}, \frac{1}{\log_b m}, \frac{1}{\log_c m}$ are in A.P.

3. It is given that $k+9, k-6, 4$ are in G.P.

$\Rightarrow \quad (k-6)^2 = (k+9)\times 4 \Rightarrow k = 0, 16.$

14. Let the first term and common ratio of the G.P. be a and r respectively. It is given that

$x = ar^3, y = ar^9$ and $z = ar^{15}$

$\therefore \quad y^2 = a^2 r^{18}$ and $xz = a^2 r^{18}$

$\Rightarrow \quad y^2 = xz$

$\Rightarrow \quad x, y, z$ are in G.P.

16. Let the first term and the common difference of the AP be a and d respectively.
It is given that its p^{th}, r^{th} and s^{th} terms are in G.P. Let A be the first term and R be the common ratio of the G.P. Then,

$$a + (p-1)d = A \quad \text{...(i)}$$
$$a + (q-1)d = AR \quad \text{...(ii)}$$
$$a + (r-1)d = AR^2 \quad \text{...(iii)}$$
$$a + (s-1)d = AR^3 \quad \text{...(iv)}$$

Subtracting (ii) from (i), we get

$$\{a + (p-1)d\} - \{a + (q-1)d\} = A - AR$$

$\Rightarrow \quad (p-q)\,d = A(1-R)$ ...(v)

Subtracting (iii) from (ii), we get

$$\left\{a + (q-1)\,d\right\} - \left\{a + (r-1)\,d\right\} = AR - AR^2$$

$\Rightarrow \quad (q-r)\,d = AR\,(1-R)$ ...(vi)

Subtracting (iv) from (iii), we get

$$\left\{a + (r-1)\,d\right\} - \left\{a + (s-1)\,d\right\} = AR^2 - AR^3$$

$\Rightarrow \quad (r-s)\,d = AR^2\,(1-R)$ ...(vii)

From (v), (vi) and (vii), we obtain that

$$(q-r)^2\,d^2 = (p-q)\,d\,(r-s)\,d$$

$\Rightarrow \quad (q-r)^2 = (p-q)\,(r-s)$

$\Rightarrow \quad (p-q), (q-r), (r-s)$ are in G.P.

17. It is given that $\dfrac{1}{a+b}, \dfrac{1}{2b}, \dfrac{1}{b+c}$ are in A.P.

$\therefore \quad \dfrac{2}{2b} = \dfrac{1}{a+b} + \dfrac{1}{b+c} \Rightarrow b^2 = ac.$

19. We have,

$2b = a + c$ ...(i) $\qquad c^2 = bd$ ...(ii) $\qquad$ and, $\dfrac{2}{d} = \dfrac{1}{c} + \dfrac{1}{e}$ ...(iii)

We have to eliminate b and d from these relations. Substitute b and d obtained from (i) and (iii) in (ii) to get $c^2 = ae$.

22. Let r be the common ratio of the G.P. Then, $b = ar$ and $c = ar^2$.

Now, $a + b + c = xb \Rightarrow a + ar + ar^2 = xar \Rightarrow r^2 + (1-x)\,r + 1 = 0.$

But, r is real.

$\therefore \quad$ Disc $\geq 0 \Rightarrow x^2 - 2x - 3 > 0 \Rightarrow x < -1$ or $x > 3$

20.7 INSERTION OF GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS

GEOMETRIC MEANS *Let a and b be two given numbers. If n numbers $G_1, G_2, \ldots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \ldots, G_n, b$ is a G.P. Then the numbers $G_1, G_2, \ldots, G_n$ are known as n geometric means (G.M.'s) between a and b.*

GEOMETRIC MEAN *If a single geometric mean G is inserted between two given numbers a and b, then G is known as the geometric mean between a and b.*

Thus,

G is the G.M. between a and b. $\Leftrightarrow a, G, b$ are in G.P. $\Leftrightarrow G^2 = ab \Leftrightarrow G = \sqrt{ab}$.

The geometric mean G between 4 and 9 is given by $G = \sqrt{4 \times 9} = 6$.

The geometric mean G between –9 and –4 is given by $G = \sqrt{-9 \times -4} = -6$.

<u>NOTE</u> *If a and b are two numbers of opposite signs, then geometric mean between them does not exist.*

20.7.1 INSERTION OF GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS

Let $G_1, G_2, \ldots, G_n$ be n geometric means between two given numbers a and b. Then,

$a, G_1, G_2, \ldots, G_n, b$ is a G.P. consisting of $(n+2)$ terms. Let r be the common ratio of this G.P. Then,

$$b = (n+2)th \text{ term } = ar^{n+1}$$

$$\Rightarrow \quad r^{n+1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$\therefore \quad G_1 = ar = a\left(\frac{b}{a}\right)^{1/(n+1)}, G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/(n+1)} \ldots, G_n = ar^n = a\left(\frac{b}{a}\right)^{n/n+1}.$$

THEOREM *If n geometric means are inserted between two quantities, then the product of n geometric means is the nth power of the single geometric mean between the two quantities.*

PROOF Let $G_1, G_2, G_3, \ldots, G_n$ be n geometric means between two quantities a and b. Then, $a, G_1, G_2, \ldots, G_n, b$ is a G.P. Let r be the common ratio of this G.P. Then,

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \text{ and, } G_1 = ar, G_2 = ar^2, G_3 = ar^3, \ldots, G_n = ar^n.$$

$$\therefore \quad G_1 . G_2 . G_3 \ldots . G_n = (ar)(ar^2)(ar^3)\ldots(ar^n) = a^n r^{1+2+3+\ldots+n}$$

$$= a^n r^{\frac{n(n+1)}{2}} = a^n \left\{\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right\}^{\frac{n(n+1)}{2}} = a^n \left(\frac{b}{a}\right)^{n/2} = a^{n/2} b^{n/2}$$

$$= \left\{\sqrt{ab}\right\}^n$$

$= G^n$, where $G = \sqrt{ab}$ is the single geometric mean between a and b.

Q.E.D.

20.7.2 SOME IMPORTANT PROPERTIES OF ARITHMETIC AND GEOMETRIC MEANS

THEOREM 1 *If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then A > G.*

PROOF We have,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

$$\therefore \quad A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{1}{2}(\sqrt{a}-\sqrt{b})^2 > 0$$

$$\Rightarrow \quad A > G.$$

Q.E.D.

THEOREM 2 *If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$.*

PROOF We have,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

The equation having a and b as its roots is

$$x^2 - x(a+b) + ab = 0 \text{ or, } x^2 - 2Ax + G^2 = 0 \qquad \left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right]$$

Q.E.D.

THEOREM 3 *If A and G be the A.M. and G.M. between two positive numbers, then the numbers are* $A \pm \sqrt{A^2 - G^2}$. **[NCERT]**

PROOF The equation having its roots as the given numbers is

$$x^2 - 2Ax + G^2 = 0 \Rightarrow x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2} \Rightarrow x = A \pm \sqrt{A^2 - G^2} \qquad \text{Q.E.D.}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **INSERTION OF GEOMETRIC MEANS BETWEEN TWO NUMBERS**

EXAMPLE 1 *Insert 5 geometric means between 576 and 9.*

SOLUTION Let G_1, G_2, G_3, G_4, G_5 be 5 geometric means between $a = 576$ and $b = 9$. Then, $576, G_1, G_2, G_3, G_4, G_5, 9$ is a G.P. with common ratio r given by

$$r = \left(\frac{9}{576}\right)^{\frac{1}{5+1}} = \left(\frac{1}{64}\right)^{\frac{1}{6}} = \frac{1}{2}. \qquad \left[\text{Using: } r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]$$

$$\therefore \quad G_1 = ar = 576 \times \frac{1}{2} = 288, \qquad G_2 = ar^2 = 576 \times \frac{1}{4} = 144,$$

$$G_3 = ar^3 = 576 \times \frac{1}{8} = 72, \qquad G_4 = ar^4 = 576 \times \frac{1}{16} = 36 \text{ and, } G_5 = ar^5 = 576 \times \frac{1}{32} = 18$$

Hence, 288, 144, 72, 36, 18 are the required geometric means between 576 and 9.

Type II **PROBLEMS BASED UPON ARITHMETIC AND GEOMETRIC MEANS**

EXAMPLE 2 *Find the value of n so that* $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ *may be the geometric mean between a and b.* **[NCERT]**

SOLUTION It is given that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the G.M. between a and b.

$$\therefore \quad \frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \sqrt{ab}$$

$$\Rightarrow \quad a^{n+1} + b^{n+1} = a^{\left(n+\frac{1}{2}\right)} b^{1/2} + a^{1/2} b^{\left(n+\frac{1}{2}\right)}$$

$$\Leftrightarrow \quad a^{n+1} - a^{\left(n+\frac{1}{2}\right)} b^{1/2} = a^{1/2} b^{\left(n+\frac{1}{2}\right)} - b^{n+1}$$

$$\Leftrightarrow \quad a^{\left(n+\frac{1}{2}\right)} (a^{1/2} - b^{1/2}) = b^{\left(n+\frac{1}{2}\right)} (a^{1/2} - b^{1/2})$$

$$\Leftrightarrow \quad a^{\left(n+\frac{1}{2}\right)} = b^{\left(n+\frac{1}{2}\right)} \qquad [\because a^{1/2} - b^{1/2} \neq 0, \text{ as } a \neq b]$$

$$\Leftrightarrow \quad \left(\frac{a}{b}\right)^{\left(n+\frac{1}{2}\right)} = 1 \Leftrightarrow \left(\frac{a}{b}\right)^{\left(n+\frac{1}{2}\right)} = \left(\frac{a}{b}\right)^0 \Leftrightarrow n + \frac{1}{2} = 0 \Leftrightarrow n = -\frac{1}{2}$$

EXAMPLE 3 *Find two numbers whose arithmetic mean is 34 and the geometric mean is 16.*

SOLUTION Let the two numbers be a and b such that $a > b$. It is given that AM and GM of a and b are 34 and 16 respectively.

i.e. $\frac{a+b}{2} = 34$ and $\sqrt{ab} = 16$

$\Rightarrow$ $a+b = 68$ and $ab = 256$ ...(i)

$\therefore$ $(a-b)^2 = (a+b)^2 - 4ab$

$\Rightarrow$ $(a-b)^2 = (68)^2 - 4 \times 256 = 3600$

$\Rightarrow$ $a-b = 60$ $[\because a > b \therefore a-b > 0]$

Solving $a+b = 68$ and $a-b = 60$ simultaneously, we get $a = 64$ and $b = 4$.

Hence, the required numbers are 64 and 4.

ALITER Here, $A = 34$ and $G = 16$.

So, the numbers are $A + \sqrt{A^2 - G^2}$ and $A - \sqrt{A^2 - G^2}$

i.e. $34 + \sqrt{34^2 - 16^2}$ and $34 - \sqrt{34^2 - 16^2}$ or, 64 and 4.

EXAMPLE 4 *If the A.M. and G.M. between two numbers are in the ratio $m : n$, then prove that the numbers are in the ratio $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$.* [NCERT]

SOLUTION Let the two numbers be a and b. Let A and G be respectively the arithmetic and geometric means between a and b. Then,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab} \Rightarrow a+b = 2A \text{ and } G^2 = ab \quad \text{...(i)}$$

The equation having a and b as its roots is

$$x^2 - (a+b)x + ab = 0$$

or, $$x^2 - 2Ax + G^2 = 0 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2} \Rightarrow x = A \pm \sqrt{A^2 - G^2}$$

So, the two numbers are $a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$.

It is given that

$$A : G = m : n \Rightarrow A = \lambda m \text{ and } G = \lambda n \text{ for some } \lambda$$

Substituting the values of A and G in $a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$, we get

$$\frac{a}{b} = \frac{\lambda m + \sqrt{\lambda^2 m^2 - \lambda^2 n^2}}{\lambda m - \sqrt{\lambda^2 m^2 - \lambda^2 n^2}} \Rightarrow \frac{a}{b} = \frac{m + \sqrt{m^2 - n^2}}{m - \sqrt{m^2 - n^2}} \Rightarrow a : b = \left(m + \sqrt{m^2 - n^2}\right) : \left(m - \sqrt{m^2 - n^2}\right)$$

LEVEL-2

Type III ON GEOMETRIC AND ARITHMETIC MEANS

EXAMPLE 5 *Find two positive numbers whose difference is 12 and whose A.M. exceeds the G.M. by 2.*

SOLUTION Let the two numbers be a and b such that $a > b$. It is given that

$$a - b = 12 \quad \text{...(i)}$$

It is also given that

$$AM - GM = 2$$

$$\Rightarrow \quad \frac{a+b}{2} - \sqrt{ab} = 2 \quad \left[\because AM = \frac{a+b}{2} \text{ and } GM = \sqrt{ab}\right]$$

$$\Rightarrow \quad a+b-2\sqrt{ab} = 4$$

$$\Rightarrow \quad (\sqrt{a}-\sqrt{b})^2 = 4$$

$$\Rightarrow \quad \sqrt{a}-\sqrt{b} = 2 \qquad \text{...(ii)}$$

Now, $\quad a-b = 12$

$$\Rightarrow \quad (\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = 12$$

$$\Rightarrow \quad (\sqrt{a}+\sqrt{b})\times(2) = 12 \qquad \text{...(iii)}$$

$$\Rightarrow \quad \sqrt{a}+\sqrt{b} = 6 \qquad \text{[Using (ii)]}$$

Solving (ii) and (iii), we get $a = 16$, $b = 4$. Hence, the required numbers are 16 and 4.

EXAMPLE 6 *If a, b, c are in G.P and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then show that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.* **[NCERT]**

SOLUTION It is given that a, b, c are in G.P. Therefore, $b^2 = ac$

Now, $\quad ax^2 + 2bx + c = 0$

$$\Rightarrow \quad ax^2 + 2\sqrt{ac}\,x + c = 0 \Rightarrow (\sqrt{a}x + \sqrt{c})^2 = 0 \Rightarrow \sqrt{a}x + \sqrt{c} = 0 \Rightarrow x = -\sqrt{\frac{c}{a}}$$

It is given that the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root and the equation $ax^2 + 2bx + c = 0$ has equal roots both equal to $-\sqrt{\frac{c}{a}}$.

$\therefore \quad -\sqrt{\frac{c}{a}}$ is a root of the equation $dx^2 + 2ex + f = 0$

$$\Rightarrow \quad d\frac{c}{a} - 2e\sqrt{\frac{c}{a}} + f = 0$$

$$\Rightarrow \quad \frac{d}{a} - 2e\sqrt{\frac{1}{ac}} + \frac{f}{c} = 0 \qquad \text{[Dividing through out by } c\text{]}$$

$$\Rightarrow \quad \frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0 \qquad [\because b^2 = ac]$$

$$\Rightarrow \quad 2\frac{e}{b} = \frac{d}{a} + \frac{f}{c} \quad \Rightarrow \frac{d}{a}, \frac{e}{b}, \frac{f}{c} \text{ are in A.P.}$$

EXAMPLE 7 *Let x be the arithmetic mean and y, z be two geometric means between any two positive numbers. Then, prove that $\frac{y^3 + z^3}{xyz} = 2$.*

SOLUTION Let a and b be two positive numbers. Then,

$$x = \text{A.M. of } a \text{ and } b \Rightarrow x = \frac{a+b}{2} \qquad \text{...(i)}$$

It is given that y and z are two geometric means between a and b. Then, a, y, z, b is a G.P. with common ratio $r = \left(\frac{b}{a}\right)^{\frac{1}{2+1}} = \left(\frac{b}{a}\right)^{1/3}$

$$\therefore \quad y = ar \Rightarrow y = a\left(\frac{b}{a}\right)^{1/3} \Rightarrow y = b^{1/3}a^{2/3} \text{ and, } z = ar^2 \Rightarrow z = a\left(\frac{b}{a}\right)^{2/3} \Rightarrow z = b^{2/3}a^{1/3}$$

$\therefore \quad y^3 + z^3 = (b^{1/3} a^{2/3})^3 + (b^{2/3} a^{1/3})^3 = ba^2 + b^2 a = ab(a+b)$

and, $\quad yz = (b^{1/3} a^{2/3})(b^{2/3} a^{1/3}) = ab.$

Now, $\quad y^3 + z^3 = ab(a+b)$ and $yz = ab$

$\Rightarrow \quad y^3 + z^3 = yz(a+b)$

$\Rightarrow \quad y^3 + z^3 = yz(2x)$ [Using (i)]

$\Rightarrow \quad \dfrac{y^3 + z^3}{xyz} = 2.$

EXAMPLE 8 *If a is the A.M. of b and c and the two geometric means are G_1 and G_2, then prove that* $G_1^3 + G_2^3 = 2abc$.

SOLUTION It is given that a is the A.M. of b and c.

$\therefore \quad a = \dfrac{b+c}{2} \Rightarrow b + c = 2a$ ...(i)

Since G_1 and G_2 are two geometric means between b and c. Therefore, b, G_1, G_2, c is a G.P. with common ratio $r = \left(\dfrac{c}{b}\right)^{1/3}$.

$\therefore \quad G_1 = br = b\left(\dfrac{c}{b}\right)^{1/3} = c^{1/3} b^{2/3}$ and $G_2 = br^2 = b\left(\dfrac{c}{b}\right)^{2/3} = b^{1/3} c^{2/3}$

$\Rightarrow \quad G_1^3 = b^2 c$ and $G_2^3 = bc^2$

$\Rightarrow \quad G_1^3 + G_2^3 = b^2 c + bc^2 = bc(b+c) = 2abc$ [Using (i)]

EXAMPLE 9 *If one geometric mean G and two arithmetic means A_1 and A_2 be inserted between two given quantities, prove that* $G^2 = (2A_1 - A_2)(2A_2 - A_1)$.

SOLUTION Let a and b be two given quantities. It is given that G is the geometric mean of a and b

$\therefore \quad G = \sqrt{ab} \Rightarrow G^2 = ab$ ...(i)

It is also given that A_1, A_2 are two arithmetic means between a and b. Therefore,, a, A_1, A_2, b is an A.P. with common difference $d = \dfrac{b-a}{3}$.

$\therefore \quad A_1 = a + d = a + \dfrac{b-a}{3} = \dfrac{2a+b}{3}, \quad A_2 = a + 2d = a + \dfrac{2(b-a)}{3} = \dfrac{a+2b}{3}$

So, $\quad 2A_1 - A_2 = 2\left(\dfrac{2a+b}{3}\right) - \left(\dfrac{a+2b}{3}\right) = a$ and $2A_2 - A_1 = 2\left(\dfrac{a+2b}{3}\right) - \left(\dfrac{2a+b}{3}\right) = b$

$\therefore \quad (2A_1 - A_2)(2A_2 - A_1) = ab$

$\Rightarrow \quad (2A_1 - A_2)(2A_2 - A_1) = G^2$ [Using (i)]

Type III **PROBLEMS ON A.M. > G.M.**

EXAMPLE 10 *If x, y, z are distinct positive numbers, then prove that* $(x+y)(y+z)(z+x) > 8xyz$.

SOLUTION Using A.M. > G.M., we obtain [NCERT EXEMPLAR]

$$\frac{x+y}{2} > \sqrt{xy},\ \frac{y+z}{2} > \sqrt{yz} \text{ and } \frac{z+x}{2} > \sqrt{zx}$$

$\Rightarrow \quad x + y > 2\sqrt{xy},\ y + z > 2\sqrt{yz}$ and $z + x > 2\sqrt{zx}$

$\Rightarrow \quad (x+y)(y+z)(z+x) > 2\sqrt{xy} \times 2\sqrt{yz} \times 2\sqrt{zx}$

$\Rightarrow \quad (x+y)(y+z)(z+x) > 8xyz.$

EXAMPLE 11 *If $x \in R$, find the minimum value of the expression $3^x + 3^{1-x}$.* **[NCERT EXEMPLAR]**

SOLUTION We know that A.M. > G.M.

$$\therefore \quad \frac{3^x + 3^{1-x}}{2} \geq \sqrt{3^x \times 3^{1-x}} \text{ for all } x \in R$$

$$\Rightarrow \quad \frac{3^x + 3^{1-x}}{2} \geq \sqrt{3} \text{ for all } x \in R$$

$$\Rightarrow \quad 3^x + 3^{1-x} \geq 2\sqrt{3} \text{ for all } x \in R$$

Hence, the minimum value of $3^x + 3^{1-x}$ for any $x \in R$ is $2\sqrt{3}$.

EXAMPLE 12 *If a, b, c, d are four distinct positive numbers in A.P. then show that $bc > ad$.*

SOLUTION It is given that a, b, c, d are in A.P. Therefore, a, b, c are in A.P.

$\Rightarrow$ b is the A.M. of a and c

The G.M. of a and c is $\sqrt{ac}$.

$\therefore$ A.M. of a and c > G.M. of a and c

$\Rightarrow$ $b > \sqrt{ac}$

$\Rightarrow$ $b^2 > ac$ ...(i)

Again, a, b, c, d are in A.P.

$\Rightarrow$ b, c, d are in A.P.

$\Rightarrow$ c is the A.M of b and d.

The G.M. of b and d is $\sqrt{bd}$.

$\therefore$ A.M. of b and d > G.M. of b and d

$\Rightarrow$ $c > \sqrt{bd}$

$\Rightarrow$ $c^2 > bd$ ...(ii)

From (i) and (ii), we obtain

$$b^2c^2 > (ac)(bd) \Rightarrow bc > ad.$$

EXAMPLE 13 *If a, b, c, d are four distinct positive numbers in G.P. then show that $a + d > b + c$.* **[NCERT EXEMPLAR]**

SOLUTION It is given that a, b, c, d are in G.P.

$\therefore$ a, b, c are in G.P.

$\Rightarrow$ b is the G.M of a and c

But, A.M. of a and c is $\frac{a+c}{2}$.

$\therefore$ A.M. of a and c > G.M. of a and c

$\Rightarrow$ $\frac{a+c}{2} > b$

$\Rightarrow$ $a + c > 2b$ ...(i)

Again, a, b, c, d are in G.P.

$\Rightarrow$ b, c, d are in G.P.

$\Rightarrow$ c is the G.M. of b and d.

But, A.M. of b and d is $\frac{b+d}{2}$

$\therefore$ A.M. of b and d > G.M. of b and d

$\Rightarrow$ $\frac{b+d}{2} > c$

$\Rightarrow \quad b+d>2c$ ...(ii)

Adding (i) and (ii), we obtain

$$a+c+b+d>2b+2c$$

$$\Rightarrow \quad a+d>b+c$$

EXERCISE 20.6

LEVEL-1

1. Insert 6 geometric means between 27 and $\frac{1}{81}$.
2. Insert 5 geometric means between 16 and $\frac{1}{4}$.
3. Insert 5 geometric means between $\frac{32}{9}$ and $\frac{81}{2}$.
4. Find the geometric means of the following pairs of numbers:
   (i) 2 and 8 (ii) a^3b and ab^3 (iii) -8 and -2
5. If a is the G.M. of 2 and $\frac{1}{4}$, find a.
6. Find the two numbers whose A.M. is 25 and GM is 20.
7. Construct a quadratic in x such that A.M. of its roots is A and G.M. is G.
8. The sum of two numbers is 6 times their geometric means, show that the numbers are in the ratio $(3+2\sqrt{2}):(3-2\sqrt{2})$. [NCERT]
9. If AM and GM of roots of a quadratic equation are 8 and 5 respectively, then obtain the quadratic equation. [NCERT]
10. If AM and GM of two positive numbers a and b are 10 and 8 respectively, find the numbers [NCERT]

LEVEL-2

11. Prove that the product of n geometric means between two quantities is equal to the nth power of a geometric mean of those two quantities.
12. If the A.M. of two positive numbers a and b $(a>b)$ is twice their geometric mean. Prove that: $a:b = (2+\sqrt{3}):(2-\sqrt{3})$.
13. If one A.M., A and two geometric means G_1 and G_2 inserted between any two positive numbers, show that $\frac{G_1^2}{G_2}+\frac{G_2^2}{G_1} = 2A$.

ANSWERS

1. $9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ 2. $8, 4, 2, 1, \frac{1}{2}$ 3. $\frac{16}{3}, 8, 12, 18, 27$

4.(i) 4 (ii) a^2b^2 (iii) -4 5. $\frac{1}{\sqrt{2}}$ 6. 40, 7. $x^2-2Ax+G^2=0$

9. $x^2-16x+25 = 0$ 10. 4, 16 or 16, 4

HINTS TO NCERT & SELECTED PROBLEMS

8. Let the numbers be a and b. Further, let A and G denote their arithmetic and geometric means respectively. It is given that

$$a+b = 6G \Rightarrow \frac{a+b}{2} = 3G \Rightarrow A = 3G.$$

Numbers a and b are roots of the quadratic equation

$$x^2 - x(a+b) + ab = 0$$

or, $x^2 - 2Ax + G^2 = 0$

or, $x^2 - 6Gx + G^2 = 0$ $[\because A = 3G]$

$$\Rightarrow \quad x = \frac{6G \pm \sqrt{36G^2 - 4G^2}}{2}$$

$\Rightarrow \quad x = 3G \pm 2\sqrt{2}\,G$

$\Rightarrow \quad x = 3G \pm 2\sqrt{2}\,G$

$\Rightarrow \quad x = (3 \pm 2\sqrt{2})G$

$\Rightarrow \quad a = (3 + 2\sqrt{2})G$ and $b = (3 - 2\sqrt{2})G$

Hence, $a : b = (3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$

9. Let a and b be the roots of the quadratic equation. Then, the quadratic equation is

$$x^2 - (a+b)x + ab = 0 \quad \text{...(i)}$$

It is given that AM = 8 and GM = 5.

i.e. $\frac{a+b}{2} = 8$ and $\sqrt{ab} = 5 \Rightarrow a + b = 16$ and $ab = 25$

Substituting these values in (i), we obtain $x^2 - 16x + 25 = 0$ as the required equation.

10. We have,

$$\frac{a+b}{2} = 10 \text{ and } \sqrt{ab} = 8 \Rightarrow a + b = 20 \text{ and } ab = 64$$

Clearly, a and b are roots of the equation

$$x^2 - (a+b)x + ab = 0$$

or, $x^2 - 20x + 64 = 0$

$\Rightarrow \quad (x-16)(x-4) = 0 \Rightarrow x = 4, 16 \Rightarrow a = 4, b = 16$ or $a = 16, b = 4$.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If the fifth term of a G.P. is 2, then write the product of its 9 terms.
2. If $(p+q)^{th}$ and $(p-q)^{th}$ terms of a G.P. are m and n respectively, then write its pth term.
3. If $\log_x a, a^{x/2}$ and $\log_b x$ are in G.P., then write the value of x.
4. If the sum of an infinite decreasing G.P. is 3 and the sum of the squares of its term is $\frac{9}{2}$, then write its first term and common difference.
5. If p^{th}, q^{th} and r^{th} terms of a G.P. are x, y, z respectively, then write the value of $x^{q-r} y^{r-p} z^{p-q}$.
6. If A_1, A_2 be two AM's and G_1, G_2 be two GM's between a and b, then find the value of $\frac{A_1 + A_2}{G_1 G_2}$.
7. If second, third and sixth terms of an A.P. are consecutive terms of a G.P., write the common ratio of the G.P.

8. Write the quadratic equation the arithmetic and geometric means of whose roots are A and G respectively.
9. Write the product of n geometric means between two numbers a and b.
10. If $a = 1 + b + b^2 + b^3 + \ldots$ to ∞, then write b in terms of a given that $|b| < 1$.

ANSWERS

1. 512 2. $\sqrt{mn}$ 3. $\log_a (\log_b a)$ 4. $a = 2, r = \frac{1}{3}$ 5. 1 6. $\frac{a+b}{ab}$
7. 3 8. $x^2 - 2Ax + G^2 = 0$ 9. $(ab)^{n/2}$ 10. $\frac{a-1}{a}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If in an infinite G.P., first term is equal to 10 times the sum of all successive terms, then its common ratio is
(a) 1/10 (b) 1/11 (c) 1/9 (d) 1/20
2. If the first term of a G.P. $a_1, a_2, a_3, \ldots$ is unity such that $4a_2 + 5a_3$ is least, then the common ratio of G.P. is
(a) $-2/5$ (b) $-3/5$ (c) 2/5 (d) none of these
3. If a, b, c are in A.P. and x, y, z are in G.P., then the value of $x^{b-c} y^{c-a} z^{a-b}$ is
(a) 0 (b) 1 (c) xyz (d) $x^a y^b z^c$
4. The first three of four given numbers are in G.P. and their last three are in A.P. with common difference 6. If first and fourth numbers are equal, then the first number is
(a) 2 (b) 4 (c) 6 (d) 8
5. If a, b, c are in G.P. and $a^{1/x} = b^{1/y} = c^{1/z}$, then xyz are in
(a) AP (b) GP (c) HP (d) none of these
6. If S be the sum, P the product and R be the sum of the reciprocals of n terms of a GP, then P^2 is equal to
(a) S/R (b) R/S (c) $(R/S)^n$ (d) $(S/R)^n$
7. The fractional value of $2.\dot{3}5\dot{7}$ is
(a) 2355/1001 (b) 2379/997 (d) 2355/999 (d) none of these
8. If pth, qth and rth terms of an A.P. are in G.P., then the common ratio of this G.P. is
(a) $\frac{p-q}{q-r}$ (b) $\frac{q-r}{p-q}$ (c) pqr (d) none of these
9. The value of $9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27} \ldots$ to ∞, is
(a) 1 (b) 3 (c) 9 (d) none of these
10. The sum of an infinite G.P. is 4 and the sum of the cubes of its terms is 92. The common ratio of the original G.P. is
(a) 1/2 (b) 2/3 (c) 1/3 (d) $-1/2$
11. If the sum of first two terms of an infinite GP is 1 and every term is twice the sum of all the successive terms, then its first term is
(a) 1/3 (b) 2/3 (c) 1/4 (d) 3/4

12. The nth term of a G.P. is 128 and the sum of its n terms is 225. If its common ratio is 2, then its first term is

(a) 1 (b) 3 (c) 8 (d) none of these

13. If second term of a G.P. is 2 and the sum of its infinite terms is 8, then its first term is

(a) 1/4 (b) 1/2 (c) 2 (d) 4

14. If a, b, c are in G.P. and x, y are AM's between a, b and b, c respectively, then

(a) $\frac{1}{x}+\frac{1}{y}=2$ (b) $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$ (c) $\frac{1}{x}+\frac{1}{y}=\frac{2}{a}$ (d) $\frac{1}{x}+\frac{1}{y}=\frac{2}{b}$

15. If A be one A.M. and p, q be two G.M.'s between two numbers, then 2 A is equal to

(a) $\frac{p^3+q^3}{pq}$ (b) $\frac{p^3-q^3}{pq}$ (c) $\frac{p^2+q^2}{2}$ (d) $\frac{pq}{2}$

16. If p, q be two A.M.'s and G be one G.M. between two numbers, then G^2 =

(a) $(2p-q)(p-2q)$ (b) $(2p-q)(2q-p)$ (c) $(2p-q)(p+2q)$ (d) none of these

17. If x is positive, the sum to infinity of the series $\frac{1}{1+x}-\frac{1-x}{(1+x)^2}+\frac{(1-x)^2}{(1+x)^3}-\frac{(1-x)^3}{(1+x)^4}+\ldots$ is

(a) 1/2 (b) 3/4 (c) 1 (d) none of these

18. If $(4^3)(4^6)(4^9)(4^{12})\ldots(4^{3x})=(0.0625)^{-54}$, the value of x is

(a) 7 (b) 8 (c) 9 (d) 10

19. Given that $x>0$, the sum $\sum_{n=1}^{\infty}\left(\frac{x}{x+1}\right)^{n-1}$ equals

(a) x (b) $x+1$ (c) $\frac{x}{2x+1}$ (d) $\frac{x+1}{2x+1}$

20. In a G.P. of even number of terms, the sum of all terms is five times the sum of the odd terms. The common ratio of the G.P. is

(a) $-\frac{4}{5}$ (b) $\frac{1}{5}$ (c) 4 (d) none of these

21. Let x be the A.M. and y, z be two G.M.s between two positive numbers. Then, $\frac{y^3+z^3}{xyz}$ is equal to

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) none of these

22. The product $(32), (32)^{1/6}(32)^{1/36}\ldots$ to ∞ is equal to

(a) 64 (b) 16 (c) 32 (d) 0

23. The two geometric means between the numbers 1 and 64 are

(a) 1 and 64 (b) 4 and 16 (c) 2 and 16 (d) 8 and 16

24. In a G.P. if the $(m+n)^{th}$ term is p and $(m-n)^{th}$ term is q, then its m^{th} term is

(a) 0 (b) pq (c) $\sqrt{pq}$ (d) $\frac{1}{2}(p+q)$

25. Let S be the sum, P be the product and R be the sum of the reciprocals of 3 terms of a G.P. then $P^2R^3 : S^3$ is equal to

(a) $1:1$ (b) (Common ratio)$^n : 1$

(c) (First term)^2 (Common ratio)^2 (d) None of these

ANSWERS

1. (b)	2. (a)	3. (b)	4. (d)	5. (a)	6. (d)	7. (c)	8. (b)
9. (b)	10. (a)	11. (d)	12. (a)	13. (d)	14. (d)	15. (a)	16. (b)
17. (a)	18. (b)	19. (b)	20. (c)	21. (b)	22. (a)	23. (b)	24. (c)
25. (a)							

SUMMARY

1. A sequence of non-zero numbers is called a geometric progression if the ratio of a term and the term preceding to it is always a constant quantity. The constant ratio is called the common ratio of the G.P.
2. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P., then the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called a geometric series.
3. The nth term of a G.P. with first term 'a' and common ratio 'r' is given by $a_n = a\,r^{n-1}$.
4. If a G.P. consists of m terms, then n^{th} term from the end is $(m-n+1)^{\text{th}}$ term from the beginning and is given by ar^{m-n}.

   If l is the last term of a G.P., then nth term from the end is given by $l\left(\frac{1}{r}\right)^{n-1}$.
5. In a G.P., the product of the terms equidistant from the beginning and the end is always same and is equal to the product of first and last term.
6. It is always convenient to select the terms of a G.P. in the following manner:

No. of terms	*Terms*	*Common ratio*
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

7. If sum of n terms of a G.P. with first term 'a' and common ratio is given by

$$S_n = a\left(\frac{r^n-1}{r-1}\right) \text{ or, } S_n = a\left(\frac{1-r^n}{1-r}\right), \text{ if } r \neq 1$$

$$S_n = n, \text{ if } r = 1$$

Also, $$S_n = \frac{a-lr}{1-r} \text{ or, } S_n = \frac{lr-a}{r-1}, \text{ where } l \text{ is the last term.}$$

8. If all the terms of G.P. be multiplied or divided by the same non-zero constant, then it remains a G.P. with the same common ratio.

9. The reciprocals of the terms of a given G.P. form a G.P.
10. If each term of a G.P. be raised to the same power the resulting sequence also forms a G.P.
11. Three numbers a, b, c are in G.P. iff $b^2 = ac$. If a, b, c are in G.P., then b is known as the geometric mean of a and c.
12. If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.
13. Let a and b be two given numbers. If n numbers $G_1, G_2, G_3, \dots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \dots, G_n, b$ is a G.P., then the numbers $G_1, G_2, G_3, \dots, G_n$ are known as n geometric means between a and b.

    The common ratio of the G.P. is given by $r = \left(\frac{b}{a}\right)^{1/n+1}$.
14. The geometric mean of a and b is given by $\sqrt{ab}$.
15. If n geometric means are inserted between two quantities, then the product of n geometric means is n^{th} power of the single geometric mean between the two quantities.
16. If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then
    (i) $A > G$
    (ii) the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$
    (iii) $a : b = \left(A + \sqrt{A^2 - G^2}\right) : \left(A - \sqrt{A^2 - G^2}\right)$
17. If AM and GM between two numbers are in the ratio $m : n$, then the numbers are in the ratio $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$.

CHAPTER 21

SOME SPECIAL SERIES

21.1 SUM TO n TERMS OF SOME SPECIAL SERIES

In this chapter, we intend to discuss the sum to n terms of some other special series viz. series of natural numbers, series of square of natural numbers, series of cubes of natural numbers etc.

21.1.1 SUM OF FIRST n NATURAL NUMBERS

THEOREM *Prove that* : $1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$.

PROOF Let $S_n = 1+2+3+\ldots+n = \sum\limits_{k=1}^{n} k$

Clearly, it is an arithmetic series with first term $a=1$, common difference $d=1$ and last term $l=n$.

$$\therefore \quad S_n = \frac{n}{2}(1+n) = \frac{n(n+1)}{2} \qquad \left[\text{Using: } S_n = \frac{n}{2}(a+l)\right]$$

Hence, $\sum\limits_{k=1}^{n} k = 1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$

21.1.2 SUM OF THE SQUARES OF FIRST n NATURAL NUMBERS

THEOREM *Prove that* : $1^2+2^2+3^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

PROOF Consider the identity $(x+1)^3 - x^3 = 3x^2+3x+1$

Putting $x = 1, 2, 3, \ldots, (n-1)$ and n successively, we get

$$2^3 - 1^3 = 3.1^2 + 3.1 + 1$$
$$3^3 - 2^3 = 3.2^2 + 3.2 + 1$$
$$4^3 - 3^3 = 3.3^2 + 3.3 + 1$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$n^3 - (n-1)^3 = 3.(n-1)^2 + 3.(n-1) + 1$$
$$(n+1)^3 - n^3 = 3.n^2 + 3.n + 1$$

Adding column wise, we obtain

$$(n+1)^3 - 1^3 = 3(1^2+2^2+\ldots+n^2) + 3(1+2+3+\ldots+n) + \underbrace{(1+1+\ldots+1)}_{n \text{ terms}}$$

$$\Rightarrow \quad (n+1)^3 - 1^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n$$

$$\Rightarrow \quad n^3 + 3n^2 + 3n = 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n \qquad \left[\because \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\right]$$

$$\Rightarrow \quad 3\sum_{k=1}^{n} k^2 = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n$$

$$\Rightarrow \quad 3\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{2} = \frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Hence, $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

21.1.3 SUM OF THE CUBES OF FIRST n NATURAL NUMBERS

THEOREM *Prove that* : $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$.

PROOF Consider the identity

$$(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1 \quad \ldots(i)$$

Putting $x = 1, 2, 3, \ldots, (n-1)$ and n successively, we get

$$2^4 - 1^4 = 4.1^3 + 6.1^2 + 4.1 + 1$$

$$3^4 - 2^4 = 4.2^3 + 6.2^2 + 4.2 + 1$$

$$4^4 - 3^4 = 4.3^3 + 6.3^2 + 4.3 + 1$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$

$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$$

Adding column wise, we get

$$(n+1)^4 - 1^4 = 4(1^3 + 2^3 + \ldots + n^3) + 6(1^2 + 2^2 + 3^2 + \ldots + n^2) + 4(1 + 2 + 3 + \ldots + n) + \underbrace{(1 + 1 + \ldots + 1)}_{n \text{ terms}}$$

$$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\left(\sum_{k=1}^{n} k^3\right) + 6\left(\sum_{k=1}^{n} k^2\right) + 4\left(\sum_{k=1}^{n} k\right) + n$$

$$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\left(\sum_{k=1}^{n} k^3\right) + 6\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 4\left\{\frac{n(n+1)}{2}\right\} + n$$

$$\Rightarrow \quad 4\left(\sum_{k=1}^{n} k^3\right) = n^4 + 4n^3 + 6n^2 + 4n - n(n+1)(2n+1) - 2n(n+1) - n$$

$$\Rightarrow \quad 4\left(\sum_{k=1}^{n} k^3\right) = n^4 + 2n^3 + n^2 = n^2(n+1)^2$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = \left(\sum_{k=1}^{n} k\right)^2$$

Hence, $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = \left(\sum_{k=1}^{n} k\right)^2.$$

REMARK 1 *Sometimes for the sake of convenience the sum of a sequence is also denoted by putting the Greek letter Σ (Sigma) before its general term. For example,* $1 + 2 + 3 + \ldots + n$ *can be written as* Σn, $1^2 + 2^2 + \ldots + n^2$ *is denoted by* Σn^2 *and* $1^3 + 2^3 + \ldots + n^3$ *by* Σn^3.

Thus, we have

$$\Sigma n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\Sigma n^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Sigma n^3 = 1^3 + 2^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

and, $$\Sigma a = \underbrace{a + a + \ldots + a}_{(n \text{ terms})} = na$$

REMARK 2 *Proceeding as above, we also have*

$$\Sigma n^4 = 1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

To find the sum to n terms of a given series of natural numbers, we may follow the following algorithm:

ALGORITHM

STEP I *Write nth term of the given series.*

STEP II *Simplify nth term and express it as a polynomial in n i.e.* $T_n = an^3 + bn^2 + cn + d$

STEP III *Take the summation from 1 to n.*

$$\text{i.e. } \sum_{k=1}^{n} T_k = a\left(\sum_{k=1}^{n} k^3\right) + b\left(\sum_{k=1}^{n} k^2\right) + c\left(\sum_{k=1}^{n} k\right) + \sum_{k=1}^{n} d.$$

STEP IV *Use the formulae for* $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ *and* $\sum_{k=1}^{n} k^3$ *and obtain the sum.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the sum to n terms of the series* $1^2 + 3^2 + 5^2 + \ldots$ *to n terms.*

SOLUTION Let T_n be the nth term of this series and S_n denote the sum of its n terms. Then,

$$T_n = [1 + (n-1) \times 2]^2 = (2n-1)^2 = 4n^2 - 4n + 1$$

and, $$S_n = \sum_{k=1}^{n} T_k$$

$\Rightarrow \quad S_n = \sum_{k=1}^{n} (4k^2 - 4k + 1)$

$\Rightarrow \quad S_n = 4\left(\sum_{k=1}^{n} k^2\right) - 4\left(\sum_{k=1}^{n} k\right) + \sum_{k=1}^{n} 1$

$\Rightarrow \quad S_n = 4\,\frac{n(n+1)(2n+1)}{6} - 4\left\{\frac{n(n+1)}{2}\right\} + n$

$\Rightarrow \quad S_n = \frac{n}{3}\{2(n+1)(2n+1) - 6(n+1) + 3\} = \frac{n}{3}\{4n^2 + 6n + 2 - 6n - 6 + 3\} = \frac{n}{3}(4n^2 - 1)$

EXAMPLE 2 *Find the sum of the series* $2^2 + 4^2 + 6^2 + \ldots + (2n)^2$

SOLUTION Let T_n be the nth term of this series and S_n denote the sum of its n terms. Then,

$T_n = (2n)^2 = 4n^2$

$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} 4k^2 = 4\sum_{k=1}^{n} k^2 = 4\left\{\frac{n(n+1)(2n+1)}{6}\right\} = \frac{2}{3}n(n+1)(2n+1)$

EXAMPLE 3 *Find the sum to n terms of the series* $1.2.3 + 2.3.4 + 3.45 + \ldots$ [NCERT]

SOLUTION Let T_n the nth term of the given series. Then,

T_n = (nth term of the sequence formed by first digits in each term)
× (nth term of the sequence of second digits in each term)
× (nth term of the sequence of third digits in each term)

$\Rightarrow \quad T_n$ = (nth term of 1, 2, 3, ...) × (nth term of 2, 3, 4, ...) × (nth term of 3, 4, 5, ...)

$\Rightarrow \quad T_n = \{1 + (n-1) \times 1\} \times \{2 + (n-1) \times 1\} \times \{3 + (n-1) \times 1\}$

$\Rightarrow \quad T_n = n(n+1)(n+2)$

Let S_n denote the sum to n terms of the given series. Then,

$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+1)(k+2)$

$\Rightarrow \quad S_n = \sum_{k=1}^{n} (k^3 + 3k^2 + 2k)$

$\Rightarrow \quad S_n = \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)$

$\Rightarrow \quad S_n = \left\{\frac{n(n+1)}{2}\right\}^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$

$\Rightarrow \quad S_n = \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\} = \frac{n(n+1)}{4}\left\{n^2 + n + 4n + 2 + 4\right\}$

$\Rightarrow \quad S_n = \frac{n(n+1)}{4}(n^2 + 5n + 6) = \frac{n(n+1)(n+2)(n+3)}{4}.$

EXAMPLE 4 *Find the sum of n terms of the series* $1.2^2 + 2.3^2 + 3.4^2 + \ldots$

SOLUTION Let T_n be the nth term of the given series. Then,

T_n = (nth term of the sequence formed by first digits in each term)
× (nth term of the sequence formed by second digits in each term)

$\Rightarrow \quad T_n$ = (nth term of 1, 2, 3, ...) × (nth term of $2^2, 3^2, 4^2 \ldots$)

$\Rightarrow \quad T_n = n(n+1)^2 = n^3 + 2n^2 + n$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (k^3 + 2k^2 + k)$$

$$\Rightarrow \quad S_n = \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$\Rightarrow \quad S_n = \left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \left\{\frac{n(n+1)}{2}\right\}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1\right\} = \frac{n(n+1)}{2}\left\{\frac{3n^2 + 3n + 8n + 4 + 6}{6}\right\}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{2}\left\{\frac{3n^2 + 11n + 10}{6}\right\} = \frac{n(n+1)(n+2)(3n+5)}{12}$$

EXAMPLE 5 *Sum the series* $3.8 + 6.11 + 9.14 + \ldots$ *to* n *terms.* [NCERT]

SOLUTION Let T_n be the nth term of the given series. Then,

T_n = (nth term of 3, 6, 9, ...) × (nth term of 8, 11, 14, ...)

$$\Rightarrow \quad T_n = [3 + (n-1)\times 3]\times[8 + (n-1)\times 3] = 3n(3n+5) = 9n^2 + 15n$$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (9k^2 + 15k) = 9\sum_{k=1}^{n} k^2 + 15\sum_{k=1}^{n} k$$

$$\Rightarrow \quad S_n = 9\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 15\left\{\frac{n(n+1)}{2}\right\} = \frac{3}{2}n(n+1)[2n+1+5] = 3n(n+1)(n+3)$$

EXAMPLE 6 *Find the sum of* n *terms of the series whose* n*th term is*

(i) $2n^2 - 3n + 5$ (ii) $n^2 + 2^n$ [NCERT]

SOLUTION (i) We have, $T_n = 2n^2 - 3n + 5$.

Let S_n denote the sum of n terms of the series whose nth term is T_n. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (2k^2 - 3k + 5) = 2\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 5$$

$$\Rightarrow \quad S_n = 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} - 3\left\{\frac{n(n+1)}{2}\right\} + 5n$$

$$\Rightarrow \quad S_n = \frac{n}{6}\left\{2(n+1)(2n+1) - 9(n+1) + 30\right\} = \frac{n}{6}\left(4n^2 + 6n + 2 - 9n - 9 + 30\right)$$

$$\Rightarrow \quad S_n = \frac{n}{6}(4n^2 - 3n + 23)$$

(ii) We have, $T_n = n^2 + 2^n$

Let S_n denote the sum of n terms of the series having T_n as its nth term. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (k^2 + 2^k) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k$$

$$\Rightarrow \quad S_n = \frac{n(n+1)(2n+1)}{6} + (2^1 + 2^2 + 2^3 + \ldots + 2^n)$$

$$\Rightarrow \quad S_n = \frac{n(n+1)(2n+1)}{6} + 2\left(\frac{2^n - 1}{2-1}\right) = \frac{n(n+1)(2n+1)}{6} + 2(2^n - 1)$$

EXAMPLE 7 *Find the sum of the following series to n terms:*

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\dots$$ [NCERT]

SOLUTION Let T_n be the nth term of the given series. Then,

$$T_n=\frac{1^3+2^3+3^3+\dots+n^3}{1+3+5+\dots+(2n-1)}=\frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\left\{1+(2n-1)\right\}}=\frac{(n+1)^2}{4}=\frac{1}{4}(n^2+2n+1)$$

Let S_n denote the sum of n terms of the given series. Then,

$$S_n=\sum_{k=1}^{n}T_k=\sum_{k=1}^{n}\frac{1}{4}(k^2+2k+1)=\frac{1}{4}\left\{\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k+\sum_{k=1}^{n}1\right\}$$

$$\Rightarrow\quad S_n=\frac{1}{4}\left\{\frac{n(n+1)(2n+1)}{6}+2\left(\frac{n(n+1)}{2}\right)+n\right\}=\frac{n}{24}(2n^2+9n+13)$$

EXAMPLE 8 *Show that:* $\dfrac{1\times2^2+2\times3^2+\dots+n\times(n+1)^2}{1^2\times2+2^2\times3+\dots+n^2\times(n+1)}=\dfrac{3n+5}{3n+1}$. [NCERT]

SOLUTION Let T_n and T_n' be the nth terms of the series in numerator and denominator of LHS. Then,

$$T_n=n\text{th term of the series } 1\times2^2+2\times3^2+\dots+n\times(n+1)^2$$

$$T_n=n(n+1)^2=n^3+2n^2+n$$

and, $$T_n'=n\text{th term of the series } 1^2\times2+2^2\times3+\dots+n^2\times(n+1)$$

$$T_n'=n^2(n+1)=n^3+n^2$$

$$\therefore\quad \text{LHS}=\frac{\sum_{k=1}^{n}T_k}{\sum_{k=1}^{n}T_k'}=\frac{\sum_{k=1}^{n}(k^3+2k^2+k)}{\sum_{k=1}^{n}(k^3+k^2)}=\frac{\sum_{k=1}^{n}k^3+2\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k}{\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k^2}$$

$$\Rightarrow\quad \text{LHS}=\frac{\left\{\frac{n(n+1)}{2}\right\}^2+2\left\{\frac{n(n+1)(2n+1)}{6}\right\}+\left\{\frac{n(n+1)}{2}\right\}}{\left\{\frac{n(n+1)}{2}\right\}^2+\left\{\frac{n(n+1)(2n+1)}{6}\right\}}$$

$$\Rightarrow\quad \text{LHS}=\frac{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2}+\frac{2(2n+1)}{3}+1\right\}}{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2}+\frac{(2n+1)}{3}\right\}}$$

$$\Rightarrow\quad \text{LHS}=\frac{\frac{3n^2+3n+8n+4+6}{6}}{\frac{3n^2+3n+4n+2}{6}}=\frac{3n^2+11n+10}{3n^2+7n+2}=\frac{(3n+5)(n+2)}{(3n+1)(n+2)}=\frac{3n+5}{3n+1}=\text{RHS}$$

EXAMPLE 9 *If S_1, S_2, S_3 are the sums of first n natural numbers, their squares, their cubes respectively, show that* $9S_2^2 = S_3(1+8S_1)$. **[NCERT]**

SOLUTION We have,

$$S_1 = \sum_{k=1}^{n} k = 1+2+\ldots+n = \frac{n(n+1)}{2}$$

$$S_2 = \sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

and, $$S_3 = \sum_{k=1}^{n} k^3 = 1^3+2^3+\ldots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\therefore \quad 9S_2^2 = 9\left\{\frac{n(n+1)(2n+1)}{6}\right\}^2 = \frac{9}{36}\left\{n(n+1)(2n+1)\right\}^2 = \frac{1}{4}\left\{n(n+1)(2n+1)\right\}^2 \quad \ldots (i)$$

and, $$S_3(1+8S_1) = \left\{\frac{n(n+1)}{2}\right\}^2\left\{1+8\times\frac{n(n+1)}{2}\right\} = \left\{\frac{n(n+1)}{2}\right\}^2\left(4n^2+4n+1\right)$$

$$\Rightarrow \quad S_3(1+8S_1) = \frac{n^2(n+1)^2(2n+1)^2}{4} = \frac{1}{4}\left\{n(n+1)(2n+1)\right\}^2 \quad \ldots (ii)$$

From (i) and (ii), we obtain

$$9S_2^2 = S_3(1+8S_1).$$

EXAMPLE 10 *Find the sum to n terms of the series:* $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots$ **[NCERT]**

SOLUTION Let T_n be the nth term of the given series. Then,

$$T_n = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}(2n^3+3n^2+n)$$

Let S_n be the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{6}(2k^3+3k^2+k)$$

$$\Rightarrow \quad S_n = \frac{2}{6}\left(\sum_{k=1}^{n} k^3\right) + \frac{3}{6}\sum_{k=1}^{n} k^2 + \frac{1}{6}\sum_{k=1}^{n} k$$

$$\Rightarrow \quad S_n = \frac{1}{3}\left(\sum_{k=1}^{n} k^3\right) + \frac{1}{2}\left(\sum_{k=1}^{n} k^2\right) + \frac{1}{6}\left(\sum_{k=1}^{n} k\right)$$

$$\Rightarrow \quad S_n = \frac{1}{3}\left\{\frac{n(n+1)}{2}\right\}^2 + \frac{1}{2}\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \frac{1}{6}\left\{\frac{n(n+1)}{2}\right\}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{12}\left\{n(n+1)+(2n+1)+1\right\} = \frac{n(n+1)}{12}(n^2+3n+2) = \frac{n}{12}(n+1)^2(n+2).$$

LEVEL-2

EXAMPLE 11 *The sequence N of natural numbers is divided into classes as follows:*

$$\begin{array}{cccccc} & & 1 & 2 & & \\ & 3 & 4 & 5 & 6 & \\ 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$

....

....

Show that the sum of the numbers in nth row is $n(2n^2+1)$.

SOLUTION Since the first row consists of 2 natural numbers, second row 4 natural numbers, third row 6 natural numbers and so on. So, the total number of natural numbers in nth row is $2n$. Now,

Total number of natural numbers upto the end of nth row

$$= 2 + 4 + 6 + \ldots + 2n = 2(1 + 2 + \ldots + n) = \frac{2n(n+1)}{2} = n(n+1).$$

$\therefore$ Total number of natural numbers upto the end of $(n-1)$th row $= (n-1)(n-1+1) = n(n-1)$.

Let S_n denote the sum of first n natural numbers. Then,

Sum of the natural numbers in nth row

= Sum of the natural numbers upto the end of nth row

− Sum of the natural numbers upto the end of $(n-1)$th row

$$= S_{n(n+1)} - S_{n(n-1)}$$

$$= S_m - S_p, \text{ where } m = n(n+1),\ p = n(n-1).$$

$$= \frac{m(m+1)}{2} - \frac{p(p+1)}{2} \qquad \left[\text{Using: } S_n = \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)\{n(n+1)+1\}}{2} - \frac{n(n-1)\{n(n-1)+1\}}{2}$$

$$= \frac{n}{2}\left\{(n+1)(n^2+n+1) - (n-1)(n^2-n+1\right\} = \frac{n}{2}(4n^2+2) = n(2n^2+1).$$

EXAMPLE 12 *If* $S_k = \dfrac{1+2+\ldots+k}{k}$, *find the value of* $S_1^{\,2} + S_2^{\,2} + \ldots + S_n^{\,2}$.

SOLUTION We have,

$$S_k = \frac{1+2+\ldots+k}{k} = \frac{k(k+1)}{2k} = \frac{k+1}{2}$$

$$\therefore \quad S_1^{\,2} + S_2^{\,2} + \ldots + S_n^{\,2} = \sum_{k=1}^{n} S_k^{\,2} = \sum_{k=1}^{n}\left(\frac{k+1}{2}\right)^2 = \frac{1}{4}\sum_{k=1}^{n}(k+1)^2$$

$$= \frac{1}{4}\left\{2^2 + 3^2 + \ldots + (n+1)^2\right\} = \frac{1}{4}\left\{1^2 + 2^2 + \ldots + (n+1)^2 - 1^2\right\}$$

$$= \frac{1}{4}\left\{\frac{(n+1)(n+1+1)\{2(n+1)+1\}}{6} - 1\right\}$$

$$= \frac{1}{4}\left\{\frac{(n+1)(n+2)(2n+3)}{6} - 1\right\} = \frac{n}{24}(2n^2+9n+13)$$

EXAMPLE 13 *Sum to n terms the series:* $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots$

SOLUTION Clearly, nth term of the given series is negative or positive according as n is even or odd respectively.

<u>CASE I</u> *When n is even* : In this case the given series is

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots + (n-1)^2 - n^2$$

$$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \ldots + ((n-1)^2 - n^2)$$

$$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \ldots + \left\{(n-1)-(n)\right\}(n-1+n)$$

$$= -(1+2+3+4+\ldots+(n-1)+n) = -\frac{n(n+1)}{2}.$$

CASE II *When n is odd :* In this case the given series is

$$(1^2-2^2)+(3^2-4^2)+\ldots+\{(n-2)^2-(n-1)^2\}+n^2$$

$$= (1-2)(1+2)+(3-4)(3+4)+\ldots+\left\{(n-2)-(n-1)\right\}\left\{(n-2)+(n-1)\right\}+n^2$$

$$= -\{1+2+3+4+\ldots+(n-2)+(n-1)\}+n^2 = -\frac{(n-1)(n-1+1)}{2}+n^2 = \frac{n(n+1)}{2}.$$

EXAMPLE 14 *Find the sum of all possible products of the first n natural numbers taken two by two.*

SOLUTION We have,

$(x_1+x_2+\ldots+x_n)^2 = (x_1{}^2+x_2{}^2+\ldots+x_n{}^2) + 2$ (Sum of all possible products taken two at a time)

or, $$\left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n} x_i^2\right) + 2\left(\sum_{i=1, i<j}^{n} \sum_{j=1}^{n} x_i x_j\right)$$

$$\Rightarrow \quad \sum_{i=1, i<j}^{n} \sum_{j=1}^{n} x_i x_j = \frac{1}{2}\left\{\left(\sum_{i=1}^{n} x_i\right)^2 - \left(\sum_{j=1}^{n} x_i^2\right)\right\}$$

$$\therefore \text{ Required sum} = \frac{1}{2}\left\{\left(\sum_{k=1}^{n} k\right)^2 - \left(\sum_{k=1}^{n} k^2\right)\right\} = \frac{1}{2}\left[\left\{\frac{n(n+1)}{2}\right\}^2 - \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{1}{2}\left[\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} - \frac{2n+1}{3}\right\}\right] = \frac{n(n+1)}{4}\left\{\frac{3n^2+3n-4n-2}{6}\right\}$$

$$= \frac{n(n+1)(3n^2-n-2)}{24} = \frac{n(n+1)(n-1)(3n+2)}{24}$$

EXAMPLE 15 *Find the sum of the series:* $1.n + 2.(n-1) + 3.(n-2) + \ldots + (n-1).2 + n.1$.

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = r\{n-(r-1)\} = r(n-r+1) = r\{(n+1)-r\} = (n+1)r - r^2$$

$$\therefore \quad \text{Required sum} = \sum_{r=1}^{n} T_r$$

$$= \sum_{r=1}^{n}\left\{(n+1)r - r^2\right\}$$

$$= (n+1)\left(\sum_{r=1}^{n} r\right) - \left(\sum_{r=1}^{n} r^2\right)$$

$$= (n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(n+2)}{6}$$

EXAMPLE 16 *Find the sum of the series*

$$(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\ldots\ldots \text{ to } n \text{ terms}$$ [NCERT EXEMPLAR]

SOLUTION Let S be the sum of the given series. Then,

$S=(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\ldots\ldots$ to n terms.

$\Rightarrow \quad S=\sum_{r=1}^{n}\left\{(2r+1)^3-(2r)^3\right\}$

$\Rightarrow \quad S=\sum_{r=1}^{n}\left\{(2r+1)-(2r)\right\}\left\{(2r+1)^2+(2r+1)(2r)+(2r)^2\right\}$

$\Rightarrow \quad S=\sum_{r=1}^{n}(12r^2+6r+1)$

$\Rightarrow \quad S=12\sum_{r=1}^{n}r^2+6\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$

$\Rightarrow \quad S=12\times\dfrac{n(n+1)(2n+1)}{6}+6\times\dfrac{n(n+1)}{2}+n$

$\Rightarrow \quad S=2n(n+1)(2n+1)+3n(n+1)+n$

$\Rightarrow \quad S=n(4n^2+6n+2+3n+3+1)=n(4n^2+9n+6)$

EXERCISE 21.1

LEVEL-1

Find the sum of the following series to n terms: (1-7)

1. $1^3+3^3+5^3+7^3+\ldots$
2. $2^3+4^3+6^3+8^3+\ldots$
3. $1.2.5+2.3.6+3.4.7+\ldots$
4. $1.2.4+2.3.7+3.4.10+\ldots$
5. $1+(1+2)+(1+2+3)+(1+2+3+4)+\ldots$
6. $1\times2+2\times3+3\times4+4\times5+\ldots$ [NCERT]
7. $3\times1^2+5\times2^2+7\times3^2+\ldots$ [NCERT]
8. Find the sum of the series whose nth term is:

   (i) $2n^3+3n^2-1$ (ii) n^3-3^n

   (iii) $n(n+1)(n+4)$ [NCERT] (iv) $(2n-1)^2$ [NCERT]
9. Find the 20th term and the sum of 20 terms of the series:

   $2\times4+4\times6+6\times8+\ldots$ [NCERT]

ANSWERS

1. $n^2(2n^2-1)$ 2. $2\{n(n+1)\}^2$ 3. $\dfrac{n}{12}(n+1)(3n^2+23n+34)$

4. $\dfrac{n}{12}(n+1)(9n^2+25n+14)$ 5. $\dfrac{n(n+1)(n+2)}{6}$

6. $\dfrac{n}{3}(n+1)(n+2)$ 7. $\dfrac{n}{6}(n+1)(3n^2+5n+1)$

8. (i) $\dfrac{n}{2}(n^3+4n^2+4n-1)$ (ii) $\left\{\dfrac{n(n+1)}{2}\right\}^2-\dfrac{3}{2}(3^n-1)$

   (iii) $\dfrac{n(n+1)}{12}(3n^2+23n+34)$ (iv) $\dfrac{n}{3}(2n+1)(2n-1)$ 9. 1680, 12320

HINTS TO NCERT & SELECTED PROBLEMS

1. Clearly, $T_n = (2n)^2 = 4n^2$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = 4 \sum_{k=1}^{n} k^2 = 4\,\frac{n(n+1)(2n+1)}{6} = \frac{2}{3}n(n+1)(2n+1)$$

2. Clearly, $T_n = (2n-1)^3$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (2k-1)^3 = 8\sum_{k=1}^{n} k^3 - 12\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

3. Clearly, $T_n = (2n)^3$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (2k)^3 = 8\sum_{k=1}^{n} k^3 = 8\left\{\frac{n(n+1)}{2}\right\}^2 = 2n^2(n+1)^2$$

8. Let T_r be the r^{th} term of the given series and S_n denote the sum of its n terms. Then,

$$T_r = r(r+1),\ r = 1, 2, 3, \ldots$$

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} r(r+1)$$

$$\Rightarrow \quad S_n = \sum_{r=1}^{n} (r^2 + r) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$

$$\Rightarrow \quad S_n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(2n+4)}{6} = \frac{n(n+1)(n+2)}{3}$$

9. Let T_r be the r^{th} term of the series and S_n be the sum of its n terms. Then,

$$T_r = (2r+1)\,r^2,\ r = 1, 2, 3, \ldots n$$

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (2r+1)\,r^2 = \sum_{r=1}^{n} (2r^3 + r^2) = 2\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2$$

$$\Rightarrow \quad S_n = 2\left\{\frac{n(n+1)}{2}\right\}^2 + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(3n^2+5n+1)}{6}$$

10. (iv) We have,

$$T_n = n(n+1)(n+4) = n^3 + 5n^2 + 4n$$

Let S_n be the sum of n terms. Then,

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (r^3 + 5r^2 + 4r)$$

$$\Rightarrow \quad S_n = \sum_{r=1}^{n} r^3 + 5\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r$$

$$\Rightarrow \quad S_n = \left\{\frac{n(n+1)}{2}\right\}^2 + \frac{5n(n+1)(2n+1)}{6} + 4\,\frac{n(n+1)}{2} = \frac{n(n+1)(3n^2+23n+34)}{12}$$

(v) We have, $T_n = (2n-1)^2$

Let S_n be the sum of n terms of the given series. Then,

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (2r-1)^2 = \sum_{r=1}^{n} (4r^2 - 4r + 1)$$

$$\Rightarrow \quad S_n = 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$\Rightarrow \quad S_n = 4\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n = \frac{n(2n-1)(2n+1)}{3}$$

11. Let T_r be the r^{th} term of the given series. Then,

$$T_r = 2r(2r+2),\ r = 1, 2, 3, \dots$$

$$\Rightarrow \quad T_r = 4r^2 + 4r$$

$$\therefore \quad T_{20} = 4 \times 20^2 + 4 \times 20 = 1600 + 80 = 1680$$

Let S_{20} be the sum of 20 terms. Then,

$$S_{20} = \sum_{r=1}^{20} T_r = \sum_{r=1}^{20} (4r^2 + 4r) = 4\left(\sum_{r=1}^{20} r^2\right) + \left(\sum_{r=1}^{20} r\right)$$

$$\Rightarrow \quad S_{20} = 4 \times \frac{20(20+1)(40+1)}{6} + 4 \times \frac{20(20+1)}{2} = 12320$$

21.2 METHOD OF DIFFERENCE

Sometimes the *nth* term of a series can not be determined by the methods discussed so far. If a series is such that the difference between successive terms are either in A.P. or in G.P., then we determine its *n*th term by the method of difference and then find the sum of the series by using the formulas for Σn, Σn^2 and Σn^3. The method of difference is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the sum to n terms of the series:* $3 + 15 + 35 + 63 + \dots$

SOLUTION The difference between the successive terms are $15 - 3 = 12$, $35 - 15 = 20$, $63 - 35 = 28, \dots$. Clearly, these differences are in A.P.

Let T_n be the *nth* term and S_n denote the sum to n terms of the given series. Then,

$$S_n = 3 + 15 + 35 + 63 + \dots + T_{n-1} + T_n \quad \dots(i)$$

$$\text{Also,} \quad S_n = \quad 3 + 15 + 35 + \dots \quad + T_{n-1} + T_n \quad \dots(ii)$$

Subtracting (ii) from (i), we get

$$0 = 3 + \left\{12 + 20 + 28 + \dots + (T_n - T_{n-1})\right\} - T_n$$

$$\Rightarrow \quad T_n = 3 + \frac{(n-1)}{2}\left\{2 \times 12 + (n-1-1) \times 8\right\} = 3 + (n-1)(12 + 4n - 8)$$

$$\Rightarrow \quad T_n = 3 + (n-1)(4n+4) = 4n^2 - 1$$

$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (4k^2 - 1)$

$\Rightarrow \quad S_n = 4 \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} 1 = 4\left\{\frac{n(n+1)(2n+1)}{6}\right\} - n = \frac{n}{3}(4n^2 + 6n - 1)$

REMARK *Instead of determining the nth term of a series by the method of difference as discussed in the above example, we can use the following steps to obtain the same.*

STEP I *Obtain the terms of the series and compute the differences between $T_2 - T_1, T_3 - T_2, T_4 - T_3, ..$ etc. If these are in A.P., then take the nth term as $T_n = an^2 + bn + c$, where a, b, c are constants. Determine constants a, b, c by putting $n = 1, 2, 3$ and equating them with the values of corresponding terms of the given series.*

STEP II *If the differences $T_2 - T_1, T_3 - T_2, T_4 - T_3, ...$ are in G.P. with common ratio r, then take $T_n = ar^{n-1} + bn + c$ and determine constants by putting $n = 1, 2, 3$ in T_n.*

STEP III *If the differences of the differences computed in step I are in A.P., then take $T_n = an^3 + bn^2 + cn + d$ and find the values of a, b, c, d by putting $n = 1, 2, 3, 4$.*

STEP IV *If the differences of the differences computed in step I are in G.P. with common ratio r, then take $T_n = ar^{n-1} + bn^2 + cn + d$ and find the values of a, b, c, d by putting $n = 1, 2, 3, 4$.*

EXAMPLE 2 *Find the sum to n terms of the series:* $1 + 5 + 12 + 22 + 35 + ...$

SOLUTION The sequence of differences between successive terms is 4, 7, 10, 13, ..., which is clearly an A.P. Let T_n be the nth term of the sequence and S_n be the sum of its n terms. Then,

$$S_n = 1 + 5 + 12 + 22 + 35 + ... + T_{n-1} + T_n \quad ...(i)$$

Also, $$S_n = \quad 1 + 5 + 12 + 22 + ... \quad + T_{n-1} + T_n \quad ...(ii)$$

Subtracting (ii) from (i), we get

$$0 = 1 + \left\{4 + 7 + 10 + 13 + ... + (T_n - T_{n-1})\right\} - T_n$$

$$\Rightarrow \quad T_n = 1 + \frac{(n-1)}{2}\left\{2 \times 4 + (n - 1 - 1) \times 3\right\} = 1 + \left(\frac{n-1}{2}\right)(3n + 2)$$

$$\Rightarrow \quad T_n = \frac{1}{2}(3n^2 - n)$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{2}(3k^2 - k) = \frac{3}{2}\sum_{k=1}^{n} k^2 - \frac{1}{2}\sum_{k=1}^{n} k$$

$$\Rightarrow \quad S_n = \frac{3}{2}\left\{\frac{n(n+1)(2n+1)}{6}\right\} - \frac{1}{2}\left\{\frac{n(n+1)}{2}\right\} = \frac{n^2(n+1)}{2}$$

ALITER The given series is : $1 + 5 + 12 + 22 + 35 + ...$

The sequence of differences between successive terms is : $4, 7, 10, 13, ...$

Clearly, it is an A.P. So, let the nth term T_n of the given series be

$$T_n = an^2 + bn + c \quad ...(i)$$

Putting $n = 1, 2, 3$, we get

$T_1 = a + b + c \Rightarrow a + b + c = 1 \quad [\because T_1 = 1]$

$T_2 = 4a + 2b + c \Rightarrow 4a + 2b + c = 5 \quad [\because T_2 = 5]$

$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 12 \quad [\because T_3 = 12]$

Solving these equations, we get

$$a = \frac{3}{2}, b = -\frac{1}{2} \text{ and } c = 0$$

Substituting the values of a, b, c in (i), we get

$$T_n = \frac{3}{2}n^2 - \frac{1}{2}n = \frac{1}{2}(3n^2 - n)$$

$\therefore$ Sum of the given series $= \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} \frac{1}{2}(3r^2 - r) = \frac{3}{2}\sum_{r=1}^{n} r^2 - \frac{1}{2}\sum_{r=1}^{n} r$

$$= \frac{3}{2}\left\{\frac{n(n+1)(2n+1)}{6}\right\} - \frac{1}{2}\left\{\frac{n(n+1)}{2}\right\} = \frac{n^2(n+1)}{2}$$

EXAMPLE 3 *Find the sum of first n terms of the following series:*

(i) $3 + 7 + 13 + 21 + 31 + ...$ [NCERT]

(ii) $5 + 11 + 19 + 29 + 41 + ...$ [NCERT]

SOLUTION (i) The given series is: $3 + 7 + 13 + 21 + 31 + ...$

The sequence of the differences between the successive terms of this series is

$4, 6, 8, 10, ...$

Clearly, it is an A.P. with common difference 2. So, let the nth term of the given series be

$$T_n = an^2 + bn + c \quad ...(i)$$

Putting $n = 1, 2, 3$, we get

$T_1 = a + b + c \Rightarrow a + b + c = 3$ $[\because T_1 = 3]$

$T_2 = 4a + 2b + c \Rightarrow 4a + 2b + c = 7$ $[\because T_2 = 7]$

$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 13$ $[\because T_3 = 13]$

Solving these equations, we get: $a = b = c = 1$

$\therefore \quad T_n = n^2 + n + 1$

Required sum $= \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (r^2 + r + 1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n = \frac{n}{3}(n^2 + 3n + 5)$$

(ii) The given series is: $5 + 11 + 19 + 29 + 41 + ...$

The sequence of the differences between the successive terms is: $6, 8, 10, 12, ...$

Clearly, it is an A.P. So, nth term of the given series is given by

$$T_n = an^2 + bn + c \quad ...(i)$$

Putting $n = 1, 2, 3$, we get

$T_1 = a + b + c \Rightarrow a + b + c = 5$ $[\because T_1 = 5]$

$T_2 = 4a + 2b + c \Rightarrow 4a + 2b + c = 11$ $[\because T_2 = 11]$

$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 19$ $[\because T_3 = 19]$

Solving these three equations, we get: $a = 1, b = 3$ and $c = 1$.

$\therefore \quad T_n = n^2 + 3n + 1$

Required sum $= \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (r^2 + 3r + 1) = \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + n = \frac{n(n+2)(n+4)}{3}$$

LEVEL-2

EXAMPLE 4 *Sum the following series to n terms:* $5 + 7 + 13 + 31 + 85 + \ldots$

SOLUTION The sequence of differences between successive terms is $2, 6, 18, 54, \ldots$

Clearly, it is a G.P. Let T_n be the nth term of the given series and S_n be the sum of its n terms. Then,

$$S_n = 5 + 7 + 13 + 31 + 85 + \ldots + T_{n-1} + T_n \quad \ldots\text{(i)}$$

Also, $$S_n = \quad 5 + \ 7 + 13 + 31 + \ldots \quad + T_{n-1} + T_n \quad \ldots\text{(ii)}$$

Subtracting (ii) from (i), we get

$$0 = 5 + \left\{2 + 6 + 18 + 54 + \ldots + (T_n - T_{n-1})\right\} - T_n$$

$$\Rightarrow \quad 0 = 5 + \frac{2(3^{n-1} - 1)}{(3-1)} - T_n$$

$$\Rightarrow \quad T_n = 5 + (3^{n-1} - 1) = 4 + 3^{n-1}$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (4 + 3^{k-1}) = \sum_{k=1}^{n} 4 + \sum_{k=1}^{n} 3^{k-1}$$

$$\Rightarrow \quad S_n = 4n + (1 + 3 + 3^2 + \ldots + 3^{n-1})$$

$$\Rightarrow \quad S_n = 4n + 1 \times \left(\frac{3^n - 1}{3-1}\right) = 4n + \left(\frac{3^n - 1}{2}\right) = \frac{1}{2}\left(3^n + 8n - 1\right)$$

<u>ALITER</u> The given series is: $5 + 7 + 13 + 31 + 85 + \ldots$

The sequence of the differences between successive terms is $2, 6, 18, 54, \ldots$

Clearly, it is a G.P. with common ratio 3. So, let the nth term of the given series be

$$T_n = a \cdot 3^{n-1} + bn + c \quad \ldots\text{(i)}$$

Putting $n = 1, 2, 3$, we get

$T_1 = a + b + c \Rightarrow a + b + c = 5$ $\quad [\because T_1 = 5]$

$T_2 = 3a + 2b + c \Rightarrow 3a + 2b + c = 7$ $\quad [\because T_2 = 7]$

$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 13$ $\quad [\because T_3 = 13]$

Solving these equations, we get: $a = 1, b = 0$ and $c = 4$

Substituting the values of a, b, c in (i), we get

$$T_n = 3^{n-1} + 4$$

$$\therefore \quad \text{Sum of the series} = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (3^{r-1} + 4) = \sum_{r=1}^{n} 3^{r-1} + \sum_{r=1}^{n} 4$$

$$= (1 + 3 + 3^2 + \ldots + 3^{n-1}) + 4n$$

$$= \frac{3^n - 1}{3-1} + 4n = \frac{3^n - 1}{2} + 4n = \frac{1}{2}(3^n - 1 + 8n)$$

21.3 SUMMATION OF SOME SPECIAL SERIES

In this section, we shall discuss some problems for finding the sum of some series of the form

$$\frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \frac{1}{(a+2d)(a+3d)} + \ldots + \ldots + \frac{1}{(a+(n-2)d)(a+(n-1)d)}$$

In order to find the sum of a finite number of terms of such series, we write its each term as the difference of two terms as given below

$$\frac{1}{a(a+d)} = \frac{1}{d}\left(\frac{1}{a}-\frac{1}{a+d}\right),$$

$$\frac{1}{(a+d)(a+2d)} = \frac{1}{d}\left(\frac{1}{a+d}-\frac{1}{a+2d}\right),$$

$$\frac{1}{(a+2d)(a+3d)} = \frac{1}{d}\left(\frac{1}{a+2d}-\frac{1}{a+3d}\right) \text{ and so on.}$$

$$\therefore \frac{1}{a(a+d)}+\frac{1}{(a+d)(a+2d)}+\frac{1}{(a+2d)(a+3d)}+\ldots+\frac{1}{\{a+(n-2)d\}\{a+(n-1)d\}}$$

$$= \frac{1}{d}\left\{\left(\frac{1}{a}-\frac{1}{a+d}\right)+\left(\frac{1}{a+d}-\frac{1}{a+2d}\right)+\left(\frac{1}{a+2d}-\frac{1}{a+3d}\right)+\ldots+\left(\frac{1}{a+(n-2)d}-\frac{1}{a+(n-1)d}\right)\right\}$$

$$= \frac{1}{d}\left\{\frac{1}{a}-\frac{1}{a+(n-1)d}\right\} = \frac{n-1}{a\{a+(n-1)d\}}$$

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the sum to n terms of the series:*

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n.(n+1)}$$ [NCERT]

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{1}{r(r+1)}, r = 1, 2, \ldots, n$$

$$\Rightarrow \quad T_r = \frac{1}{r}-\frac{1}{r+1}, \ r=1, 2, \ldots, n$$

$$\therefore \quad \text{Required sum} = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1}$$

EXAMPLE 2 *Find the sum to n terms of the series:* $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{1}{(2r-1)(2r+1)}, \ r = 1, 2, 3 \ldots n$$

$$\Rightarrow \quad T_r = \frac{1}{2}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right), \ r=1, 2, 3, \ldots n$$

$$\therefore \quad \text{Required sum} = \sum_{r=1}^{n} T_r = \frac{1}{2}\left\{\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\ldots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right\}$$

$$= \frac{1}{2}\left\{1-\frac{1}{2n+1}\right\}=\frac{n}{2n+1}$$

EXAMPLE 3 *Find the sum :* $\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)}$

SOLUTION We have,

$$\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)}$$

$$= \sum_{r=1}^{n} \frac{1}{a}\left(\frac{1}{ar+b}-\frac{1}{ar+a+b}\right)$$

$$= \frac{1}{a}\sum_{r=1}^{n}\left(\frac{1}{ar+b}-\frac{1}{ar+a+b}\right)$$

$$= \frac{1}{b}\left[\left(\frac{1}{a+b}-\frac{1}{2a+b}\right)+\left(\frac{1}{2a+b}-\frac{1}{3a+b}\right)+\ldots+\left(\frac{1}{na+b}-\frac{1}{(n+1)a+b}\right)\right]$$

$$= \frac{1}{a}\left\{\frac{1}{a+b}-\frac{1}{(n+1)a+b}\right\}=\frac{n}{(a+b)\left\{(n+1)a+b\right\}}$$

LEVEL-2

EXAMPLE 4 *Find the sum to n terms of the series:* $\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+\ldots$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{(2r+1)}{r^2(r+1)^2}, r = 1, 2, 3, \ldots$$

$$\Rightarrow \quad T_r = \frac{(r+1)^2-r^2}{r^2(r+1)^2}, r=1, 2, 3, \ldots$$

$$\Rightarrow \quad T_r = \left\{\frac{1}{r^2}-\frac{1}{(r+1)^2}\right\}, \quad r = 1, 2, 3, \ldots$$

Let S_n be the sum to n terms of the given series. Then,

$$\therefore \quad \text{Required sum} = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left\{\frac{1}{r^2}-\frac{1}{(r+1)^2}\right\} = 1-\frac{1}{(n+1)^2} = \frac{2n+n^2}{(n+1)^2}$$

EXAMPLE 5 *Find the sum to n terms of the series:*

$$\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+\ldots$$

SOLUTION Let T_r be the rth term of the given series. Then,

$$T_r = \frac{r}{1+r^2+r^4}, r = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad T_r = \frac{r}{(r^2+r+1)(r^2-r+1)} = \frac{1}{2}\left\{\frac{2r}{(r^2+r+1)(r^2-r+1)}\right\} = \frac{1}{2}\left\{\frac{(r^2+r+1)-(r^2-r+1)}{(r^2+r+1)(r^2-r+1)}\right\}$$

$$\Rightarrow \quad T_r = \frac{1}{2}\left\{\frac{1}{r^2-r+1}-\frac{1}{r^2+r+1}\right\}, r=1, 2, \ldots, n$$

Let S_n be the sum to n terms of the given series. Then,

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \frac{1}{2}\left\{\sum_{r=1}^{n}\left(\frac{1}{r^2-r+1}-\frac{1}{r^2+r+1}\right)\right\}$$

$$= \frac{1}{2}\left\{\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{13}\right)+\ldots+\left(\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right)\right\}$$

$$= \frac{1}{2}\left\{1-\frac{1}{n^2+n+1}\right\} = \frac{n^2+n}{2(n^2+n+1)}$$

EXERCISE 21.2

LEVEL-1

Sum the following series to n terms:

1. $3+5+9+15+23+\ldots$
2. $2+5+10+17+26+\ldots$
3. $1+3+7+13+21+\ldots$
4. $3+7+14+24+37+\ldots$
5. $1+3+6+10+15+\ldots$
6. $1+4+13+40+121+\ldots$
7. $4+6+9+13+18+\ldots$
8. $2+4+7+11+16+\ldots$
9. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots$
10. $\frac{1}{1.6}+\frac{1}{6.11}+\frac{1}{11.16}+\frac{1}{16.21}+\ldots+\frac{1}{(5n-4)(5n+1)}$

ANSWERS

1. $\frac{n}{3}(n^2+8)$
2. $\frac{n}{6}(2n^2+3n+7)$
3. $\frac{n(n^2+2)}{3}$
4. $\frac{n}{2}(n^2+n+4)$
5. $\frac{n}{6}(n+1)(n+2)$
6. $\frac{1}{4}(3^{n+1}-2n-3)$
7. $\frac{n}{6}(n^2+3n+20)$
8. $\frac{n}{6}(n^2+3n+8)$
9. $\frac{n}{3n+1}$
10. $\frac{n}{5n+1}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the sum of the series: $2+4+6+8+\ldots+2n$.
2. Write the sum of the series: $1^2-2^2+3^2-4^2+5^2-6^2+\ldots+(2n-1)^2-(2n)^2$.

3. Write the sum to n terms of a series whose r^{th} term is: $r + 2^r$.

4. If $\sum_{r=1}^{n} r = 55$, find $\sum_{r=1}^{n} r^3$.

5. If the sum of first n even natural numbers is equal to k times the sum of first n odd natural numbers, then write the value of k.

6. Write the sum of 20 terms of the series: $1 + \frac{1}{2}(1+2) + \frac{1}{3}(1+2+3) + ...$

7. Write the 50th term of the series $2+3+6+11+18+....$

8. Let S_n denote the sum of the cubes of first n natural numbers and s_n denote the sum of first n natural numbers. Then, write the value of $\sum_{r=1}^{n} \frac{S_r}{s_r}$.

ANSWERS

1. $n(n+1)$
2. $-n(2n+1)$
3. $\frac{n(n+1)}{2} + 2^{n+1} - 2$
4. 3025
5. $\frac{n+1}{n}$
6. 115
7. $49^2 + 2$
8. $\frac{n(n+1)(n+2)}{6}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The sum to n terms of the series $\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + ...$ is

(a) $\sqrt{2n+1}$ (b) $\frac{1}{2}\sqrt{2n+1}$ (c) $\sqrt{2n+1}-1$ (d) $\frac{1}{2}\left\{\sqrt{2n+1}-1\right\}$

2. The sum of the series : $\frac{1}{\log_2 4} + \frac{1}{\log_4 4} + \frac{1}{\log_8 4} + ... + \frac{1}{\log_{2^n} 4}$ is

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{12}$ (c) $\frac{n(n+1)}{4}$ (d) none of these

3. The value of $\sum_{r=1}^{n} \left\{(2r-1)a + \frac{1}{b^r}\right\}$ is equal to

(a) $an^2 + \frac{b^{n-1}-1}{b^{n-1}(b-1)}$ (b) $an^2 + \frac{b^n - 1}{b^n(b-1)}$

(c) $an^3 + \frac{b^{n-1}-1}{b^n(b-1)}$ (d) none of these

4. If $\Sigma n = 210$, then $\Sigma n^2 =$

(a) 2870 (b) 2160 (c) 2970 (d) none of these

5. If $S_n = \sum_{r=1}^{n} \frac{1+2+2^2+\ldots \text{Sum to } r \text{ terms}}{2^r}$, then S_n is equal to

(a) $2^n - n - 1$ (b) $1 - \frac{1}{2^n}$ (c) $n - 1 + \frac{1}{2^n}$ (d) $2^n - 1$

6. If $1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \ldots$ to n terms is S. Then, S is equal to

(a) $\frac{n(n+3)}{4}$ (b) $\frac{n(n+2)}{4}$ (c) $\frac{n(n+1)(n+2)}{6}$ (d) n^2

7. Sum of n terms of the series $\sqrt{2} + \sqrt{8} + \sqrt{18} + \sqrt{32} + \ldots$ is

(a) $\frac{n(n+1)}{2}$ (b) $2n(n+1)$ (c) $\frac{n(n+1)}{\sqrt{2}}$ (d) 1

8. The sum of 10 terms of the series $\sqrt{2} + \sqrt{6} + \sqrt{18} + \ldots$ is

(a) $121(\sqrt{6} + \sqrt{2})$ (b) $243(\sqrt{3} + 1)$ (c) $\frac{121}{\sqrt{3} - 1}$ (d) $242(\sqrt{3} - 1)$

9. The sum of the series $1^2 + 3^2 + 5^2 + \ldots$ to n terms is

(a) $\frac{n(n+1)(2n+1)}{2}$ (b) $\frac{n(2n-1)(2n+1)}{3}$ (c) $\frac{(n-1)^2(2n+1)}{6}$ (d) $\frac{(2n+1)^3}{3}$

10. The sum of the series $\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} + \ldots$ to n terms is

(a) $n - \frac{1}{2}(3^{-n} - 1)$ (b) $n - \frac{1}{2}(1 - 3^{-n})$ (c) $n + \frac{1}{2}(3^n - 1)$ (d) $n - \frac{1}{2}(3^n - 1)$

ANSWERS

1. (d) 2. (c) 3. (b) 4. (a) 5. (c) 6. (a) 7. (c) 8. (a)
9. (b) 10. (b)

SUMMARY

1. For any $n \in N$, we have

(i) $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

(ii) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

(iv) $\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$

2. In a series $a_1 + a_2 + a_3 + a_4 + \ldots + a_n + \ldots$,

(i) if the differences $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$ are in A.P., then the nth term is given by

$$a_n = an^2 + bn + c, \text{ where } a, b, c \text{ are constants.}$$

(ii) if the differences $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$ are in G.P. with common ratio r, then

$$a_n = ar^{n-1} + bn + c, \text{ where } a, b, c \text{ are constants.}$$

To determine constants a, b, c we put $n = 1, 2, 3$ and equate them with the values of corresponding terms of the given series.

CHAPTER 22

BRIEF REVIEW OF CARTESIAN SYSTEM OF RECTANGULAR CO-ORDINATES

22.1 CARTESIAN CO-ORDINATE SYSTEM

RECTANGULAR CO-ORDINATE AXES Let $X'OX$ and $Y'OY$ be two mutually perpendicular lines through any point O in the plane of the paper. We call the point O, the origin. Now choose a convenient unit of length and starting from the origin as zero, mark off a number scale on the horizontal line $X'OX$, positive to the right of the origin O and negative to the left of origin O. Also, mark off the same scale on the vertical line $Y'OY$, positive upwards and negative downwards of the origin O.

The line $X'OX$ is called the x-axis or axis of x, the line $Y'OY$ is known as the y-axis or axis of y, and the two lines taken together are called the co-ordinate axes or the axes of coordinates.

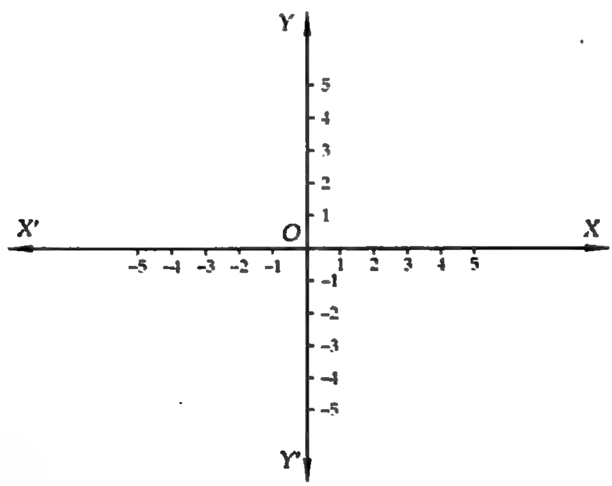


Fig. 22.1

CARTESIAN CO-ORDINATES OF A POINT Let $X'OX$ and $Y'OY$ be the co-ordinate axes, and let P be any point in the plane. Draw perpendiculars PM and PN from P on x *and* y-axis respectively. The length of the directed line segment OM in the units of scale chosen is called the *x-coordinate or abscissa of point P*. Similarly, the length of the directed line segment ON on the same scale is called the *y-coordinate or ordinate* of point P. Let $OM = x$ and $ON = y$. Then the position of the point P in the plane with respect to the coordinate axes is represented by the ordered (x, y). The ordered pair (x, y) is called the coordinates of point P.

Thus, for a given point, the abscissa and ordinate are the distances of the given point from y-axis and x-axis respectively.

The above system of coordinating an ordered pair (x, y) with every point in a plane is called the *Rectangular Cartesian coordinate system.*

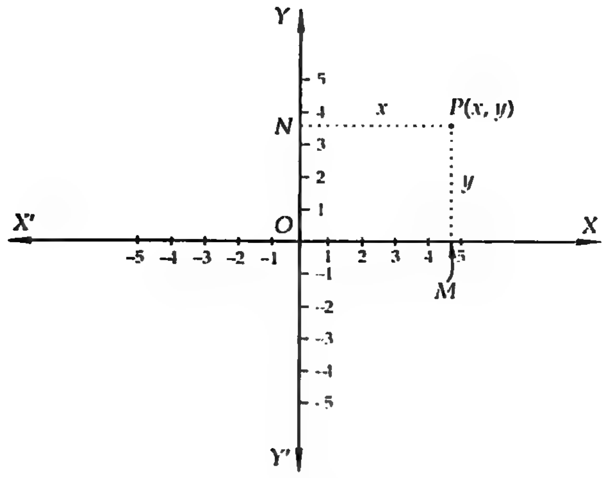


Fig. 22.2

It follows from the above discussion that corresponding to every point P in the Euclidean plane there is a unique ordered pair (x, y) of real numbers called its Cartesian coordinates. Conversely, when we are given an ordered pair (x, y) and a Cartesian coordinate system, we can determine a point in the Euclidean plane having its coordinates (x, y). For this we mark off a directed line segment $OM = x$ on the x-axis and another directed line segment $ON = y$ on y-*axis*. Now, draw perpendiculars at M and N to X and Y axes respectively. The point of intersection of these two perpendiculars determines point P in the Euclidean space having coordinates (x, y).

Thus, there is one-to-one correspondence between the set of all ordered pairs (x, y) of real numbers and the points in the Euclidean plane. The set of all ordered pairs (x, y) of real numbers is called the Cartesian plane and is denoted by R^2.

QUADRANTS Let $X'OX$ and $Y'OY$ be the coordinate axes. We observe that the two axes divide the Euclidean plane into four regions, called the quadrants. The regions XOY, $X'OY$, $X'OY'$ and $Y'OX$ are known as the first, the second, the third and the fourth quadrants respectively. The ray OX is taken as positive x-*axis*, OX' as negative x-axis, OY as positive y-axis and OY' as negative y-axis. In view of the above sign convention the four quadrants are characterised by the following signs of abscissa and ordinate.

I Quadrant : $x > 0, y > 0$
II Quadrant : $x < 0, y > 0$
III Quadrant : $x < 0, y < 0$
IV Quadrant : $x > 0, y < 0$

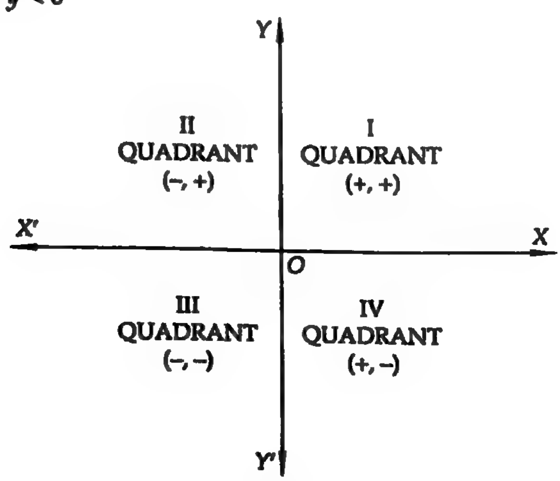


Fig. 22.3

The coordinates of the origin are taken as (0, 0). The coordinates of any point on x-axis are of the form (x, 0) and the coordinates of any point on y-axis are of the form (0, y). Thus, if the abscissa of a point is zero, it would lie somewhere on the y-axis and if its ordinate is zero it would lie on x-axis.

It follows from the above discussion that by simply looking at the coordinates of a point we can tell in which quadrant it would lie.

22.2 DISTANCE BETWEEN TWO POINTS

The distance between any two points in the plane is the length of the line segment joining them.

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i.e. $$PQ = \sqrt{(\textit{Difference of abscisse})^2 + (\textit{Difference of ordinates})^2}$$

SOME USEFUL POINTS

(I) In order to prove that a given figure is a

- (i) square, prove that the four sides are equal and the diagonals are also equal.
- (ii) rhombus, prove that the four sides are equal.
- (iii) rectangle, prove that opposite sides are equal and the diagonals are also equal.
- (iv) a parallelogram, prove that the opposite sides are equal.
- (v) parallelogram but not a rectangie, prove that its opposite sides are equal but the diagonals are not equal.
- (vi) a rhombus but not a square, prove that its all sides are equal but the diagonals are not equal.

(II) For three points to be collinear, prove that the sum of the distances between two pairs of points is equal to the third pair of points.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the distance between the points*

(i) $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$ (ii) $L(a\cos\alpha, a\sin\alpha)$ and $M(a\cos\beta, a\sin\beta)$

SOLUTION (i) Clearly,

$$AB = \sqrt{(at_2^2 - at_1^2)^2 + (2at_2 - 2at_1)^2} = \sqrt{a^2(t_2 - t_1)^2(t_2 + t_1)^2 + 4a^2(t_2 - t_1)^2}$$

$$\Rightarrow \quad AB = a(t_2 - t_1)\sqrt{(t_2 + t_1)^2 + 4}$$

(ii)
$$\begin{aligned}
LM &= \sqrt{(a\cos\beta - a\cos\alpha)^2 + (a\sin\beta - a\sin\alpha)^2} \\
&= \sqrt{a^2(\cos\beta - \cos\alpha)^2 + a^2(\sin\beta - \sin\alpha)^2} \\
&= a\sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2} \\
&= a\sqrt{\cos^2\beta + \cos^2\alpha + \sin^2\beta + \sin^2\alpha - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\
&= a\sqrt{(\cos^2\beta + \sin^2\beta) + (\cos^2\alpha + \sin^2\alpha) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)} \\
&= a\sqrt{1 + 1 - 2\cos(\alpha - \beta)} = a\sqrt{2\{1 - \cos(\alpha - \beta)\}} \\
&= a\sqrt{2 \times 2\sin^2\left(\frac{\alpha - \beta}{2}\right)} = 2a\sin\left(\frac{\alpha - \beta}{2}\right)
\end{aligned}$$

EXAMPLE 2 *Show that four points (0, – 1), (6, 7), (– 2, 3) and (8, 3) are the vertices of a rectangle.*

SOLUTION Let A (0, – 1), B (6, 7), C (– 2, 3) and D (8, 3) be the given points. Then,

$$AD = \sqrt{(8-0)^2 + (3+1)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$BC = \sqrt{(6+2)^2 + (7-3)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$AC = \sqrt{(-2-0)^2 + (3+1)^2} = \sqrt{4+16} = 2\sqrt{5}$$

and, $$BD = \sqrt{(8-6)^2 + (3-7)^2} = \sqrt{4+16} = 2\sqrt{5}$$

$\therefore$ $AD = BC$ and $AC = BD$.

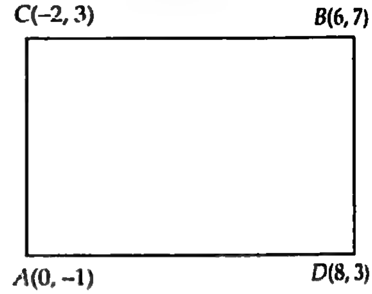


Fig. 22.4

So, $ADBC$ is a parallelogram.

Now, $AB = \sqrt{(6-0)^2 + (7+1)^2} = \sqrt{36+64} = 10$ and, $CD = \sqrt{(8+2)^2 + (3-3)^2} = 10$.

Clearly, $AB^2 = AD^2 + DB^2$ and $CD^2 = CB^2 + BD^2$. Hence, $ADBC$ is a rectangle.

LEVEL-2

EXAMPLE 3 *If the two vertices of an equilateral triangle be (0, 0), (3, $\sqrt{3}$), find the third vertex.*

SOLUTION O (0, 0) and A (3, $\sqrt{3}$) be the given points and let $B\,(x, y)$ be the third vertex of equilateral $\Delta\,OAB$. Then,

$$OA = OB = AB \Rightarrow OA^2 = OB^2 = AB^2 \qquad \text{...(i)}$$

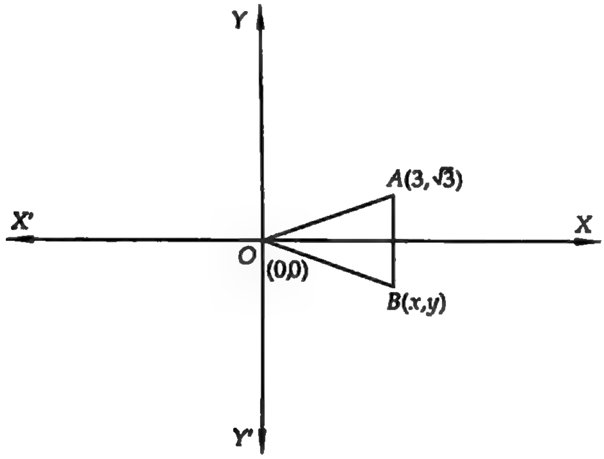


Fig. 22.5

Clearly, $OA^2 = (3-0)^2 + (\sqrt{3}-0)^2 = 12$, $OB^2 = x^2 + y^2$

and, $AB^2 = (x-3)^2 + (y-\sqrt{3})^2 = x^2 + y^2 - 6x - 2\sqrt{3}y + 12$

Now, $OA^2 = OB^2 = AB^2$ [From (i)]

$\Rightarrow$ $OA^2 = OB^2$ and $OB^2 = AB^2$

$\Rightarrow$ $x^2 + y^2 = 12$ and $x^2 + y^2 = x^2 + y^2 - 6x - 2\sqrt{3}\,y + 12$

$\Rightarrow$ $x^2 + y^2 = 12$ and $6x + 2\sqrt{3}\,y = 12$

$\Rightarrow$ $x^2 + y^2 = 12$ and $3x + \sqrt{3}\,y = 6$

$\Rightarrow$ $x^2 + \left(\dfrac{6-3x}{\sqrt{3}}\right)^2 = 12$ $\left[\because\ 3x + \sqrt{3}\,y = 6 \therefore\ y = \dfrac{6-3x}{\sqrt{3}}\right]$

$\Rightarrow \quad 3x^2 + (6-3x)^2 = 36 \Rightarrow 12x^2 - 36x = 0 \Rightarrow x = 0, 3$

Putting $x = 0$ and 3 respectively in $y = \frac{6-3x}{\sqrt{3}}$, we get $y = 2\sqrt{3}$, and $y = -\sqrt{3}$ respectively.

Hence, the coordinates of the third vertex B are $(0, 2\sqrt{3},)$ or $(3, -\sqrt{3})$.

EXAMPLE 4 *If the segments joining the points $A(a, b)$ and $B(c, d)$ subtends an angle θ at the origin, prove that:* $\cos\theta = \frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$.

SOLUTION Let O be the origin. Then,

$$OA^2 = a^2 + b^2, OB^2 = c^2 + d^2 \text{ and, } AB^2 = (c-a)^2 + (d-b)^2$$

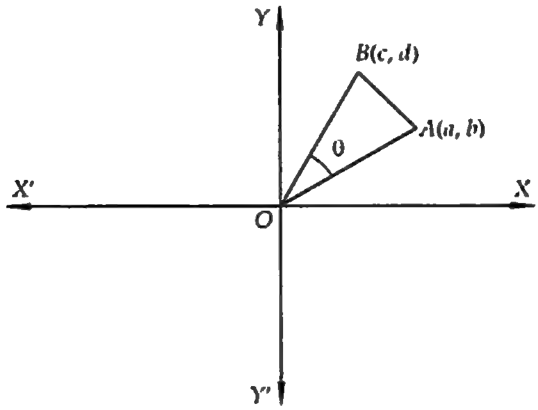


Fig. 22.6

Using cosine formula in ΔOAB, we have

$$AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos\theta$$

$\Rightarrow \quad (c-a)^2 + (d-b)^2 = a^2 + b^2 + c^2 + d^2 - 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos\theta$

$\Rightarrow \quad c^2 + a^2 - 2ac + d^2 + b^2 - 2bd = a^2 + b^2 + c^2 + d^2 - 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos\theta$

$\Rightarrow \quad 2(ac+bd) = 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos\theta$

$\Rightarrow \quad \cos\theta = \frac{ac+bd}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$.

EXAMPLE 5 *Find the coordinates of the circumcentre of the triangle whose vertices are (8, 6), (8, –2) and (2, –2). Also, find its circum-radius.*

SOLUTION Let $A(8, 6)$, $B(8, -2)$ and $C(2, -2)$ be the vertices of the given triangle and let $P(x, y)$ be the circumcentre of this triangle. Then, $PA^2 = PB^2 = PC^2$

Now, $\quad PA^2 = PB^2$

$\Rightarrow \quad (x-8)^2 + (y-6)^2 = (x-8)^2 + (y+2)^2$

$\Rightarrow \quad x^2 + y^2 - 16x - 12y + 100 = x^2 + y^2 - 16x + 4y + 68$

$\Rightarrow \quad 16y = 32 \Rightarrow y = 2$

Again, $\quad PB^2 = PC^2$

$\Rightarrow \quad (x-8)^2 + (y+2)^2 = (x-2)^2 + (y+2)^2$

$\Rightarrow \quad x^2 + y^2 - 16x + 4y + 68 = x^2 + y^2 - 4x + 4y + 8$

$\Rightarrow \quad 12x = 60 \Rightarrow x = 5.$

So, the coordinates of the circumcentre P are $(5, 2)$.

Also, Circum-radius $= PA = PB = PC = \sqrt{(5-8)^2 + (2-6)^2} = 5.$

EXAMPLE 6 *The vertices of a triangle are A (1, 1), B (4, 5) and C (6, 13). Find cos A.*

SOLUTION We know that: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, where $a = BC, b = CA$ and $c = AB$ are the sides of the triangle ABC.

Here,

$$a = BC = \sqrt{(4-6)^2 + (5-13)^2} = \sqrt{68},\ b = CA = \sqrt{(6-1)^2 + (13-1)^2} = \sqrt{169} = 13,$$

and, $$c = AB = \sqrt{(4-1)^2 + (5-1)^2} = 5$$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{169 + 25 - 68}{2 \times 13 \times 5} = \frac{63}{65}$$

EXAMPLE 7 *Let the opposite angular points of a square be (3, 4) and (1, –1). Find the coordinates of the remaining angular points.*

SOLUTION Let $ABCD$ be a square and let $A(3, 4)$ and $C(1, -1)$ be the given angular points. Let $B(x, y)$ be the unknown vertex.

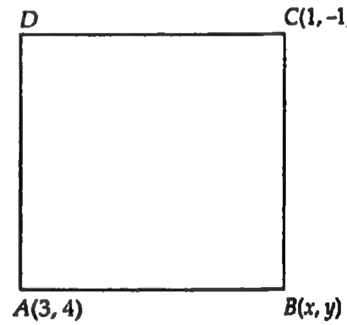


Fig. 22.7

Then, $AB = BC$

$$\Rightarrow \quad AB^2 = BC^2$$

$$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-1)^2 + (y+1)^2$$

$$\Rightarrow \quad 4x + 10y - 23 = 0$$

$$\Rightarrow \quad x = \frac{23 - 10y}{4} \quad \text{...(i)}$$

Applying Pythagoras Theorem in triangle ABC, we obtain

$$AB^2 + BC^2 = AC^2$$

$$\Rightarrow \quad (x-3)^2 + (y-4)^2 + (x-1)^2 + (y+1)^2 = (3-1)^2 + (4+1)^2$$

$$\Rightarrow \quad x^2 + y^2 - 4x - 3y - 1 = 0 \quad \text{...(ii)}$$

Substituting the value of x from (i) into (ii), we get

$$\left(\frac{23 - 10y}{4}\right)^2 + y^2 - (23 - 10y) - 3y - 1 = 0$$

$$\Rightarrow \quad 4y^2 - 12y + 5 = 0 \Rightarrow (2y-1)(2y-5) = 0 \Rightarrow y = \frac{1}{2} \text{ or, } \frac{5}{2}$$

Putting $y = \frac{1}{2}$ and $y = \frac{5}{2}$ respectively in (i), we get $x = \frac{9}{2}$ and $x = -\frac{1}{2}$ respectively.

Hence, the required vertices of the square are $(9/2, 1/2)$ and $(-1/2, 5/2)$.

22.3 AREA OF A TRIANGLE

THEOREM *The area of a triangle, the coordinates of whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is the absolute value of*

$$\frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right] \text{ or, } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

NOTE 1 *Sign of area: If the points A, B, C are plotted in the two dimensional plane and three points are taken in the anticlockwise sense then the area calculated of the triangle ABC will be positive while if the points are taken in clockwise sense then the area calculated will be negative. But, if the points are taken arbitrarily, then the area calculated may be positive or negative, the numerical value being the same in both cases. In case the area calculated is negative we will consider it positive.*

NOTE 2 *To find the area of a polygon we divide it into triangles and take numerical value of the area of each of the triangles.*

CONDITION OF COLLINEARITY OF THREE POINTS Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear iff

(i) Area of $\Delta ABC = 0$

i.e. $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ or, $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

OR

(ii) $AB + BC = AC$ or, $AC + BC = AB$ or, $AC + AB = BC$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE AREA OF A TRIANGLE WHEN COORDINATES OF ITS VERTICES ARE GIVEN**

EXAMPLE 1 *Prove that the area of the triangle whose vertices are $(t, t-2)$, $(t+2, t+2)$ and $(t+3, t)$ is independent of t.*

SOLUTION Let $A = (x_1, y_1) = (t, t-2)$, $B = (x_2, y_2) = (t+2, t+2)$ and $C = (x_3, y_3) = (t+3, t)$ be the vertices of the given triangle. Then,

$$\begin{aligned} \text{Area of } \Delta\, ABC &= \frac{1}{2}\left|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\right| \\ &= \frac{1}{2}\left|\{t(t+2-t) + (t+2)(t-t+2) + (t+3)(t-2-t-2)\}\right| \\ &= \frac{1}{2}\left|\{2t + 2t + 4 - 4t - 12\}\right| = |-4| = 4 \text{ sq. units.} \end{aligned}$$

Clearly, area of ΔABC is independent of t.

Type II **ON FINDING THE AREA OF A QUADRILATERAL WHEN COORDINATES OF ITS VERTICES ARE GIVEN**

EXAMPLE 2 *Find the area of the quadrilateral ABCD whose vertices are respectively A (1, 1), B (7, –3), C (12, 2) and D (7, 21).*

SOLUTION Clearly, Area of quadrilateral $ABCD = |\text{Area of } \Delta ABC| + |\text{Area of } \Delta ACD|$

Now,

$$\begin{aligned} \text{Area of } \Delta ABC &= \frac{1}{2}\left|\{1 \times (-3-2) + 7 \times (2-1) + 12 \times (1+3)\}\right| = \frac{1}{2}\left|(-5 + 7 + 48)\right| \\ &= 25 \text{ sq. units} \\ \text{Area of } \Delta ACD &= \frac{1}{2}\left|\{1 \times (2-21) + 12 \times (21-1) + 7 \times (1-2)\}\right| = \frac{1}{2}\left|(-19 + 240 - 7)\right| \\ &= 107 \text{ sq. units} \end{aligned}$$

$\therefore$ Area of quadrilateral $ABCD = 25 + 107 = 132$ sq. units.

Type III **ON COLLINEARITY OF THREE POINTS**

EXAMPLE 3 *Prove that the points $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$ are collinear.*

SOLUTION Let $A = (x_1, y_1) = (a, b+c)$, $B = (x_2, y_2) = (b, c+a)$ and $C = (x_3, y_3) = (c, a+b)$ be three given points. Then,

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$$

$$= a\{(c+a)-(a+b)\} + b\{(a+b)-(b+c)\} + c\{(b+c)-(c+a)\}$$
$$= a(c-b) + b(a-c) + c(b-a) = 0$$

Hence, the given points are collinear.

Type IV **ON FINDING THE DESIRED RESULT OR UNKNOWN WHEN THREE POINTS ARE COLLINEAR**

EXAMPLE 4 *For what value of k are the points $(k, 2-2k)$ $(-k+1, 2k)$ and $(-4-k, 6-2k)$ are collinear?*

SOLUTION Let three given points be $A=(x_1, y_1)=(k, 2-2k)$, $B=(x_2, y_2)=(-k+1, 2k)$ and $C=(x_3, y_3)=(-4-k, 6-2k)$.

If the given points are collinear, then

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2) = 0$$
$$\Rightarrow \quad k(2k-6+2k)+(-k+1)(6-2k-2+2k)+(-4-k)(2-2k-2k) = 0$$
$$\Rightarrow \quad k(4k-6)-4(k-1)+(4+k)(4k-2)=0$$
$$\Rightarrow \quad 4k^2-6k-4k+4+4k^2+14k-8=0$$
$$\Rightarrow \quad 8k^2+4k-4 = 0 \Rightarrow 2k^2+k-1 = 0 \Rightarrow (2k-1)(k+1) = 0 \Rightarrow k=1/2 \text{ or } k=-1.$$

Hence, the given points are collinear for $k=1/2$ or $k=-1$.

LEVEL-2

Type V **MIXED PROBLEMS BASED UPON THE CONCEPT OF AREA OF A TRIANGLE**

EXAMPLE 5 *If the vertices of a triangle have integral coordinates, prove that the triangle cannot be equilateral.*

SOLUTION Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC, where $x_i, y_i, i=1,2,3$ are integers. Then, the area of ΔABC is given by

$$\Delta = \frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

$\Rightarrow \quad \Delta =$ A rational number $\quad [\because x_i, y_i$ are integers$]$

If possible, let the triangle ABC be an equilateral triangle, then its area is given by

$$\Delta = \frac{\sqrt{3}}{4}(\text{Side})^2 = \frac{\sqrt{3}}{4}(AB)^2 \quad [\because AB=BC=CA]$$

$$\Rightarrow \quad \Delta = \frac{\sqrt{3}}{4}(\text{A positive integer}) \quad [\because \text{Vertices are integers} \therefore AB^2 \text{ is a positive integer}]$$

$\Rightarrow \quad \Delta =$ An irrational number

This is a contradiction to the fact that the area is a rational number. Hence, the triangle cannot be equilateral.

EXAMPLE 6 *If the coordinates of two points A and B are (3, 4) and (5, – 2) respectively. Find the coordinates of any point P, if PA = PB* and Area of $\Delta PAB=10$.

SOLUTION Let the coordinates of P be (x, y). Then,

$$PA = PB$$
$$\Rightarrow \quad PA^2 = PB^2$$
$$\Rightarrow \quad (x-3)^2+(y-4)^2=(x-5)^2+(y+2)^2 \Rightarrow x-3y-1=0 \quad \text{...(i)}$$

Now, Area of $\Delta PAB = 10$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \pm 10$$

$$\Rightarrow \quad 6x + 2y - 26 = \pm 20$$

$\Rightarrow \quad 6x + 2y - 46 = 0 \quad$ or, $\quad 6x + 2y - 6 = 0$

$\Rightarrow \quad 3x + y - 23 = 0 \quad$ or, $\quad 3x + y - 3 = 0$ ...(ii)

Solving $x - 3y - 1 = 0$ and $3x + y - 23 = 0$, we get: $x = 7, y = 2$.

Solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get: $x = 1, y = -0$.

Thus, the coordinates of P are (7, 2) or (1, 0).

EXAMPLE 7 *The coordinates of A, B, C are (6, 3), (– 3, 5) and (4, – 2) respectively and P is any point (x, y). Show that the ratio of the areas of triangles PBC and ABC is* $\left|\dfrac{x+y-2}{7}\right|$.

SOLUTION We have,

$$\frac{\text{Area of } \Delta\, PBC}{\text{Area of } \Delta\, ABC} = \frac{\frac{1}{2}\left| x(5+2) + (-3)(-2-y) + 4(y-5) \right|}{\frac{1}{2}\left| 6(5+2) + (-3)(-2-3) + 4(3-5) \right|}$$

$$\Rightarrow \quad \frac{\text{Area of } \Delta\, PBC}{\text{Area of } \Delta\, ABC} = \frac{|7x + 7y - 14|}{|42 + 15 - 8|} = \frac{7|x+y-2|}{49} = \left|\frac{x+y-2}{7}\right|$$

22.4 SECTION FORMULAE

(i) The coordinates of the point P which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

NOTE 1 *If P is the mid-point of AB, then it divides AB in the ratio 1 : 1, so its coordinates are* $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

Fig. 22.8

NOTE 2 *The following diagram will help to remember the section formula.*

(ii) The coordinates of the point which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m : n$ are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE SECTION POINT WHEN THE SECTION RATIO IS GIVEN**

EXAMPLE 1 *Find the coordinates of points on the line joining the point P (3, – 4) and Q (– 2, 5) that is twice as far from P as from Q.*

SOLUTION Let $R(x, y)$ be the required point. Then, $PR = 2 \cdot RQ$ (given) i.e. $PR : RQ = 2 : 1$. Thus, R divides PQ internally or externally in the ratio 2 : 1.

If R divides PQ internally in the ratio 2 : 1, then the coordinates of R are given by

$$x = \frac{2 \times -2 + 1 \times 3}{2+1} \text{ and } y = \frac{2 \times 5 + 1 \times -4}{2+1} \Rightarrow x = -\frac{1}{3} \text{ and } y = 2.$$

So, the coordinates of R are $(-1/3, 2)$

If R divides PQ externally in the ratio $2:1$, then the coordinates of R are given by

$$x = \frac{2\times -2 - 1\times 3}{2-1} \text{ and } y = \frac{2\times 5 - 1\times -4}{2-1} \Rightarrow x=-7 \text{ and } y=14.$$

Hence, the coordinates of R are $(-7, 14)$.

Type II **ON FINDING THE SECTION RATIO OR AN END POINT OF THE SEGMENT WHEN THE SECTION POINT IS GIVEN**

EXAMPLE 2 *Determine the ratio in which the line $3x + y - 9 = 0$ divides the segment joining the points $(1, 3)$ and $(2, 7)$.*

SOLUTION Suppose the line $3x + y - 9 = 0$ divides the line segment joining A $(1, 3)$ and B $(2, 7)$ in the ratio $k:1$ at point C. Then, the coordinates of C are $\left(\frac{2k+1}{k+1}, \frac{7k+3}{k+1}\right)$. But, C lies on the line $3x + y - 9 = 0$.

$$\therefore \quad 3\left(\frac{2k+1}{k+1}\right) + \frac{7k+3}{k+1} - 9 = 0 \Rightarrow 6k + 3 + 7k + 3 - 9k - 9 = 0 \Rightarrow k = \frac{3}{4}$$

So, the required ratio is $3:4$ internally.

Type III **ON DETERMINATION OF THE TYPE OF A GIVEN QUADRILATERAL**

EXAMPLE 3 *Prove that: $(4, -1)$, $(6, 0)$, $(7, 2)$ and $(5, 1)$ are the vertices of a rhombus. Is it a square?*

SOLUTION Let the given points be A, B, C and D respectively. Then,

Coordinates of the mid-point of AC are $\left(\frac{4+7}{2}, \frac{-1+2}{2}\right) = \left(\frac{11}{2}, \frac{1}{2}\right)$

Coordinates of the mid-point of BD are $\left(\frac{6+5}{2}, \frac{0+1}{2}\right) = \left(\frac{11}{2}, \frac{1}{2}\right)$

Thus, AC and BD have the same mid-point. Hence, $ABCD$ is a parallelogram.

Now, $AB = \sqrt{(6-4)^2 + (0+1)^2} = \sqrt{5}$, $BC = \sqrt{(7-6)^2 + (2-0)^2} = \sqrt{5}$

$\therefore \quad AB = BC$

So, $ABCD$ is a parallelogram whose adjacent sides are equal. Hence, $ABCD$ is a rhombus.

Now, $AC = \sqrt{(7-4)^2 + (2+1)^2} = 3\sqrt{2}$, and $BD = \sqrt{(6-5)^2 + (0-1)^2} = \sqrt{2}$.

Clearly, $AC \neq BD$. So, $ABCD$ is not a square.

Type IV **ON FINDING THE UNKNOWN VERTEX FROM GIVEN POINTS**

EXAMPLE 4 *If the coordinates of the mid-points of the sides of a triangle are $(1, 2)$ $(0, -1)$ and $(2, -1)$. Find the coordinates of its vertices.*

SOLUTION Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC. Let $D(1, 2)$, $E(0, -1)$, and $F(2, -1)$ be the mid-points of sides BC, CA and AB respectively.

Since D is the mid-point of BC.

$$\therefore \quad \frac{x_2+x_3}{2} = 1 \text{ and } \frac{y_2+y_3}{2} = 2 \Rightarrow x_2 + x_3 = 2 \text{ and } y_2 + y_3 = 4 \quad \text{...(i)}$$

Similarly, E and F are the mid-points of CA and AB respectively.

$$\therefore \quad \frac{x_1+x_3}{2} = 0 \text{ and } \frac{y_1+y_3}{2} = -1 \Rightarrow x_1 + x_3 = 0 \text{ and } y_1 + y_3 = -2 \quad \text{...(ii)}$$

$$\text{and,} \quad \frac{x_1+x_2}{2} = 2 \text{ and } \frac{y_1+y_2}{2} = -1 \Rightarrow x_1 + x_2 = 4 \text{ and } y_1 + y_2 = -2 \quad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$(x_2 + x_3) + (x_1 + x_3) + (x_1 + x_2) = 2 + 0 + 4 \text{ and, } (y_2 + y_3) + (y_1 + y_3) + (y_1 + y_2) = 4 - 2 - 2$$

$\Rightarrow x_1 + x_2 + x_3 = 3$ and $y_1 + y_2 + y_3 = 0$ ...(iv)

From (i) and (iv), we get

$$x_1 + 2 = 3 \text{ and } y_1 + 4 = 0 \Rightarrow x_1 = 1 \text{ and } y_1 = -4$$

So, the coordinates of A are $(1, -4)$.

From (ii) and (iv), we get

$$x_2 + 0 = 3 \text{ and } y_2 - 2 = 0 \Rightarrow x_2 = 3 \text{ and } y_2 = 2$$

So, coordinates of B are (3, 2).

From (iii) and (iv), we get

$$x_3 + 4 = 3 \text{ and } y_3 - 2 = 0 \Rightarrow x_3 = -1 \text{ and } y_3 = 2$$

So, coordinates of C are (–1, 2).

Hence, the vertices of the triangle ABC are $A(1, -4)$, $B(3, 2)$ and $C(-1, 2)$.

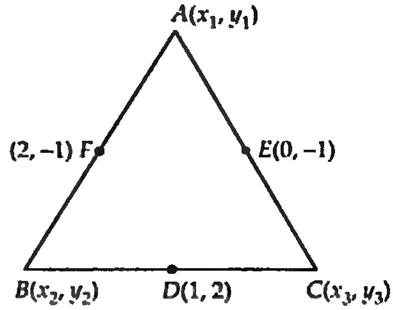


Fig. 22.9

22.5 CENTROID, IN-CENTRE AND EX-CENTRES OF A TRIANGLE

(i) The coordinates of the centroid of the triangle whose vertices are (x_1, y_1) and (x_2, y_2) are (x_3, y_3) are $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$

(ii) The coordinates of the in-centre of a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ $C(x_3, y_3)$ are $\left(\dfrac{ax_1 + bx_2 + cx_3}{a + b + c}, \dfrac{ay_1 + by_2 + cy_3}{a + b + c}\right)$, where $a = BC$, $b = CA$ and $C = AB$

(iii) Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC, and let a, b, c be the lengths of the sides BC, CA, AB respectively. The circle which touches the side BC and two sides AB and AC produced is called the escribed circle opposite to the angle A. The bisectors of the external angles B and C meet at a point I_1 which is the centre of the escribed circle opposite to the angle A. The coordinates of I_1 are given by

$$\left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right)$$

The coordinates of I_2 and I_3 (centres of escribed circles opposite to the angles B and C respectively) are given by

$$I_2\left(\frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c}\right) \text{ and } I_3\left(\frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c}\right)$$

respectively.

LEVEL-1

EXAMPLE 1 *If the coordinates of the mid-points of the sides of a triangle are (1, 1), (2, – 3) and (3, 4). Find its (i) centroid (ii) in-centre.*

SOLUTION Let $P(1, 1)$, $Q(2, -3)$, $R(3, 4)$ be the mid-points of sides AB, BC and CA respectively of triangle ABC. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of triangle ABC. Then, P is the mid-point of AB.

$$\therefore \quad \frac{x_1 + x_2}{2} = 1, \frac{y_1 + y_2}{2} = 1 \Rightarrow x_1 + x_2 = 2 \text{ and } y_1 + y_2 = 2 \quad \text{...(i)}$$

Q is the mid-point of BC.

$\therefore \quad \dfrac{x_2+x_3}{2}=2, \dfrac{y_2+y_3}{2}=-3 \Rightarrow x_2+x_3=4$ and $y_2+y_3=-6$ ...(ii)

R is the mid-point of AC

$\therefore \quad \dfrac{x_1+x_3}{2}=3$ and $\dfrac{y_1+y_3}{2}=4 \Rightarrow x_1+x_3=6$ and $y_1+y_3=8$ ...(iii)

From (i), (ii) and (iii), we get

$$x_1+x_2+x_2+x_3+x_1+x_3 = 2+4+6 \text{ and, } y_1+y_2+y_2+y_3+y_1+y_3 = 2-6+8$$

$\Rightarrow \quad x_1+x_2+x_3 = 6$ and, $y_1+y_2+y_3 = 2$ ...(iv)

From (i) and (iv), we get

$$x_3+2=6 \text{ and } 2+y_3=2 \Rightarrow x_3=4, \; y_3=0$$

So, the coordinates of C are (4, 0).

From (ii) and (iv), we get

$$x_1+4=6 \text{ and } y_1-6=2 \Rightarrow x_1=2, \; y_1=8$$

So, coordinates of A are (2, 8).

From (iii) and (iv), we get

$$x_2+6=6 \text{ and } y_2+8=2 \Rightarrow x_2=0 \text{ and } y_2=-6$$

So, coordinates of B are (0, – 6).

Now, $a = BC = \sqrt{(4-0)^2+(0+6)^2} = \sqrt{52} = 2\sqrt{13}$

$b = AC = \sqrt{(4-2)^2+(0-8)^2} = \sqrt{68} = 2\sqrt{17}$

and, $c = AB = \sqrt{(2-0)^2+(8+6)^2} = \sqrt{200} = 10\sqrt{2}$

The coordinates of the in-centre of the triangle ABC are

$$\left(\frac{a x_1+b x_2+c x_3}{a+b+c}, \frac{a y_1+b y_2+c y_3}{a+b+c}\right)$$

or, $$\left(\frac{(2\sqrt{13})2+(2\sqrt{17})0+(10\sqrt{2})4}{2\sqrt{13}+2\sqrt{17}+10\sqrt{2}}, \frac{(2\sqrt{13})8+(2\sqrt{17})(-6)+(10\sqrt{2})0}{2\sqrt{13}+2\sqrt{17}+10\sqrt{2}}\right)$$

or, $$\left(\frac{4\sqrt{13}+40\sqrt{2}}{2\sqrt{13}+2\sqrt{17}+10\sqrt{2}}, \frac{16\sqrt{13}-12\sqrt{17}}{2\sqrt{13}+2\sqrt{17}+10\sqrt{2}}\right)$$

or, $$\left(\frac{2\sqrt{13}+20\sqrt{2}}{\sqrt{13}+\sqrt{17}+5\sqrt{2}}, \frac{8\sqrt{13}-6\sqrt{17}}{\sqrt{13}+\sqrt{17}+5\sqrt{2}}\right)$$

EXAMPLE 2 *Two vertices of a triangle are (3, – 5) and (– 7, 4). If its centroid is (2, – 1), find the third vertex.*

SOLUTION Let the coordinates of the third vertex be (x, y). Then,

$$\frac{x+3-7}{3}=2 \text{ and } \frac{y-5+4}{3}=-1$$

$\Rightarrow \quad x-4=6$ and $y-1=-3 \Rightarrow x=10$ and $y=-2$

Thus, the coordinates of the third vertex are (10, – 2).

EXERCISE 22.1

LEVEL-1

1. If the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ subtends an angle α at the origin O, prove that: $OP \cdot OQ \cos\alpha = x_1 x_2 + y_1 y_2$.

2. The vertices of a triangle ABC are $A\ (0, 0)$, $B\ (2, -1)$ and $C\ (9, 2)$. Find $\cos B$.

3. Four points $A\ (6, 3)$, $B\ (-3, 5)$, $C\ (4, -2)$ and $D\ (x, 3x)$ are given in such a way that $\dfrac{\Delta DBC}{\Delta ABC} = \dfrac{1}{2}$, find x.

4. The points $A\ (2, 0)$, $B\ (9, 1)$, $C\ (11, 6)$ and $D\ (4, 4)$ ar the vertices of a quadrilateral $ABCD$. Determine whether $ABCD$ is a rhombus or not.

5. Find the coordinates of the centre of the circle inscribed in a triangle whose vertices are $(-36, 7)$, $(20, 7)$ and $(0, -8)$.

6. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find the vertices of the triangle. [NCERT]

7. Find the distance between $P\ (x_1, y_1)$ and $Q\ (x_2, y_2)$ when (i) PQ is parallel to the y-axis (ii) PQ is parallel to the x-axis. [NCERT]

8. Find a point on the x-axis, which is equidistant from the point $(7, 6)$ and $(3, 4)$. [NCERT]

ANSWERS

2. $\dfrac{-11}{\sqrt{290}}$ 3. $\dfrac{11}{8}$ 4. No 5. $(-1, 0)$

6. $(0, a), (0, -a)$ and $(-\sqrt{3}a, 0)$ or $(0, a), (0, -a)$ and $(\sqrt{3}a, 0)$

7. (i) $|y_2 - y_1|$ (ii) $|x_2 - x_1|$ 8. $(15/2, 0)$

HINTS TO NCERT & SELECTED PROBLEM

6. Let BC be the base of equilateral triangle ABC. It is given that $BC = 2a$ and mid-point of BC is at the origin. Therefore, coordinates of B and C are $(0, a)$ and $(0, -a)$ respectively. As the triangle ABC is equilateral. So, vertex A lies on x-axis. Let its coordinates be $(x, 0)$.

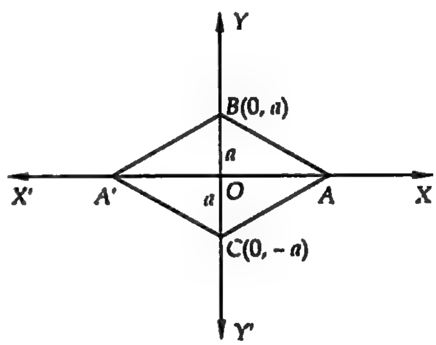


Fig. 22.10

Also, $AB = BC = AC \Rightarrow AB = AC = 2a$

Using Pythagoras theorem in $\Delta\ AOB$, we get

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow \quad (2a)^2 = OA^2 + a^2$$

$\Rightarrow \quad OA = \sqrt{3}a$ So, coordinates of A are $(-\sqrt{3}a, 0)$. Similarly, coordinates of A' are $(-\sqrt{3}a, 0)$.

Hence, the coordinates of the vertices of triangle are $A\ (\sqrt{3}a, 0)$, $B\ (0, a)$ and $C\ (0, -a)$ or, $A'\ (-\sqrt{3}a, 0)$, $B\ (0, a)$ and $C\ (0, -a)$.

7. (i) If PQ is parallel to y-axis, then $x_1 = x_2$

and, PQ = Absolute value of the difference of y-coordinates of P and $Q = |y_2 - y_1|$.

(ii) If PQ is parallel to x-axis, then $y_1 = y_2$

and, PQ = Absolute value of the difference of x-coordinates of P and $= |x_2 - x_1|$.

8. Let $P(x, 0)$ be a point on x-axis which is equidistant from the points $A\ (7, 6)$ and $B\ (3, 4)$. Then,

$$PA = PB$$

$$\Rightarrow \quad PA^2 = PB^2$$

$$\Rightarrow \quad (x-7)^2+(0-6)^2=(x-3)^2+(0-4)^2 \Rightarrow -14x+85 = -6x+25 \Rightarrow 8x=60 \Rightarrow x=\frac{15}{2}$$

Hence, $P\left(\frac{15}{2}, 0\right)$ is the required point.

22.6 LOCUS AND EQUATION TO A LOCUS

LOCUS *The curve described by a point which moves under given condition or conditions is called its locus.*

For example, suppose C is a point in the plane of the paper and P is a variable point in the plane of the paper such that its distance from C is always equal to a (say). It is clear that all the positions of the moving point P lie on the circumference of a circle whose centre is C and whose radius is a. The circumference of this circle is therefore the "Locus" of point P when it moves under the condition that its distance from the point C is always equal to constant a.

Let A and B be two fixed points in the plane of the paper and P be a variable point in the plane of the paper which moves in such a way that its distances from A and B are always equal. Obviously all the positions of the moving point P lie on the perpendicular bisector of AB. Thus, the "locus" of P is the perpendicular bisector of AB when it moves under the condition that its distances from A and B are always equal.

EQUATION TO THE LOCUS OF A POINT *The equation to the locus of a point is the relation which is satisfied by the coordinates of every point on the locus of the point.*

In order to find the locus of a point, we may use the following algorithm.

ALGORITHM

STEP I *Assume the coordinates of the point say, (h, k) whose locus is to be found.*

STEP II *Write the given condition in mathematical form involving h, k.*

STEP III *Eliminate the variable (s), if any.*

STEP IV *Replace h by x and k by y in the result obtained in step III.*

The equation so obtained is the locus of the point which moves under some stated condition (s).

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE LOCUS OF A POINT WHEN GIVEN GEOMETRICAL CONDITIONS DO NOT INVOLVE A VARIABLE**

EXAMPLE 1 *The sum of the squares of the distances of a moving point from two fixed points $(a, 0)$ and $(-a, 0)$ is equal to a constant quantity $2c^2$. Find the equation to its locus.*

SOLUTION Let $P(h, k)$ be any position of the moving point and let $A(a, 0)$, $B(-a, 0)$ be the given points. Then,

$$PA^2+PB^2=2c^2 \qquad \text{[Given]}$$

$$\Rightarrow \quad (h-a)^2+(k-0)^2+(h+a)^2+(k-0)^2=2c^2$$

$$\Rightarrow \quad h^2-2ah+a^2+k^2+h^2+2ah+a^2+k^2=2c^2$$

$$\Rightarrow \quad 2h^2+2k^2+2a^2=2c^2$$

$$\Rightarrow \quad h^2+k^2=c^2-a^2$$

Hence, locus of (h, k) is $x^2+y^2=c^2-a^2$.

EXAMPLE 2 *Find the equation to the locus of a point equidistant from the points $A(1, 3)$ and $B(-2, 1)$.*

SOLUTION Let $P(h, k)$ be any point on the locus. Then,

$$PA = PB \qquad \text{[Given]}$$

$\Rightarrow \quad PA^2 = PB^2 \Rightarrow (h-1)^2 + (k-3)^2 = (h+2)^2 + (k-1)^2 \Rightarrow 6h + 4k = 5$

Hence, locus of (h, k) is $6x + 4y = 5$.

EXAMPLE 3 *Find the locus of a point such that the sum of its distances from the points (0, 2) and (0, – 2) is 6.*

SOLUTION Let $P(h, k)$ be any point on the locus and let $A(0, 2)$ and $B(0, -2)$ be the given points. By the given condition

$$PA + PB = 6$$

$$\Rightarrow \quad \sqrt{(h-0)^2 + (k-2)^2} + \sqrt{(h-0)^2 + (k+2)^2} = 6$$

$$\Rightarrow \quad \sqrt{h^2 + (k-2)^2} = 6 - \sqrt{h^2 + (k+2)^2}$$

$$\Rightarrow \quad h^2 + (k-2)^2 = 36 - 12\sqrt{h^2 + (k+2)^2} + h^2 + (k+2)^2$$

$$\Rightarrow \quad -8k - 36 = -12\sqrt{h^2 + (k+2)^2}$$

$$\Rightarrow \quad (2k+9) = 3\sqrt{h^2 + (k+2)^2}$$

$$\Rightarrow \quad (2k+9)^2 = 9(h^2 + (k+2)^2)$$

$$\Rightarrow \quad 4k^2 + 36k + 81 = 9h^2 + 9k^2 + 36k + 36$$

$$\Rightarrow \quad 9h^2 + 5k^2 = 45$$

Hence, locus of (h, k) is $9x^2 + 5y^2 = 45$.

EXAMPLE 4 *Find the locus of a point, so that the join of (– 5, 1) and (3, 2) subtends a right angle at the moving point.*

SOLUTION Let $P(h, k)$ be a moving point and let $A(-5, 1)$ and $B(3, 2)$ be given points. By the given condition

$$\angle APB = 90^\circ$$

$\therefore \quad \Delta APB$ is a right angle triangle

$$\Rightarrow \quad AB^2 = AP^2 + PB^2$$

$$\Rightarrow \quad (3+5)^2 + (2-1)^2 = (h+5)^2 + (k-1)^2 + (h-3)^2 + (k-2)^2$$

$$\Rightarrow \quad 65 = 2(h^2 + k^2 + 2h - 3k) + 39$$

$$\Rightarrow \quad h^2 + k^2 + 2h - 3k - 13 = 0$$

Hence, locus of (h, k) is $x^2 + y^2 + 2x - 3y - 13 = 0$.

EXAMPLE 5 *A point moves so that the sum of its distances from* $(ae, 0)$ and $(-ae, 0)$ *is* $2a$*, prove that the equation to its locus is* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$*, where* $b^2 = a^2(1 - e^2)$.

SOLUTION Let $P(h, k)$ be the moving point such that the sum of its distances from $A(ae, 0)$ and $B(-ae, 0)$ is $2a$. Then,

$$PA + PB = 2a$$

$$\Rightarrow \quad \sqrt{h - ae)^2 + (k-0)^2} + \sqrt{(h+ae)^2 + (k-0)^2} = 2a$$

$$\Rightarrow \quad \sqrt{(h-ae)^2 + k^2} = 2a - \sqrt{(h+ae)^2 + k^2}$$

$$\Rightarrow \quad (h-ae)^2 + k^2 = 4a^2 + (h+ae)^2 + k^2 - 4a\sqrt{(h+ae)^2 + k^2} \qquad \text{[Squaring both sides]}$$

$$\Rightarrow \quad -4aeh - 4a^2 = -4a\sqrt{(h+ae)^2 + k^2}$$

$\Rightarrow \quad (eh + a) = \sqrt{(h + ae)^2 + k^2}$

$\Rightarrow \quad (eh + a)^2 = (h + ae)^2 + k^2$

$\Rightarrow \quad e^2 h^2 + 2aeh + a^2 = h^2 + a^2 e^2 + 2aeh + k^2$

$\Rightarrow \quad h^2 (1 - e^2) + k^2 = a^2 (1 - e^2)$

$\Rightarrow \quad \dfrac{h^2}{a^2} + \dfrac{k^2}{a^2 (1 - e^2)} = 1$

Hence, locus of (h, k) is $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 (1 - e^2)} = 1$ or, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b^2 = a^2 (1 - e^2)$

EXAMPLE 6 *Find the equation to the locus of a point which moves so that the sum of its distances from (3, 0) and (–3, 0) is less than 9.*

SOLUTION Let $P(h, k)$ be the moving point such that the sum of its distances from $A(3, 0)$ and $B(-3, 0)$ is less than 9. Then,

$PA + PB < 9$

$\Rightarrow \quad \sqrt{(h - 3)^2 + (k - 0)^2} + \sqrt{(h + 3)^2 + (k - 0)^2} < 9$

$\Rightarrow \quad \sqrt{(h - 3)^2 + k^2} < 9 - \sqrt{(h + 3)^2 + k^2}$

$\Rightarrow \quad (h - 3)^2 + k^2 < \{9 - \sqrt{(h + 3)^2 + k^2}\}^2 \qquad [\because x < y \Rightarrow x^2 < y^2 \text{ for } x, y > 0]$

$\Rightarrow \quad (h - 3)^2 + k^2 < 81 + (h + 3)^2 + k^2 - 18\sqrt{(h + 3)^2 + k^2}$

$\Rightarrow \quad -12h - 81 < -18\sqrt{(h + 3)^2 + k^2}$

$\Rightarrow \quad 4h + 27 > 6\sqrt{(h + 3)^2 + k^2}$

$\Rightarrow \quad (4h + 27)^2 > 36[(h + 3)^2 + k^2]$

$\Rightarrow \quad 20h^2 + 36k^2 < 405$

Hence, the locus of (h, k) is $20x^2 + 36y^2 < 405$.

LEVEL-2

Type II ON FINDING THE LOCUS OF A POINT WHEN GIVEN GEOMETRICAL CONDITIONS INVOLVE SOME VARIABLE(S)

EXAMPLE 7 *Find the locus of the point of intersection of the lines* $x \cos \alpha + y \sin \alpha = a$ and $x \sin \alpha - y \cos \alpha = b$, where α is a variable.

SOLUTION Let $P(h, k)$ be the point of intersection of the given lines. Then,

$h \cos \alpha + k \sin \alpha = a \qquad$ ...(i)

and, $h \sin \alpha - k \cos \alpha = b \qquad$ ...(ii)

Here α is a variable. So, we have to eliminate α. Squaring and adding (i) and (ii), we get

$$(h \cos \alpha + k \sin \alpha)^2 + (h \sin \alpha - k \cos \alpha)^2 = a^2 + b^2 \Rightarrow h^2 + k^2 = a^2 + b^2$$

Hence, the locus of (h, k) is $x^2 + y^2 = a^2 + b^2$.

EXAMPLE 8 *A rod of length l slides with its ends on two perpendicular lines. Find the locus of its mid-point.*

SOLUTION Let the two perpendicular lines be the coordinate axes. Let AB be a rod of length l. Let the coordinates of A and B be $(a, 0)$ and $(0, b)$ respectively. As the rod slides the values of a and b change. So a and b are two variables. Let $P(h, k)$ be the mid-point of the rod AB in one of the infinite positions it attains. Then,

$$h = \frac{a+0}{2} \text{ and } k = \frac{0+b}{2}$$

$$\Rightarrow \quad h = \frac{a}{2} \text{ and } k = \frac{b}{2} \quad \text{...(i)}$$

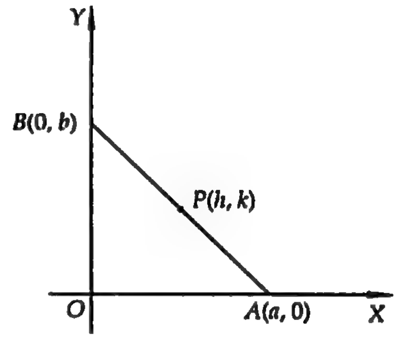


Fig. 22.11

From ΔOAB, we have

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow \quad a^2 + b^2 = l^2$$

$$\Rightarrow \quad (2h)^2 + (2k)^2 = l^2 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad 4h^2 + 4k^2 = l^2$$

Hence, the locus of (h, k) is $4x^2 + 4y^2 = l^2$.

EXAMPLE 9 *AB is a variable line sliding between the coordinate axes in such a way that A lies on x-axis and B lies on y-axis. If P is a variable point on AB such that $PA = b$, $PB = a$ and $AB = a + b$, find the equation of the locus of P.*

SOLUTION Let $P(h, k)$ be a variable point on AB such that $\angle OAB = \theta$, where θ is a variable. From triangles ALP and PMB, we have

$$\sin\theta = \frac{k}{b} \quad \text{...(i)} \qquad \cos\theta = \frac{h}{a} \quad \text{...(ii)}$$

Here, θ is a variable. So, we have to eliminate θ.

Squaring (i) and (ii) and adding, we get: $\dfrac{k^2}{b^2} + \dfrac{h^2}{a^2} = 1$

Hence, the locus of (h, k) is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

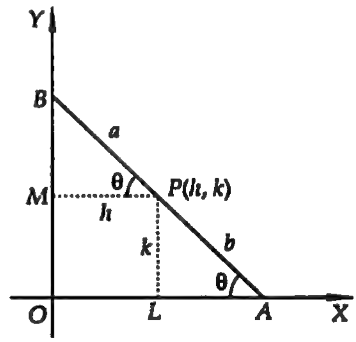


Fig. 22.12

EXAMPLE 10 *If O is the origin and Q is a variable point on $x^2 = 4y$. Find the locus of the mid-point of OQ.*

SOLUTION Let the coordinates of Q be (a, b) and let $P(h, k)$ be the mid-point of OQ. Then,

$$h = \frac{a+0}{2} = \frac{a}{2} \text{ and } k = \frac{0+b}{2} = \frac{b}{2} \Rightarrow a = 2h \text{ and } b = 2k \quad \text{...(i)}$$

Here, a and b are two variables which are to be eliminated. Since, (a, b) lies on $x^2 = 4y$.

$$\therefore \quad a^2 = 4b$$

$$\Rightarrow \quad (2h)^2 = 4(2k) \quad \text{[Using (i)]}$$

$$\Rightarrow \quad h^2 = 2k$$

Hence, the locus of (h, k) is $x^2 = 2y$.

EXERCISE 22.2

LEVEL-1

1. Find the locus of a point equidistant from the point (2, 4) and the y-axis.
2. Find the equation of the locus of a point which moves such that the ratio of its distances from (2, 0) and (1, 3) is 5 : 4.
3. A point moves as so that the difference of its distances from $(ae, 0)$ and $(-ae, 0)$ is $2a$, prove that the equation to its locus is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2 (e^2 - 1)$.
4. Find the locus of a point such that the sum of its distances from (0, 2) and (0, –2) is 6.
5. Find the locus of a point which is equidistant from (1, 3) and x-axis.
6. Find the locus of a point which moves such that its distance from the origin is three times its distance from x-axis.
7. A (5, 3), B (3, – 2) are two fixed points; find the equation to the locus of a point P which moves so that the area of the triangle PAB is 9 units.
8. Find the locus of a point such that the line segments having end points (2, 0) and (–2, 0) subtend a right angle at that point.
9. If A (–1, 1) and B (2, 3) are two fixed points, find the locus of a point P so that the area of $\Delta PAB = 8$ sq. units.

LEVEL-2

10. A rod of length l slides between the two perpendicular lines. Find the locus of the point on the rod which divides it in the ratio 1 : 2.
11. Find the locus of the mid-point of the portion of the line $x \cos \alpha + y \sin \alpha = p$ which is intercepted between the axes.
12. If O is the origin and Q is a variable point on $y^2 = x$. Find the locus of the mid-point of OQ.

ANSWERS

1. $y^2 - 8y - 4x + 20 = 0$
2. $9x^2 + 9y^2 + 14x - 150y + 186 = 0$
4. $9x^2 + 5y^2 = 45$
5. $x^2 - 2x - 6y + 10 = 0$
6. $x^2 = 8y^2$
7. $5x - 2y - 1 = 0$ or $5x - 2y - 37 = 0$
8. $x^2 + y^2 = 4$
9. $2x - 3y - 11 = 0$, $2x - 3y + 21 = 0$
10. $\frac{x^2}{4} + y^2 = \frac{l^2}{9}$
11. $p^2 (x^2 + y^2) = 4x^2 y^2$
12. $2y^2 = x$

HINTS TO SELECTED PROBLEM

8. Let $P(h, k)$ be the variable point and let $A(2, 0)$ and $B(-2, 0)$ be the given points. Then, $\angle APB = \pi/2 \Rightarrow AB^2 = PA^2 + PB^2$.

22.7 SHIFTING OF ORIGIN

Let O be the origin and let $x'Ox$ and $y'Oy$ be the axis of x and y respectively. Let O' and P be two points in the plane having coordinates (h, k) and (x, y) respectively referred to $X'OX$ and $Y'OY$ as the coordinate axes. Let the origin be transferred to O' and let $X'O'X$ and $Y'O'Y$ be new rectangular axes. Let the coordinates of P referred to new axes as the coordinate axes be (X, Y). Then,

$$O'N = X, \; PN = Y, OM = x, PM = y, OL = h \text{ and, } O'L = k.$$

Now, $x = OM = OL + LM = OL + O'N = h + X$

and, $y = PM = PN + NM = PN + O'L = Y + k$

$\therefore$ $x = X + h$ and $y = Y + k$.

Thus, if (x, y) are coordinates of a point referred to old axes and (X, Y) are the coordinates of the same point referred to new axes, then

$$x = X + h \text{ and } y = Y + k$$

i.e., (Old x-coordinate) = (New x-coordinate) + h

and, (Old y-coordinate) = (New y-coordinate) + k.

If therefore the origin is shifted at a point (h, k) we must substitute $X + h$ and $Y + k$ for x and y respectively.

The transformation formula from new axes to old axes is: $X = x - h$, $Y = y - k$

The coordinates of the old origin referred to the new axes are $(-h, -k)$.

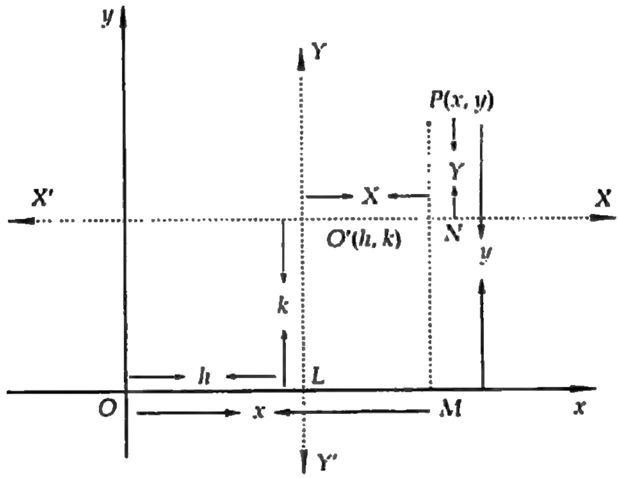


Fig. 22.13

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If the axes are shifted to the point $(1, -2)$ without rotation, what do the following equations become?*

(i) $2x^2 + y^2 - 4x + 4y = 0$ (ii) $y^2 - 4x + 4y + 8 = 0$

SOLUTION (i) Substituting $x = X + 1$, $y = Y + (-2) = Y - 2$ in the equation $2x^2 + y^2 - 4x + 4y = 0$, we get

$$2(X+1)^2 + (Y-2)^2 - 4(X+1) + 4(Y-2) = 0 \Rightarrow 2X^2 + Y^2 = 6.$$

(ii) Substituting $x = X + 1$, $y = Y - 2$ in the equation $y^2 - 4x + 4y + 8 = 0$, we get

$$(Y-2)^2 - 4(X+1) + 4(Y-2) + 8 = 0 \Rightarrow Y^2 = 4X$$

EXAMPLE 2 *At what point the origin be shifted, if the coordinates of a point (4, 5) become (–3, 9)?*

SOLUTION Let (h, k) be the point to which the origin is shifted. Then,

$$x = 4, \ y = 5, \ X = -3, \ Y = 9$$

$\therefore$ $x = X + h$ and $y = Y + k \Rightarrow 4 = -3 + h$ and $5 = 9 + k \Rightarrow h = 7$ and $k = -4$

Hence, the origin must be shifted to $(7, -4)$

EXAMPLE 3 *Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain term in y and the constant term.*

SOLUTION Let the origin be shifted to (h, k). Then, $x = X + h$ and $y = Y + k$.
Substituting $x = X + h$, $y = Y + k$ in the equation $y^2 + 4y + 8x - 2 = 0$, we get

$$(Y + k)^2 + 4(Y + k) + 8(X + h) - 2 = 0$$

$$\Rightarrow \quad Y^2 + (4 + 2k)Y + 8X + (k^2 + 4k + 8h - 2) = 0$$

For this equation to be free from the term containing Y and the constant term, we must have

$$4 + 2k = 0 \text{ and } k^2 + 4k + 8h - 2 = 0 \Rightarrow k = -2 \text{ and } h = \frac{3}{4}.$$

Hence, the origin is shifted at the point $(3/4, -2)$.

EXAMPLE 4 *Find the point to which the origin should be shifted so that the equation* $y^2 - 6y - 4x + 13 = 0$ *is transformed to the form* $y^2 + Ax = 0$.

SOLUTION Let the origin be shifted to the point (h, k). Then, $x = X + h$ and $y = Y + k$.
Substituting $x = X + h$ and $y = Y + k$ in the equation $y^2 - 6y - 4x + 13 = 0$, we get

$$(Y + k)^2 - 6(Y + k) - 4(X + h) + 13 = 0$$

$$\Rightarrow \quad Y^2 + (2k - 6)Y - 4X + (k^2 - 6k + 13 - 4h) = 0$$

This equation should be of the form $Y^2 + AX = 0$. This means that it should not contain term containing Y and constant term.

$$\therefore \quad 2k - 6 = 0 \text{ and } k^2 - 4h + 13 - 6k = 0 \Rightarrow k = 3 \text{ and } h = 1$$

Hence, the coordinates of the required point are (1, 3).

LEVEL-2

EXAMPLE 5 *Prove that the area of a triangle is invariant under the translation of the axes.*

SOLUTION Let ABC be a triangle having the coordinates of its vertices as $A(x_1, y_1)$, $B(x_2, y_1)$ and $C(x_3, y_3)$. Then, area of triangle ABC is given by

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \qquad \text{...(i)}$$

Let the origin be shifted at (h, k). Then, the new coordinates of the vertices are
$A(x_1 + h, y_1 + k)$, $B(x_2 + h, y_2 + k)$ and $C(x_3 + h, y_3 + k)$.

Therefore, the area of the triangle in the new-coordinate system is given by

$$\Delta_1 = \frac{1}{2}\Big[(x_1 + h)\{(y_2 + k) - (y_3 + k)\} + (x_2 + h)\{(y_3 + k) - (y_1 + k)\} + (x_3 + h)\{(y_1 + k) - (y_2 + k)\}\Big]$$

$$\Rightarrow \quad \Delta_1 = \frac{1}{2}\Big[(x_1 + h)(y_2 - y_3) + (x_2 + h)(y_3 - y_1) + (x_3 + h)(y_1 - y_2)\Big]$$

$$\Rightarrow \quad \Delta_1 = \frac{1}{2}\Big[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) + h(y_2 - y_3 + y_3 - y_1 + y_1 - y_2)\Big]$$

$$\Rightarrow \quad \Delta_1 = \frac{1}{2}\Big[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\Big] \qquad \text{...(ii)}$$

From (i) and (ii), we get $\Delta = \Delta_1$.
Hence, the area of a triangle is invariant under the translation of the axes.

EXERCISE 22.3

LEVEL-1

1. What does the equation $(x-a)^2+(y-b)^2=r^2$ become when the axes are transferred to parallel axes through the point $(a-c, b)$?
2. What does the equation $(a-b)(x^2+y^2)-2abx=0$ become if the origin is shifted to the point $\left(\frac{ab}{a-b}, 0\right)$ without rotation?
3. Find what the following equations become when the origin is shifted to the point (1, 1)?
   (i) $x^2+xy-3x-y+2=0$ (ii) $x^2-y^2-2x+2y=0$
   (iii) $xy-x-y+1=0$ (iv) $xy-y^2-x+y=0$
4. At what point the origin be shifted so that the equation $x^2+xy-3x-y+2=0$ does not contain any first degree term and constant term?
5. Verify that the area of the triangle with vertices (2, 3), (5, 7) and (– 3 – 1) remains invariant under the translation of axes when the origin is shifted to the point (–1, 3).
6. Find, what the following equations become when the origin is shifted to the ponit (1, 1).
   (i) $x^2+xy-3y^2-y+2=0$ (ii) $xy-y^2-x+y=0$
   (iii) $xy-x-y+1=0$ (iv) $x^2-y^2-2x+2y=0$
7. Find the point to which the origin should be shifted after a translation of axes so that the following equations will have no first degree terms:
   (i) $y^2+x^2-4x-8y+3=0$ (ii) $x^2+y^2-5x+2y-5=0$ (iii) $x^2-12x+4=0$
8. Verify that the area of the triangle with vertices (4, 6), (7, 10) and (1, – 2) remains invariant under the translation of axes when the origin is shifted to the point (– 2, 1). [NCERT]

ANSWERS

1. $X^2+Y^2-2CX=r^2-c^2$ 2. $(a-b)^2(X^2+Y^2)=a^2b^2$
3. (i) $x^2+xy=0$ (ii) $x^2-y^2=0$ (iii) $xy=0$ (iv) $xy-y^2=0$
4. (1, 1) 6. (i) $x^2-3y^2+xy+3x-6y=0$ (ii) $xy-y^2=0$
(iii) $xy=0$ (iv) $x^2-y^2=0$ 7. (i) (2, 4) (ii) $(5/2, -1)$ (iii) $(6, k), k \in R$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. The vertices of a triangle are $O(0, 0)$, $A(a, 0)$ and $B(0, b)$. Write the coordinates of its circumcentre.
2. In Q.No. 1, write the distance between the circumcentre and orthocentre of ΔOAB.
3. Write the coordinates of the orthocentre of the triangle formed by points (8, 0), (4, 6) and (0, 0).
4. Three vertices of a parallelogram, taken in order, are (–1, – 6), (2, –5) and (7, 2). Write the coordinates of its fourth vertex.

5. If the points $(a, 0)$, $(at_1^{\;2}, 2at_1)$ and $(at_2^{\;2}, 2at_2)$ are collinear, write the value of $t_1 t_2$.
6. If the coordinates of t (ides AB and AC of a $\Delta\, ABC$ are $(3, 5)$ and $(-3, -3)$ respectively, then write the length of side BC.
7. Write the coordinates of the circumcentre of a triangle whose centroid and orthocentre are at $(3, 3)$ and $(-3, 5)$ respectively.
8. Write the coordinates of the incentre of the triangle having its vertices at $(0, 0)$, $(5, 0)$ and $(0, 12)$.
9. If the points $(1, -1)$, $(2, -1)$ and $(4, -3)$ are the mid-points of the sides of a triangle, then write the coordinates of its centroid.
10. Write the area of the triangle having vertices at $(a, b + c)$, $(b, c + a)$, $(c, a + b)$.

ANSWERS

1. $\left(\frac{a}{2}, \frac{b}{2}\right)$	2. $\frac{1}{2}\sqrt{a^2+b^2}$	3. $\left(\frac{8}{3}, 4\right)$	4. $(4, 1)$	5. -1
6. 20 units	7. $(6, 2)$	8. $(2, 2)$	9. $\left(\frac{7}{3}, \frac{-5}{3}\right)$	10. 0

SUMMARY

1. The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by
$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
i.e. $$PQ = \sqrt{(\text{Difference of abscissae})^2 + (\text{Difference of ordinates})^2}$$
2. The distance of a point $P(x, y)$ from the origin $O(0, 0)$ is given by $OP = \sqrt{x^2 + y^2}$.
3. The area of the triangle, the coordinates of whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is the absolute value of
$$\frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right] \text{ or, } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$
4. If the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$
5. The coordinates of the point dividing the line segment joining (x_1, y_1) and (x_2, y_2) in the ratio $m : n$ are
$$\left(\frac{m x_2 + n x_1}{m+n}, \frac{m y_2 + n y_1}{m+n}\right), \text{ internally; } \left(\frac{m x_2 - n x_1}{m-n}, \frac{m y_2 - n y_1}{m-n}\right), \text{ externally}$$
6. The coordinates of the mid-point of the line segment joining (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
7. The coordinates of the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_2, y_3) are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

CHAPTER 23

THE STRAIGHT LINES

23.1 DEFINITION OF A STRAIGHT LINE

A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

THEOREM *Every first degree equation in x, y represents a straight line.*

[NCERT EXEMPLAR]

PROOF Let $ax + by + c = 0$ be a first degree equation in x, y where a, b, c are constants. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on the curve represented by $ax + by + c = 0$. Then,

$$ax_1 + by_1 + c = 0 \text{ and } ax_2 + by_2 + c = 0 \quad \text{...(i)}$$

Let R be any point on the line segment joining P and Q. Suppose R divides PQ in the ratio $\lambda : 1$. Then, the coordinates of R are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$. In order to prove that the curve represented by $ax + by + c = 0$ is a straight line, it is sufficient to show that R lies on it for all values of λ.

Now,

$$a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c = \frac{\lambda (ax_2 + by_2 + c) + (ax_1 + by_1 + c)}{\lambda + 1}$$

$$= \lambda 0 + 0 = 0 \quad \text{[Using (i)]}$$

$\therefore \quad R\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ lies on the curve represented by $ax + by + c = 0$.

Thus, every point on the line segment joining P and Q lies on $ax + by + c = 0$.

Hence, $ax + by + c = 0$ represents a straight line.

Q.E.D.

NOTE *When we say that a first degree equation in x, y i.e., $ax + by + c = 0$ represents a line, it means that all points (x, y) satisfying $ax + by + c = 0$ lie along a line. Thus, a line is also defined as the locus of a point satisfying the condition $ax + by + c = 0$ where a, b, c are constants.*

It follows from the above discussion that $ax + by + c = 0$ is the general equation of a line.

It should be noted that there are only two unknowns in the equation of a straight line because equation of every straight line can be put in the form $ax + by + 1 = 0$ where a, b are two unknowns. Note that x, y are not unknowns. In fact these are the coordinates of any point on the line and are known as the current coordinates. Thus, to determine a line we will need two conditions to determine the two unknowns. In the further discussion on straight line you will find that whenever it will be asked to find a straight line there will always be two conditions connecting the two unknowns.

23.2 SLOPE (GRADIENT) OF A LINE

The trigonometrical tangent of the angle that a line makes with the positive direction of the x-axis in anticlockwise sense is called the slope or gradient of the line.

The slope of a line is generally denoted by m. Thus, $m = \tan \theta$.

Since a line parallel to x-axis makes an angle of $0°$ with x-axis, therefore its slope is $\tan 0° = 0$. A line parallel to y-axis i.e., perpendicular to x-axis makes an angle of $90°$ with x-axis, so its slope is

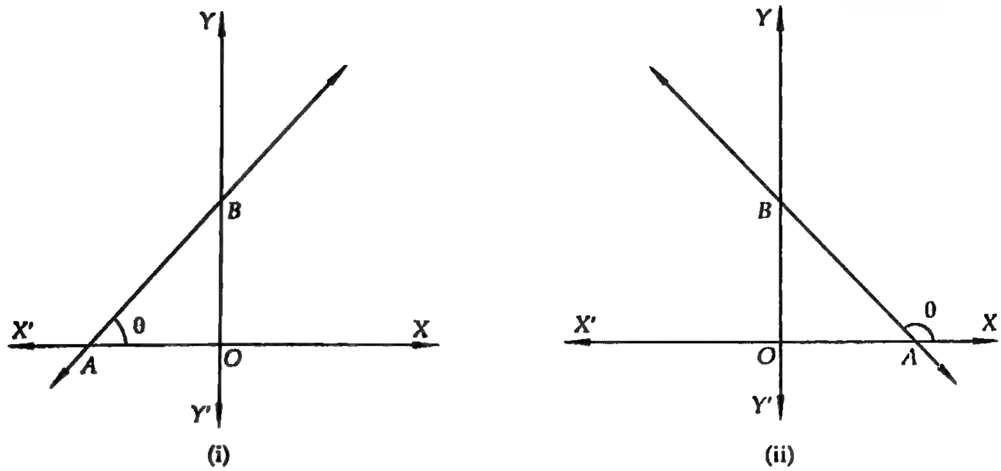


Fig. 23.1

$\tan \pi/2 = \infty$. Also, the slope of a line equally inclined with axes is 1 or -1 as it makes 45° or 135° angle with x-axis.

REMARK *The angle of inclination of a line with the positive direction of x-axis in anticlockwise sense always lies between* 0° *and* 180°.

ILLUSTRATION 1 *Find the slope of a line whose inclination to the positive direction of x-axis in anticlockwise sense is (i)* 60° *(ii)* 0° *(iii)* 150° *(iv)* 120°. [NCERT]

SOLUTION (i) Slope $= \tan 60° = \sqrt{3}$.

(ii) Slope $= \tan 0° = 0$.

(iii) Slope $= \tan 150° = -\cot 60° = -\frac{1}{\sqrt{3}}$.

(iv) Slope $= \tan 120° = -\cot 30° = -\sqrt{3}$.

ILLUSTRATION 2 *What can be said regarding a line if its slope is (i) positive (ii) zero (iii) negative?*

SOLUTION Let θ be the angle of inclination of the given line with the positive direction of x-axis in anticlockwise sense. Then, its slope is given by $m = \tan\theta$.

(i) If the slope of the line is positive, then

$$m = \tan\theta > 0 \Rightarrow \theta \text{ lies between } 0° \text{ and } 90° \Rightarrow \theta \text{ is an acute angle.}$$

Thus, a line of positive slope makes an acute angle with the positive direction of x-axis.

(ii) If the slope of the line is zero, then

$$m = \tan\theta = 0 \Rightarrow \theta = 0° \Rightarrow \text{ either the line is } x\text{-axis or it is parallel to } x\text{-axis.}$$

Thus, a line of zero slope is parallel or coincident to x-axis.

(iii) If the slope of the line is negative, then

$$m = \tan\theta < 0 \Rightarrow \theta \text{ lies between } 90° \text{ and } 180° \Rightarrow \theta \text{ is an obtuse angle.}$$

Thus, a line of negative slope makes an obtuse angle with the positive direction of x-axis in anticlockwise direction.

23.2.1 SLOPE OF A LINE IN TERMS OF COORDINATES OF ANY TWO POINTS ON IT

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on a line making an angle θ with the positive direction of x-axis. Draw PL, QM perpendiculars on x-axis and $PN \perp$ on QM. Then,

$$PN = LM = OM - OL = x_2 - x_1 \text{ and, } QN = QM - NM = QM - PL = y_2 - y_1$$

In $\Delta\, PQN$, we have

$$\tan\theta = \frac{QN}{PN} = \frac{y_2 - y_1}{x_2 - x_1}$$

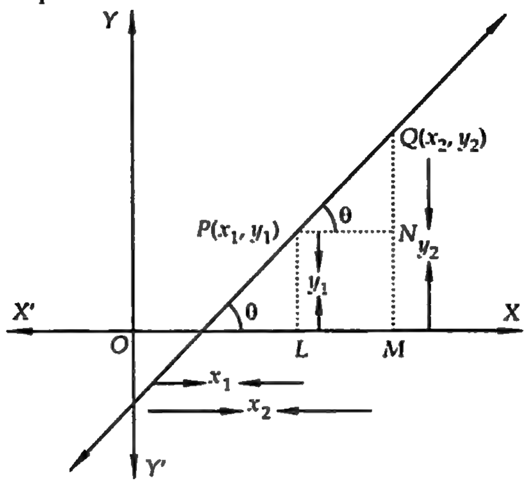


Fig. 23.2

Thus, if (x_1, y_1) and (x_2, y_2) are coordinates of any two points on a line, then its slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

ILLUSTRATION 3 *Find the slope of a line which passes through points (3, 2) and (– 1, 5).* [NCERT]

SOLUTION We know that the slope of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$. Here, the line passes through (3, 2) and (–1, 5).

So, its slope is given by $m = \frac{5-2}{-1-3} = -\frac{3}{4}$.

23.3 ANGLE BETWEEN TWO LINES

THEOREM *The angle θ between the lines having slopes m_1 and m_2 is given by* $\tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2}$.

PROOF Let m_1 and m_2 be the slopes of two given lines AB and CD which intersect at a point P and make angles θ_1 and θ_2 respectively with the positive direction of x-axis. Then, $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.

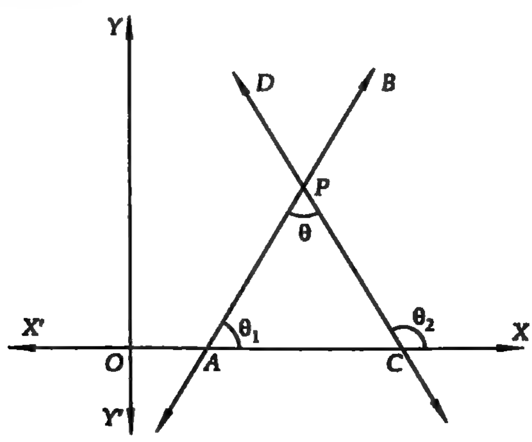


Fig. 23.3

Let $\angle APC = \theta$ be the angle between the given lines. Then,

$$\theta_2 = \theta + \theta_1 \qquad \text{[See Fig. 23.3]}$$

$$\Rightarrow \quad \theta = \theta_2 - \theta_1$$

$$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1)$$

$$\Rightarrow \quad \tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1} \qquad \left[\text{Using : } \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right]$$

$$\Rightarrow \quad \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{...(i)}$$

Since $\angle APD = \pi - \theta$ is also the angle between AB and CD. Therefore,

$$\tan\angle APD = \tan(\pi - \theta) = -\tan\theta = -\frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{[Using (i)]} \quad \text{...(ii)}$$

From (i) and (ii), we find that the angles between two lines of slopes m_1 and m_2 are given by

$$\tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2} \Rightarrow \theta = \tan^{-1}\left(\pm\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

The acute angle between the lines is given by $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$.

Q.E.D.

ILLUSTRATION *If $A(-2, 1)$, $B(2, 3)$ and $C(-2, -4)$ are three points, find the angle between BA and BC.*

SOLUTION Let m_1 and m_2 be the slopes of BA and BC respectively. Then,

$$m_1 = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}, \quad \text{and} \quad m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$$

Let θ be the acute angle between BA and BC. Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4}\times\frac{1}{2}}\right| = \left|\frac{\frac{10}{8}}{\frac{15}{8}}\right| = \frac{2}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{2}{3}\right).$$

CONDITION OF PARALLELISM OF LINES If two lines of slopes m_1 and m_2 are parallel, then the angle θ between them is of $0°$.

$$\therefore \quad \tan\theta = \tan 0° = 0$$

$$\Rightarrow \quad \frac{m_2 - m_1}{1 + m_1 m_2} = 0 \qquad \left[\text{Using : } \tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2}\right]$$

$$\Rightarrow \quad m_2 = m_1$$

Thus, when two lines are parallel, their slopes are equal.

CONDITION OF PERPENDICULARITY OF TWO LINES If two lines of slopes m_1 and m_2 are perpendicular, then the angle θ between them is of $90°$.

From Fig. 23.3, we have

$$\theta_2 = \theta + \theta_1$$

$$\Rightarrow \quad \theta_2 = 90° + \theta_1 \qquad [\because \theta = 90°]$$

$$\Rightarrow \quad \tan\theta_2 = \tan(90° + \theta_1)$$

$$\Rightarrow \quad \tan\theta_2 = -\cot\theta_1$$

$\Rightarrow \quad \tan\theta_1 \tan\theta_2 = -1 \Rightarrow m_1 m_2 = -1$

Thus, when two lines are perpendicular, the product of their slopes is – 1. If m is the slope of a line, then the slope of a line perpendicular to it is – (1/m).

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Determine x so that the line passing through (3, 4) and (x, 5) makes 135° angle with the positive direction of x-axis.*

SOLUTION Since the line passing through (3, 4) and (x, 5) makes an angle of 135° with x-axis. Therefore, its slope is $\tan 135° = -1$. But, the slope of the line is also equal to

$$\frac{5-4}{x-3} \qquad \left[\text{Using}: m = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

$$\therefore \quad -1 = \frac{5-4}{x-3} \Rightarrow -x+3 = 1 \Rightarrow x = 2.$$

EXAMPLE 2 *Find the angle between the lines joining the points (0, 0), (2, 3) and the points (2, – 2), (3, 5).*

SOLUTION Let θ be the angle between the given lines.

We have,

$$m_1 = \text{Slope of the line joining } (0, 0) \text{ and } (2, 3) = \frac{3-0}{2-0} = \frac{3}{2}$$

$$m_2 = \text{Slope of the line joining } (2, -2) \text{ and } (3, 5) = \frac{5+2}{3-2} = 7$$

$$\therefore \quad \tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2} = \pm\frac{7 - 3/2}{1 + 7(3/2)} = \pm\frac{11/2}{23/2} = \pm\frac{11}{23} \Rightarrow \theta = \tan^{-1}\left(\pm\frac{11}{23}\right).$$

EXAMPLE 3 *Let A (6, 4) and B (2, 12) be two given points. Find the slope of a line perpendicular to AB.*

SOLUTION Let m be the slope of AB. Then,

$$m = \frac{12-4}{2-6} = \frac{8}{-4} = -2$$

So, the slope of a line perpendicular to AB is $-\frac{1}{m} = \frac{1}{2}$

EXAMPLE 4 *Determine x so that 2 is the slope of the line through (2, 5) and (x, 3).*

SOLUTION The slope of the line through (2, 5) and (x, 3) is $\frac{3-5}{x-2}$. But, the slope of the line is given as 2.

$$\therefore \quad \frac{3-5}{x-2} = 2 \Rightarrow 2x - 4 = -2 \Rightarrow x = 1$$

EXAMPLE 5 *What is the value of y so that the line through (3, y) and (2, 7) is parallel to the line through (–1, 4) and (0, 6)?*

SOLUTION Let $A(3, y)$, $B(2, 7)$, $C(-1, 4)$ and $D(0, 6)$ be the given points. Then,

$$m_1 = \text{Slope of the line } AB = \frac{7-y}{2-3} = y - 7$$

and, $$m_2 = \text{Slope of the line } CD = \frac{6-4}{0-(-1)} = 2$$

Since AB and CD are parallel.

$$\therefore \quad m_1 = m_2 \Rightarrow y - 7 = 2 \Rightarrow y = 9.$$

EXAMPLE 6 *Without using Pythagoras theorem, show that A (4, 4), B (3, 5) and C (–1, –1) are the vertices of a right-angled triangle.* [NCERT]

SOLUTION In ΔABC, we have,

$$m_1 = \text{Slope of } AB = \frac{4-5}{4-3} = -1 \text{ and, } m_2 = \text{Slope of } AC = \frac{4-(-1)}{4-(-1)} = 1$$

Clearly, $m_1 m_2 = -1$. This shows that AB is perpendicular to AC i.e. $\angle CAB = \pi/2$.

Hence, the given points are the vertices of a right-angled triangle.

EXAMPLE 7 *A quadrilateral has the vertices at the points $(-4, 2), (2, 6), (8, 5)$ and $(9, -7)$. Show that the mid-points of the sides of this quadrilateral are the vertices of a parallelogram.*

SOLUTION Let $A(-4, 2), B(2, 6), C(8, 5)$ and $D(9, -7)$ be the vertices of the given quadrilateral. Let P, Q, R and S be the mid-points of AB, BC, CD and DA respectively. Then, the coordinates of P, Q, R and S are $P(-1, 4)$, $Q(5, 11/2)$, $R(17/2, -1)$ and $S(5/2, -5/2)$ respectively. In order to prove that $PQRS$ is a parallelogram, it is sufficient to show that PQ is parallel to RS and $PQ = RS$. Let m_1 and m_2 be the slope of PQ and RS respectively. Then,

$$m_1 = \frac{11/2-4}{5-(-1)} = \frac{1}{4} \text{ and, } m_2 = \frac{-5/2+1}{5/2-17/2} = \frac{1}{4}$$

Clearly, $m_1 = m_2$. Therefore, PQ is parallel to RS.

Now,

$$PQ = \sqrt{(5+1)^2 + \left(\frac{11}{2}-4\right)^2} = \frac{\sqrt{153}}{2} \text{ and, } RS = \sqrt{\left(\frac{5}{2}-\frac{17}{2}\right)^2 + \left(-\frac{5}{2}+1\right)^2} = \frac{\sqrt{153}}{2}$$

$$\therefore \quad PQ = RS$$

Thus, $PQ \parallel RS$ and $PQ = RS$. Hence, $PQRS$ is a parallelogram.

EXAMPLE 8 *Prove that $A(4, 3), B(6, 4), C(5, 6)$ and $D(3, 5)$ are the angular points of a square.*

SOLUTION Clearly,

$$AB = \sqrt{(6-4)^2 + (4-3)^2} = \sqrt{5},\ BC = \sqrt{(6-4)^2 + (5-4)^2} = \sqrt{5},$$

$$CD = \sqrt{(5-6)^2 + (3-5)^2} = \sqrt{5} \text{ and, } DA = \sqrt{(5-3)^2 + (3-4)^2} = \sqrt{5}$$

$$\therefore \quad AB = BC = CD = DA.$$

Now, $m_1 = \text{Slope of } AB = \dfrac{4-3}{6-4} = \dfrac{1}{2}$, $m_2 = \text{Slope of } BC = \dfrac{6-4}{5-6} = -2$

and, $m_3 = \text{Slope of } CD = \dfrac{5-6}{3-5} = \dfrac{1}{2}$

Clearly, $m_1 m_2 = (1/2)(-2) = -1$ and $m_1 = m_3$. Therefore, AB is perpendicular to BC and it is parallel to CD. Thus, $AB = BC = CA = AD$, $AB \perp BC$ and AB is parallel to CD. Hence, $ABCD$ is a square.

EXAMPLE 9 *If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the line $\frac{1}{2}$, find the slope of the other line.* [NCERT]

SOLUTION We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \quad \text{...(i)}$$

Let $m_1 = \frac{1}{2}$ and $m_2 = m$ = slope of the other line. It is given that $\theta = \frac{\pi}{4}$.

Substituting $m_1 = \frac{1}{2}$, $m_2 = m$ and $\theta = \frac{\pi}{4}$ in (i), we obtain

$\therefore \qquad \tan\frac{\pi}{4} = \left|\dfrac{m-\frac{1}{2}}{1+m\times\frac{1}{2}}\right|$

$\Rightarrow \qquad 1 = \left|\dfrac{2m-1}{2+m}\right|$

$\Rightarrow \qquad \dfrac{2m-1}{m+2} = \pm 1 \Rightarrow 2m-1 = m+2$ or, $2m-1 = -(m+2) \Rightarrow m = 3$ or, $m = -\dfrac{1}{3}$

Hence, the slope of the other line is 3 or, $-\dfrac{1}{3}$.

EXAMPLE 10 *If the points $P(h, k)$, $Q(x_1, y_1)$ and $R(x_2, y_2)$ lie on a line. Show that:*

$(h-x_1)(y_2-y_1) = (k-y_1)(x_2-x_1)$ **[NCERT]**

SOLUTION It is given that the points $P(h, k)$, $Q(x_1, y_1)$ and $R(x_2, y_2)$ are collinear.

$\therefore \qquad$ Slope of PQ = Slope of QR

$\Rightarrow \qquad \dfrac{k-y_1}{h-x_1} = \dfrac{y_2-y_1}{x_2-x_1} \Rightarrow (k-y_1)(x_2-x_1) = (h-x_1)(y_2-y_1)$

EXAMPLE 11 *In Fig. 23.4, time and distance graph of a linear motion is given. Two positions of time and distance recorded as, when $T=0$, $D=2$ and when $T=3$, $D=8$. Using the concept of slope, find law of motion i.e. how distance depends upon time.* **[NCERT]**

SOLUTION Let $P(T, D)$ be any point on the line, where D denotes the distance at any time T. Clearly, points $A(0, 2)$, $B(3, 8)$ and $P(T, D)$ are collinear.

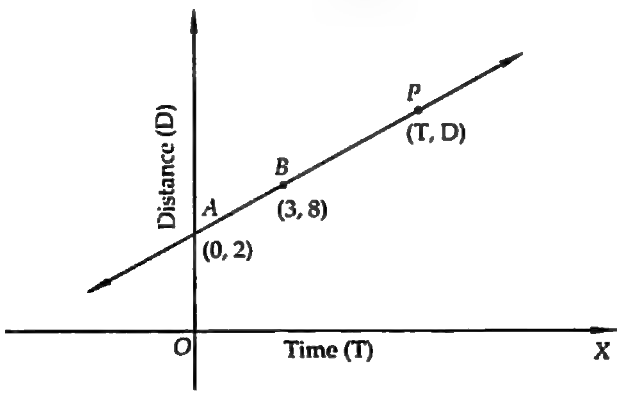


Fig. 23.4

$\therefore \qquad$ Slope of AB = Slope of BP

$\Rightarrow \qquad \dfrac{8-2}{3-0} = \dfrac{D-8}{T-3}$

$\Rightarrow \qquad 2 = \dfrac{D-8}{T-3} \Rightarrow D = 2(T+1)$, which is the required relation.

EXAMPLE 12 *If points $(a, 0)$, $(0, b)$ and (x, y) are collinear, using the concept of slope, prove that $\dfrac{x}{a}+\dfrac{y}{b} = 1$.*

SOLUTION Let $A(a, 0)$, $B(0, b)$ and $P(x, y)$ be the given collinear points. Then,

Slope of AB = Slope of BP

$\Rightarrow \qquad \dfrac{b-0}{0-a} = \dfrac{y-b}{x-0}$

$$\Rightarrow \quad \frac{-b}{a} = \frac{y-b}{x}$$

$$\Rightarrow \quad -bx = ay - ab \Rightarrow bx + ay = ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 1 \qquad \text{[On dividing both sides by } ab]$$

LEVEL-2

EXAMPLE 13 *A ray of light passing through the point* (1, 2) *reflects on the x-axis at point A and the reflected ray passes through the point (5, 3). Find the co-ordinates of A.*

[NCERT EXEMPLAR]

SOLUTION Let the coordinates of A be $(h, 0)$. Let AN be the normal at A. Then,

$$\angle PAN = \angle QAN = \theta \text{ (say)}$$

Clearly, AQ makes angle $90°-\theta$ with OX. Therefore, slope of AQ is $\tan(90°-\theta) = \cot\theta$. Also, the coordinates of A and Q are $(h, 0)$ and $(5, 3)$ respectively.

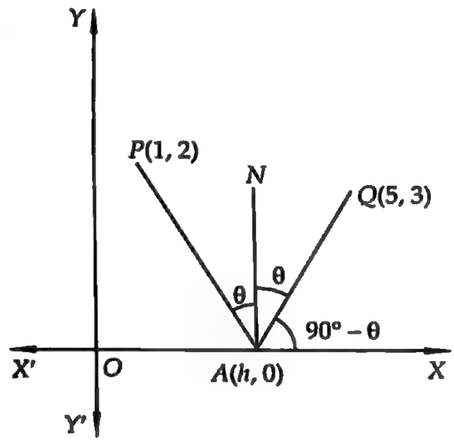


Fig. 23.5

$$\therefore \quad \text{Slope of } AQ = \frac{3-0}{5-h}$$

Thus, $$\cot\theta = \frac{3-0}{5-h}$$

$$\Rightarrow \quad \cot\theta = \frac{3}{5-h} \qquad \text{...(i)}$$

AP makes angle $90°+\theta$ with OX. Therefore, slope of AP is $\tan(90°+\theta) = -\cot\theta$. Also, AP passes through $A(h, 0)$ and $P(1, 2)$. Therefore,

$$\text{Slope of } AP = \frac{2-0}{1-h}$$

Thus $$-\cot\theta = \frac{2}{1-h}$$

$$\Rightarrow \quad \cot\theta = \frac{2}{h-1} \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$\frac{3}{5-h} = \frac{2}{h-1}$$

$$\Rightarrow \quad 3h - 3 = 10 - 2h$$

$$\Rightarrow \quad 5h = 13$$

$$\Rightarrow \qquad h = \frac{13}{5}$$

Hence, the coordinates of A are (13/5 , 0).

EXAMPLE 14 *Prove that the line joining the mid-points of the two sides of a triangle is parallel to the third side.*

SOLUTION Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a ΔABC and D and E be the mid-points of sides AB and AC respectively. Then, the coordinates of D and E are $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$ and $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$ respectively.

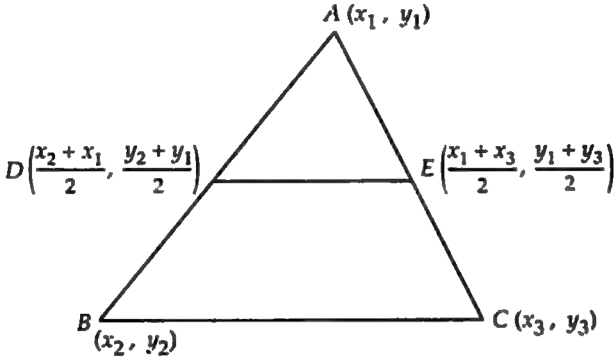


Fig. 23.6

$$\therefore \quad m_1 = \text{Slope of } DE = \frac{\frac{y_1+y_3}{2} - \frac{y_2+y_1}{2}}{\frac{x_1+x_3}{2} - \frac{x_2+x_1}{2}} = \frac{y_3-y_2}{x_3-x_2} \text{ and, } m_2 = \text{Slope of } BC = \frac{y_3-y_2}{x_3-x_2}$$

Clearly, $m_1 = m_2$. Hence, DE is parallel to BC.

EXAMPLE 15 *If $A(2, 0)$, $B(0, 2)$ and $C(0, 7)$ are three vertices, taken in order, of an isosceles trapezium $ABCD$ in which $AB \parallel DC$. Find the coordinates of D.*

SOLUTION Let the coordinates of D be (x, y). It is given that $AB \parallel DC$.

$$\therefore \quad \text{Slope of } AB = \text{Slope of } DC \Rightarrow \frac{2-0}{0-2} = \frac{7-y}{0-x} \Rightarrow -1 = \frac{y-7}{x} \Rightarrow x+y-7=0 \qquad \text{...(i)}$$

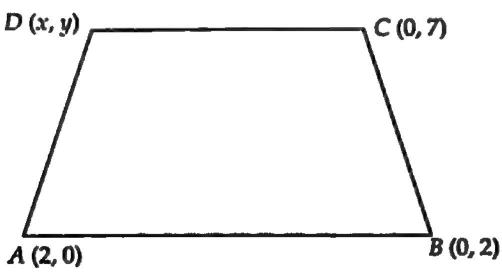


Fig. 23.7

Since $ABCD$ is an isosceles trapezium. Therefore,

$$AD = BC$$

$$\Rightarrow \quad AD^2 = BC^2$$

$$\Rightarrow \quad (x-2)^2 + (y-0)^2 = (0-0)^2 + (2-7)^2$$

$$\Rightarrow \quad (x-2)^2 + y^2 = 25$$

$\Rightarrow \quad (x-2)^2+(7-x)^2 = 25 \qquad$ [From (i), $y = 7-x$]

$\Rightarrow \quad 2x^2-18x+28 = 0 \Rightarrow x^2-9x+14 = 0 \Rightarrow (x-2)(x-7) = 0 \Rightarrow x = 2, 7$

From (i), $x = 2 \Rightarrow y = 5$ and $x = 7 \Rightarrow y = 0$.

Hence, the coordinates of D are (2, 5) or (7, 0).

EXAMPLE 16 *By using the concept of slope, prove that the diagonals of a rhombus are at right angles.*

SOLUTION Let $OABC$ be a rhombus whose each side is of length a such that O is the origin and OA is along x-axis. Let b be the height of the rhombus. Let BL and CM be perpendiculars drawn from B and C respectively on x-axis.

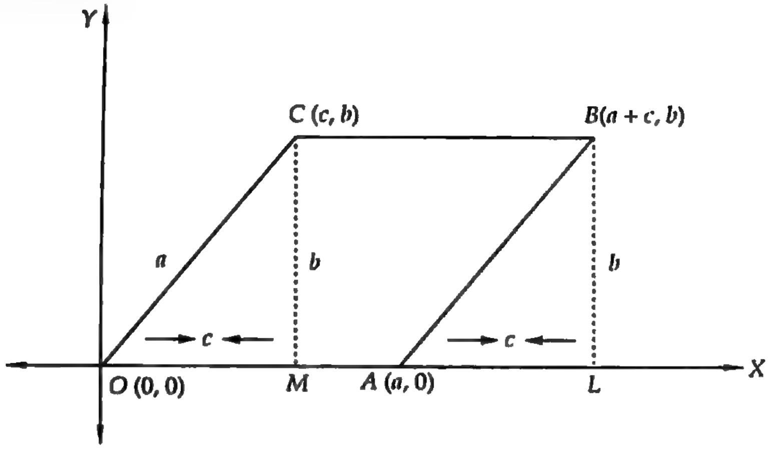


Fig. 23.8

Further, let $OM = c$.

Clearly, $\Delta OMC \cong \Delta ALB$

$\therefore \quad OM = AL \Rightarrow AL = c$.

Thus, we have

$$OM = c, CM = b, OA = a, OL = a+c \text{ and } LB = b$$

So, the coordinates of the vertices of the rhombus are O (0, 0), A (a, 0), B ($a+c$, b) and C (c, b)

In right triangle OMC, we have

$$OC^2 = OM^2+MC^2 \Rightarrow a^2 = c^2+b^2$$

Now,

$$m_1 = \text{Slope of diagonal } OB = \frac{b-0}{a+c-0} = \frac{b}{a+c}$$

$$m_2 = \text{Slope of diagonal } AC = \frac{b-0}{c-a} = \frac{b}{c-a}$$

$$\therefore \quad m_1 m_2 = \frac{b}{a+c} \times \frac{b}{c-a} = \frac{b^2}{c^2-a^2} = \frac{b^2}{-b^2} = -1 \qquad [\because a^2 = c^2+b^2]$$

Hence, OB is perpendicular to AC.

EXAMPLE 17 *Using the concept of slope, prove that medians of an equilateral triangle are perpendicular to the corresponding sides.*

SOLUTION Let ABC be an equilateral triangle such that $AB = AC = BC = 2a$.

Let BC be along X-axis, mid-point of BC be the origin and a line passing through O and perpendicular to BC be Y-axis. Then, the coordinates of B and C are $(-a, 0)$ and $(a, 0)$ respectively. Let the coordinates of A be (α, β).

Since ABC is an equilateral triangle.

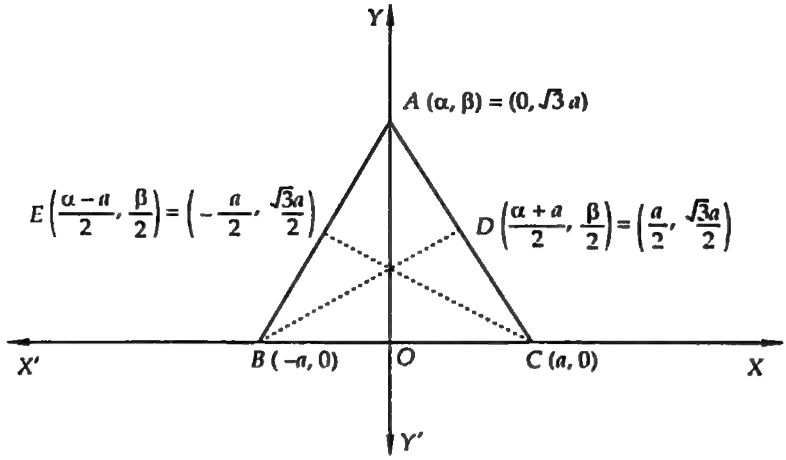


Fig. 23.9

$\therefore \quad AB = AC = BC$

Now, $\quad AB = AC$

$\Rightarrow \quad AB^2 = AC^2$

$\Rightarrow \quad (\alpha + a)^2 + (\beta - 0)^2 = (\alpha - a)^2 + (\beta - 0)^2 \Rightarrow 4a\alpha = 0 \Rightarrow \alpha = 0.$

In right triangle COA, we have

$$AC^2 = OA^2 + OC^2 \Rightarrow (2a)^2 = \beta^2 + a^2 \Rightarrow \beta = \sqrt{3}a$$

Thus, the coordinates of A are $(0, \sqrt{3}a)$. Consequently, A lies on y-axis.

$\therefore \quad OA \perp BC$ $\quad$ [$\because$ OA is along y-axis and BC is along x-axis]

Let D and E be the mid-points of AC and AB respectively. Then, the coordinates of D and E are $\left(\frac{\alpha + a}{2}, \frac{\beta}{2}\right) = \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$ and $\left(\frac{\alpha - a}{2}, \frac{\beta}{2}\right) = \left(-\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$ respectively.

Now, $\quad m_1 = \text{Slope of } AC = \dfrac{\sqrt{3}a - 0}{0 - a} = -\sqrt{3}, \; m_2 = \text{Slope of } BD = \dfrac{\frac{\sqrt{3}a}{2} - 0}{\frac{a}{2} + a} = \dfrac{1}{\sqrt{3}}$

Clearly, $m_1 m_2 = -1$. Therefore, $BD \perp AC$.

Also $\quad \text{Slope of } AB \times \text{Slope of } CE = \dfrac{\sqrt{3}a - 0}{0 + a} \times \dfrac{\frac{\sqrt{3}a}{2} - 0}{\frac{-a}{2} - a} = \sqrt{3} \times -\dfrac{\sqrt{3}}{3} = -1$

$\therefore \quad AB \perp CE$

Thus, AO, BD and CE are medians of an equilateral triangle ABC such that $AO \perp BC$, $BD \perp CA$ and $CE \perp AB$.

Hence, medians of an equilateral triangle are perpendicular to the corresponding sides.

EXAMPLE 18 *Prove that a triangle which has one of the angle as 30°, cannot have all vertices with integral coordinates.*

SOLUTION Let ABC be a triangle the coordinates of whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, where $x_1, x_2, x_3, y_1, y_2, y_3$ are integers. Let $\angle BAC = 30°$.

We have,

$$m_1 = \text{Slope of } AB = \frac{y_1 - y_2}{x_1 - x_2}, \text{ and } m_2 = \text{Slope of } AC = \frac{y_1 - y_3}{x_1 - x_3}$$

Now, $\angle BAC = 30°$

$$\Rightarrow \quad \tan 30° = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \left|\frac{\frac{y_1 - y_2}{x_1 - x_2} - \frac{y_1 - y_3}{x_1 - x_3}}{1 + \frac{y_1 - y_2}{x_1 - x_2} \times \frac{y_1 - y_3}{x_1 - x_3}}\right|$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \left|\frac{(y_1 - y_2)(x_1 - x_3) - (y_1 - y_3)(x_1 - x_2)}{(x_1 - x_2)(x_1 - x_3) + (y_1 - y_2)(y_1 - y_3)}\right|$$

This is not possible as LHS is an irrational number and RHS is a rational number. Hence, $x_1, x_2, x_3, y_1, y_2, y_3$ cannot be all integers.

EXAMPLE 19 *The vertices of a triangle are $A(x_1, x_1 \tan \theta_1)$, $B(x_2, x_2 \tan \theta_2)$ and $C(x_3, x_3 \tan \theta_3)$. If the circumcentre of ΔABC coincides with the origin and $H(\bar{x}, \bar{y})$ is the orthocentre, show that*

$$\frac{\bar{y}}{\bar{x}} = \frac{\sin \theta_1 + \sin \theta_2 + \sin \theta_3}{\cos \theta_1 + \cos \theta_2 + \cos \theta_3}$$

SOLUTION The circumcentre of ΔABC is at the origin O. Let the circum-radius be r. Then,

$$OA = OB = OC = r$$

$$\Rightarrow \quad OA^2 = OB^2 = OC^2 = r^2$$

$$\Rightarrow \quad x_1^2 + x_1^2 \tan^2 \theta_1 = x_2^2 + x_2^2 \tan^2 \theta_2 = x_3^2 + x_3^2 \tan^2 \theta_3 = r^2$$

$$\Rightarrow \quad x_1^2 \sec^2 \theta_1 = x_2^2 \sec^2 \theta_2 = x_3^2 \sec^2 \theta_3 = r^2$$

$$\Rightarrow \quad x_1 = r \cos \theta_1,\ x_2 = r \cos \theta_2,\ x_3 = r \cos \theta_3$$

So, the coordinates of the vertices of ΔABC are

$$A \equiv (x_1, x_1 \tan \theta_1) = (r \cos \theta_1, r \sin \theta_1),\ B \equiv (x_2, x_2 \tan \theta_2) = (r \cos \theta_2, r \sin \theta_2)$$

and, $C \equiv (x_3, x_3 \tan \theta_3) = (r \cos \theta_3, r \sin \theta_3)$.

So, the coordinates of the centroid G are

$$\left(\frac{r \cos \theta_1 + r \cos \theta_2 + r \cos \theta_3}{3}, \frac{r \sin \theta_1 + r \sin \theta_2 + r \sin \theta_3}{3}\right)$$

We know that the circumcentre (O), Centroid (G) and orthocentric (H) of a triangle are collinear.

$\therefore$ Slope of OH = Slope of OG

$$\Rightarrow \quad \frac{\bar{y} - 0}{\bar{x} - 0} = \frac{\frac{r \sin \theta_1 + r \sin \theta_2 + r \sin \theta_3}{3} - 0}{\frac{r \cos \theta_1 + r \cos \theta_2 + r \cos \theta_3}{3} - 0} \Rightarrow \frac{\bar{y}}{\bar{x}} = \frac{\sin \theta_1 + \sin \theta_2 + \sin \theta_3}{\cos \theta_1 + \cos \theta_2 + \cos \theta_3}$$

EXERCISE 23.1

LEVEL-1

1. Find the slopes of the lines which make the following angles with the positive direction of x-axis:

(i) $-\frac{\pi}{4}$ (ii) $\frac{2\pi}{3}$ (iii) $\frac{3\pi}{4}$ (iv) $\frac{\pi}{3}$

2. Find the slope of a line passing through the following points:
   (i) $(-3, 2)$ and $(1, 4)$ (ii) $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ (iii) $(3, -5)$, and $(1, 2)$
3. State whether the two lines in each of the following are parallel, perpendicular or neither:
   (i) Through $(5, 6)$ and $(2, 3)$; through $(9, -2)$ and $(6, -5)$
   (ii) Through $(9, 5)$ and $(-1, 1)$; through $(3, -5)$ and $(8, -3)$
   (iii) Through $(6, 3)$ and $(1, 1)$; through $(-2, 5)$ and $(2, -5)$
   (iv) Through $(3, 15)$ and $(16, 6)$; through $(-5, 3)$ and $(8, 2)$.
4. Find the slope of a line (i) which bisects the first quadrant angle (ii) which makes an angle of 30° with the positive direction of y-axis measured anticlockwise. [NCERT]
5. Using the method of slope, show that the following points are collinear:
   (i) $A(4, 8)$, $B(5, 12)$, $C(9, 28)$ (ii) $A(16, -18)$, $B(3, -6)$, $C(-10, 6)$
6. What is the value of y so that the line through $(3, y)$ and $(2, 7)$ is parallel to the line through $(-1, 4)$ and $(0, 6)$?
7. What can be said regarding a line if its slope is
   (i) zero (ii) positive (iii) negative?
8. Show that the line joining $(2, -3)$ and $(-5, 1)$ is parallel to the line joining $(7, -1)$ and $(0, 3)$.
9. Show that the line joining $(2, -5)$ and $(-2, 5)$ is perpendicular to the line joining $(6, 3)$ and $(1, 1)$.
10. Without using Pythagoras theorem, show that the points $A(0, 4)$, $B(1, 2)$ and $C(3, 3)$ are the vertices of a right angled triangle.
11. Prove that the points $(-4, -1)$, $(-2, -4)$, $(4, 0)$ and $(2, 3)$ are the vertices of a rectangle.
12. If three points $A(h, 0)$, $P(a, b)$ and $B(0, k)$ lie on a line, show that: $\frac{a}{h} + \frac{b}{k} = 1$. [NCERT]
13. The slope of a line is double of the slope of another line. If tangents of the angle between them is $\frac{1}{3}$, find the slopes of the other line. [NCERT]
14. Consider the following population and year graph:
    Find the slope of the line AB and using it, find what will be the population in the year 2010. [NCERT]

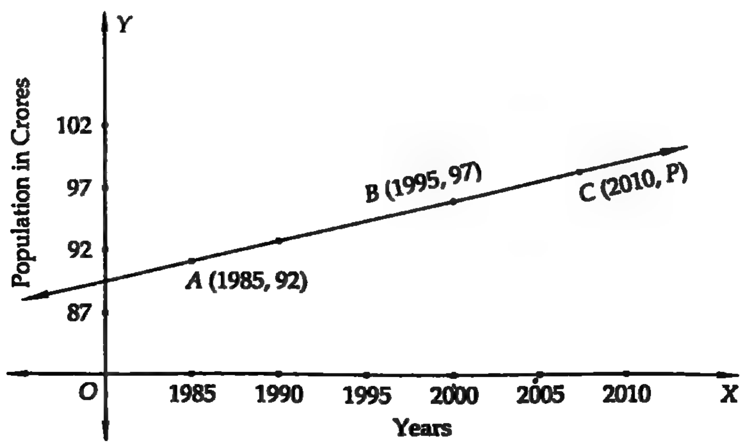


Fig. 23.10

15. Without using the distance formula, show that points $(-2, -1)$, $(4, 0)$, $(3, 3)$ and $(-3, 2)$ are the vertices of a parallelogram.

16. Find the angle between the X-axis and the line joining the points $(3, -1)$ and $(4, -2)$. [NCERT]

17. Line through the points $(-2, 6)$ and $(4, 8)$ is perpendicular to the line through the points $(8, 12)$ and $(x, 24)$. Find the value of x. [NCERT]

18. Find the value of x for which the points $(x, -1)$, $(2, 1)$ and $(4, 5)$ are collinear. [NCERT]

19. Find the angle between X-axis and the line joining the points $(3, -1)$ and $(4, -2)$.

20. By using the concept of slope, show that the points $(-2, -1)$, $(4, 0)$, $(3, 3)$ and $(-3, 2)$ are the vertices of a parallelogram. [NCERT]

21. A quadrilateral has vertices $(4, 1)$, $(1, 7)$, $(-6, 0)$ and $(-1, -9)$. Show that the mid-points of the sides of this quadrilateral form a parallelogram.

ANSWERS

1. (i) -1 (ii) $-\sqrt{3}$ (iii) -1 (iv) $\sqrt{3}$
2. (i) $\frac{1}{2}$ (ii) $\frac{2}{t_2 + t_1}$ (iii) $-\frac{7}{2}$
3. (i) parallel (ii) parallel (iii) perpendicular (iv) neither.
4. (i) 1 (ii) $-\sqrt{3}$ 6. 9
7. (i) The line is either x-axis or it is parallel to x-axis.
   (ii) The line makes an acute angle with positive direction of x-axis.
   (iii) The line makes an obtuse angle with the positive direction of x-axis.

13. $1, \frac{1}{2}$ 14. 104.50 crores 16. $135°$ 17. 4 18. 1 19. $\frac{3\pi}{4}$

HINTS TO NCERT & SELECTED PROBLEMS

12. It is given that points $A(h, 0)$, $B(0, k)$ and $P(a, b)$ are collinear. Therefore,

Slope of PA = Slope of PB

$$\Rightarrow \quad \frac{b-0}{a-h} = \frac{b-k}{a-0}$$

$$\Rightarrow \quad ab = (a-h)(b-k) \Rightarrow ab = ab - ak - bh + hk \Rightarrow hk = ak + bh \Rightarrow \frac{a}{h} + \frac{b}{k} = 1$$

13. Let m be the slop of first line. Then the slope of the second line is $2m$. Let θ be the angle between the lines. Then,

$$\tan\theta = \left|\frac{2m - m}{1 + 2m^2}\right|$$

$$\Rightarrow \quad \frac{1}{3} = \pm\frac{m}{1 + 2m^2}$$

$$\Rightarrow \quad 2m^2 \pm 3m + 1 = 0$$

$$\Rightarrow \quad 2m^2 + 3m + 1 = 0 \text{ or, } 2m^2 - 3m + 1 = 0$$

$$\Rightarrow \quad (2m+1)(m+1) = 0 \text{ or, } (2m-1)(m-1) = 0 \Rightarrow m = \pm\frac{1}{2}, \pm 1.$$

14. Let P be the population in the year 2010. Then, $C(2010, P)$ lies on the line.

Since A, B, C are collinear points. Therefore, Slope of AB = Slope of BC

$$\Rightarrow \quad \frac{97-92}{1995-1985} = \frac{P-97}{2010-1995} \Rightarrow \frac{5}{10} = \frac{P-97}{15} \Rightarrow 7.5 = P - 97 = P = 104.5$$

16. Let θ be the angle between the line joining the points $(3, -1)$ and $(4, -2)$ and x-axis. Then,

$$\tan\theta = \text{Slope of the line} \Rightarrow \tan\theta = \frac{-2+1}{4-3} \Rightarrow \tan\theta = -1 \Rightarrow \theta = 135^\circ.$$

17. It is given that the line through the points $A(-2, 6)$ and $B(4, 8)$ is perpendicular to the line through the points $C(8, 12)$ and $D(x, 24)$. Therefore,

Slope of $AB \times$ Slope of $CD = -1$

$$\Rightarrow \frac{8-6}{4+2} \times \frac{24-12}{x-8} = -1 \Rightarrow \frac{1}{3} \times \frac{12}{x-8} = -1 \Rightarrow -4 = x-8 \Rightarrow x = 4$$

18. It is given that points $A(x, -1)$, $B(2, 1)$ and $C(4, 5)$ are collinear.

$\therefore$ Slope of AB = Slope of BC

$$\Rightarrow \frac{1+1}{2-x} = \frac{5-1}{4-2} \Rightarrow \frac{2}{2-x} = 2 \Rightarrow 2-x = 1 \Rightarrow x = 1.$$

19. Let the required angle be θ. Then,

$$\tan\theta = \text{Slope of the line} \Rightarrow \tan\theta = \frac{-2-(-1)}{4-3} = 1 \Rightarrow \theta = \frac{\pi}{4}$$

20. Given points are $A(-2, -1)$, $B(4, 0)$, $C(3, 3)$ and $D(-3, 2)$. Therefore,

$$m_1 = \text{Slope of } AB = \frac{0+1}{4+2} = \frac{1}{6}, \quad m_2 = \text{Slope of } CD = \frac{2-3}{-3-2} = \frac{1}{6}$$

$$m_3 = \text{Slope of } BC = \frac{3-0}{3-4} = -3 \text{ and, } m_4 = \text{Slope of } AD = \frac{2+1}{-3+2} = -3$$

Clearly, $m_1 = m_2$ and $m_3 = m_4$. Therefore, $AB \parallel CD$ and $BC \parallel AD$.

Hence, $ABCD$ is a parallelogram.

23.4 INTERCEPTS OF A LINE ON THE AXES

If a straight line cuts x-axis at A and the y-axis at B, then OA and OB are known as the intercepts of the line on x-axis and y-axis respectively.

The intercepts are positive or negative according as the line meets with positive or negative directions of the coordinate axes.

In Fig. 23.11, OA = x-intercept, OB = y-intercept.

OA is positive or negative according as A lies on OX or OX' respectively. Similarly, OB is positive or negative according as B lies on OY or OY' respectively.

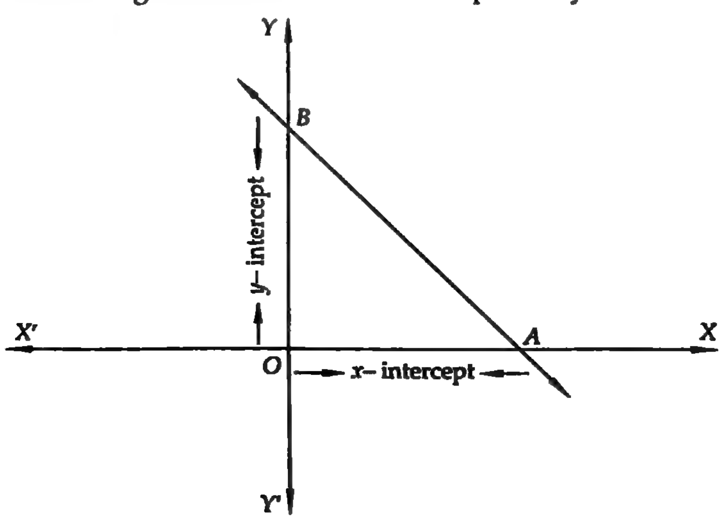


Fig. 23.11

23.5 EQUATIONS OF LINES PARALLEL TO THE COORDINATE AXES

23.5.1 EQUATION OF A LINE PARALLEL TO x-AXIS

Let AB be a straight line parallel to x-axis at a distance b from it. Then, clearly the ordinate of each point on AB is b. Thus, AB can be considered as the locus of a point at a distance b from x-axis. Thus, if $P(x, y)$ is any point on AB, then $y = b$. (See Fig. 23.12).

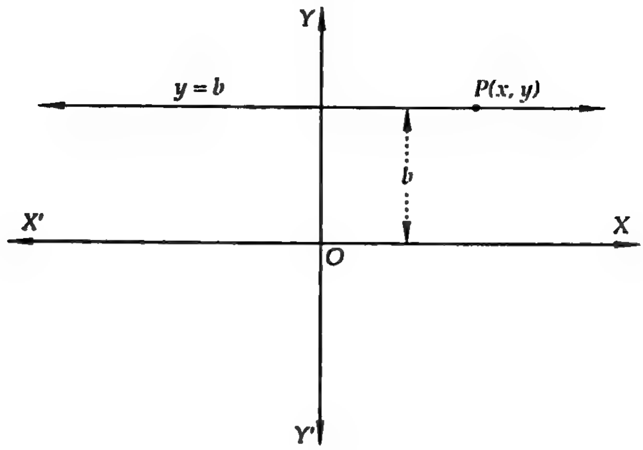


Fig. 23.12

Hence, the equation of a line parallel to x-axis at a distance b from it is $y = b$.

Since x-axis is a parallel to itself at a distance 0 from it, therefore the equation of x-axis is $y = 0$.

If a line is parallel to x-axis at a distance b and below x-axis, then its equation is $y = -b$.

23.5.2 EQUATION OF A LINE PARALLEL TO y-AXIS

Let AB be a line parallel to y-axis and at a distance a from it. Then, the abscissa of every point on AB is a. So it can be treated as the locus of a point a distance a from y-axis. Thus, if $P(x, y)$ is any point on AB, then $x = a$. (See Fig. 23.13)

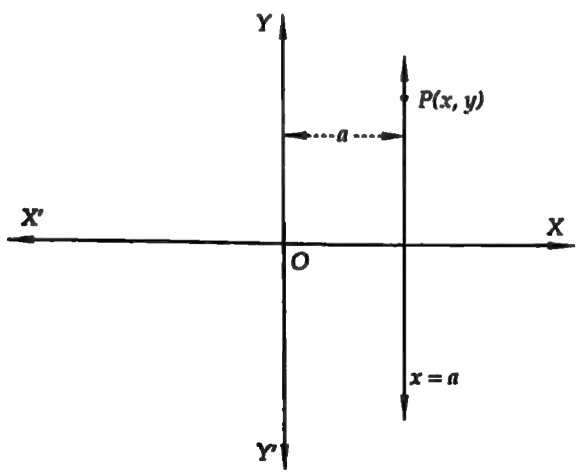


Fig. 23.13

Hence, the equation of a line parallel to y-axis at a distance a from it, is $x = a$.

Since y-axis is parallel to itself at a distance 0 from it, therefore the equation of y-axis is $x = 0$.

If a line is parallel to y-axis at a distance a and to the left of y-axis, then its equation is $x = -a$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Write down the equations of the following lines:*

(i) *x-axis* (ii) *y-axis* [NCERT]

(iii) *A line parallel to x-axis at a distance of 3 units below x-axis.*

(iv) *A line parallel to y-axis at a distance of 5 units on the left hand side of it.*

SOLUTION (i) $y = 0$ (ii) $x = 0$ (iii) $y = -3$ (iv) $x = -5$.

EXAMPLE 2 *Find the equation of a line which is parallel to x-axis and passes through (3, –5).*

SOLUTION The equation of a line parallel to x-axis is $y = b$. Since it passes through $(3, -5)$. So, $-5 = b \Rightarrow b = -5$. Hence, the equation of the required line is $y = -5$.

ALITER Since y-coordinate of every point on a line parallel to x-axis is always same, it follows that the equation of the required line is $y = -5$.

EXAMPLE 3 *Find the equation of a line which is parallel to y-axis and passes through (–4, 3).*

SOLUTION The equation of a line parallel to y-axis is $x = a$. Since, it passes through $(-4, 3)$. So $-4 = a \Rightarrow a = -4$. Hence, the equation of the required line is $x = -4$.

ALITER Since the abscissa of every point on a line parallel to *y-axis* is always same. So, the equation of the required line is $x = -4$.

EXAMPLE 4 *Find the equation of a line which is equidistant from the lines $x = -4$ and $x = 8$.*

SOLUTION Since the given lines are both parallel to y-axis and the required line is equidistant from these lines, so it is also parallel to y-axis and its distance from y-axis is $\frac{1}{2}(-4 + 8) = 2$ units.

Hence, its equation is $x = 2$.

EXERCISE 23.2

LEVEL-1

1. Find the equation of the line parallel to x-axis and passing through $(3, -5)$.
2. Find the equation of the line perpendicular to x-axis and having intercept -2 on x-axis.
3. Find the equation of the line parallel to x-axis and having intercept -2 on y-axis.
4. Draw the lines $x = -3$, $x = 2$, $y = -2$, $y = 3$ and write the coordinates of the vertices of the square so formed.
5. Find the equations of the straight lines which pass through $(4, 3)$ and are respectively parallel and perpendicular to the x-axis.
6. Find the equation of a line which is equidistant from the lines $x = -2$ and $x = 6$.
7. Find the equation of a line equidistant from the lines $y = 10$ and $y = -2$.

ANSWERS

1. $y = -5$ 2. $x = -2$ 3. $y = -2$ 4. $(2, 3), (-3, 3), (-3, -2), (2, -2)$
5. $y = 3, x = 4$ 6. $x = 2$ 7. $y = 4$

HINTS TO NCERT & SELECTED PROBLEMS

6. Since the given lines are parallel to y-axis and the required line is equidistant from the given lines. Therefore, it is parallel to y-axis at a distance $\frac{1}{2}(-2 + 6) = 2$ units from it. So its equation is $x = 2$.

23.6 DIFFERENT FORMS OF THE EQUATION OF A STRAIGHT LINE

In section 23.1, we have seen that a first degree equation in x, y represents a straight line. The equation of a straight line can be written in different forms depending on the data given. In this section, we shall learn about these forms.

23.6.1 SLOPE INTERCEPT FORM OF A LINE

THEOREM *The equation of a line with slope m and making an intercept c on y-axis is* $y = mx + c$.

PROOF Let the given line intersects y-axis at Q and makes an angle θ with x-axis. Then, $m = \tan\theta$. Let $P(x, y)$ be any point on the line. Draw PL perpendicular to x-axis and $QM \perp PL$.

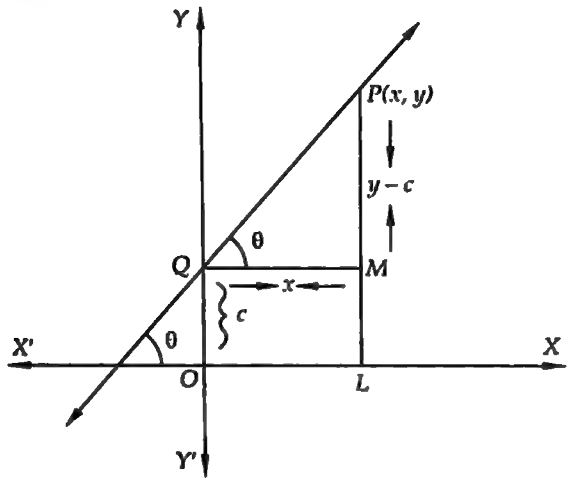


Fig. 23.14

Clearly, $\angle MQP = \theta$, $QM = OL = x$

and, $PM = PL - ML = PL - OQ = y - c$.

From triangle PMQ, we have

$$\tan\theta = \frac{PM}{QM} = \frac{y-c}{x}$$

$$\Rightarrow \quad m = \frac{y-c}{x}$$

$\Rightarrow \quad y = mx + c$, which is the required equation of the line. Q.E.D.

REMARK 1 *If the line passes through the origin, then* $0 = m0 + c \Rightarrow c = 0$. *Therefore, the equation of a line passing through the origin is* $y = mx$, *where m is the slope of the line.*

REMARK 2 *If the line is parallel to x-axis, then* $m = 0$, *therefore the equation of a line parallel to x-axis is* $y = c$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of a line with slope* -1 *and cutting off an intercept of 4 units on negative direction of y-axis.*

SOLUTION Here, $m = -1$ and $c = -4$

Substituting these values in $y = mx + c$, we obtain that the equation of the line is

$$y = -x - 4 \text{ or, } x + y + 4 = 0$$

EXAMPLE 2 *Find the equation of a straight line which cuts off an intercept of 5 units on negative direction of y-axis and makes an angle of* $120°$ *with the positive direction of x-axis.*

SOLUTION Here, $m = \tan 120° = \tan(90 + 30°) = -\cot 30° = -\sqrt{3}$ and $c = -5$.

Substituting these values in $y = mx + c$, we obtain that the equation of the line is

$$y = -\sqrt{3}x - 5 \text{ or, } \sqrt{3}\,x + y + 5 = 0$$

EXAMPLE 3 *Find the equation of a straight line cutting off an intercept –1 from y-axis and being equally inclined to the axes.*

SOLUTION Since the required line is equally inclined with the coordinate axes, therefore it makes either an angle of 45° or 135° with the x-axis.
So, its slope is either $m = \tan 45°$ or, $m = \tan 135°$ i.e. $m = 1$ or, -1. It is given that $c = -1$.

Substituting these values in $y = mx + c$, we obtain that the equations of the lines are $y = x - 1$ and $y = -x - 1$.

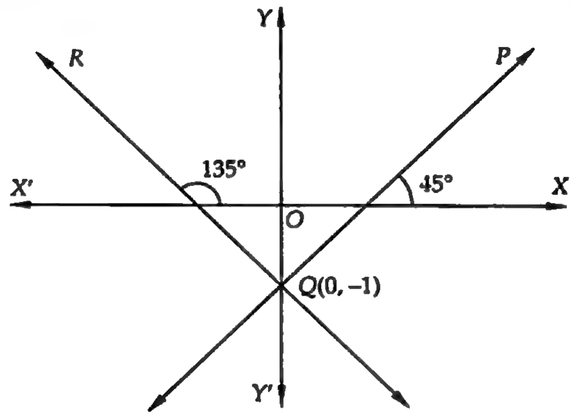


Fig. 23.15

EXAMPLE 4 *Find the equation of a straight line which makes an angle of* $\tan^{-1}\sqrt{2}$ *with the x-axis and cuts off an intercept of* $-\frac{3}{\sqrt{2}}$ *with the y-axis.*

SOLUTION Here, $m = \tan\theta = \tan(\tan^{-1}\sqrt{2}) = \sqrt{2}$ and $c = -\frac{3}{\sqrt{2}}$.

Substituting these values in $y = mx + c$, we obtain that the equation of the required line is

$$y = \sqrt{2}\,x - \frac{3}{\sqrt{2}} \text{ or, } \sqrt{2}y = 2x - 3$$

EXAMPLE 5 *Find the equation of a straight line which cuts off an intercept of length 3 on y-axis and is parallel to the line joining the points* $(3, -2)$ *and* $(1, 4)$.

SOLUTION Let m be the slope of the required line. Since the required line is parallel to the line joining the points $A(3, -2)$ and $B(1, 4)$.

$$\therefore \quad m = \text{Slope of the line } AB = \frac{4-(-2)}{1-3} = -3$$

It is given that $c = 3$. Substituting these values in $y = mx + c$, we obtain that the equation of the required line is $y = -3x + 3$ or, $3x + y - 3 = 0$.

EXAMPLE 6 *Find the equation of a line that has y-intercept 4 and is perpendicular to the line joining (2, –3) and (4, 2).*

SOLUTION Let m be the slope of the required line. Since the required line is perpendicular to the line joining $A(2, -3)$ and $B(4, 2)$.

$$\therefore \quad m \times \text{Slope of } AB = -1 \Rightarrow m \times \frac{2+3}{4-2} = -1 \Rightarrow m = -\frac{2}{5}$$

The required line cuts off an intercept of length 4 on y-axis. So, $c = 4$

Substituting these values in $y = mx + c$, we obtain that the equation of the required line is

$$y = -\frac{2}{5}x + 4 \text{ or, } 2x + 5y - 20 = 0.$$

EXAMPLE 7 *Find the equation of the straight line which makes angle of 15° with the positive direction of x-axis and which cuts an intercept of length 4 on the negative direction of Y-axis.*

SOLUTION Let m be the slope of the line. Then,

$$m = \tan 15° = \tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\Rightarrow \quad m = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

It is given that the line cuts an intercept of length 4 on the negative direction of y-axis.

$\therefore \quad c = -4$

Substituting these values in $y = mx + c$, we get

$y = (2-\sqrt{3})\,x - 4$ as the equation of the required line.

LEVEL-2

EXAMPLE 8 *P_1, P_2 are points on either of the two lines $y - \sqrt{3}\,|x| = 2$ at a distance of 5 units from their point of intersection. Find the coordinates of the foot of perpendiculars drawn from P_1, P_2 on the bisector of the angle between the given lines.* [NCERT EXEMPLAR]

SOLUTION The equations of the given lines are

$\therefore \quad y - \sqrt{3}x = 2$ for $x \geq 0$ ... (i)

and, $\quad y + \sqrt{3}x = 2$ for $x < 0$ ... (ii)

The slopes of these two lines are $\sqrt{3}$ and $-\sqrt{3}$ respectively. So, they make angles of 60° and 120° respectively with x-axis. Consequently, each makes 30° angles with the positive direction of y-axis as shown in Fig. 23.16.

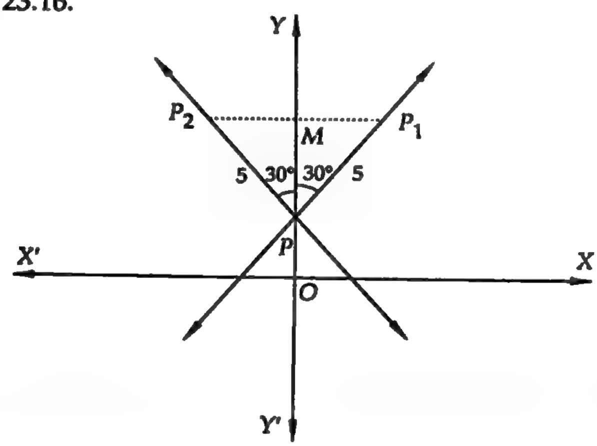


Fig. 23.16

Clearly, lines (i) and (ii) intersect at $P\,(0, 2)$ and y-axis is the bisector of the acute angle between them. It is given that $PP_1 = 5 = PP_2$.

Let M be the foot of the perpendiculars drawn from P_1 and P_2 on y-axis.

In right triange PMP_1, we have

$$\cos 30° = \frac{PM}{PP_1} \Rightarrow \frac{\sqrt{3}}{2} = \frac{PM}{5} \Rightarrow PM = \frac{5\sqrt{3}}{2}$$

$$\therefore \quad OM = OP + PM = 2 + \frac{5\sqrt{3}}{2} = \frac{4+5\sqrt{3}}{2}$$

Hence, the coordinates of M are $\left(\frac{4+5\sqrt{3}}{2}, 0\right)$.

EXERCISE 23.3

LEVEL-1

1. Find the equation of a line making an angle of 150° with the x-axis and cutting off an intercept 2 from y-axis.
2. Find the equation of a straight line:
   (i) with slope 2 and y-intercept 3; (ii) with slope $-1/3$ and y-intercept -4.
   (iii) with slope -2 and intersecting the x-axis at a distance of 3 units to the left of origin.
3. Find the equations of the bisectors of the angles between the coordinate axes.
4. Find the equation of a line which makes an angle of $\tan^{-1}(3)$ with the x-axis and cuts off an intercept of 4 units on negative direction of y-axis.
5. Find the equation of a line that has y-intercept -4 and is parallel to the line joining $(2, -5)$ and $(1, 2)$.
6. Find the equation of a line which is perpendicular to the line joining $(4, 2)$ and $(3, 5)$ and cuts off an intercept of length 3 on y-axis.
7. Find the equation of the perpendicular to the line segment joining $(4, 3)$ and $(-1, 1)$ if it cuts off an intercept -3 from y-axis.
8. Find the equation of the straight line intersecting y-axis at a distance of 2 units above the origin and making an angle of 30° with the positive direction of the x-axis. [NCERT]

ANSWERS

1. $x+\sqrt{3}\,y=2\sqrt{3}$ 2. (i) $y=2x+3$ (ii) $x+3y+12=0$ (iii) $2x+y+6=0$
3. $x \pm y=0$ 4. $y=3x-4$ 5. $7x+y+4=0$ 6. $x-3y+9=0$
7. $5x+2y+6=0$ 8. $x-\sqrt{3}y+2\sqrt{3}=0$

HINTS TO NCERT & SELECTED PROBLEM

8. Clearly, c = y-intercept = 2 and, $m=\tan 30°=\frac{1}{\sqrt{3}}$.

   So, the equation of the line is $y=\frac{1}{\sqrt{3}}x+2$ or, $x-\sqrt{3}y+2\sqrt{3}=0$.

23.6.2 POINT-SLOPE FORM OF A LINE

THEOREM *The equation of a line which passes through the point (x_1, y_1) and has the slope 'm' is* $y-y_1=m(x-x_1)$.

PROOF Let the line pass through the point $Q(x_1, y_1)$ and let $P(x, y)$ be any point on the line. Then,

Slope of the line is $=\frac{y-y_1}{x-x_1}$

But, the slope of the line is m.

$$\therefore \quad m=\frac{y-y_1}{x-x_1} \Rightarrow y-y_1=m(x-x_1)$$

Hence, $y-y_1=m(x-x_1)$ is the required equation of the line. Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of a line passing through $(2, -3)$ and inclined at an angle of 135° with the positive direction of x-axis.*

SOLUTION Here, m = Slope of the line = $\tan 135° = \tan(90°+45°) = -\cot 45° = -1$.

$x_1 = 2, y_1 = -3$

So, the equation of the line is $y - y_1 = m(x - x_1)$

i.e. $y - (-3) = -1(x-2) \Rightarrow y + 3 = -x + 2 \Rightarrow x + y + 1 = 0.$

EXAMPLE 2 *Determine the equation of line through the point $(-4, -3)$ and parallel to x-axis.*

SOLUTION Here, m = Slope = 0, $x_1 = -4$, $y_1 = -3$.

So, the equation of the line is $y - y_1 = m(x - x_1)$

or, $y + 3 = 0(x+4) \Rightarrow y + 3 = 0.$

EXAMPLE 3 *Find the equation of the line passing through $(1, 2)$ and making angle of 30° with y-axis.* [NCERT EXEMPLAR]

SOLUTION The required line makes 30° with the positive direction of y-axis as shown in Fig. 23.17. So, it makes 60° with the positive direction of x-axis. Therefore, its slope m is given by $m = \tan 60° = \sqrt{3}$.

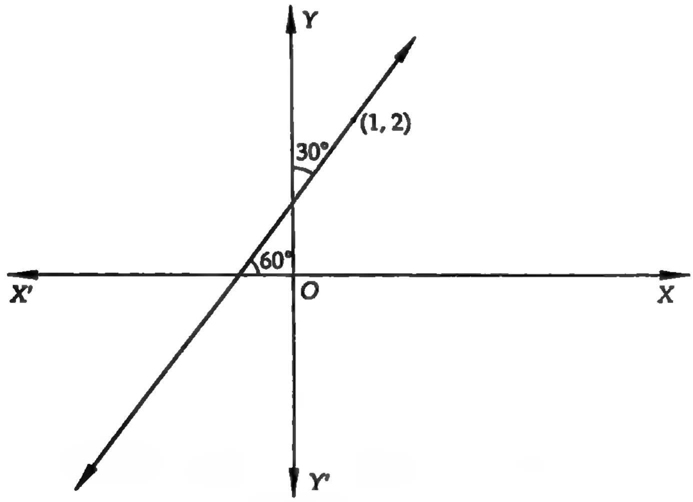


Fig. 23.17

Thus, the required line passes through $(1, 2)$ and has $\sqrt{3}$ as its slope.
Hence, its equation is

$$y - 2 = \sqrt{3}(x - 1) \quad \text{or} \quad \sqrt{3}x - y + 2 - \sqrt{3} = 0$$

EXAMPLE 4 *Find the equation of the perpendicular bisector of the line segment joining the points $A(2, 3)$ and $B(6, -5)$.*

SOLUTION The slope of AB is given by

$$m = \frac{-5-3}{6-2} = -2 \qquad \left[\text{Using: } m = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

$\therefore$ Slope of a line perpendicular to $AB = -\frac{1}{m} = \frac{1}{2}$

Let P be the mid-point of AB. Then, the coordinates of P are $\left(\frac{2+6}{2}, \frac{3-5}{2}\right)$ i.e. $(4, -1)$.

Thus, the required line passes through $P(4, -1)$ and has slope $\frac{1}{2}$. So its equation is

$$y + 1 = \frac{1}{2}(x - 4) \qquad [\text{Using: } y - y_1 = m(x - x_1)]$$

$$\Rightarrow \quad x - 2y - 6 = 0.$$

EXAMPLE 5 *Find the equation of the line for which* $\tan\theta = \frac{1}{2}$, *where* θ *is the inclination of the line and (i) x-intercept equal to 4. (ii) y-intercept is* $-\frac{3}{2}$. **[NCERT]**

SOLUTION (i) Clearly, the line passes through (4, 0) and has slope $= \frac{1}{2}$.

So, the equation of the line is

$$y - 0 = \frac{1}{2}(x-4) \qquad \left[\text{Putting } x_1 = 4,\ y_1 = 0 \text{ and } m = \frac{1}{2} \text{ in } y - y_1 = m(x - x_1)\right]$$

$$\Rightarrow \quad x - 2y - 4 = 0$$

(ii) The line passes through $\left(0, -\frac{3}{2}\right)$ and has slope $= \frac{1}{2}$.

So, it equation is

$$y - \left(-\frac{3}{2}\right) = \frac{1}{2}(x-0) \qquad \left[\text{Putting } x_1 = 0,\ y_1 = -\frac{3}{2} \text{ and } m = \frac{1}{2} \text{ in } y - y_1 = m(x - x_1)\right]$$

$$\Rightarrow \quad 2y + 3 = x \text{ or, } x - 2y - 3 = 0$$

EXAMPLE 6 *The perpendicular from the origin to a line meets it at the point* (– 2, 9), *find the equation of the line.* **[NCERT]**

SOLUTION We have,

$$m_1 = \text{Slope of } OP = \frac{9-0}{-2-0} = -\frac{9}{2}$$

Let m be the slope of the line AB. Then,

Slope of $AB \times$ Slope of $OP = -1$

$$\Rightarrow \quad m \times -\frac{9}{2} = -1$$

$$\Rightarrow \quad m = \frac{2}{9}$$

The equation of AB is

$$y - 9 = \frac{2}{9}[x - (-2)]$$

$$\Rightarrow \quad 9y - 81 = 2x + 4$$

$$\Rightarrow \quad 2x - 9y + 85 = 0$$

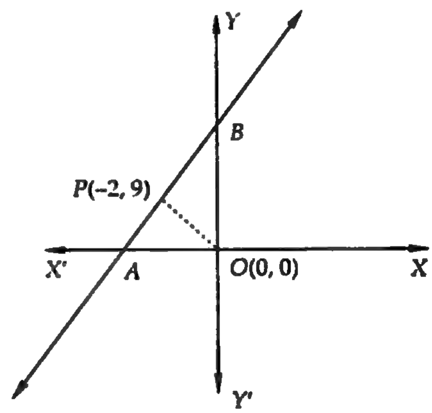


Fig. 23.18

EXAMPLE 7 *Find the equation of the line passing through the point (0, 2) making an angle* $\frac{2\pi}{3}$ *with the positive x-axis. Also, find the equation of line parallel to it and crossing the y-axis at a distance of 2 units below the origin.* **[NCERT]**

SOLUTION The equation of the line passing through (0, 2) and making an angle $\frac{2\pi}{3}$ with the positive x-axis is

$$y - 2 = \tan\frac{2\pi}{3}(x-0) \qquad [\text{Using: } y - y_1 = m(x - x_1)]$$

$$\Rightarrow \quad y - 2 = -\sqrt{3}x \Rightarrow \sqrt{3}x + y - 2 = 0$$

A line parallel to this line crosses y-axis at a distance of 2 units below the origin. So, it passes through (0, – 2) and makes an angle $\frac{2\pi}{3}$ with the x-axis. Hence, its equation is

$$y + 2 = \tan\frac{2\pi}{3}(x-0) \Rightarrow y + 2 = -\sqrt{3}x \Rightarrow \sqrt{3}x + y + 2 = 0$$

EXAMPLE 8 *Two lines passing through the point (2, 3) intersect each other at an angle of 60°. If slope of one line is 2, find the equation of the other line.* [NCERT]

SOLUTION Let the slope of the other line be m. It is given that the angle between the two lines is 60°.

$$\therefore \quad \tan 60^\circ = \left|\frac{m-2}{1+2m}\right|$$

$$\Rightarrow \quad \sqrt{3} = \left|\frac{m-2}{1+2m}\right|$$

$$\Rightarrow \quad \frac{m-2}{1+2m} = \pm\sqrt{3}$$

$$\Rightarrow \quad m-2 = \pm\sqrt{3} \pm 2\sqrt{3}m$$

$$\Rightarrow \quad m(1+-2\sqrt{3}) = 2 \pm \sqrt{3}$$

$$\Rightarrow \quad m = \frac{2\pm\sqrt{3}}{1+-2\sqrt{3}} \Rightarrow m = \frac{2+\sqrt{3}}{1-2\sqrt{3}}, \frac{2-\sqrt{3}}{1+2\sqrt{3}} \Rightarrow m = -\frac{2+\sqrt{3}}{2\sqrt{3}-1}, \frac{2-\sqrt{3}}{2\sqrt{3}+1}$$

Substituting $x_1 = 2$, $y_1 = 3$ and the values of m in $y - y_1 = m(x - x_1)$, we obtain that the equations of the required lines are

$$y - 3 = -\frac{2+\sqrt{3}}{2\sqrt{3}-1}(x-2) \text{ and } y - 3 = \frac{2-\sqrt{3}}{2\sqrt{3}+1}(x-2)$$

EXAMPLE 9 *Find the equation of the line passing through (– 3, 5) and perpendicular to the line through the points (2, 5) and (– 3, 6).* [NCERT]

SOLUTION The slope of the line passing through (2, 5) and (– 3, 6) is $\frac{6-5}{-3-2} = \frac{-1}{5}$

Let m be the slope of the line perpendicular to the line passing through (2, 5) and (–3, 6). Then,

$$m \times -\frac{1}{5} = -1 \Rightarrow m = 5$$

The required line passes through (– 3, 5) and has slope $m = 5$. So, its equation is

$y - 5 = 5(x + 3)$ or, $5x - y + 20 = 0$

EXAMPLE 10 *A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1 : n. Find the equation of the line.* [NCERT]

SOLUTION The slope of the line joining A (1, 0) and B (2, 3) is $\frac{3-0}{2-1} = 3$ and the coordinates of the point dividing it in the ratio $1 : n$ are $\left(\frac{n+2}{n+1}, \frac{3}{n+1}\right)$. The slope of the line perpendicular to the line segment AB is $-\frac{1}{3}$.

Hence, the equation of the required line is

$$y - \frac{3}{n+1} = -\frac{1}{3}\left(x - \frac{n+2}{n+1}\right) \text{ or, } (n+1)x + 3(n+1)y = n + 11$$

EXAMPLE 11 *Find the equation of a line which divides the join of (1, 0) and (3, 0) in the ratio 2 : 1 and perpendicular to it.*

SOLUTION Let C be the point which divides the join of A (1, 0) and B (3, 0) in the ratio 2 : 1. Then, the coordinates of C are

$$\left(\frac{2\times 3+1\times 1}{2+1}, \frac{2\times 0+1\times 0}{2+1}\right)=\left(\frac{7}{3}, 0\right)$$

Since AB is along x-axis, therefore a line perpendicular to AB is parallel to y-axis. As it passes through $C\ (7/3, 0)$, therefore its equation is

$$x=\frac{7}{3} \Rightarrow 3x=7$$

Hence, the equation of the required line is $3x=7$.

EXAMPLE 12 *The vertices of a triangle are A (10, 4), B (−4, 9) and C (−2, −1). Find the equation of its altitudes. Also, find its orthocentre.*

SOLUTION Let AD, BE and CF be three altitudes of ΔABC. Clearly, $AD \perp BC$, $BE \perp CA$ and $CF \perp AB$.

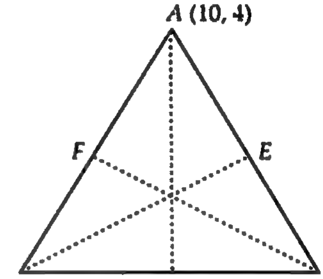


Fig. 23.19

We have,

$$\text{Slope of } BC=\frac{-1-9}{-2+4}=-5$$

$$\therefore \quad \text{Slope of } AD=\frac{1}{5} \qquad [\because AD \perp BC]$$

Since AD passes through A (10, 4). Therefore, equation of AD is

$$y-4=\frac{1}{5}(x-10) \Rightarrow x-5y+10=0$$

...(i)

$$\text{Slope of } AC=\frac{4+1}{10+2}=\frac{5}{12}$$

$$\therefore \quad \text{Slope of } BE=-\frac{12}{5} \qquad [\because BE \perp AC]$$

Clearly, BE passes through B (−4, 9) and has slope −12/5. So, the equation of BE is

$$y-9=-\frac{12}{5}(x+4) \Rightarrow 12x+5y+3=0 \qquad \text{...(ii)}$$

$$\text{Slope of } AB=\frac{4-9}{10+4}=-\frac{5}{14} \Rightarrow \text{Slope of } CF=\frac{14}{5} \qquad [\because CF \perp AB]$$

Clearly, CF passes through C (−2, −1) and has slope $\frac{14}{5}$. So, the equation of CF is

$$y+1=\frac{14}{5}(x+2) \Rightarrow 14x-5y+23=0 \qquad \text{...(iii)}$$

Thus, the altitudes of ΔABC are

$$x-5y+10=0, 12x+5y+3=0 \text{ and, } 14x-5y+23=0.$$

The orthocentre of ΔABC is the point of intersection of its altitudes.

Solving (i) and (ii) by cross-multiplication, we get

$$\frac{x}{-65}=\frac{y}{117}=\frac{1}{65} \Rightarrow x=-1, y=\frac{9}{5}$$

Hence, the coordinates of the orthocentre are $(-1, 9/5)$.

EXAMPLE 13 *Find the equations of the altitudes of the triangle whose vertices are* $A(7,-1)$, $B(-2,8)$ *and* $C(1,2)$.

SOLUTION Let AD, BE and CF be three altitudes of triangle ABC. Let m_1, m_2 and m_3 be the slopes of AD, BE and CF respectively. Then,

$$AD \perp BC \Rightarrow \text{Slope of } AD \times \text{Slope of } BC = -1$$

$$\Rightarrow m_1 \times \left(\frac{2-8}{1+2}\right) = -1 \Rightarrow m_1 = \frac{1}{2}$$

$$BE \perp AC \Rightarrow \text{Slope of } BE \times \text{Slope of } AC = -1$$

$$\Rightarrow m_2 \times \left(\frac{-1-2}{7-1}\right) = -1 \Rightarrow m_2 = 2$$

and, $$CF \perp AB \Rightarrow \text{Slope of } CF \times \text{Slope of } AB = -1$$

$$\Rightarrow m_3 \times \frac{-1-8}{7+2} = -1 \Rightarrow m_3 = 1.$$

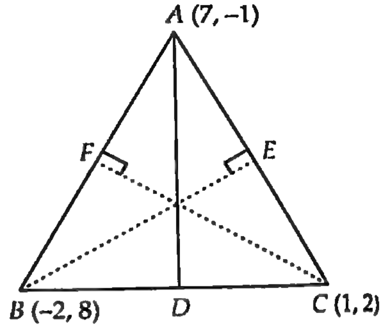


Fig. 23.20

Since AD passes through $A(7,-1)$ and has slope $m_1 = \frac{1}{2}$.

So, its equation is

$$y+1 = \frac{1}{2}(x-7) \Rightarrow x-2y-9 = 0$$

Similarly, equation of BE is

$$y-8 = 2(x+2) \Rightarrow 2x-y+12 = 0$$

Equation of CF is $y-2 = 1(x-1) \Rightarrow x-y+1 = 0$

EXAMPLE 14 *The mid-points of the sides of a triangle are* $(2,1)$, $(-5,7)$ *and* $(-5,-5)$. *Find the equations of the sides of the triangle.*

SOLUTION Let $D(2,1)$, $E(-5,7)$ and $F(-5,-5)$ be the mid-points of sides BC, CA and AB respectively of ΔABC.

We know that the line joining the mid-points of two sides of a triangle is parallel to the third two side.

$\therefore$ $DE \parallel AB$, $EF \parallel BC$ and $DF \parallel AC$

$\therefore$ Slope of AB = Slope of DE

Slope of BC = Slope of EF and, Slope of AC = Slope of DF

Let m_1, m_2 and m_3 be the slopes of AB, BC and CA respectively. Then,

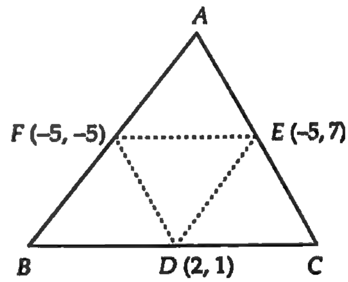


Fig. 23.21

$$m_1 = \text{Slope of } AB = \text{Slope of } DE = \frac{7-1}{-5-2} = \frac{-6}{7}$$

$$m_2 = \text{Slope of } BC = \text{Slope of } EF = \frac{7+5}{-5+5} \text{ (Undefined)}$$

$$m_3 = \text{Slope of } CA = \text{Slope of } DF = \frac{1+5}{2+5} = \frac{6}{7}$$

Side AB passes through $F(-5,-5)$ and has slope $m_1 = \frac{-6}{7}$. So, its equation is

$$y+5 = \frac{-6}{7}(x+5) \Rightarrow 6x+7y+65 = 0$$

Side BC is parallel to Y-axis and passes through $D(2,1)$. So, its equation is $x = k$. As it passes through $(2,1)$.

$\therefore$ $2 = k$.

Hence, equation of BC is $x = 2$.

Side CA passes through $E(-5, 7)$ and has slope $m_3 = \frac{6}{7}$. So, its equation is

$$y - 7 = \frac{6}{7}(x+5) \Rightarrow 6x - 7y + 79 = 0$$

EXAMPLE 15 *Find the equation of the perpendicular bisector of the line segment joining the points* $(1, 1)$ *and* $(2, 3)$. **[NCERT]**

SOLUTION Let P be the mid-point of the line segment joining points $A(1, 1)$ and $B(2, 3)$. Then, the coordinates of P are $\left(\frac{3}{2}, 2\right)$

Let m be the slope of perpendicular bisector of AB. Then,

$$m \times \text{Slope of } AB = -1$$

$$\Rightarrow \quad m \times \frac{3-1}{2-1} = -1$$

$$\Rightarrow \quad m = \frac{-1}{2}$$

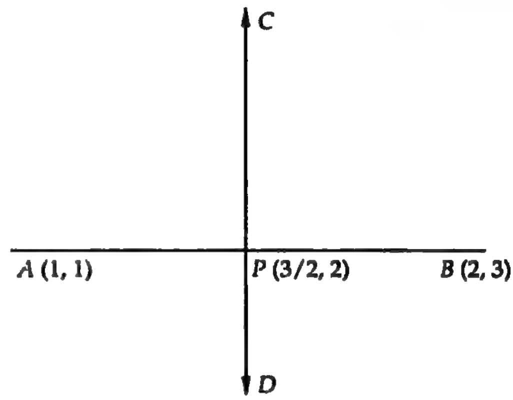


Fig. 23.22

Clearly, perpendicular bisector of AB passes through $P\left(\frac{3}{2}, 2\right)$ and has slope $m = -\frac{1}{2}$. So, its equation is

$$y - 2 = -\frac{1}{2}\left(x - \frac{3}{2}\right) \text{ or, } 2x + 4y - 11 = 0.$$

EXAMPLE 16 *Show that the perpendicular drawn from the point* $(4, 1)$ *on the line segment joining* $(6, 5)$ *and* $(2, -1)$ *divides it internally in the ratio* $8:5$.

SOLUTION Suppose perpendicular drawn from $P(4, 1)$ on the line joining $A(6, 5)$ and $B(2, -1)$ meets AB at M. Let m be the slope of PM. Then,

$$PM \perp AB$$

$$\Rightarrow \quad m \times \text{Slope of } AB = -1$$

$$\Rightarrow \quad m \times \frac{-1-5}{2-6} = -1$$

$$\Rightarrow \quad m \times \frac{3}{2} = -1$$

$$\Rightarrow \quad m = -\frac{2}{3}$$

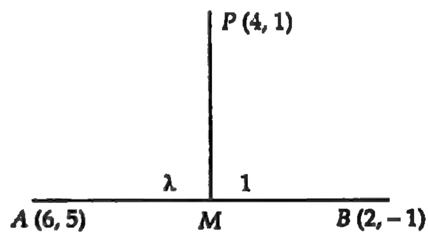


Fig. 23.23

Clearly, PM passes through $P(4, 1)$ and has slope $m = -\frac{2}{3}$. So, its equation is

$$y - 1 = -\frac{2}{3}(x - 4) \text{ or, } 2x + 3y - 11 = 0 \qquad \text{...(i)}$$

Suppose M divides line segment AB in the ratio $\lambda : 1$. Then, coordinates of M are

$$\left(\frac{2\lambda + 6}{\lambda + 1}, \frac{-\lambda + 5}{\lambda + 1}\right)$$

Since M lies on line PM whose equation is $2x + 3y - 11 = 0$

$$\therefore \quad 2\left(\frac{2\lambda + 6}{\lambda + 1}\right) + 3\left(\frac{-\lambda + 5}{\lambda + 1}\right) - 11 = 0$$

$$\Rightarrow \quad 4\lambda + 12 - 3\lambda + 15 - 11\lambda - 11 = 0 \Rightarrow -10\lambda + 16 = 0 \Rightarrow \lambda = \frac{8}{5}$$

Hence, M divides AB internally in the ratio $8:5$.

LEVEL-2

EXAMPLE 17 *One side of a square makes an angle α with x-axis and one vertex of the square is at the origin. Prove that the equations of its diagonals are $x(\sin\alpha+\cos\alpha)=y(\cos\alpha-\sin\alpha)$ and $x(\cos\alpha-\sin\alpha)+y(\sin\alpha+\cos\alpha)=a$, where a is the length of the side of the square.*

SOLUTION Let $OABC$ be the square such that its side OA makes an angle α with x-axis. Since $OA=a$, therefore coordinates of A are $(a\cos\alpha, a\sin\alpha)$. Clearly, the diagonal OB makes an angle $(\pi/4+\alpha)$ with x-axis and passes through $(0, 0)$. So, equation of OB is

$$y-0 = \tan(\pi/4+\alpha)(x-0)$$

or, $$y = \frac{1+\tan\alpha}{1-\tan\alpha}x$$

$$\Rightarrow \quad y(\cos\alpha-\sin\alpha) = x(\cos\alpha+\sin\alpha) \quad ...(i)$$

Since the diagonal AC is perpendicular to OB. Therefore,

$$\text{Slope of } AC = \frac{-1}{\tan\left(\frac{\pi}{4}+\alpha\right)}.$$

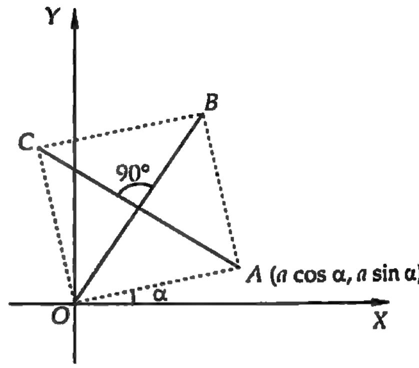


Fig. 23.24

Clearly, diagonal AC passes through $(a\cos\alpha, a\sin\alpha)$. So, equation of AC is

$$y-a\sin\alpha = \frac{-1}{\tan\left(\frac{\pi}{4}+\alpha\right)}(x-a\cos\alpha)$$

$$\Rightarrow \quad y-a\sin\alpha = -\frac{1-\tan\alpha}{1+\tan\alpha}(x-a\cos\alpha)$$

$$\Rightarrow \quad y-a\sin\alpha = -\left(\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\right)(x-a\cos\alpha)$$

$$\Rightarrow \quad x(\cos\alpha-\sin\alpha)+y(\cos\alpha+\sin\alpha) = a.$$

EXAMPLE 18 *A line passing through the point $A(3, 0)$ makes $30°$ angle with the positive direction of x-axis. If this line is rotated through an angle of $15°$ in clockwise direction, find its equation in new position.* [NCERT EXEMPLAR]

SOLUTION Let AB be the given line and AC be its new position. Clearly, AC makes an angle of $15°$ with the positive direction of X-axis.

$$\therefore \quad m = \text{Slope of } AC = \tan 15°$$

$$\Rightarrow \quad m = \tan(45°-30°)$$

$$\Rightarrow \quad m = \frac{\tan 45°-\tan 30°}{1+\tan 45°\tan 30°}$$

$$\Rightarrow \quad m = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\Rightarrow \quad m = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

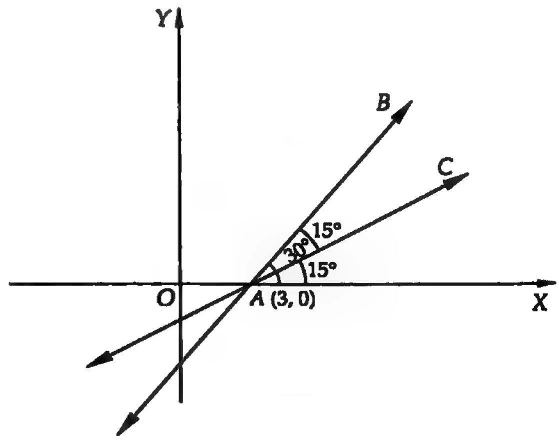


Fig. 23.25

Clearly, AC passes through $A(3, 0)$ and has slope $m = 2-\sqrt{3}$. So, its equation is

$$y-0 = (2-\sqrt{3})(x-3) \text{ or, } (2-\sqrt{3})x-y-3(2-\sqrt{3})=0$$

EXERCISE 23.4

LEVEL-1

1. Find the equation of the straight line passing through the point (6, 2) and having slope -3.
2. Find the equation of the straight line passing through $(-2, 3)$ and inclined at an angle of $45°$ with the x-axis.
3. Find the equation of the line passing through (0, 0) with slope m. [NCERT]
4. Find the equation of the line passing through $(2, 2\sqrt{3})$ and inclined with x-axis at an angle of $75°$. [NCERT]
5. Find the equation of the straight line which passes through the point (1, 2) and makes such an angle with the positive direction of x-axis whose sine is $\frac{3}{5}$.
6. Find the equation of the straight line passing through $(3, -2)$ and making an angle of $60°$ with the positive direction of y-axis.
7. Find the lines through the point (0, 2) making angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ with the x-axis. Also, find the lines parallel to them cutting the y-axis at a distance of 2 units below the origin.
8. Find the equations of the straight lines which cut off an intercept 5 from the y-axis and are equally inclined to the axes.
9. Find the equation of the line which intercepts a length 2 on the positive direction of the x-axis and is inclined at an angle of $135°$ with the positive direction of y-axis.
10. Find the equation of the straight line which divides the join of the points (2, 3) and $(-5, 8)$ in the ratio 3 : 4 and is also perpendicular to it.
11. Prove that the perpendicular drawn from the point (4, 1) on the join of $(2, -1)$ and (6, 5) divides it in the ratio 5 : 8.
12. Find the equations to the altitudes of the triangle whose angular points are $A(2, -2)$, $B(1, 1)$ and $C(-1, 0)$.
13. Find the equation of the right bisector of the line segment joining the points (3, 4) and $(-1, 2)$. [NCERT]
14. Find the equation of the line passing through the point $(-3, 5)$ and perpendicular to the line joining (2, 5) and $(-3, 6)$.
15. Find the equation of the right bisector of the line segment joining the points $A(1, 0)$ and $B(2, 3)$.

ANSWERS

1. $3x + y - 20 = 0$
2. $x - y + 5 = 0$
3. $y = mx$
4. $(2 + \sqrt{3})x - y - 4 = 0$
5. $3x - 4y + 5 = 0$
6. $x - \sqrt{3}y - 3 - 2\sqrt{3} = 0$
7. $\sqrt{3}x - y + 2 = 0$, $\sqrt{3}x + y - 2 = 0$, $\sqrt{3}x + y + 2 = 0$
8. $y = x + 5$ or $x + y = 5$
9. $x - y - 2 = 0$
10. $49x - 35y + 229 = 0$
12. $2x + y - 2 = 0$, $3x - 2y - 1 = 0$, $x - 3y + 1 = 0$
13. $2x + y = 5$
14. $5x - y + 20 = 0$
15. $x + 3y - 6 = 0$

HINTS TO NCERT & SELECTED PROBLEMS

3. The equation of the line passing through (0, 0) and slope m is $y - 0 = m(x - 0)$ or, $y = mx$.

4. The equation of the required line is

$$y-2\sqrt{3} = \tan 75°(x-2)$$

or, $$y-2\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1}(x-2) \qquad \left[\because \tan 75° = \tan(45°+30°) = \frac{\tan 45° + \tan 30°}{1-\tan 45° \tan 30°}\right]$$

or, $$y-2\sqrt{3} = (2+\sqrt{3})(x-2)$$

13. The right bisector of the segment joining $A(3, 4)$ and $B(-1, 2)$ passes through the mid-point $C(1, 3)$ of AB and is perpendicular to AB. Let m be the slope of AB. Then,

$$m = \frac{2-4}{-1-3} = \frac{1}{2}$$

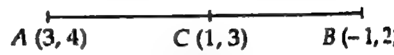


Fig. 23.25

So, m_1 = Slope of a line perpendicular to $AB = -2$.
Hence, the equation of the right bisector of AB is $y-3 = -2(x-1)$ or, $2x+y-5 = 0$.

23.6.3 TWO-POINT FORM OF A LINE

THEOREM *The equation of a line passing through two points* (x_1, y_1) and (x_2, y_2) *is*

$$y-y_1 = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1).$$

PROOF Let m be the slope of the line passing through (x_1, y_1) and (x_2, y_2). Then,

$$m = \frac{y_2-y_1}{x_2-x_1}$$

So, the equation of the line is

$$y - y_1 = m(x-x_1) \qquad \text{[Using point-slope form]}$$

Substituting the value of m, we obtain

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

This is the required equation of the line in two point form. Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the line joining the points* $(-1, 3)$ *and* $(4, -2)$.

SOLUTION Here, the two points are $(x_1, y_1) \equiv (-1, 3)$ and $(x_2, y_2) \equiv (4, -2)$.
So, the equation of the line in two-point form is

$$y - 3 = \frac{3-(-2)}{-1-4}(x+1) \Rightarrow y - 3 = -x - 1 \Rightarrow x + y - 2 = 0.$$

EXAMPLE 2 *Find the equation of the line joining the points* $(at_1^2, 2at_1)$ *and* $(at_2^2, 2at_2)$.

SOLUTION Here, $x_1 = at_1^2$, $y_1 = 2at_1$, $x_2 = at_2^2$, $y_2 = 2at_2$.
So, the equation of the required line is

$$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$

$$\Rightarrow \quad y - 2at_1 = \frac{2}{t_1+t_2}(x-at_1^2)$$

$$\Rightarrow \quad y(t_1+t_2) - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$$

$$\Rightarrow \quad y(t_1+t_2) = 2x + 2at_1t_2.$$

EXAMPLE 3 *Find the equations of the medians of the triangle ABC whose vertices are* $A(2, 5)$, $B(-4, 9)$ *and* $C(-2, -1)$.

SOLUTION Let D, E, F be the mid-points of BC, CA and AB respectively. Then, the coordinates of these points are $D(-3, 4)$, $E(0, 2)$ and $F(-1, 7)$ respectively. The median AD passes through points $A(2, 5)$ and $D(-3, 4)$.

So, equation of AD is

$$y - 5 = \frac{4-5}{-3-2}(x-2)$$

$$\Rightarrow \quad y - 5 = \frac{1}{5}(x-2)$$

$$\Rightarrow \quad x - 5y + 23 = 0$$

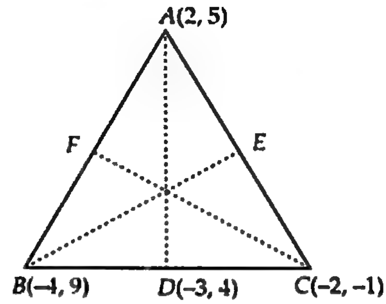


Fig. 23.26

The median BE passes through points $B(-4, 9)$ and $E(0, 2)$.

So, equation of median BE is

$$(y-9) = \left(\frac{2-9}{0+4}\right)(x+4) \Rightarrow 7x + 4y - 8 = 0.$$

Similarly, the equation of the median CF is

$$(y+1) = \frac{7+1}{-1+2}(x+2) \Rightarrow 8x - y + 15 = 0.$$

EXAMPLE 4 *In what ratio is the line joining the points* $(2, 3)$ *and* $(4, 1)$ *divides the segment joining the points* $(1, 2)$ *and* $(4, 3)$?

SOLUTION The equation of the line joining the points $(2, 3)$ and $(4, 1)$ is

$$y - 3 = \frac{1-3}{4-2}(x-2) \Rightarrow y - 3 = -x + 2 \Rightarrow x + y - 5 = 0 \qquad \text{...(i)}$$

Suppose the line joining $(2, 3)$ and $(4, 1)$ divides the segment joining $(1, 2)$ and $(4, 3)$ at point P in the ratio $\lambda : 1$. Then, the coordinates of P are $\left(\frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+2}{\lambda+1}\right)$.

Clearly P lies on line (i).

$$\therefore \quad \frac{4\lambda+1}{\lambda+1} + \frac{3\lambda+2}{\lambda+1} - 5 = 0 \Rightarrow \lambda = 1.$$

Hence, the required ratio is $\lambda : 1$ i.e., $1 : 1$.

EXAMPLE 5 *In what ratio, the line joining* $(-1, 1)$ *and* $(5, 7)$ *is divided by the line* $x + y = 4$?

SOLUTION Suppose the line $x + y = 4$ divides the join of $A(-1, 1)$ and $B(5, 7)$ in the ratio $\lambda : 1$. The coordinates of the point of division are $\left(\frac{5\lambda+1}{\lambda+1}, \frac{7\lambda+1}{\lambda+1}\right)$. It lies on $x + y = 4$.

$$\therefore \quad \frac{5\lambda-1}{\lambda+1} + \frac{7\lambda+1}{\lambda+1} = 4 \Rightarrow 5\lambda - 1 + 7\lambda + 1 = 4(\lambda+1) \Rightarrow 12\lambda = 4\lambda + 4 \Rightarrow 8\lambda = 4 \Rightarrow \lambda = \frac{1}{2}$$

Hence, the required ratio are $1 : 2$.

EXAMPLE 6 *Prove that the points* $(5, 1)$, $(1, -1)$ *and* $(11, 4)$ *are collinear. Also ,find the equation of the straight line on which these points lie.*

SOLUTION Let the given points be $A(5, 1)$, $B(1, -1)$ and $C(11, 4)$. Then, the equation of the line passing through A and B is

$$y - 1 = \frac{-1-1}{1-5}(x-5) \Rightarrow x - 2y - 3 = 0$$

Clearly, point C (11, 4) satisfies the equation $x - 2y - 3 = 0$. Hence, the given points lie on the same straight line, whose equation is $x - 2y - 3 = 0$.

EXAMPLE 7 *The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that K = 273 when F = 32 and that K = 373 when F = 212. Express K in terms of F and find the value of F, when K = 0.* [NCERT]

SOLUTION Assuming F along X-axis and K along Y-axis, we have two points (32, 273) and (212, 373) in xy-plane or FK-plane.

As F and K satisfy a linear equation. The equation of the line passing through (32, 273) and (212, 373) is

$$K - 273 = \frac{373-273}{212-32}(F-32)$$

$$\Rightarrow \quad K - 273 = \frac{100}{180}(F-32)$$

$$K = \frac{5}{9}(F-32) + 273 \quad \text{...(i)}$$

ting $K = 0$ in (i), we get

$$0 = \frac{5}{9}(F-32) + 273 \Rightarrow F - 32 = -\frac{273 \times 9}{5} \Rightarrow F = 32 - 491.4 \Rightarrow F = -459.4$$

LEVEL-2

EXAMPLE 8 *Find the equation of the internal bisector of angle BAC of the triangle ABC whose vertices A, B, C are (5, 2), (2, 3) and (6, 5) respectively.*

SOLUTION We have,

$$AB = \sqrt{(5-2)^2 + (2-3)^2} = \sqrt{10}$$

and, $$AC = \sqrt{(5-6)^2 + (2-5)^2} = \sqrt{10}$$

$$\therefore \quad AB : AC = \sqrt{10} : \sqrt{10} = 1 : 1$$

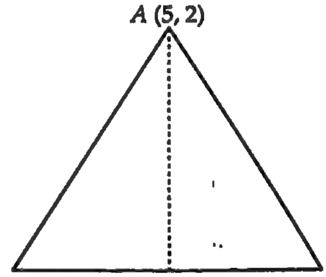


Fig. 23.27

The internal bisector AD of $\angle BAC$ divides BC in the ratio $AB : AC$ i.e. 1 : 1. So, coordinates of D are

$$\left(\frac{2+6}{2}, \frac{3+5}{4}\right) = (4, 4).$$

Equation of AD is $y - 2 = \dfrac{4-2}{4-5}(x-5)$ or, $2x + y - 12 = 0$

EXAMPLE 9 *A rectangle has two opposite vertices at the points (1, 2) and (5, 5). If the other vertices lie on the line x = 3, find the equations of the sides of the rectangle.*

SOLUTION Let $ABCD$ be a rectangle whose two opposite vertices are A (1, 2) and C (5, 5).

Let the coordinates of other two vertices B and D of rectangle $ABCD$ be $B(3, y_1)$ and $D(3, y_2)$.

Since diagonals AC and BD bisect each other. Therefore, mid-points of AC and BD are same.

$\therefore \quad \dfrac{y_1 + y_2}{2} = \dfrac{2+5}{2} \Rightarrow y_1 + y_2 = 7$ ...(i)

Since $ABCD$ is a rectangle.

$\therefore \quad AC = BD$

$\Rightarrow \quad AC^2 = BD^2$

$\Rightarrow \quad (1-5)^2 + (2-5)^2 = (3-3)^2 + (y_1 - y_2)^2$

$\Rightarrow \quad 16 + 9 = (y_1 - y_2)^2$

$\Rightarrow \quad y_1 - y_2 = \pm 5$ ...(ii)

Solving (i) and (ii), we get

$$y_1 = 6 \text{ and } y_2 = 1 \text{ or, } y_1 = 1 \text{ and } y_2 = 6$$

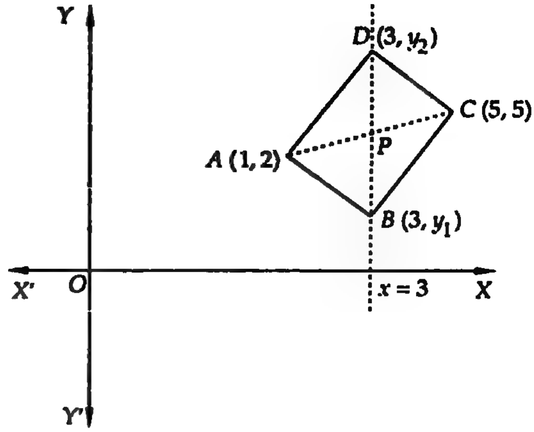


Fig. 23.28

Thus, the coordinates of B and D are $B(3, 1)$ and $D(3, 6)$.

The equation of side AB is

$$y - 2 = \frac{1-2}{3-1}(x-1) \text{ or, } y - 2 = -\frac{1}{2}(x-1) \text{ or, } x + 2y - 5 = 0$$

The equation of side BC is

$$y - 1 = \frac{5-1}{5-3}(x-3) \text{ or, } y - 1 = 2(x-3) \text{ or, } 2x - y - 5 = 0$$

The equation of side CD is

$$y - 5 = \frac{6-5}{3-5}(x-5) \text{ or, } y - 5 = -\frac{1}{2}(x-5) \text{ or, } x + 2y - 15 = 0$$

The equation of side AD is

$$y - 2 = \frac{6-2}{3-1}(x-1) \text{ or, } y - 2 = 2(x-1) \text{ or, } 2x - y = 0$$

EXAMPLE 10 *Find the coordinates of the vertices of a square inscribed in the triangle with vertices $A(0, 0)$, $B(2, 1)$ and $C(3, 0)$; given that two of its vertices are on the side AC.*

SOLUTION Let $PQRS$ be the square inscribed in the triangle ABC such that its vertices P and S lie on side AC which is along X-axis. Let the length of each side of the square be l and the coordinates of P be $(a, 0)$. Then, the coordinates of other vertices are $P(a, 0)$, $S(a + l, 0)$, $Q(a, l)$ and $R(a + l, l)$.

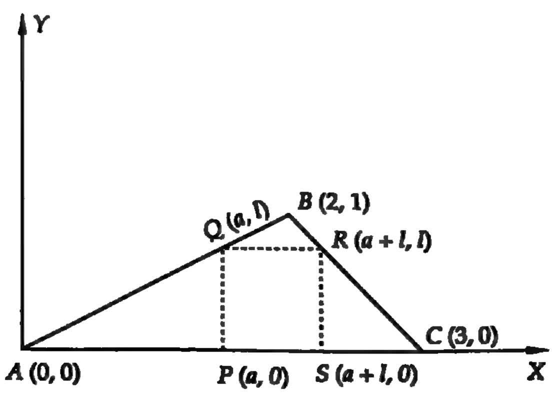


Fig. 23.29

The equations of sides AB and BC are

$$y-0 = \frac{1-0}{2-0}(x-0) \text{ and } y-0 = \frac{1-0}{2-3}(x-3) \text{ respectively}$$

or, $\quad x-2y = 0$ and $x+y-3 = 0$ respectively.

Since Q and R lie on AB and BC respectively.

$\therefore \quad a-2l = 0$ and $a+l+l-3 = 0$

$\Rightarrow \quad a = 2l$ and $a+2l-3 = 0$

$\Rightarrow \quad a = \frac{3}{2}$ and $l = \frac{3}{4}$

Hence, the coordinates of the vertices of the square are

$$P(3/2, 0), Q(3/2, 3/4), R(9/4, 3/4) \text{ and } S(9/4, 0)$$

EXAMPLE 11 *A line is such that its segment between the lines $5x-y+4=0$ and $3x+4y-4=0$ is bisected at the point $(1, 5)$. Obtain its equation.* [NCERT]

SOLUTION Suppose the required line intersects the lines $5x-y+4=0$ and $3x+4y-4=0$ at $P(x_1, y_1)$ and $Q(x_2, y_2)$ respectively. Clearly, $P(x_1, y_1)$ lies on $5x-y+4=0$ and $Q(x_2, y_2)$ lies on $3x+4y-4=0$.

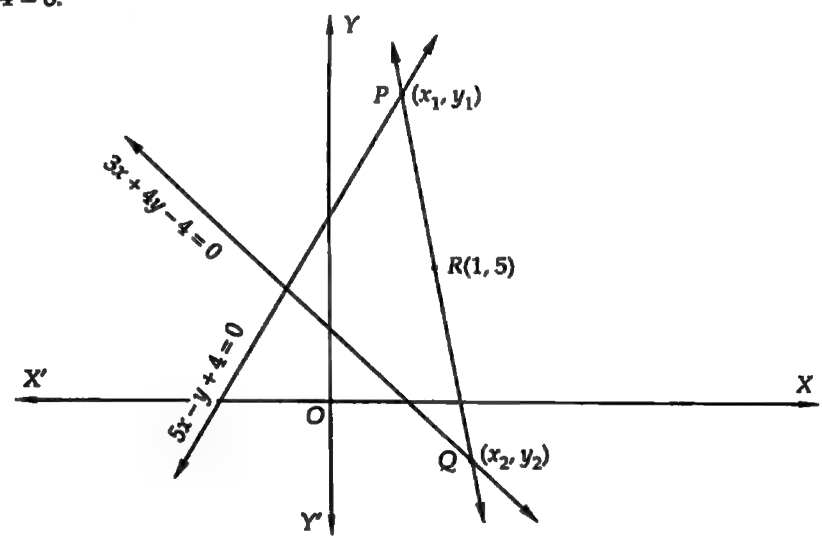


Fig. 23.30

$\therefore \quad 5x_1-y_1+4 = 0$ and, $3x_2+4y_2-4 = 0$

$\Rightarrow \quad y_1 = 5x_1+4$ and $y_2 = \frac{4-3x_2}{4}$ ...(i)

Since R is the mid-point of PQ. Therefore,

$$\frac{x_1+x_2}{2} = 1 \text{ and } \frac{y_1+y_2}{2} = 5$$

$\Rightarrow \quad x_1+x_2 = 2$ and $y_1+y_2 = 10$

$\Rightarrow \quad x_1+x_2 = 2$ and $5x_1+4+\frac{4-3x_2}{4} = 10$ [Using (i)]

$\Rightarrow \quad x_1+x_2 = 2$ and $20x_1-3x_2 = 20$

Solving these two equations, we get

$$x_1 = \frac{26}{23} \text{ and } x_2 = \frac{20}{23}$$

Substituting these values in (i), we get

$$y_1 = \frac{222}{23} \text{ and } y_2 = \frac{8}{23}$$

Thus, the coordinates of P and Q are $\left(\frac{26}{23}, \frac{222}{23}\right)$ and $\left(\frac{20}{23}, \frac{8}{23}\right)$ respectively.

Hence, the equation of PQ is

$$y - \frac{222}{23} = \frac{\frac{8}{23} - \frac{222}{23}}{\frac{20}{23} - \frac{26}{23}} \left(x - \frac{26}{23}\right)$$

$$\Rightarrow \quad 23y - 222 = \frac{-214}{-6}(23x - 26) \Rightarrow 23y - 222 = \frac{107}{3}(23x - 26) \Rightarrow 107x - 3y - 92 = 0.$$

EXERCISE 23.5

LEVEL-1

1. Find the equation of the straight lines passing through the following pair of points:
   (i) $(0, 0)$ and $(2, -2)$
   (ii) (a, b) and $(a + c\sin\alpha, b + c\cos\alpha)$
   (iii) $(0, -a)$ and $(b, 0)$
   (iv) (a, b) and $(a + b, a - b)$
   (v) $(at_1, a/t_1)$ and $(at_2, a/t_2)$
   (vi) $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$
2. Find the equations to the sides of the triangles the coordinates of whose angular points are respectively: (i) $(1, 4), (2, -3)$ and $(-1, -2)$ (ii) $(0, 1), (2, 0)$ and $(-1, -2)$.
3. Find the equations of the medians of a triangle, the coordinates of whose vertices are $(-1, 6), (-3, -9)$ and $(5, -8)$.
4. Find the equations to the diagonals of the rectangle the equations of whose sides are $x = a, x = a', y = b$ and $y = b'$.
5. Find the equation of the side BC of the triangle ABC whose vertices are $A(-1, -2)$, $B(0, 1)$ and $C(2, 0)$ respectively. Also, find the equation of the median through $A(-1, -2)$.
6. By using the concept of equation of a line, prove that the three points $(-2, -2), (8, 2)$ and $(3, 0)$ are collinear. [NCERT]
7. Prove that the line $y - x + 2 = 0$ divides the join of points $(3, -1)$ and $(8, 9)$ in the ratio $2:3$.
8. Find the equation to the straight line which bisects the distance between the points $(a, b), (a', b')$ and also bisects the distance between the points $(-a, b)$ and $(a', -b')$.
9. In what ratio is the line joining the points $(2, 3)$ and $(4, -5)$ divided by the line passing through the points $(6, 8)$ and $(-3, -2)$. [NCERT]
10. The vertices of a quadrilateral are $A(-2, 6)$, $B(1, 2)$, $C(10, 4)$ and $D(7, 8)$. Find the equations of its diagonals.
11. The length L (in centimeters) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if $L = 124.942$ when $C = 20$ and $L = 125.134$ when $C = 110$, express L interms of C. [NCERT]
12. The owner of a milk store finds that he can sell 980 liters milk each week at Rs 14 per liter and 1220 liters of milk each week at ₹ 16 per liter. Assuming a linear relationship between selling price and demand, how many liters could he sell weekly at Rs 17 per liter. [NCERT]

LEVEL-2

13. Find the equation of the bisector of angle A of the triangle whose vertices are $A(4, 3)$, $B(0, 0)$ and $C(2, 3)$.
14. Find the equations to the straight lines which go through the origin and trisect the portion of the straight line $3x + y = 12$ which is intercepted between the axes of coordinates.

15. Find the equations of the diagonals of the square formed by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 1$. [NCERT EXEMPLAR]

ANSWERS

1. (i) $y = -x$ (ii) $y - b = \cot\alpha\,(x - a)$ (iii) $ax - by = ab$
(iv) $(a - 2b)\,x - by + b^2 + 2ab - a^2 = 0$ (v) $t_1 t_2\, y + x = a\,(t_1 + t_2)$
(vi) $x \cos\left(\frac{\alpha+\beta}{2}\right) + y \sin\left(\frac{\alpha+\beta}{2}\right) = a \cos\left(\frac{\alpha-\beta}{2}\right)$

2. (i) $x + 3y + 7 = 0,\ y - 3x = 1,\ y + 7x = 11$ (ii) $2x - 3y = 4,\ y - 3x = 1,\ x + 2y = 2.$
3. $29x + 4y + 5 = 0,\ 8x - 5y - 21 = 0,\ 13x + 14y + 47 = 0$
4. $y\,(a' - a) - x\,(b' - b) = a'b - ab',\ y\,(a' - a) + x\,(b' - b) = a'b' - ab$
5. $x + 2y - 2 = 0$, Median: $5x - 4y - 3 = 0$
6. $2ay - 2b'x = ab - a'b'$
9. $5 : 97$
10. $x + 6y - 34 = 0,\ x - y + 1 = 0$
11. $L = \frac{4}{1875} C + 124.899$
12. 1340 liters
13. $x - 3y + 5 = 0$
14. $y = 6x,\ 2y = 3x$
15. $y = x,\ x + y = 1$

HINTS TO NCERT & SELECTED PROBLEMS

6. The equation of the line passing through points $(-2, -2)$ and $(8, 2)$ is
$$y + 2 = \frac{2+2}{8+2}(x + 2) \text{ or, } 2x - 5y - 6 = 0$$
Clearly, $(3, 0)$ satisfies this equation which means that the line passing through $(-2, -2)$ and $(8, 2)$ also passes through $(3, 0)$. Hence, these points are collinear.

9. The equation of the line passing through $(6, 8)$ and $(-3, -2)$ is
$$y + 2 = \frac{8+2}{6+3}(x + 3) \text{ or, } 10x - 9y + 12 = 0 \qquad \text{...(i)}$$
Suppose this line divides the line segment joining $(2, 3)$ and $(4, -5)$ in the ratio $\lambda : 1$, then the point of division $\left(\frac{4\lambda + 2}{\lambda + 1}, \frac{-5\lambda + 3}{\lambda + 1}\right)$ lies on (i).
$$\therefore \quad 10\left(\frac{4\lambda + 2}{\lambda + 1}\right) - 9\left(\frac{-5\lambda + 3}{\lambda + 1}\right) + 12 = 0$$
$$\Rightarrow \quad 40\lambda + 20 + 45\lambda - 27 + 12\lambda + 12 = 0 \Rightarrow 97\lambda + 5 = 0 \Rightarrow \lambda = \frac{-5}{97}.$$
Hence, the required ratio is $5 : 97$ externally.

11. The equation of the line passing through $(C_1 = 20, L_1 = 124.942)$ and $(C_2 = 110, L_2 = 125.134)$ is
$$L - 124.942 = \frac{125.134 - 124.942}{110 - 20}(C - 20) \Rightarrow L = \frac{4}{1875} C + 124.899$$

12. Let x denote the price per liter and y denote the quantity of the milk sold at this price. Since there is linear relationship between the price per liter and quantity solved. So, the line representing the relationship passes through $(14, 980)$ and $(16, 1220)$. So, its equation is
$$y - 980 = \frac{1220 - 980}{16 - 14}(x - 14) \Rightarrow y - 980 = 120(x - 14) \Rightarrow 120x - y - 700 = 0$$
When $x = 17$, we obtain
$$120 \times 17 - y - 700 \Rightarrow y = 1340$$

23.6.4 THE INTERCEPT FORM OF A LINE

THEOREM *The equation of a line which cuts off intercepts a and b respectively from the x and y-axes is* $\frac{x}{a}+\frac{y}{b}=1$.

PROOF Let AB be the line which cuts off intercepts $OA=a$ and $OB=b$ on the x and y axes respectively. Let $P(x, y)$ be any point on the line. Draw $PL \perp OX$. Then, $OL = x$ and $PL = y$.

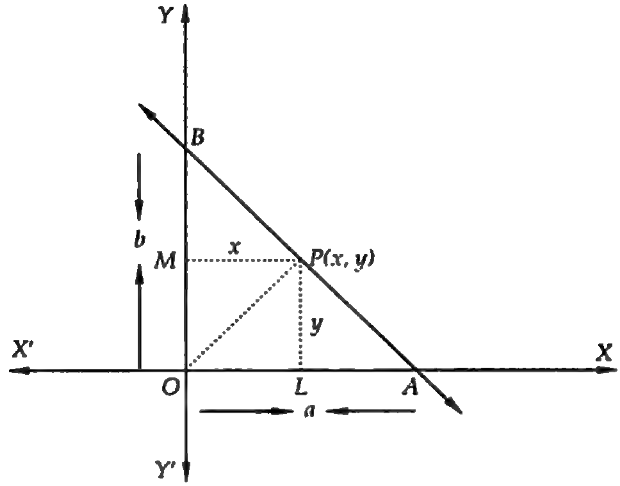


Fig. 23.31

Clearly,

Area of ΔOAB = Area of ΔOPA + Area of ΔOPB

$$\Rightarrow \quad \frac{1}{2} OA \cdot OB = \frac{1}{2} OA \cdot PL + \frac{1}{2} OB \cdot PM$$

$$\Rightarrow \quad \frac{1}{2} ab = \frac{1}{2} ay + \frac{1}{2} bx$$

$$\Rightarrow \quad ab = ay + bx$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

This is the equation of the line in the intercept form.

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the line which cuts off an intercept 4 on the positive direction of x-axis and an intercept 3 on the negative direction of y-axis.*

SOLUTION Here $a = 4$, $b = -3$. So, the equation of the line is

$$\frac{x}{a}+\frac{y}{b}=1 \text{ or, } \frac{x}{4}+\frac{y}{-3}=1 \text{ or, } 3x-4y=12.$$

EXAMPLE 2 *Find the equation of the straight line which makes equal intercepts on the axes and passes through the point (2, 3).* [NCERT]

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$. Since it makes equal intercepts on the coordinate axes, therefore $a=b$. So, the equation of the line is

$$\frac{x}{a}+\frac{y}{a}=1 \text{ or, } x+y=a \quad \text{... (i)}$$

This passes through the point (2, 3).

$\therefore \quad 2+3=a \Rightarrow a=5.$

Thus, the equation of the required line is $x+y=5$. [Putting $a=5$ in (i)]

EXAMPLE 3 *Find the equation of the line which cuts off equal and positive intercepts from the axes and passes through the point* (α, β).

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$ which cuts off intercepts a and b with the coordinate axes. It is given that $a=b$. Therefore, the equation of the line is

$$\frac{x}{a}+\frac{y}{a}=1 \Rightarrow x+y=a \quad \text{...(i)}$$

It is given that the line (i) passes through (α, β).

$\therefore \quad \alpha+\beta=a.$

Putting the value of a in (i), we obtain the equation of the line as $x+y=\alpha+\beta$.

EXAMPLE 4 *Find the equation of a straight line which passes through the point* $(4,-2)$ *and whose intercept on y-axis is twice that on x-axis.*

SOLUTION Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{...(i)}$$

It is given that its y-intercept is twice the x-intercept.

$\therefore \quad b=2a$

Putting $b=2a$ in (i), we get

$$\frac{x}{a}+\frac{y}{2a}=1 \text{ or, } 2x+y=2a \quad \text{...(ii)}$$

It passes through the point $(4,-2)$. Therefore, putting $x=4$, $y=-2$ in (ii), we get

$$8-2=2a \Rightarrow a=3.$$

Substituting $a=3$ in (ii), we get

$2x+y=6$ as the required equation of the line.

EXAMPLE 5 *Find the equation of the straight line whose intercepts on X-axis and Y-axis are respectively twice and thrice of those by the line* $3x+4y=12$.

SOLUTION The equation of the given line is $3x+4y=12$. This can be written as $\frac{x}{4}+\frac{y}{3}=1$.

Clearly, its intercepts on X and Y-axes are 4 and 3 respectively.

$\therefore$ x-intercept of the required line $=2\times 4=8$ and, y-intercept of the required line $=3\times 3=9$

Hence, the equation of the required line is $\frac{x}{8}+\frac{y}{9}=1$ or, $9x+8y=72$.

EXAMPLE 6 *Find the equation of the line through (2, 3) so that the segment of the line intercepted between the axes is bisected at this point.*

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$ which meets the x and y axes at $A(a, 0)$ and $B(0, b)$ respectively. The coordinates of the mid-point of AB are $(a/2, b/2)$. It is given that the point (2, 3) bisects AB.

$$\therefore \quad \frac{a}{2}=2 \text{ and } \frac{b}{2}=3 \Rightarrow a=4 \text{ and } b=6.$$

Puting $a=4$ and $b=6$ in $\frac{x}{a}+\frac{y}{b}=1$, we obtain

$$\frac{x}{4}+\frac{y}{6}=1 \text{ or, } 3x+2y=12$$

Hence, the equation of the required line is $3x+2y=12$.

EXAMPLE 7 *If the intercept of a line between the coordinate axes is divided by the point (–5, 4) in the ratio 1 : 2, then find the equation of the line.*

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$. It meets the coordinate axes at $A(a, 0)$ and $B(0, b)$. It is given that $P(-5, 4)$ divides AB in the ratio 1 : 2.

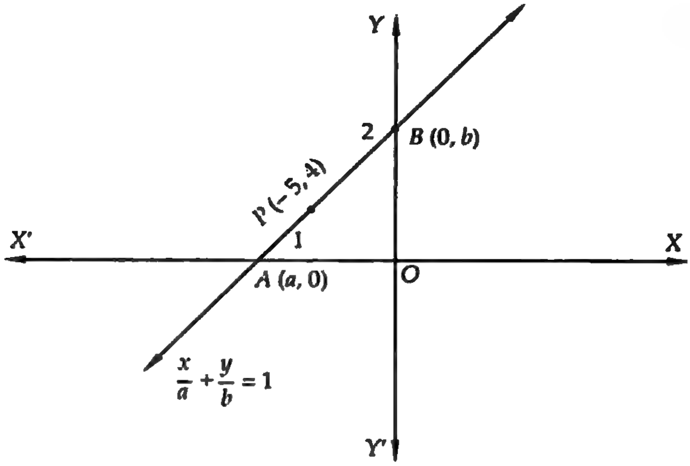


Fig. 23.32

Using section formula the coordinates of P are

$$\left(\frac{1\times 0+2\times a}{1+2}, \frac{1\times b+2\times 0}{1+2}\right)=\left(\frac{2a}{3}, \frac{b}{3}\right).$$

$$\therefore \quad -5=\frac{2a}{3}, 4=\frac{b}{3} \Rightarrow a=-\frac{15}{2}, b=12.$$

Substituting the values of a and b in $\frac{x}{a}+\frac{y}{b}=1$, we obtain

$-\frac{2x}{15}+\frac{y}{12}=1$ or, $-8x+5y=60$ or, $8x-5y+60=0$ as the equation of the line.

EXAMPLE 8 *A straight line cuts intercepts from the axes of coordinates the sum of whose reciprocals is a constant. Show that it always passes through a fixed point.* **[NCERT EXEMPLAR]**

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$ ...(i)

Its intercepts on x and y axes are a and b respectively. It is given that

$$\frac{1}{a}+\frac{1}{b}=\text{Constant}=k \text{ (say)}$$

$$\therefore \quad \frac{1}{ka}+\frac{1}{kb}=1$$

$$\Rightarrow \quad \frac{\frac{1}{k}}{a}+\frac{\frac{1}{k}}{b}=1$$

$$\Rightarrow \quad \left(\frac{1}{k}, \frac{1}{k}\right) \text{ satisfies the equation } \frac{x}{a}+\frac{y}{b}=1$$

Hence, line (i) passes through the fixed point $\left(\frac{1}{k}, \frac{1}{k}\right)$.

EXAMPLE 9 *A line passes through the point (3, – 2). Find the locus of the middle point of the portion of the line intercepted between the axes.*

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$ ...(i)

It passes through (3, − 2).

$\therefore \quad \frac{3}{a}-\frac{2}{b}=1$ ...(ii)

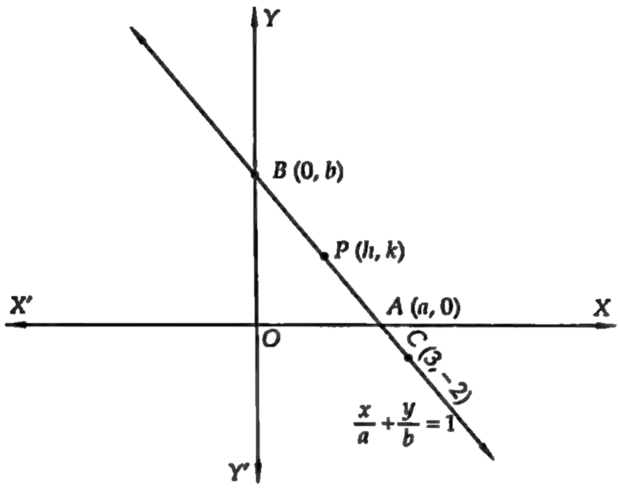


Fig. 23.33

The line (i) cuts the coordinate axes at $A\ (a, 0)$ and $B\ (0, b)$. Let $P\ (h, k)$ be the mid-point of the portion AB. Then,

$$h=\frac{a+0}{2}, k=\frac{0+b}{2} \Rightarrow a=2h \text{ and } b=2k$$

Substituting the values of a and b in (ii), we get

$$\frac{3}{2h}-\frac{2}{2k}=1$$

Hence, locus of $P\ (h, k)$ is $\frac{3}{2x}-\frac{1}{y}=1$ or, $3y-2x=2xy$.

LEVEL-2

EXAMPLE 10 *Find the equation of the line which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.*

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$ ...(i)

This passes through (3, 4).

$\therefore \quad \frac{3}{a}+\frac{4}{b}=1$ ...(ii)

It is given that $a+b=14$

$\therefore \quad b=14-a$

Putting $b=14-a$ in (ii), we get

$$\frac{3}{a}+\frac{4}{14-a}=1 \Rightarrow 3(14-a)+4a=a(14-a) \Rightarrow a^2-13a+42=0 \Rightarrow (a-7)(a-6)=0 \Rightarrow a=7, 6$$

When $a=7, b=14-a \Rightarrow b=14-7=7$ and for $a=6$, $b=14-a \Rightarrow b=14-6=8$.

Thus, we obtain

$$a=7, b=7 \text{ or, } a=6, b=8$$

Putting the values of a and b in (i), we obtain that the equations of the lines are

$$\frac{x}{7}+\frac{y}{7}=1 \text{ and } \frac{x}{6}+\frac{y}{8}=1 \text{ or, } x+y=7 \text{ and } 4x+3y=24.$$

EXAMPLE 11 *Find the equations of the lines which cut-off intercepts on the axes whose sum and product are 1 and – 6 respectively.* **[NCERT]**

SOLUTION Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \qquad ...(i)$$

Clearly, it cuts off intercepts a and b on x and y-axes respectively. It is given that

$$a+b=1 \text{ and } ab=-6$$

$$\therefore \quad (a-b)^2=(a+b)^2-4ab \Rightarrow (a-b)^2=1-4\times-6=25 \Rightarrow a-b=\pm 5$$

Solving $a+b=1$ and $a-b=5$, we get: $a=3$ and $b=-2$

Solving $a+b=1$ and $a-b=-5$, we get: $a=-2$ and $b=3$.

Substituting these values in (i), we obtain the equations of the required line as

$$\frac{x}{3}-\frac{y}{2}=1 \text{ and } -\frac{x}{2}+\frac{y}{3}=1 \text{ or, } 2x-3y-6=0 \text{ and } -3x+2y-6=0$$

EXAMPLE 12 *Find the equations of the straight lines which pass through the origin and trisect the intercept of the line $3x+4y=12$ between the axes.*

SOLUTION The equation of the given line is

$$3x+4y=12 \text{ or, } \frac{x}{4}+\frac{y}{3}=1$$

It cuts the coordinate axes at $A(4, 0)$ and $B(0, 3)$.

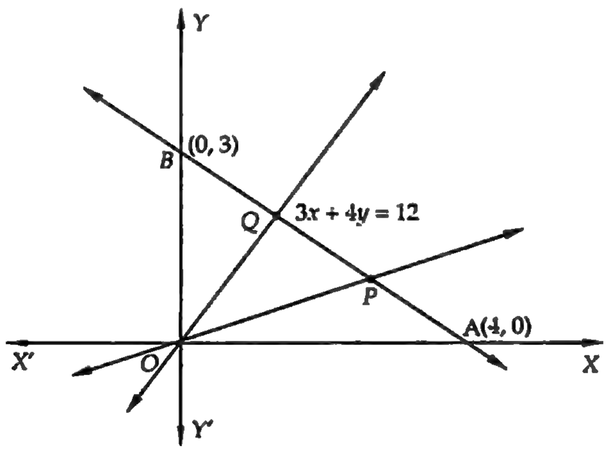


Fig. 23.34

The portion AB of the given line intercepted between the axes is trisected by points P and Q.

$$\therefore \quad \frac{AP}{PB}=\frac{1}{2} \text{ and } \frac{AQ}{QB}=\frac{2}{1}$$

$\Rightarrow$ P and Q divide AB internally in the ratio 1:2 and 2:1 respectively.

So, coordinates P and Q are

$$P\left(\frac{1\times 0+2\times 4}{1+2}, \frac{1\times 3+2\times 0}{1+2}\right)\equiv P\left(\frac{8}{3}, 1\right), \quad Q\left(\frac{2\times 0+1\times 4}{2+1}, \frac{2\times 3+1\times 0}{2+1}\right)\equiv Q\left(\frac{4}{3}, 2\right)$$

Hence, the equation of OQ is

$$y-0=\frac{2-0}{\frac{4}{3}-0}(x-0) \text{ or, } 3x-2y=0.$$

EXAMPLE 13 *The area of the triangle formed by the coordinates axes and a line is 6 square units and the length of the hypotenuse is 5 units. Find the equation of the line.*

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$ ...(i)

It cuts the coordinates axes at $A\,(a,\,0)$ and $B(0,\,b)$ such that area of ΔOAB is 6 square units and $AB=5$ units.

Now,

$$\text{Area of } \Delta OAB = 6 \text{ sq.units}$$

$$\Rightarrow \quad \frac{1}{2}(OA\times OB)=6$$

$$\Rightarrow \quad |a||b|=12 \Rightarrow |ab|=12 \Rightarrow ab=\pm 12 \quad \text{...(ii)}$$

and,

$$AB=5 \Rightarrow AB^2=25 \Rightarrow OA^2+OB^2=25 \Rightarrow a^2+b^2=25 \quad \text{...(iii)}$$

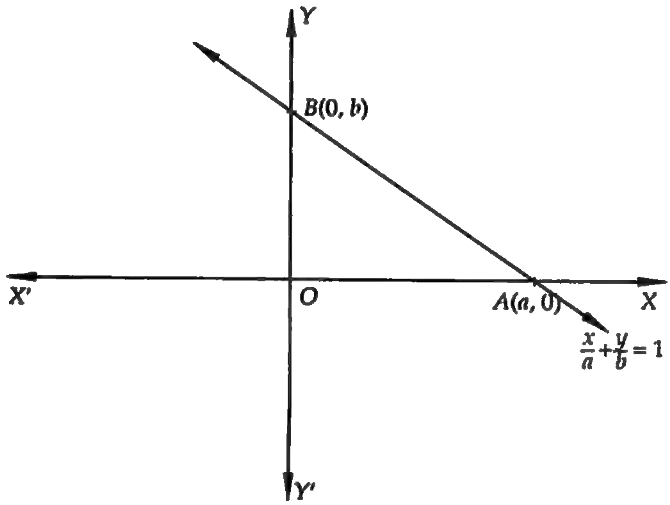


Fig. 24.35

Following cases arise:

<u>CASE I</u> When $ab=12,\, a^2+b^2=25$ and $a>0,\, b>0$.

In this case,

$$(a+b)^2=a^2+b^2+2ab \Rightarrow (a+b)^2=25+2\times 12=49 \Rightarrow a+b=7$$

and, $(a-b)^2=a^2+b^2-2ab \Rightarrow (a-b)^2=25-24=1 \quad \Rightarrow a-b=\pm 1$

Thus, we have

$$(a+b=7 \text{ and } a-b=1) \text{ or, } (a+b=7 \text{ and } a-b=-1)$$

$$\Rightarrow \quad (a=4,\, b=3) \text{ or, } (a=3,\, b=4)$$

Substituting the values of a and b in (i), we obtain

$\frac{x}{4}+\frac{y}{3}=1$ or $\frac{x}{3}+\frac{y}{4}=1$ as the equation of the line.

<u>CASE II</u> *When $ab=12,\, a^2+b^2=25$ and $a<0,\, b<0$:*

In this case, we have

$$(a+b)^2=a^2+b^2+2ab=25+24=49 \Rightarrow a+b=-7 \qquad [\because\ a<0,\, b<0]$$

and, $(a-b)^2=a^2+b^2-2ab=25-24=1 \Rightarrow a-b=\pm 1$

Thus, we have

$$(a+b=-7 \text{ and } a-b=1) \text{ or } (a+b=-7 \text{ and } a-b=-1)$$

$\Rightarrow \qquad (a=-3, b=-4)$ or $(a=-4, b=-3)$

Substituting the values of a and b in (i), we obtain

$$\frac{x}{-3}+\frac{y}{-4}=1 \text{ or } \frac{x}{-4}+\frac{y}{-3}=1 \text{ as the equation of the line.}$$

CASE III *When* $ab=-12$, $a^2+b^2=25$ *and* $a>0, b<0$:

In this case, we have

$$(a+b)^2=a^2+b^2+2ab=25-24=1 \Rightarrow a+b=\pm 1$$

and, $\quad (a-b)^2=a^2+b^2-2ab=25+24=49 \Rightarrow a-b=7 \qquad [\because a>0, b<0 \because a-b>0]$

Thus, we have

$$(a+b=1 \text{ and } a-b=7) \text{ or } (a+b=-1 \text{ and } a-b=7)$$

$\Rightarrow \qquad (a=4, b=-3)$ or $(a=3, b=-4)$

Substituting the values of a and b in (i), we obtain

$$\frac{x}{4}-\frac{y}{3}=1 \text{ or } \frac{x}{3}-\frac{y}{4}=1 \text{ as the equation of the line.}$$

CASE IV *When* $ab=-12$, $a^2+b^2=25$ *and* $a<0, b>0$:

In this case, we have

$$(a+b)^2=a^2+b^2+2ab=25-24=1 \Rightarrow a+b=\pm 1$$

and, $\quad (a-b)^2=a^2+b^2-2ab=25+24=49 \Rightarrow a-b=-7 \qquad [a<0, b>0 \therefore a-b<0]$

Thus, we have

$$(a+b=1 \text{ and } a-b=-7) \text{ or, } (a+b=-1 \text{ and } a-b=-7)$$

$\Rightarrow \qquad (a=-3 \text{ and } b=4)$ or $(a=-4, b=3)$

Substituting the values of a and b in (i), we obtain

$$\frac{x}{-3}+\frac{y}{4}=1 \text{ or } \frac{x}{-4}-\frac{y}{3}=1 \text{ as the equation of the line.}$$

EXAMPLE 14 *Find the equation of the line which passes through* $P(1, -7)$ *and meets the axes at A and B respectively so that* $4AP-3BP=0$.

SOLUTION Let the equation of the required line be $\frac{x}{a}+\frac{y}{b}=1 \qquad$ ...(i)

It passes through $P(1, -7)$.

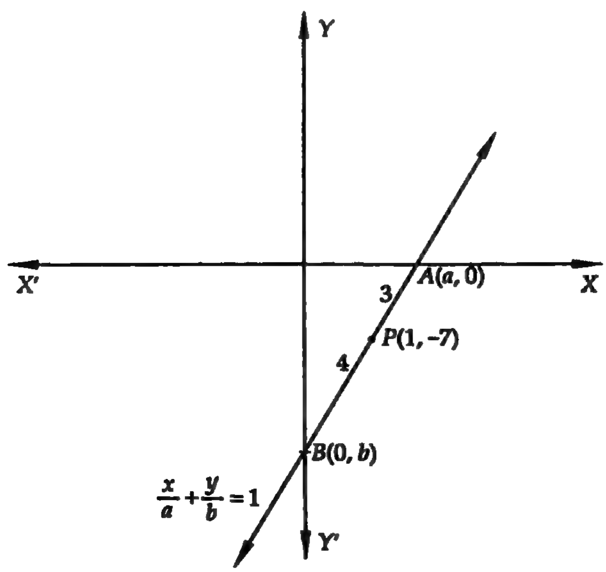


Fig. 24.36

$\therefore \qquad \frac{1}{a} - \frac{7}{b} = 1 \qquad$ ...(ii)

It is given that the point $P(1, -7)$ divides segment AB in such a way that

$$4AP - 3BP = 0 \text{ i.e. } \frac{AP}{BP} = \frac{3}{4} \text{ or, } AP : BP = 3 : 4$$

This means that P divides AB internally in the ratio 3 : 4. So, the coordinates of P are

$$\left(\frac{3\times 0 + 4\times a}{3+4}, \frac{3\times b + 4\times 0}{3+4}\right) = \left(\frac{4a}{7}, \frac{3b}{7}\right)$$

But, the coordinates of P are given as $(1, -7)$.

$$\therefore \qquad \frac{4a}{7} = 1 \text{ and } \frac{3b}{7} = -7 \Rightarrow a = \frac{7}{4} \text{ and } b = -\frac{49}{3}$$

Substituting the values of a and b in (i), we obtain

$$\frac{4x}{7} - \frac{3y}{49} = 1 \text{ or } 28x - 3y = 49 \text{ as the required equation.}$$

EXAMPLE 15 *Show that the locus of the mid-point of the segment intercepted between the axes of the variable line $x\cos\alpha + y\sin\alpha = p$ is $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$, where p is a constant.* [NCERT EXEMPLAR]

SOLUTION The given equation is $x\cos\alpha + y\sin\alpha = p$ or, $\frac{x}{p/\cos\alpha} + \frac{y}{p/\sin\alpha} = 1 \qquad$ ...(i)

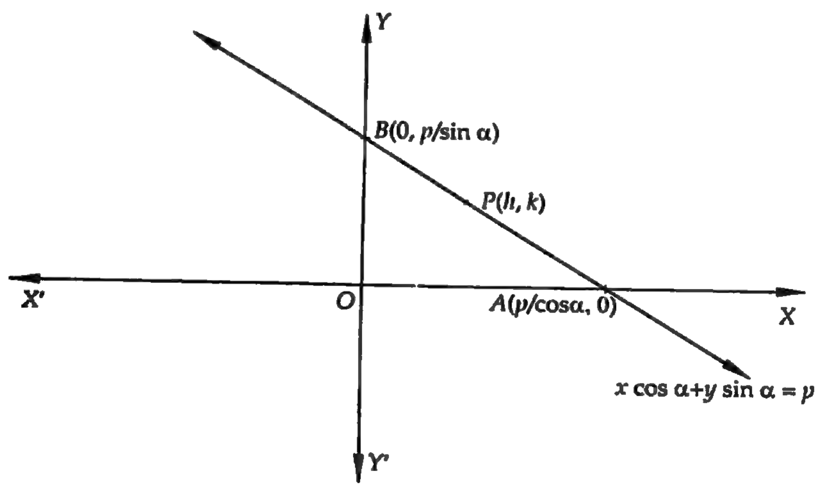


Fig. 24.37

This cuts the coordinate axes at $A(p/\cos\alpha, 0)$ and $B(0, p/\sin\alpha)$. Let $P(h, k)$ be the mid-point of the intercept AB. Then,

$$h = \frac{p/\cos\alpha + 0}{2}, k = \frac{0 + p/\sin\alpha}{2}$$

$$\Rightarrow \qquad h = \frac{p}{2\cos\alpha}, k = \frac{p}{2\sin\alpha}$$

$$\Rightarrow \qquad \cos\alpha = \frac{p}{2h}, \sin\alpha = \frac{p}{2k} \qquad \text{...(i)}$$

Here, α is a variable. To find the locus of $P(h, k)$, we have to eliminate α.
From (i), we obtain

$$\cos^2\alpha + \sin^2\alpha = \frac{p^2}{4h^2} + \frac{p^2}{4k^2} \Rightarrow 1 = \frac{p^2}{4h^2} + \frac{p^2}{4k^2} \Rightarrow \frac{4}{p^2} = \frac{1}{h^2} + \frac{1}{k^2}$$

Hence, the locus of (h, k) is $\frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{p^2}$.

EXAMPLE 16 *If the sum of the distances of a moving point in a plane from the axes is 1, then find the locus of the point.* **[NCERT EXEMPLAR]**

SOLUTION Let $P(h, k)$ be a moving point in the xy-plane. Let PL and PM be perpendiculars from P on OX and OY respectively. Then, $PL = |k|$ and $PM = |h|$.

It is given that $P(h, k)$ moves in the xy-plane such that

$$PL + PM = 1 \Rightarrow |k| + |h| = 1$$

Hence, the locus of $P(h, k)$ is $|y| + |x| = 1$ or, $|x| + |y| = 1$.

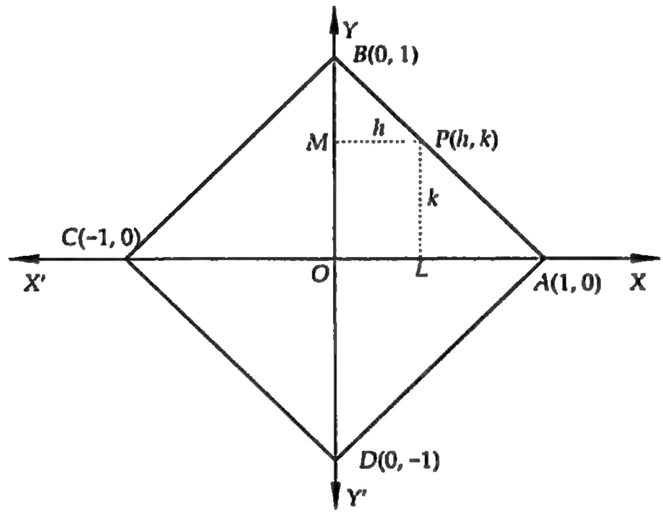


Fig. 24.38

Now,

$$|x| + |y| = 1 \Rightarrow 0 \le |x| \le 1, 0 \le |y| \le 1$$

Also,

$$|x| + |y| = 1 \Rightarrow \begin{cases} x+y=1, \text{ if } 0 \le x \le 1, 0 \le y \le 1 \\ -x+y=1 \text{ , if } -1 \le x < 0, 0 \le y \le 1 \\ -x-y=1 \text{ , if } -1 < x \le 0, -1 < y \le 0 \\ x-y=1, \text{ if } 0 \le x \le 1, -1 < y \le 0 \end{cases}$$

Thus, $|x| + |y| = 1$ gives four line segments AB, BC, CD and DA.

These line segments form a square $ABCD$ as shown in Fig. 24.38.

Thus, the locus of the variable point P is the square having vertices at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$.

EXAMPLE 17 *The line $\frac{x}{a}+\frac{x}{b}=1$ moves in such a way that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$, where c is a constant. Find the locus of the foot of the perpendicular from the origin on the given line.*

SOLUTION Let $P(h, k)$ be the foot of the perpendicular from the origin O on the line $\frac{x}{a}+\frac{y}{b}=1$ which cuts the coordinates axes at $A(a, 0)$ and $B(0, b)$. Then,

$$\text{Slope of } OP \times \text{Slope of } AB = -1$$

$$\Rightarrow \quad \frac{k-0}{h-0} \times \frac{b-0}{0-a} = -1$$

$$\Rightarrow \quad bk = ah$$

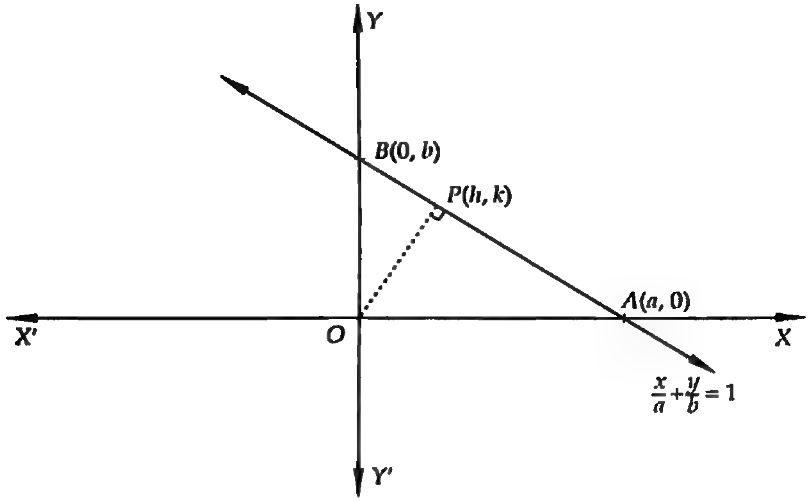


Fig. 23.39

$$\Rightarrow \quad b=\frac{ah}{k} \qquad \text{...(i)}$$

Also, $P(h, k)$ lies on $\frac{x}{a}+\frac{y}{b}=1$.

$$\therefore \quad \frac{h}{a}+\frac{k}{b}=1 \qquad \text{...(ii)}$$

$$\Rightarrow \quad \frac{h}{a}+\frac{k^2}{ah}=1 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad a=\frac{h^2+k^2}{h}$$

Substituting this values of a in (i), we obtain

$$b=\frac{h^2+k^2}{k}$$

Substituting the values of a and b in $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$, we obtain

$$\frac{h^2}{(h^2+k^2)^2}+\frac{k^2}{(h^2+k^2)^2}=\frac{1}{c^2} \text{ or, } h^2+k^2=c^2$$

Hence, the locus of (h, k) is $x^2+y^2=c^2$.

EXERCISE 23.6

LEVEL-1

1. Find the equation to the straight line
   (i) cutting off intercepts 3 and 2 from the axes.
   (ii) cutting off intercepts -5 and 6 from the axes.
2. Find the equation of the straight line which passes through $(1, -2)$ and cuts off equal intercepts on the axes [NCERT EXEMPLAR]
3. Find the equation to the straight line which passes through the point (5, 6) and has intercepts on the axes
   (i) equal in magnitude and both positive. (ii) equal in magnitude but opposite in sign.
4. For what values of a and b the intercepts cut off on the coordinate axes by the line $ax+by+8=0$ are equal in length but opposite in signs to those cut off by the line $2x-3y+6=0$ on the axes. [NCERT EXEMPLAR]

5. Find the equation to the straight line which cuts off equal positive intercepts on the axes and their product is 25.
6. Find the equation of the line which passes through the point $(-4, 3)$ and the portion of the line intercepted between the axes is divided internally in the ratio $5:3$ by this point. **[NCERT EXEMPLAR]**
7. A straight line passes through the point (α, β) and this point bisects the portion of the line intercepted between the axes. Show that the equation of the straight line is $\frac{x}{2\alpha}+\frac{y}{2\beta}=1$. **[NCERT]**
8. Find the equation of the line which passes through the point $(3, 4)$ and is such that the portion of it intercepted between the axes is divided by the point in the ratio $2:3$.

LEVEL-2

9. Point $R(h, k)$ divides a line segment between the axes in the ratio $1:2$. Find the equation of the line. **[NCERT]**
10. Find the equation of the straight line which passes through the point $(-3, 8)$ and cuts off positive intercepts on the coordinate axes whose sum is 7.
11. Find the equation to the straight line which passes through the point $(-4, 3)$ and is such that the portion of it between the axes is divided by the point in the ratio $5:3$.
12. Find the equation of a line which passes through the point $(22, -6)$ and is such that the intercept on x-axis exceeds the intercept on y-axis by 5.
13. Find the equation of the line, which passes through $P(1, -7)$ and meets the axes at A and B respectively so that $4\,AP - 3\,BP = 0$.
14. Find the equation of the line passing through the point $(2, 2)$ and cutting off intercepts on the axes whose sum is 9. **[NCERT]**
15. Find the equation of the straight line which passes through the point $P(2, 6)$ and cuts the coordinate axes at the point A and B respectively so that $\frac{AP}{BP}=\frac{2}{3}$.
16. Find the equations of the straight lines each of which passes through the point $(3, 2)$ and cuts off intercepts a and b respectively on x and y-axes such that $a-b=2$.
17. Find the equations of the straight lines which pass through the origin and trisect the portion of the straight line $2x+3y=6$ which is intercepted between the axes.
18. Find the equation of the straight line passing through the point $(2, 1)$ and bisecting the portion of the straight line $3x-5y=15$ lying between the axes.
19. Find the equation of the straight line passing through the origin and bisecting the portion of the line $ax+by+c=0$ intercepted between the coordinate axes.

ANSWERS

1. (i) $2x+3y=6$ (ii) $-6x+5y=30$ 2. $x+y=-1$
3. (i) $x+y=11$ (ii) $x-y=-1$
4. $a=-\frac{8}{3}, b=4$ 5. $x+y=5$ 6. $9x-20y+96=0$ 8. $2x+y=10$
9. $2kx+hy=3hk$ 10. $4x+3y=12$ 11. $9x-20y+96=0$
12. $6x+11y-66=0$ or $x+2y-10=0$ 13. $28x-3y=49$
14. $x+2y-6=0, 2x+y-6=0$ 15. $9x+2y=30$
16. $2x+3y=12,\ x-y=1$ 17. $x-3y=0,\ 4x-3y=0$
18. $5x+y=11$ 19. $ax-by=0$

HINTS TO NCERT & SELECTED PROBLEMS

2. The equation of a line cutting off equal intercepts 'a' on the coordinate axes is

$$\frac{x}{a}+\frac{y}{a}=1 \text{ or, } x+y=a \quad ...(i)$$

If it passes through (1,–2), then $1-2=a \Rightarrow a=-1$.
Substituting $a=1$ in (i), we get $x+y=-1$ as the equation of the line.

7. Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \quad ...(i)$$

This line cuts the coordinate axes at $A(a, 0)$ and $B(0, b)$. It is given that (α, β) bisects the segment AB.

$$\therefore \quad \alpha=\frac{a+0}{2},\ \beta=\frac{0+b}{2} \Rightarrow a=2\alpha,\ b=2\beta$$

Substituting these values in (i), we get

$$\frac{x}{2\alpha}+\frac{y}{2\beta}=1$$

9. Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \quad ...(i)$$

It cuts the axes at $A(a, 0)$ and $B(0, b)$.
It is given that the point $R(h, k)$ divides segment AB in the ratio 1 : 2.

$$\therefore \quad h=\frac{2a+0}{3} \text{ and } k=\frac{0+b}{3} \Rightarrow a=\frac{3h}{2}, b=3k$$

Substituting these values in (i), we obtain $\frac{2x}{h}+\frac{y}{k}=3$ or, $2kx+hy=3hk$ as the equation of the line.

14. Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \quad ...(i)$$

It passes through (2, 2) and the sum of the intercepts on the axes is 9. Therefore,

$$\frac{2}{a}+\frac{2}{b}=1 \text{ and } a+b=9$$

$$\Rightarrow \quad 2b+2a=ab \text{ and } a+b=9$$

$$\Rightarrow \quad 2(9-a)+2a=a(9-a) \quad \text{[On eliminating } b\text{]}$$

$$\Rightarrow \quad a^2-9a+18=0$$

$$\Rightarrow \quad (a-6)(a-3)=0$$

$$\Rightarrow \quad a=3, 6.$$

When $a=3$, $a+b=9$ gives $b=6$. When $a=6$, $a+b=9$ gives $b=3$.
Hence, the equations of the line are

$$\frac{x}{3}+\frac{y}{6}=1 \text{ and, } \frac{x}{6}+\frac{y}{3}=1 \text{ or, } 2x+y=6 \text{ and } x+2y=6$$

23.6.5 NORMAL FORM OR PERPENDICULAR FORM OF A LINE

THEOREM *The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with x-axis is $x\cos\alpha+y\sin\alpha=p$.*

PROOF Let the line AB be such that the length of the perpendicular OQ from the origin O to the line be p and $\angle XOQ=\alpha$. Let $P(x, y)$ be any point on the line. Draw $PL \perp OX$, $LM \perp OQ$ and $PN \perp LM$. Then, $OL=x$ and $LP=y$.

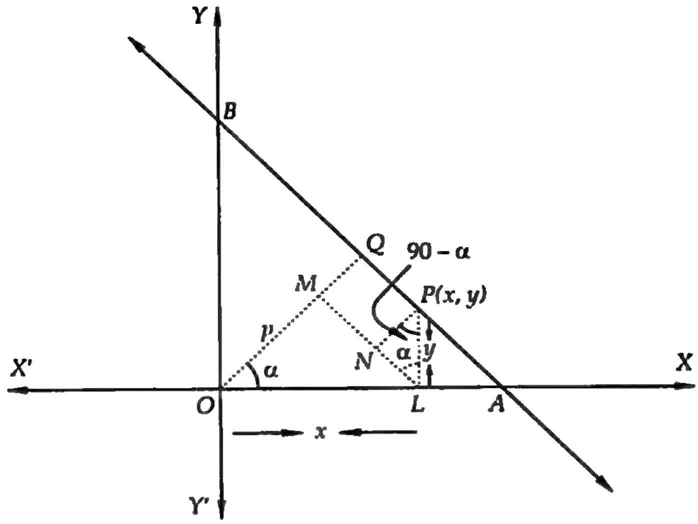


Fig. 23.40

In $\Delta\, OLM$, we have

$$\cos\alpha = \frac{OM}{OL}$$

$\Rightarrow \quad OM = OL\cos\alpha = x\cos\alpha.$

In $\Delta\, PNL$, we have

$$\sin\alpha = \frac{PN}{PL}$$

$\Rightarrow \quad PN = PL\sin\alpha = y\sin\alpha$

$\Rightarrow \quad MQ = PN = y\sin\alpha$

Now, $\quad p = OQ = OM + MQ = x\cos\alpha + y\sin\alpha$

Hence, the equation of the required line is $x\cos\alpha + y\sin\alpha = p$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the line which is at a distance 3 from the origin and the perpendicular from the origin to the line makes an angle of 30° with the positive direction of the x-axis.*

SOLUTION Here, $p = 3,\ \alpha = 30°$.

The equation of the line in the normal form is

$$x\cos 30° + y\sin 30° = 3 \Rightarrow x\frac{\sqrt{3}}{2} + \frac{y}{2} = 3 \Rightarrow \sqrt{3}\,x + y = 6.$$

EXAMPLE 2 *Find the equation of the straight line on which the length of the perpendicular from the origin is 4 units and the line makes an angle of 120° with positive direction of x-axis.*

[NCERT EXEMPLAR]

SOLUTION It is given that $\angle XAB = 120°$. Therefore, $\angle AOP = 30°$.

Thus, we have

$$p = 4 \text{ and } \alpha = 30°$$

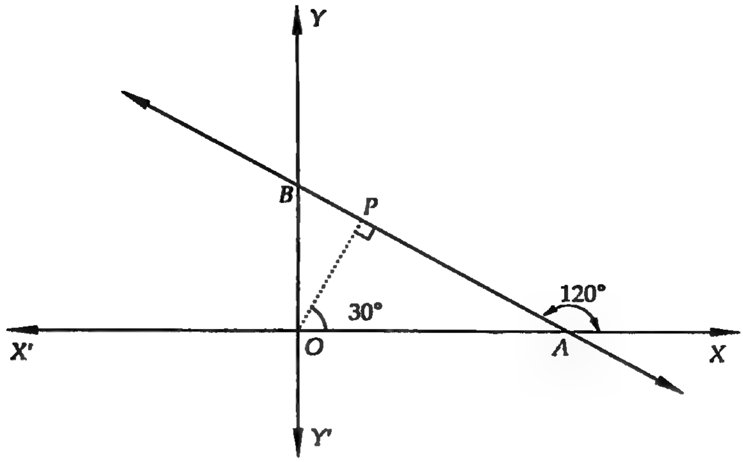


Fig. 23.41

So, the equation of the line is

$$x\cos\alpha + y\sin\alpha = p \quad \text{or, } x\cos 30° + y\sin 30° = 4 \quad \text{or, } \sqrt{3}x + y = 8$$

EXAMPLE 3 *The length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150° with the positive direction of y-axis. Find the equation of the line.*

SOLUTION It is evident from the Figure 23.42 that the perpendicular OQ from the origin on the line makes 30° angle with x-axis. Therefore, $\alpha = 30°$. It is given that $OQ = 7$. Therefore, $p = 7$.

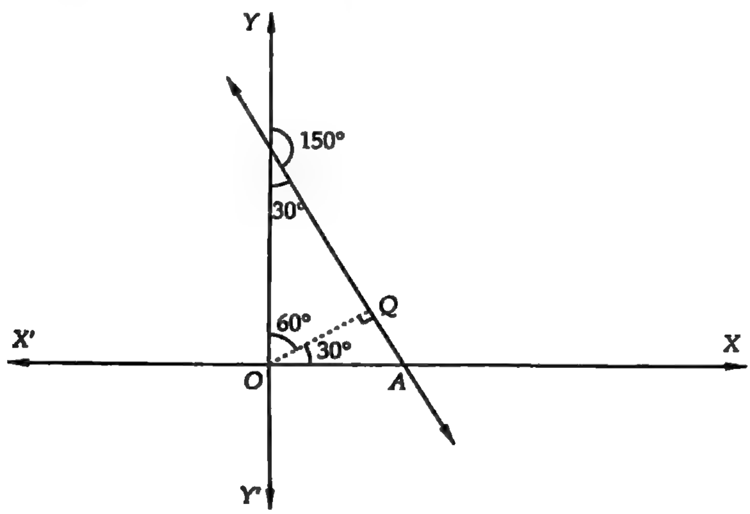


Fig. 23.42

So, the equation of the required line is

$$x\cos\alpha + y\sin\alpha = p \quad \text{or, } x\cos 30° + y\sin 30° = 7$$

$$\Rightarrow \quad \frac{\sqrt{3}x}{2} + \frac{y}{2} = 7 \Rightarrow \sqrt{3}\,x + y = 14.$$

EXAMPLE 4 *Find the equation of the straight line upon which the length of perpendicular from origin is $3\sqrt{2}$ units and this perpendicular makes an angle of 75° with the positive direction of x-axis.*

SOLUTION Let OL be the perpendicular from the origin on the required line. It is given that $OL = 3\sqrt{2}$ and $\angle XOL = 75°$ i.e. $p = 3\sqrt{2}$ and $\alpha = 75°$.

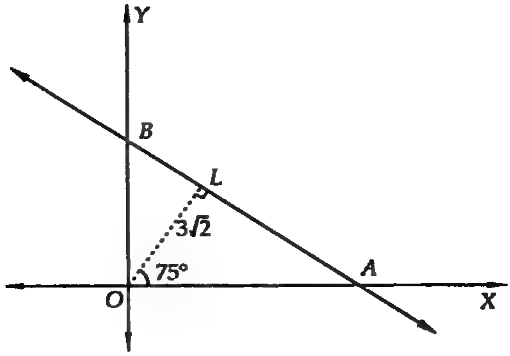


Fig. 23.43

So, the equation of the line is

$$x\cos\alpha + y\sin\alpha = p \text{ or, } x\cos 75° + y\sin 75° = 3\sqrt{2} \quad \text{...(i)}$$

or, $$\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)x + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)y = 3\sqrt{2} \quad \left[\because \cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}} \text{ and } \sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$$

or, $(\sqrt{3}-1)\,x + (\sqrt{3}+1)\,y = 12$, which is the required equation.

EXAMPLE 5 *Find the equation of the straight line upon which the length of the perpendicular from the origin is 5 and the slope of this perpendicular is* $\frac{3}{4}$.

SOLUTION Suppose the perpendicular OL drawn from the origin O on the given line makes acute angle α with x-axis. Then, the slope of OL is $\tan\alpha$. But, it is given that the slope of OL is $\frac{3}{4}$.

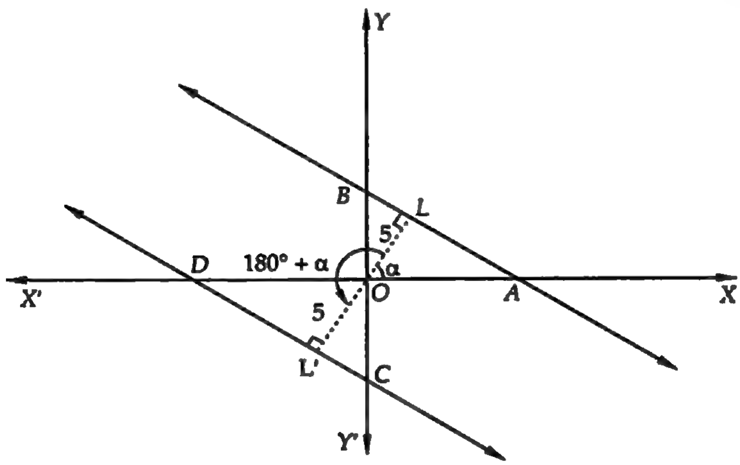


Fig. 23.44

$\therefore \quad \tan\alpha = \frac{3}{4}$ [Given]

Since $\tan(180° + \alpha) = \tan\alpha$. So, there are two possible lines AB and CD on which the perpendicular drawn from the origin has slope $\frac{3}{4}$.

Now, $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$

Here, $p = 5$

So, the equations of the required lines are

$$x \cos \alpha + y \sin \alpha = p \text{ and, } x \cos (180° + \alpha) + y \sin (180° + \alpha) = p$$

or, $$x \cos \alpha + y \sin \alpha = p \text{ and, } -x \cos \alpha - y \sin \alpha = p$$

or, $$\frac{4x}{5} + \frac{3y}{5} = 5 \text{ and } \frac{-4x}{5} - \frac{3y}{5} = 5$$

or, $$4x + 3y - 25 = 0 \text{ and } 4x + 3y + 25 = 0$$

LEVEL-2

EXAMPLE 6 *A line forms a triangle of area $54\sqrt{3}$ square units with the coordinate axes. Find the equation of the line if the perpendicular drawn from the origin to the line makes an angle of 60° with the X-axis.*

SOLUTION Let AB be the given line and $OL = p$ be the perpendicular drawn from the origin on the line.

It is given that the perpendicular OL makes 60° angle with x-axis. Therefore, $\alpha = 60°$.

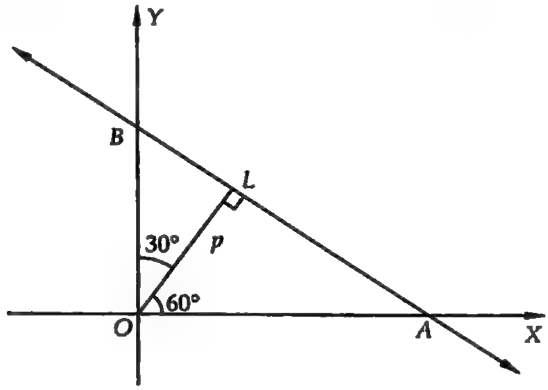


Fig. 23.45

Thus, the equation of the line AB is

$$x \cos \alpha + y \sin \alpha = p \text{ or, } x \cos 60° + y \sin 60° = p$$

or, $$x + \sqrt{3}y = 2p \text{ or, } \frac{x}{2p} + \frac{\sqrt{3}y}{2p} = 1 \quad \text{...(i)}$$

This, cuts the coordinates axes at A and B such that $OA = 2p$ and $OB = \frac{2p}{\sqrt{3}}$.

It is given that area of ΔOAB is $54\sqrt{3}$ sq. units.

$\therefore$ $$\frac{1}{2} \times OA \times OB = 54\sqrt{3}$$

$\Rightarrow$ $$\frac{1}{2} \times 2p \times \frac{2p}{\sqrt{3}} = 54\sqrt{3} \Rightarrow p^2 = 81 \Rightarrow p = 9$$

Substituting $p = 9$ in (i), we get $x + \sqrt{3}y = 18$ as the equation of the required line.

EXAMPLE 7 *A straight canal is $4\frac{1}{2}$ miles from a place and the shortest route from this place to the canal is exactly north-east. A village is 3 miles north and four miles east from the place. Does it lie by the nearest edge of the canal?*

SOLUTION Let the given place be O. Take this as the origin and the east and north directions through O as the x and y axes respectively. Let AB be the nearest edge of the canal. It is given that the canal is $4\frac{1}{2}$ miles from O. This means that the perpendicular distance of AB from O is $4\frac{1}{2}$ miles i.e. $OL \perp AB$ and $OL = 4\frac{1}{2}$. It is also given that OL is exactly north-east. Therefore, $\angle LOA = 45°$.

So, the equation of the edge AB of the canal is

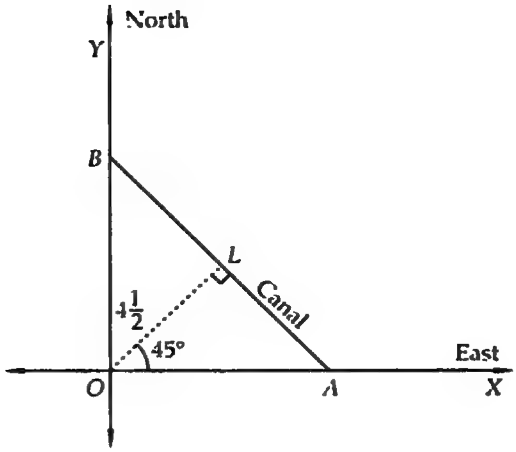


Fig. 23.46

$$x \cos 45° + y \sin 45° = 4\frac{1}{2} \quad \text{or,} \quad \sqrt{2}\,(x+y) = 9 \qquad \text{...(i)}$$

The position of the village is (4, 3). The village will lie on the edge of the canal, if (4, 3) satisfies the equation (i). Clearly, (4, 3) does not satisfy (i). Hence, the village does not lie by the nearer edge of the canal.

EXERCISE 23.7

LEVEL-1

1. Find the equation of a line for which
   (i) $p = 5$, $\alpha = 60°$ (ii) $p = 4$, $\alpha = 150°$ (iii) $p = 8$, $\alpha = 225°$ (iv) $p = 8$, $\alpha = 300°$
2. Find the equation of the line on which the length of the perpendicular segment from the origin to the line is 4 and the inclination of the perpendicular segment with the positive direction of x-axis is 30°. [NCERT EXEMPLAR]
3. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of x-axis is 15°. [NCERT]
4. Find the equation of the straight line at a distance of 3 units from the origin such that the perpendicular from the origin to the line makes an angle α given by $\tan\alpha = \frac{5}{12}$ with the positive direction of x-axis.
5. Find the equation of the straight line on which the length of the perpendicular from the origin is 2 and the perpendicular makes an angle α with x-axis such that $\sin\alpha = \frac{1}{3}$.
6. Find the equation of the straight line upon which the length of the perpendicular from the origin is 2 and the slope of this perpendicular is $\frac{5}{12}$.
7. The length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150° with the positive direction of y-axis. Find the equation of the line.
8. Find the value of θ and p, if the equation $x\cos\theta + y\sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$. [NCERT]

LEVEL-2

9. Find the equation of the straight line which makes a triangle of area $96\sqrt{3}$ with the axes and perpendicular from the origin to it makes an angle of 30° with y-axis.

10. Find the equation of a straight line on which the perpendicular from the origin makes an angle of 30° with x-axis and which forms a triangle of area $50/\sqrt{3}$ with the axes.

ANSWERS

1. (i) $x+\sqrt{3}\,y=10$ (ii) $-\sqrt{3}\,x+y=8$ (iii) $x+y+8\sqrt{2}$ (iv) $x-\sqrt{3}\,y=16$
2. $\sqrt{3}x+y=8$
3. $(\sqrt{3}+1)\,x+(\sqrt{3}-1)\,y=8\sqrt{2}$
4. $12\,x+5\,y=39$
5. $2\sqrt{2}\,x+y=6$
6. $12x+5y\pm 26=0$
7. $\sqrt{3}x+y=14$
8. $\theta=\dfrac{7\pi}{6}, p=1$
9. $x+\sqrt{3}y=24$
10. $\sqrt{3}x+y=\pm 10$

HINTS TO NCERT & SELECTED PROBLEMS

3. Here, $\alpha=15°$ and $p=4$. So, the equation of the line is
$$x\cos 15° + y\sin 15° = 4 \text{ or, } (\sqrt{3}+1)\,x+(\sqrt{3}-1)\,y=8\sqrt{2}.$$

8. We have,
$$\sqrt{3}x+y+2=0 \Rightarrow -\sqrt{3}x-y=2 \Rightarrow \left(\frac{-\sqrt{3}}{2}\right)x+\left(-\frac{1}{2}\right)y=1$$
This is same as $x\cos\theta+y\sin\theta=p$.
$$\therefore \quad \cos\theta=-\frac{\sqrt{3}}{2},\ \sin\theta=-\frac{1}{2} \text{ and } p=1$$
$$\Rightarrow \quad \theta=\frac{7\pi}{6} \text{ and } p=1$$

22.6.6 DISTANCE FORM OF A LINE

THEOREM *The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is* $\dfrac{x-x_1}{\cos\theta}=\dfrac{y-y_1}{\sin\theta}=r$, *where r is the distance of the point (x, y) on the line from the point (x_1, y_1).*

PROOF Let the given line meets x-axis at A, y-axis at B and passes through the point $Q(x_1, y_1)$. Let $P(x, y)$ be any point on the line at a distance r from $Q(x_1, y_1)$ i.e. $PQ=r$. Draw $PL\perp OX$, $QM\perp OX$ and $QN\perp PL$. Then,

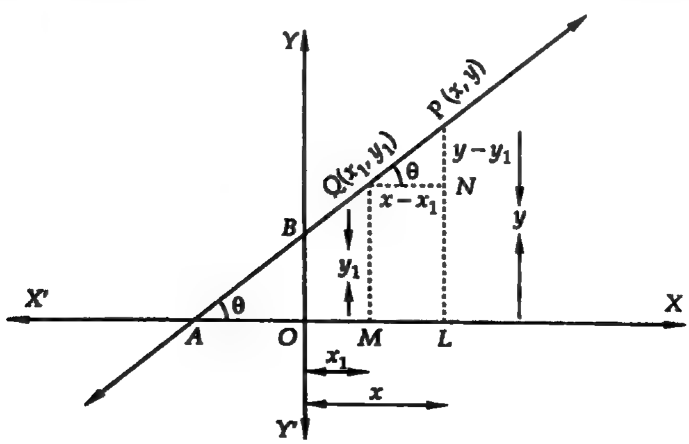


Fig. 23.47

$$QN=ML=OL-OM=x-x_1 \text{ and, } PN=PL-NL=PL-QM=y-y_1.$$

In ΔPQN, we have
$$\cos\theta=\frac{QN}{PQ}$$

$$\Rightarrow \quad \cos\theta = \frac{x - x_1}{r} \quad \ldots(i)$$

$$\text{and,} \quad \sin\theta = \frac{PN}{PQ}$$

$$\Rightarrow \quad \sin\theta = \frac{y - y_1}{r} \quad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

This is the required equation of the line in the distance form. **Q.E.D.**

NOTE 1 *The equation of the line is*

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

$$\Rightarrow \quad x - x_1 = r\cos\theta \text{ and } y - y_1 = r\sin\theta \Rightarrow x = x_1 + r\cos\theta \text{ and } y = y_1 + r\sin\theta.$$

Thus, the coordinates of any point on the line at a distance r from the given point (x_1, y_1) *are* $(x_1 + r\cos\theta,\ y_1 + r\sin\theta)$. *If P is on the right side of* (x_1, y_1), *then r is positive and if P is on the left side of* (x_1, y_1), *then r is negative. Since different values of r determine different points on the line, therefore the above form of the line is also called parametric form or symmetric form of a line.*

NOTE 2 *In the above form one can determine the coordinates of any point on the line at a given distance from the given point through which it passes. At a given distance r from the point* (x_1, y_1) *on the line* $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}$ *there are two points viz.* $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ *and* $(x_1 - r\cos\theta,\ y_1 - r\sin\theta)$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A straight line is drawn through the point P (2, 3) and is inclined at an angle of 30° with the x-axis. Find the coordinates of two points on it at a distance 4 from P on either side of P.*

SOLUTION Here, $(x_1, y_1) = (2, 3)$, $\theta = 30°$. So, the equation of the line is

$$\frac{x - 2}{\cos 30°} = \frac{y - 3}{\sin 30°}$$

$$\text{or,} \quad \frac{x - 2}{\frac{\sqrt{3}}{2}} = \frac{y - 3}{\frac{1}{2}} \quad \text{or, } x - 2 = \sqrt{3}\,(y - 3) \text{ or, } x - \sqrt{3}\,y = 2 - 3\sqrt{3}.$$

Points on the line at a distance 4 from P (2, 3) are

$$(x_1 \pm r\cos\theta,\ y_1 \pm r\sin\theta) \quad \text{or,} \quad (2 \pm 4\cos 30°,\ 3 \pm 4\sin 30°) \text{ or, } (2 \pm 2\sqrt{3},\ 3 \pm 2)$$

$$\text{or,} \quad (2 + 2\sqrt{3}, 5) \text{ and } (2 - 2\sqrt{3}, 1).$$

EXAMPLE 2 *The slope of a straight line through A (3, 2) is 3/4 Find the coordinates of the points on the line that are 5 units away from A.* **[NCERT EXEMPLAR]**

SOLUTION Suppose the given line makes an angle θ with x-axis. It is given that its slope is 3/4.

$$\therefore \quad \tan\theta = \frac{3}{4} \Rightarrow \sin\theta = \frac{3}{5} \text{ and } \cos\theta = \frac{4}{5}$$

The equation of the line in distance from is $\frac{x-3}{\cos\theta} = \frac{y-2}{\sin\theta}$ or, $\frac{x-3}{4/5} = \frac{y-2}{3/5}$ and the coordinates of two points P and Q at a distance of 5 units from A are given by

$$\frac{x-3}{4/5}=\frac{y-2}{3/5}=\pm 5$$

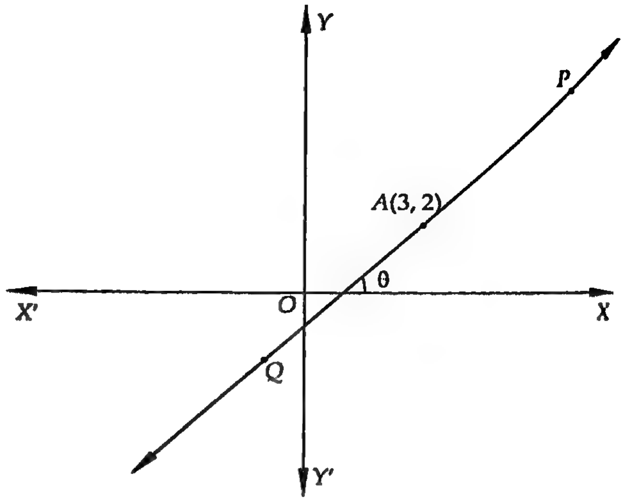


Fig. 23.48

Now,

$$\frac{x-3}{4/5}=\frac{y-2}{3/5}=5$$

$\Rightarrow \quad x-3=\frac{4}{5}\times 5$ and, $y-2=\frac{3}{5}\times 5$

$\Rightarrow \quad x=7,\ y=5$

and, $\quad \frac{x-3}{4/5}=\frac{y-2}{3/5}=-5$

$\Rightarrow \quad x-3=\frac{4}{5}\times -5$ and, $y-2=\frac{3}{5}\times -5$

$\Rightarrow \quad x=-1,\ y=-1$

Hence, the coordinates of P and Q are $(7, 5)$ and $(-1, -1)$ respectively.

REMARK *The coordinates of P and Q are* $(3\pm 5\cos\theta,\ 2\pm 5\sin\theta)$, *where* $\cos\theta=\frac{4}{5}$ *and* $\sin\theta=\frac{3}{5}$.

EXAMPLE 3 *Find the equation of the line through the point A (2, 3) and making an angle of 45° with the x-axis. Also, determine the length of intercept on it between A and the line* $x+y+1=0$.

SOLUTION The equation of a line through A and making an angle of 45° with the x-axis is

$$\frac{x-2}{\cos 45^\circ}=\frac{y-3}{\sin 45^\circ} \Rightarrow \frac{x-2}{\frac{1}{\sqrt{2}}}=\frac{y-3}{\frac{1}{\sqrt{2}}} \Rightarrow x-y+1=0$$

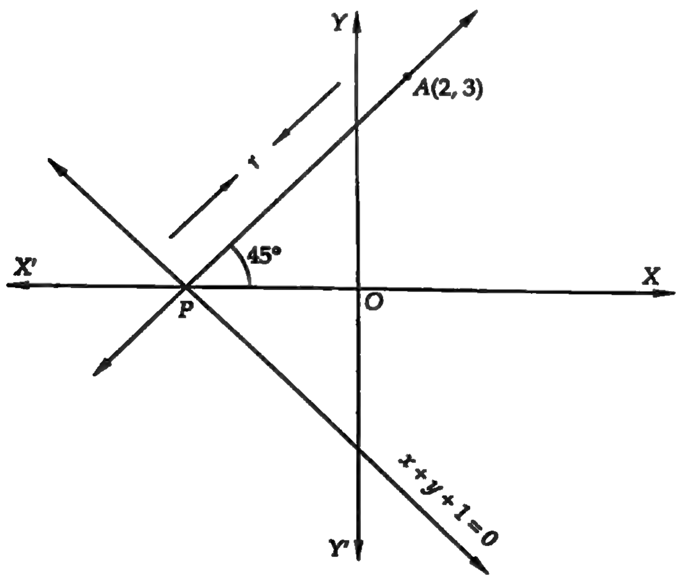


Fig. 23.49

Suppose this line meets the line $x+y+1=0$ at P such that $AP=r$. Then, the coordinates of P are given by

$$\frac{x-2}{\cos 45^\circ}=\frac{y-3}{\sin 45^\circ}=r$$

$\Rightarrow \quad x = 2 + r\cos 45°,\ y = 3 + r\sin 45°$

$\Rightarrow \quad x = 2 + \frac{r}{\sqrt{2}},\ y = 3 + \frac{r}{\sqrt{2}}$

Thus, the coordinates of P are $\left(2+\frac{r}{\sqrt{2}}, 3+\frac{r}{\sqrt{2}}\right)$.

Since P lies on $x + y + 1 = 0$. Therefore,

$$2+\frac{r}{\sqrt{2}}+3+\frac{r}{\sqrt{2}}+1 = 0$$

$\Rightarrow \quad \sqrt{2}\,r = -6 \Rightarrow r = -3\sqrt{2}$

$\therefore \quad$ Length $AP = |r| = 3\sqrt{2}$.

Thus, the length of the intercept is $3\sqrt{2}$.

ALITER The equation of the line through A (2, 3) and making an angle of 45° with x-axis is

$$y-3 = \tan 45°(x-2) \text{ or, } x-y+1=0.$$

This line intersects $x+y+1=0$ at $P(-1, 0)$.

$$\therefore \quad AP = \sqrt{(2+1)^2+(3-0)^2} = \sqrt{18} = 3\sqrt{2}$$

EXAMPLE 4 *If the straight line through the point P (3, 4) makes an angle $\frac{\pi}{6}$ with x-axis and meets the line $12x+5y+10=0$ at Q, find the length of PQ.*

SOLUTION The equation of a line passing through P (3, 4) and making an angle $\frac{\pi}{6}$ with x-axis is

$$\frac{x-3}{\cos\frac{\pi}{6}} = \frac{y-4}{\sin\frac{\pi}{6}} = r \text{ or, } \frac{x-3}{\frac{\sqrt{3}}{2}} = \frac{y-4}{\frac{1}{2}} = r$$

where r represents the distance of any point on this line from the given point P (3, 4).

The coordinates of any point Q on this line are $\left(3+\frac{\sqrt{3}}{2}r,\ 4+\frac{r}{2}\right)$. If Q lies on $12x+5y+10=0$, then

$$12\left(3+\frac{\sqrt{3}}{2}r\right)+5\left(4+\frac{r}{2}\right)+10=0 \Rightarrow r = \frac{-132}{12\sqrt{3}+5}$$

Hence, length $PQ = \dfrac{132}{12\sqrt{3}+5}$.

ALITER The equation of the line through the point $P(3, 4)$ and making an angle of $\frac{\pi}{6}$ with x-axis is

$$y-4 = \tan\frac{\pi}{6}(x-3) \text{ or, } x-\sqrt{3}y+4\sqrt{3}-3=0$$

This intersects the line $12x+5y+10=0$ at $Q\left(\frac{15-30\sqrt{3}}{5+12\sqrt{3}}, \frac{48\sqrt{3}-46}{5+12\sqrt{3}}\right)$.

$$\therefore \quad PQ = \sqrt{\left(\frac{15-30\sqrt{3}}{5+12\sqrt{3}}-3\right)^2+\left(\frac{48\sqrt{3}-46}{5+12\sqrt{3}}-4\right)^2} = \frac{132}{5+12\sqrt{3}}$$

EXAMPLE 5 ***The line joining two points A (2, 0), B (3, 1) is rotated about A in anti-clockwise direction through an angle of 15°. Find the equation of the line in the new position. If point B goes to point C in the new position, what will be the coordinates of C?***

SOLUTION The slope m of the line AB is given by $m = \frac{1-0}{3-2} = 1$. So, AB makes an angle of 45° with x-axis. Now, AB is rotated through 15° in anticlockwise direction and so it makes an angle of 60° with x-axis in its new position AC. Clearly, AC passes through A (2, 0) and makes an angle of 60° with x-axis. Therefore, the equation of AC in distance form is

$$\frac{x-2}{\cos 60°} = \frac{y-0}{\sin 60°} \quad \text{or,} \quad \frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}}$$

Clearly, $AB = \sqrt{(3-2)^2 + (1-0)^2} = \sqrt{2}$.

The point C is at a distance $\sqrt{2}$ from A. So, the coordinates of C are given by

$$\frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}} = \sqrt{2}$$

$$\Rightarrow \quad x = 2 + \frac{1}{2}\sqrt{2} = 2 + \frac{1}{\sqrt{2}} \text{ and } y = \frac{\sqrt{3}}{2} \times \sqrt{2} = \frac{\sqrt{6}}{2}$$

Hence, the coordinates of C are $\left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}\right)$.

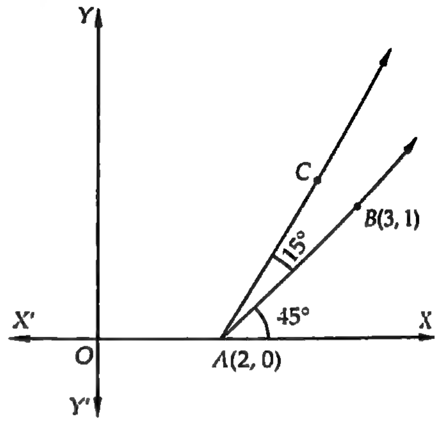


Fig. 23.50

EXAMPLE 6 *Find the distance of the line $4x - y = 0$ from the point P (4, 1) measured along the line making an angle of 135° with the positive x-axis.* [NCERT]

SOLUTION The equation in distance form of the line passing through P (4, 1) and making an angle of 135° with the positive x-axis is

$$\frac{x-4}{\cos 135°} = \frac{y-1}{\sin 135°}$$

Suppose it cuts $4x - y = 0$ at Q such that $PQ = r$. Then, the coordinates of Q are given by

$$\frac{x-4}{\cos 135°} = \frac{y-1}{\sin 135°} = r$$

$$\Rightarrow \quad \frac{x-4}{-1/\sqrt{2}} = \frac{y-1}{1/\sqrt{2}} = r$$

$$\Rightarrow \quad x = 4 - \frac{r}{\sqrt{2}}, y = 1 + \frac{r}{\sqrt{2}}$$

So, the coordinates of Q are $\left(4 - \frac{r}{\sqrt{2}}, 1 + \frac{r}{\sqrt{2}}\right)$.

Clearly, Q lies on $4x - y = 0$.

$$\therefore \quad 16 - \frac{4r}{\sqrt{2}} - 1 - \frac{r}{\sqrt{2}} = 0 \Rightarrow \frac{5r}{\sqrt{2}} = 15 \Rightarrow r = 3\sqrt{2}$$

Hence, required distance is $3\sqrt{2}$ units.

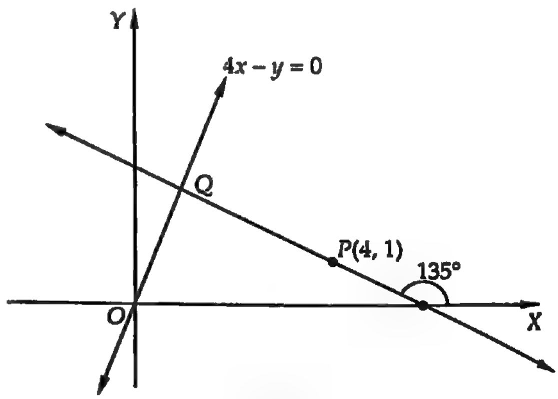


Fig. 23.51

EXAMPLE 7 *Find the distance of the point (2, 3) from the line $2x - 3y + 9 = 0$ measured along a line $x - y + 1 = 0$.*

SOLUTION The slope of the line $x - y + 1 = 0$ is 1. So it makes an angle of 45° with x-axis.

The equation of a line, in distance form, passing through P (2, 3) and making an angle of 45° with x-axis is

$$\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°}$$

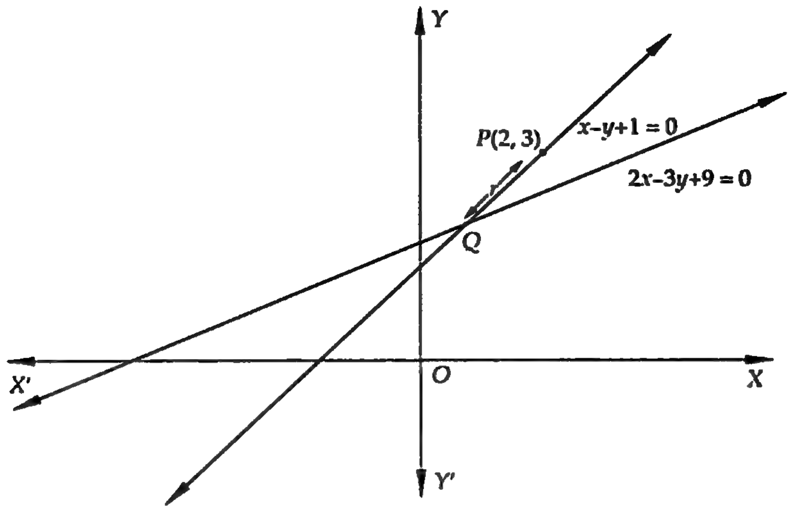


Fig. 23.52

The coordinates of any point Q on this line are given by

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r \qquad \left[\text{Using}: \frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r\right]$$

So, the coordinates of Q are $(2 + r\cos 45^\circ,\ 3 + r\sin 45^\circ)$ or, $\left(2 + \frac{r}{\sqrt{2}},\ 3 + \frac{r}{\sqrt{2}}\right)$

If this point lies on the line $2x - 3y + 9 = 0$, then

$$4 + r\sqrt{2} - 9 - \frac{3r}{\sqrt{2}} + 9 = 0 \Rightarrow r = 4\sqrt{2} \Rightarrow PQ = 4\sqrt{2}$$

Hence, the required distance is $4\sqrt{2}$ units.

EXAMPLE 8 *Find the distance of the line* $4x + 7y + 5 = 0$ *from the point* $(1, 2)$ *along the line* $2x - y = 0$. **[NCERT]**

SOLUTION Clearly, line $2x - y = 0$ passes through $P(1, 2)$ and intersects $4x + 7y + 5 = 0$ at Q. Let $PQ = r$. If $2x - y = 0$ makes an angle θ with x-axis. Then, $\tan\theta = 2$ and hence $\sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$. Thus, the line $2x - y = 0$ passes through $P(1, 2)$ and makes an angle θ with x-axis such

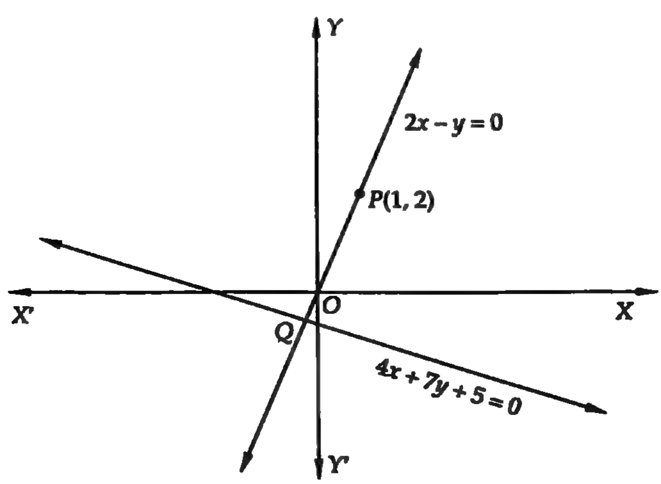


Fig. 23.53

that $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = \frac{2}{\sqrt{5}}$. So, its equation in distance form is $\frac{x-1}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}}$ and the coordinates of Q are given by $\frac{x-1}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}} = r$.

Now, $\frac{x-1}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}} = r \Rightarrow x = 1 + \frac{r}{\sqrt{5}},\ y = 2 + \frac{2r}{\sqrt{5}}$

So, the coordinates of Q are $\left(1+\frac{r}{\sqrt{5}},\ 2+\frac{2r}{\sqrt{5}}\right)$.

As Q lies on $4x+7y+5=0$.

$$\therefore \quad 4\left(1+\frac{r}{\sqrt{5}}\right)+7\left(2+\frac{2r}{\sqrt{5}}\right)+5=0$$

$$\Rightarrow \quad \frac{18r}{\sqrt{5}} = -23 \Rightarrow r = -\frac{23\sqrt{5}}{18}$$

Hence, $PQ = |r| = \frac{23\sqrt{5}}{18}$.

LEVEL-2

EXAMPLE 9 *In what direction a line be drawn through the point* $(1, 2)$ *that its point of intersection with the line* $x+y=4$ *is at a distance* $\frac{\sqrt{6}}{3}$ *from the given point.* **[NCERT EXEMPLAR]**

SOLUTION Let the line drawn through $A\ (1, 2)$ makes an angle θ with the positive direction of x-axis and intersects the line $x+y=4$ at P such that $AP = \frac{\sqrt{6}}{3}$. Then, the coordinates of P are given by

$$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = \frac{\sqrt{6}}{3} \Rightarrow x = 1+\sqrt{\frac{2}{3}}\cos\theta,\ y = 2+\sqrt{\frac{2}{3}}\sin\theta$$

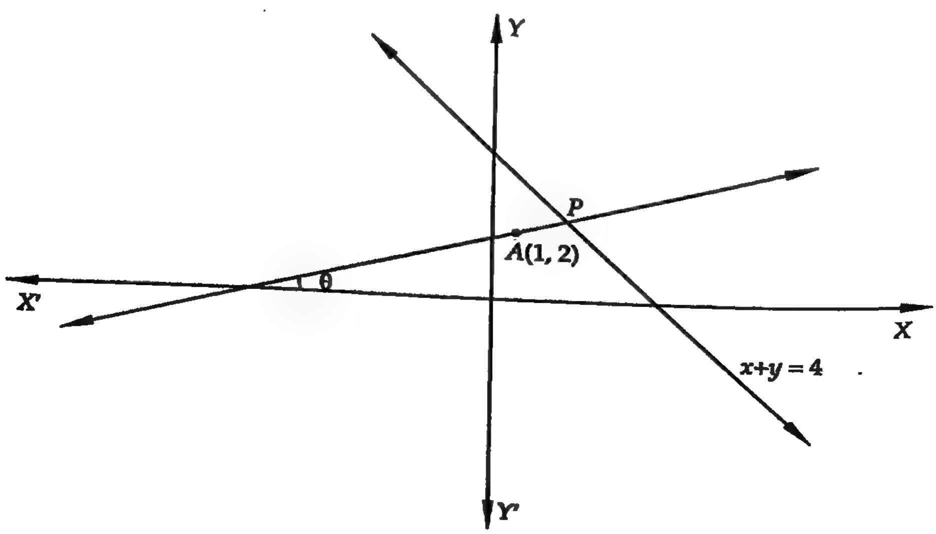


Fig. 23.54

So, the coordinates of P are $\left(1+\sqrt{\frac{2}{3}}\cos\theta,\ 2+\sqrt{\frac{2}{3}}\sin\theta\right)$.

Clearly, point P lies on the line $x+y=4$.

$\therefore \quad 1+\sqrt{\frac{2}{3}}\cos\theta+2+\sqrt{\frac{2}{3}}\sin\theta=4$

$\Rightarrow \quad \cos\theta+\sin\theta=\sqrt{\frac{3}{2}}$

$\Rightarrow \quad (\cos\theta+\sin\theta)^2=\frac{3}{2}$

$\Rightarrow \quad 1+\sin 2\theta=\frac{3}{2} \Rightarrow \sin 2\theta=\frac{1}{2} \Rightarrow 2\theta=\frac{\pi}{6}$ or, $2\theta=\frac{5\pi}{6} \Rightarrow \frac{\pi}{12}, \frac{5\pi}{12}$

Hence, the line drawn makes an angle whose measure is either $\frac{\pi}{12}$ or $\frac{5\pi}{12}$ with the x-axis.

EXAMPLE 10 *Find the direction in which a straight line must be drawn through the point $(-1, 2)$ so that its point of intersection with the line $x+y=4$ may be at a distance of 3 units from the point.* [NCERT]

SOLUTION Suppose the required line makes an angle θ with x-axis. It passes through the point $P(-1, 2)$. So, its equation is

$$\frac{x-(-1)}{\cos\theta}=\frac{y-2}{\sin\theta} \text{ or, } \frac{x+1}{\cos\theta}=\frac{y-2}{\sin\theta}$$

The coordinates of a point Q on this line at a distance of 3 units from $P(-1, 2)$ are given by

$$\frac{x+1}{\cos\theta}=\frac{y-2}{\sin\theta}=3$$

$\Rightarrow \quad x=-1+3\cos\theta,\ y=2+3\sin\theta$

So, the coordinates of Q are

$$(-1+3\cos\theta, 2+3\sin\theta)$$

If point Q lies on $x+y=4$, then

$$-1+3\cos\theta+2+3\sin\theta=4$$

$\Rightarrow \quad 3\cos\theta+3\sin\theta=3$

$\Rightarrow \quad \cos\theta+\sin\theta=1$

$\Rightarrow \quad (\cos\theta+\sin\theta)^2=1$

$\Rightarrow \quad 1+\sin 2\theta=1$

$\Rightarrow \quad \sin 2\theta=0$

$\Rightarrow \quad 2\theta=0 \text{ or } 2\theta=\pi$

$\Rightarrow \quad \theta=0 \text{ or } \theta=\frac{\pi}{2}$

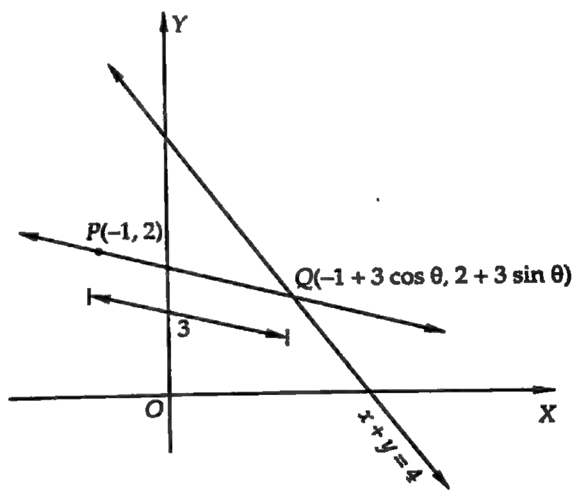


Fig. 23.55

Hence, the required line must be either parallel to x-axis or to y-axis.

EXAMPLE 11 *A line is such that its segment between the lines $5x-y+4=0$ and $3x+4y-4=0$ is bisected at the point $(1, 5)$. Obtain its equation.* [NCERT]

SOLUTION Let AB be the line making angle θ with x-axis such that its intercept AB between the lines $5x-y+4=0$ and $3x+4y-4=0$ is bisected at $P(1, 5)$. Then, the equation of the line is

$$\frac{x-1}{\cos\theta}=\frac{y-5}{\sin\theta} \text{ or, } y-5=\tan\theta(x-1) \quad \text{...(i)}$$

Let $AP=BP=r$. Then, the coordinates of A and B are given by

$$\frac{x-1}{\cos\theta}=\frac{y-5}{\sin\theta}=r \text{ and } \frac{x-1}{\cos\theta}=\frac{y-5}{\sin\theta}=-r \text{ respectively.}$$

Now, $\frac{x-1}{\cos\theta}=\frac{y-5}{\sin\theta}=r \Rightarrow x=1+r\cos\theta,\ y=5+r\sin\theta$

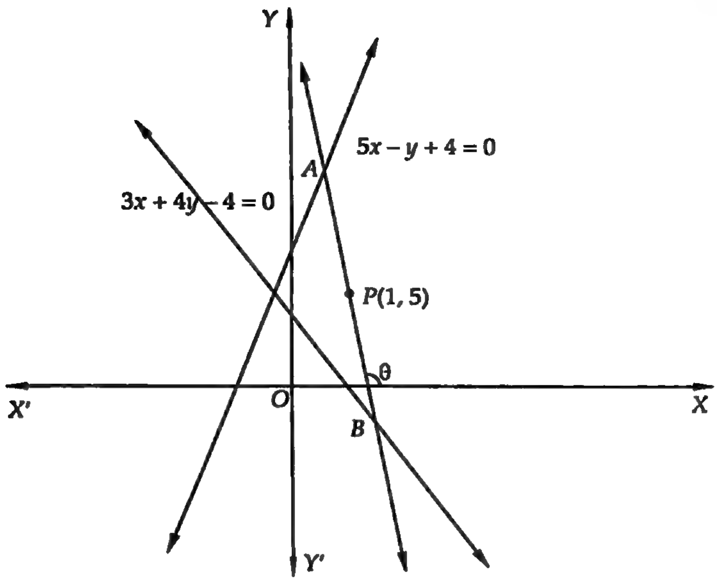


Fig. 23.56

l, $\frac{x-1}{\cos\theta}=\frac{y-5}{\sin\theta}=-r \Rightarrow x=1-r\cos\theta, y=5-r\sin\theta$

the co-ordinates of A and B are $(1+r\cos\theta, 5+r\sin\theta)$ and $(1-r\cos\theta, 5-r\sin\theta)$ respec-ely. Points A and B lie on lines $5x-y+4=0$ and $3x+4y-4=0$ respectively.

. $5(1+r\cos\theta)-(5+r\sin\theta)+4=0$ and, $3(1-r\cos\theta)+4(5-r\sin\theta)-4=0$

$\Rightarrow$ $r(5\cos\theta-\sin\theta)=-4$ and $r(3\cos\theta+4\sin\theta)=19$

$\Rightarrow$ $r=\frac{-4}{5\cos\theta-\sin\theta}$ and $r=\frac{19}{3\cos\theta+4\sin\theta}$

$\Rightarrow$ $\frac{-4}{5\cos\theta-\sin\theta}=\frac{19}{3\cos\theta+4\sin\theta}$

$\Rightarrow$ $-12\cos\theta-16\sin\theta=95\cos\theta-19\sin\theta$

$\Rightarrow$ $107\cos\theta=3\sin\theta$

$\Rightarrow$ $\tan\theta=\frac{107}{3}$

Putting the value of $\tan\theta$ in (i), we obtain

$$y-5=\tan\theta(x-1) \text{ or, } y-5=\frac{107}{3}(x-1)$$

or, $107x-3y-92=0$ as the required equation of the line.

EXAMPLE 12 *Find the equation of the line passing through the point (2, 3) and making an intercept of length 3 units between the lines $y+2x=2$ and $y+2x=5$.*

SOLUTION The equations of the given lines are

$2x+y=2$ ...(i) and, $2x+y=5$ ...(ii)

We observe that the lines given by equations (i) and (ii) are parallel. Suppose a line passing through $A(2, 3)$ and intercepting length $BC=3$ between lines (i) and (ii) makes an angle θ with x-axis. The equation of this line in distance form is

$$\frac{x-2}{\cos\theta}=\frac{y-3}{\sin\theta} \quad ...(iii)$$

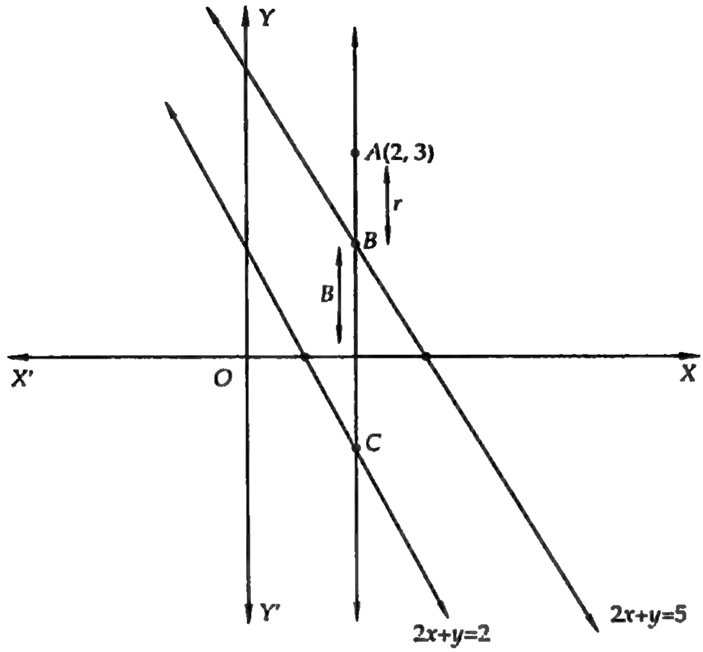


Fig. 23.57

Let $AB = r$. Then, $AC = AB + BC = (r+3)$. Clearly, B and C are points on line (iii) at distances r and $r+3$ respectively from A.

So, the coordinates of B and C are given by

$$\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r \text{ and } \frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r+3 \text{ respectively.}$$

Now,

$$\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r \Rightarrow x = 2 + r\cos\theta,\ y = 3 + r\sin\theta$$

and, $$\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r+3 \Rightarrow x = 2 + (r+3)\cos\theta,\ y = 3 + (r+3)\sin\theta$$

So, the coordinates of B and C are

$$(2 + r\cos\theta,\ 3 + r\sin\theta) \text{ and } (2 + (r+3)\cos\theta,\ 3 + (r+3)\sin\theta) \text{ respectively.}$$

We observe that points B and C lie on lines (i) and (ii) respectively.

$\therefore \quad 2(2 + r\cos\theta) + (3 + r\sin\theta) = 2$ and, $2\{2 + (r+3)\cos\theta\} + \{3 + (r+3)\sin\theta\} = 5$

$\Rightarrow \quad r(2\cos\theta + \sin\theta) = -5$ and, $(r+3)(2\cos\theta + \sin\theta) = -2$

$\Rightarrow \quad r = \dfrac{-5}{2\cos\theta + \sin\theta}$ and $r + 3 = \dfrac{-2}{2\cos\theta + \sin\theta}$

$\Rightarrow \quad \dfrac{-5}{2\cos\theta + \sin\theta} + 3 = \dfrac{-2}{2\cos\theta + \sin\theta}$ [On eliminating r]

$\Rightarrow \quad 3 = \dfrac{3}{2\cos\theta + \sin\theta}$ ...(iv)

$\Rightarrow \quad 2\cos\theta + \sin\theta = 1$

$\Rightarrow \quad 2\cos\theta = 1 - \sin\theta$ ...(v)

$\Rightarrow \quad 4\cos^2\theta = 1 + \sin^2\theta - 2\sin\theta$

$\Rightarrow \quad 4(1 - \sin^2\theta) = 1 + \sin^2\theta - 2\sin\theta$

$\Rightarrow \quad 5\sin^2\theta - 2\sin\theta - 3 = 0$

$\Rightarrow \quad (\sin\theta - 1)(5\sin\theta + 3) = 0 \Rightarrow \sin\theta = 1$ or $\sin\theta = -3/5$

Putting $\sin\theta = 1$ in (v), we obtain $\cos\theta = 0$. Putting $\sin\theta = -3/5$ in (v), we obtain $\cos\theta = 4/5$. Substituting the values of $\sin\theta$ and $\cos\theta$ in (iii), we obtain that the equations of the required lines are

$$\frac{x-2}{0} = \frac{y-3}{1} \text{ and } \frac{x-2}{4/5} = \frac{y-3}{-3/5} \quad \text{or, } x - 2 = 0 \text{ and } 3x + 4y = 18.$$

EXAMPLE 13 *A line through $A(-5, -4)$ meets the lines $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at the points B, C and D respectively, if $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$ find the equation of the line.*

SOLUTION The equations of the given lines are

$$x + 3y + 2 = 0 \quad \ldots\text{(i)}, \qquad 2x + y + 4 = 0 \quad \ldots\text{(ii) and,} \quad x - y - 5 = 0 \quad \ldots\text{(iii)}$$

The equation of a line passing through $A(-5, -4)$ and making an angle θ with x-axis is

$$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} \quad \ldots\text{(iv)}$$

Suppose this line cuts lines (i), (ii) and (iii) at B, C and D respectively such that $AB = r_1$, $AC = r_2$ and $AD = r_3$. Then, the coordinates of B, C and D are given by

$$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_1, \ \frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_2 \text{ and } \frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_3 \text{ respectively.}$$

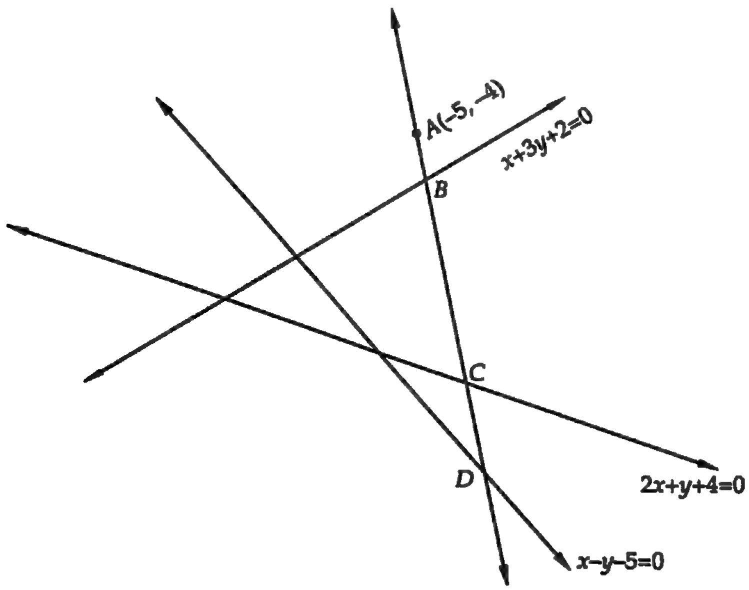


Fig. 23.58

The coordinates of B, C and D are $(-5 + r_1\cos\theta, -4 + r_1\sin\theta)$, $(-5 + r_2\cos\theta, -4 + r_2\sin\theta)$, and $(-5 + r_3\cos\theta, -4 + r_3\sin\theta)$ respectively.

Points B, C and D lie on lines (i), (ii) and (iii) respectively.

$\therefore \quad (-5 + r_1\cos\theta) + 3(-4 + r_1\sin\theta) + 2 = 0$, $2(-5 + r_2\cos\theta) + (-4 + r_2\sin\theta) + 4 = 0$

and $\quad (-5 + r_3\cos\theta) - (-4 + r_3\sin\theta) - 5 = 0$

$$\Rightarrow \quad r_1 = \frac{15}{\cos\theta + 3\sin\theta}, \ r_2 = \frac{10}{2\cos\theta + \sin\theta} \text{ and } r_3 = \frac{6}{\cos\theta - \sin\theta}$$

$$\Rightarrow \quad \frac{15}{AB} = \cos\theta + 3\sin\theta, \ \frac{10}{AC} = 2\cos\theta + \sin\theta \text{ and } \frac{6}{AD} = \cos\theta - \sin\theta$$

Substituting these values in $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$, we obtain

$$(\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$$

$$\Rightarrow \quad 4\cos^2\theta + 12\sin\theta\cos\theta + 9\sin^2\theta = 0$$

$$\Rightarrow \quad (2\cos\theta + 3\sin\theta)^2 = 0 \Rightarrow 2\cos\theta + 3\sin\theta = 0 \Rightarrow \tan\theta = -2/3$$

Putting $\tan\theta = -2/3$ in (iv), we obtain

$y + 4 = -(2/3)(x+5)$ or, $2x + 3y + 22 = 0$ as the required equation of the line.

EXAMPLE 14 *The sides AB and AC of a triangle ABC are respectively $2x + 3y = 29$ and $x + 2y = 16$ respectively. If the mid-point of BC is (5, 6) then find the equation of BC.*

SOLUTION Suppose BC makes an angle θ with the x-axis. Then, its equation is

$$\frac{x-5}{\cos\theta} = \frac{y-6}{\sin\theta} \qquad \text{...(i)}$$

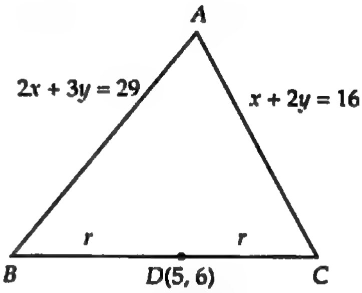


Fig. 23.59

Let $BD = CD = r$. Then, the coordinates B and C are given by

$$\frac{x-5}{\cos\theta} = \frac{y-6}{\sin\theta} = -r \text{ and, } \frac{x-5}{\cos\theta} = \frac{y-6}{\sin\theta} = r \text{ respectively.}$$

Now, $\frac{x-5}{\cos\theta} = \frac{y-6}{\sin\theta} = -r \Rightarrow x = 5 - r\cos\theta,\ y = 6 - r\sin\theta$

and, $\frac{x-5}{\cos\theta} = \frac{y-6}{\sin\theta} = r \Rightarrow x = 5 + r\cos\theta,\ y = 6 + r\sin\theta$

Thus, the coordinates of B and C are $B(5 - r\cos\theta, 6 - r\sin\theta)$ and $C(5 + r\cos\theta, 6 + r\sin\theta)$ respectively which lie on lines $2x + 3y = 29$ and $x + 2y = 16$ respectively.

$$\therefore \quad 2(5 - r\cos\theta) + 3(6 - r\sin\theta) = 29 \text{ and, } (5 + r\cos\theta) + 2(6 + r\sin\theta) = 16$$

$$\Rightarrow \quad r = \frac{-1}{2\cos\theta + 3\sin\theta} \text{ and } r = \frac{-1}{\cos\theta + 2\sin\theta}$$

$$\Rightarrow \quad \frac{-1}{2\cos\theta + 3\sin\theta} = \frac{-1}{\cos\theta + 2\sin\theta} \qquad \text{[On eliminating } r\text{]}$$

$$\Rightarrow \quad 2\cos\theta + 3\sin\theta = \cos\theta + 2\sin\theta \Rightarrow \sin\theta = -\cos\theta \Rightarrow \tan\theta = -1 \Rightarrow \theta = 3\pi/4$$

Putting $\theta = \frac{3\pi}{4}$ in (i), we obtain

$$\frac{x-5}{\cos 3\pi/4} = \frac{y-6}{\sin 3\pi/4} \text{ or, } x + y - 11 = 0 \text{ as the required equation of the line.}$$

EXERCISE 23.8

LEVEL-1

1. A line passes through a point $A(1, 2)$ and makes an angle of $60°$ with the x-axis and intersects the line $x + y = 6$ at the point P. Find AP.
2. If the straight line through the point $P(3, 4)$ makes an angle $\pi/6$ with the x-axis and meets the line $12x + 5y + 10 = 0$ at Q, find the length PQ.
3. A straight line drawn through the point $A(2, 1)$ making an angle $\pi/4$ with positive x-axis intersects another line $x + 2y + 1 = 0$ in the point B. Find length AB.
4. A line a drawn through $A(4, -1)$ parallel to the line $3x - 4y + 1 = 0$. Find the coordinates of the two points on this line which are at a distance of 5 units from A.
5. The straight line through $P(x_1, y_1)$ inclined at an angle θ with the x-axis meets the line $ax + by + c = 0$ in Q. Find the length of PQ.

6. Find the distance of the point (2, 3) from the line $2x - 3y + 9 = 0$ measured along a line making an angle of 45° with the x-axis.
7. Find the distance of the point (3, 5) from the line $2x + 3y = 14$ measured parallel to a line having slope 1/2.
8. Find the distance of the point (2, 5) from the line $3x + y + 4 = 0$ measured parallel to a line having slope 3/4.
9. Find the distance of the point (3, 5) from the line $2x + 3y = 14$ measured parallel to the line $x - 2y = 1$.
10. Find the distance of the point (2, 5) from the line $3x + y + 4 = 0$ measured parallel to the line $3x - 4y + 8 = 0$.
11. Find the distance of the line $2x + y = 3$ from the point $(-1, -3)$ in the direction of the line whose slope is 1.

LEVEL-2

12. A line is such that its segment between the straight lines $5x - y - 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1, 5). Obtain its equation.
13. Find the equation of straight line passing through $(-2, -7)$ and having an intercept of length 3 between the straight lines $4x + 3y = 12$ and $4x + 3y = 3$.

ANSWERS

1. $3(\sqrt{3} - 1)$
2. $\dfrac{132}{5 + 12\sqrt{3}}$
3. $\dfrac{5\sqrt{2}}{3}$
4. $(8, 2), (0, -4)$
5. $\left|\dfrac{ax_1 + by_1 + c}{a\cos\theta + b\sin\theta}\right|$
6. $4\sqrt{2}$
7. $\sqrt{5}$
8. 5 units
9. $\sqrt{5}$
10. 5
11. $\dfrac{8\sqrt{2}}{3}$
12. $83x - 35y + 92 = 0$
13. $x + 2 = 0, 7x + 24y + 182 = 0$

23.7 TRANSFORMATION OF GENERAL EQUATION IN DIFFERENT STANDARD FORMS

The general equation of a straight line is $Ax + By + C = 0$ which can be transformed to various standard forms as discussed below.

(i) *Transformation of $Ax + By + C = 0$ in the slope intercept form ($y = mx + c$):*
The equation of the line is

$$Ax + By + C = 0 \Rightarrow By = -Ax - C \Rightarrow y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

This is of the form $y = mx + c$, where $m = -\dfrac{A}{B}$ and, $c = -\dfrac{C}{B}$.

Thus, for the straight line $Ax + By + C = 0$, we have

$$m = \text{Slope} = -\frac{A}{B} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} \text{ and, Intercept on } y\text{-axis} = -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$$

<u>NOTE</u> *To determine the slope of a line by the formula $m = -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } y}$, we must first transfer all terms in the equation on one side.*

(ii) *Transformation of* $Ax + By + C = 0$ *in intercept form* $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$:

The equation of the line is

$$Ax + By + C = 0 \Rightarrow Ax + By = -C \Rightarrow \frac{Ax}{-C} + \frac{By}{-C} = 1 \Rightarrow \frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1$$

This is of the form $\frac{x}{a} + \frac{y}{b} = 1.$

Thus, for the straight line $Ax + By + C = 0$, we have

Intercept on x-axis $= -\frac{C}{A} = -\frac{\text{Constant term}}{\text{Coefficient of } x}$, Intercept on y-axis $= -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$

NOTE *As discussed above the intercepts made by a line with the coordinate axes can be determined by reducing its equation to intercept form. We may also use the following method to determine the intercepts on the coordinate axes:*

For intercept on x-axis : Put $y = 0$ in the equation of the line and find the value of x. Similarly to find y-intercept, put $x = 0$ in the equation of the line and find the value of y.

(iii) *Transformation of* $Ax + By + C = 0$ *in the normal form* ($x \cos \alpha + y \sin \alpha = p$):

We have, $$Ax + By + C = 0 \quad \text{...(i)}$$

Let $$x \cos \alpha + y \sin \alpha - p = 0 \quad \text{...(ii)}$$

be the normal form of $Ax + By + C = 0$.

Then, (i) and (ii) represent the same straight line.

$$\therefore \quad \frac{A}{\cos \alpha} = \frac{B}{\sin \alpha} = \frac{C}{-p}$$

$$\Rightarrow \quad \cos \alpha = -\frac{Ap}{C} \text{ and, } \sin \alpha = -\frac{Bp}{C} \quad \text{...(iii)}$$

$$\Rightarrow \quad \cos^2 \alpha + \sin^2 \alpha = \frac{A^2p^2}{C^2} + \frac{B^2p^2}{C^2}$$

$$\Rightarrow \quad 1 = \frac{p^2}{C^2}(A^2 + B^2) \Rightarrow p = \pm \frac{C}{\sqrt{A^2 + B^2}}$$

But, p denotes the length of the perpendicular from the origin to the line and is always positive.

$$\therefore \quad p = \frac{|C|}{\sqrt{A^2 + B^2}}$$

Putting the value of p in (iii), we get

$$\cos \alpha = -\frac{A}{\sqrt{A^2 + B^2}} \text{ and, } \sin \alpha = -\frac{B}{\sqrt{A^2 + B^2}}$$

So, the equation (ii) takes the form

$$-\frac{A}{\sqrt{A^2 + B^2}}x - \frac{B}{\sqrt{A^2 + B^2}}y - \frac{C}{\sqrt{A^2 + B^2}} = 0$$

or, $$-\frac{A}{\sqrt{A^2 + B^2}}x - \frac{B}{\sqrt{A^2 + B^2}}y = \frac{C}{\sqrt{A^2 + B^2}}$$

This is the required normal form of the line $Ax + By + C = 0$.

In order to transform the general equation of a line to the normal form, we use the following steps :

STEP I *Shift the constant term on the RHS and make it positive*

STEP II *Divide both sides by* $\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$

The equation so obtained is in the normal form.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Transform the equation of the line* $\sqrt{3}\,x + y - 8 = 0$ *to (i) slope intercept form and find its slope and y-intercept (ii) intercept form and find intercepts on the coordinate axes (iii) normal form and find the inclination of the perpendicular segment from the origin on the line with the axis and its length.*

SOLUTION (i) We have,

$\sqrt{3}x + y - 8 = 0 \Rightarrow y = -\sqrt{3}x + 8$, which is the slope intercept form of the given line.

$\therefore$ Slope $= -\sqrt{3}$, and y-intercept $= 8$

(ii) We have,

$\sqrt{3}\,x + y - 8 = 0 \Rightarrow \dfrac{x}{8/\sqrt{3}} + \dfrac{y}{8} = 1$, which is the intercept form of the given line.

So, x-intercept $= \dfrac{8}{\sqrt{3}}$ and, y-intercept $= 8$

(iii) We have, $\sqrt{3}\,x + y - 8 = 0$ or, $\sqrt{3}\,x + y = 8$

Dividing throughout by $\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$, we obtain

$$\Rightarrow \quad \frac{\sqrt{3}}{\sqrt{(\sqrt{3})^2 + 1^2}}\,x + \frac{1}{\sqrt{(\sqrt{3})^2 + 1^2}}\,y = \frac{8}{\sqrt{(\sqrt{3})^2 + 1^2}}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4, \text{ which is the normal form of the given line.}$$

Comparing this equation with $x\cos\alpha + y\sin\alpha = p$, we obtain

$$\cos\alpha = \frac{\sqrt{3}}{2}, \ \sin\alpha = \frac{1}{2} \text{ and } p = 4.$$

As $\sin\alpha$ and $\cos\alpha$ both are positive, therefore α is in first quadrant and is equal to $\pi/6$. Hence, for the given line, we have $\alpha = \pi/6$ and $p = 4$.

EXAMPLE 2 *Reduce the lines* $3x - 4y + 4 = 0$ *and* $4x - 3y + 12 = 0$ *to the normal form and hence determine which line is nearer to the origin.*

SOLUTION The equation of the first line is

$$3x - 4y + 4 = 0$$

$$\Rightarrow \quad -3x + 4y = 4$$

Dividing throughout by $\sqrt{(-3)^2 + (4)^2}$, we obtain

$$-\frac{3x}{\sqrt{(-3)^2 + 4^2}} + \frac{4y}{\sqrt{(-3)^2 + 4^2}} = \frac{4}{\sqrt{(-3)^2 + 4^2}}$$

$$\text{or,} \quad -\frac{3}{5}x + \frac{4}{5}y = \frac{4}{5}$$

This is the normal form of $3x - 4y + 4 = 0$ from which we find that the length of the perpendicular from the origin to it is given by $p_1 = 4/5$.

The equation of the second line is

$$4x - 3y + 12 = 0 \text{ or, } -4x + 3y = 12$$

Dividing throughout by $\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$, we obtain

$$-\frac{4x}{\sqrt{(-4)^2+3^2}} + \frac{3y}{\sqrt{(-4)^2+3^2}} = \frac{12}{\sqrt{(-4)^2+3^2}}$$

or, $$-\frac{4}{5}x + \frac{3}{5}y = \frac{12}{5}$$

This is the normal form of $4x - 3y + 12 = 0$ from which we find that the length of the perpendicular from the origin is given by $p_2 = \frac{12}{5}$.

Clearly, $p_2 > p_1$. Therefore, the line $3x - 4y + 4 = 0$ is nearer to the origin.

EXAMPLE 3 *Find the values of k for which the line $(k-3)x - (4-k^2)y + k^2 - 7k + 6 = 0$ is*

(i) *parallel to the x-axis.* (ii) *parallel to the y-axis.* (iii) *passing through the origin.*

[NCERT]

SOLUTION Let m be the slope of the line

$$(k-3)x - (4-k^2)y + k^2 - 7k + 6 = 0 \qquad \text{...(i)}$$

Then,

$$m = -\frac{(k-3)}{-(4-k^2)} = \frac{k-3}{4-k^2}$$

(i) If the line is parallel to x-axis, then

$$\text{Slope} = 0 \Rightarrow \frac{k-3}{4-k^2} = 0 \Rightarrow k - 3 = 0 \Rightarrow k = 3$$

(ii) If the line is parallel to y-axis, then

$$\frac{1}{m} = 0 \Rightarrow \frac{4-k^2}{k-3} = 0 \Rightarrow 4 - k^2 = 0 \Rightarrow k = \pm 2$$

(iii) If the line passes through the origin, then (0, 0) must satisfy the equation (i).

$$\therefore \quad (k-3)\times 0 - (4-k^2)\times 0 + k^2 - 7k + 6 = 0 \Rightarrow (k-1)(k-6) = 0 \Rightarrow k = 1, 6.$$

LEVEL-2

EXAMPLE 4 *Find the equation of a line with slope 2 and the length of the perpendicular from the origin equal to $\sqrt{5}$.*

SOLUTION Let the y-intercept of the required line be c. Then, its equation is

$$y = 2x + c \qquad \text{...(i)}$$

$$\Rightarrow \quad -2x + y = c$$

Dividing throughout by $\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$, we obtain

$$-\frac{2}{\sqrt{(-2)^2+1^2}}x + \frac{y}{\sqrt{(-2)^2+1^2}} = \frac{c}{\sqrt{(-2)^2+1^2}} \text{ or, } -\frac{2}{\sqrt{5}}x + \frac{y}{\sqrt{5}} = \frac{c}{\sqrt{5}}$$

This is the normal form of line (i). Therefore, RHS represents the length of the perpendicular from the origin. But, the length of the perpendicular from the origin is given to be $\sqrt{5}$.

$$\therefore \quad \left|\frac{c}{\sqrt{5}}\right| = \sqrt{5} \Rightarrow |c| = 5 \Rightarrow c = \pm 5$$

Putting $c = \pm 5$ in (i), we obtain the equations of the required lines as $y = 2x \pm 5$.

EXAMPLE 5 *Prove that the slope of a line is invariant under the translation of the axes.*

SOLUTION Let the equation of a straight line referred to a system of coordinate axes be

$$ax + by + c = 0 \qquad \text{...(i)}$$

The slope of this line is $m = -\frac{a}{b}$.

Now, let the origin be shifted to the point (h, k) under some translation of the axes. Then, any point (X, Y) with respect to the new system of coordinate axes is given by the relation

$$x = X + h \text{ and } y = Y + k$$

where (x, y) are the coordinates of the point in the old system of coordinate axes.

The equation of line (i) in the new system of axes is given by

$$a(X + h) + b(Y + k) + c = 0 \text{ or, } aX + bY + (ah + bk + c) = 0 \quad \text{...(ii)}$$

Let m' be the slope of this line. Then, $m' = -\frac{a}{b}$. Clearly, $m = m'$.

Hence, the slope of a straight line is invariant under the translation of coordinate axes.

EXAMPLE 6 *The line $2x - y = 5$ turns about the point on it, whose ordinate and abscissae are equal, through an angle of $45°$ in the anti-clockwise direction. Find the equation of the line in the new position.*

SOLUTION If the line $2x - y = 5$ makes an angle θ with x-axis. Then, $\tan\theta = 2$.

Let $P(\alpha, \alpha)$ be a point on the line $2x - y = 5$. Then,

$$2\alpha - \alpha = 5 \Rightarrow \alpha = 5$$

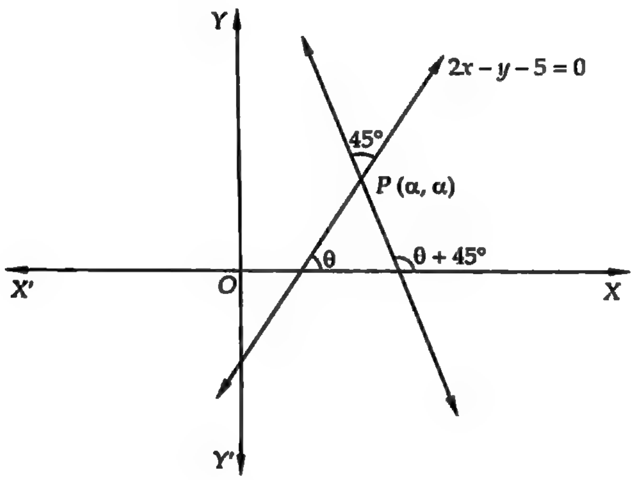


Fig. 23.60

So, the coordinates of P are $(5, 5)$.

If the line $2x - y - 5 = 0$ is rotated about point P through $45°$ in anti-clockwise direction, then the line in its new position makes angle $\theta + 45°$ with x-axis. Let m' be the slope of the line in its new position. Then,

$$m' = \tan(\theta + 45°) = \frac{\tan\theta + \tan 45°}{1 - \tan\theta \tan 45°} = \frac{2 + 1}{1 - 2 \times 1} = -3$$

Thus, the line in its new position passes through $P(5, 5)$ and has slope $m' = -3$.

So, its equation is $y - 5 = m'(x - 5)$ or, $y - 5 = -3(x - 5)$ or, $3x + y - 20 = 0$

EXAMPLE 7 *Find the coordinates of one vertex of an equilateral triangle with centroid at the origin and the opposite side $x + y - 2 = 0$.* [NCERT EXEMPLAR]

SOLUTION Let ABC be an equilateral triangle having $x + y - 2 = 0$ as the equation of side BC and opposite vertex A. Let the coordinates of its vertices be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. It is given that the centroid of ΔABC is at the origin.

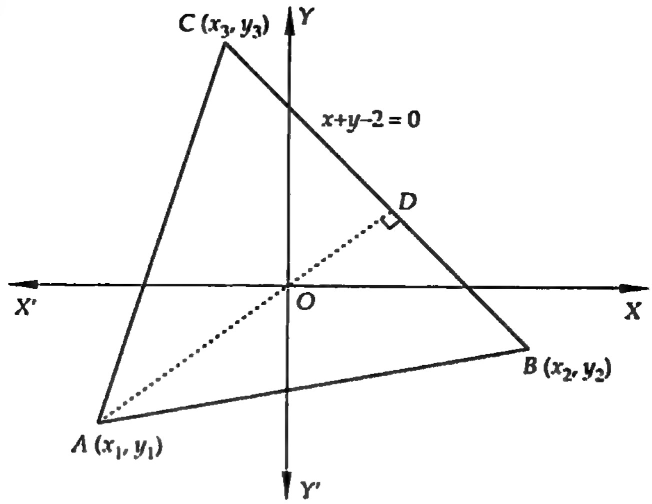


Fig. 23.61

$\therefore \quad \frac{x_1+x_2+x_3}{3}=0$ and $\frac{y_1+y_2+y_3}{3}=0$

$\Rightarrow \quad x_1+x_2+x_3=0$ and $y_1+y_2+y_3=0$

$\Rightarrow \quad x_2+x_3=-x_1$ and $y_2+y_3=-y_1$ ...(i)

It is given that ΔABC is an equilateral triangle. Therefore, median $AD \perp BC$ and so $OA \perp BC$.

$\therefore \quad$ Slope of $OA \times$ Slope of $BC=-1$

$\Rightarrow \quad \frac{y_1-0}{x_1-0} \times -1 = 1$ $\left[\text{Equation of } BC \text{ is } x+y-2=0 \therefore \text{ Slope of } BC = -\frac{1}{1}=-1\right]$

$\Rightarrow \quad y_1=x_1$ ...(ii)

Clearly, D is the mid-point of BC. So, the coordinates of D are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)=\left(-\frac{x_1}{2}, -\frac{y_1}{2}\right) \qquad \text{[Using (i)]}$$

Point D lies on BC whose equation is $x+y-2=0$.

$\therefore \quad -\frac{x_1}{2}-\frac{y_1}{2}-2=0 \Rightarrow x_1+y_1+4=0$ ...(iii)

Solving (ii) and (iii), we obtain $x_1=-2, y_1=-2$.

Hence, the coordinates of A are $(-2, -2)$.

ALITER It is given that ABC is an equilateral triangle with centroid at the origin O. Therefore, $OA \perp BC$ and so

Slope of $OA \times$ Slope of $BC=-1$

$\Rightarrow \quad$ Slope of $OA \times -1=-1$

$\Rightarrow \quad$ Slope of $OA=1$

Thus, OA makes an angle of 45° with x-axis.

So, the equation OA in distance form is

$$\frac{x-0}{\cos 45°}=\frac{y-0}{\sin 45°} \text{ or, } \frac{x-0}{1/\sqrt{2}}=\frac{y-0}{1/\sqrt{2}}$$

The equation of BC in normal form is

$$\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}=\sqrt{2}$$

$\therefore \quad OD=\sqrt{2}$

Since O is the centroid of ΔABC. Therefore, $OA=2(OD)=2\sqrt{2}$.

Thus, A is a point on OA at a distance $2\sqrt{2}$ from the origin O. So, the coordinates of A are given by

$$\frac{x-0}{1/\sqrt{2}}=\frac{y-0}{1/\sqrt{2}}=-2\sqrt{2} \Rightarrow x=-2 \text{ and } y=-2$$

Hence, the co-ordinates of A are $(-2,-2)$.

EXERCISE 23.9

LEVEL-1

1. Reduce the equation $\sqrt{3}\,x+y+2=0$ to:
   (i) slope-intercept form and find slope and y-intercept;
   (ii) intercept form and find intercept on the axes;
   (iii) the normal form and find p and α.
2. Reduce the following equations to the normal form and find p and α in each case:
   (i) $x+\sqrt{3}\,y-4=0$ (ii) $x+y+\sqrt{2}=0$ (iii) $x-y+2\sqrt{2}=0$ (iv) $x-3=0$
   (v) $y-2=0$.
3. Put the equation $\frac{x}{a}+\frac{y}{b}=1$ to the slope intercept form and find its slope and y-intercept.
4. Reduce the lines $3x-4y+4=0$ and $2x+4y-5=0$ to the normal form and hence find which line is nearer to the origin.
5. Show that the origin is equidistant from the lines $4x+3y+10=0$; $5x-12y+26=0$ and $7x+24y=50$.
6. Find the values of θ and p, if the equation $x\cos\theta+y\sin\theta=p$ is the normal form of the line $\sqrt{3}x+y+2=0$. [NCERT]
7. Reduce the equation $3x-2y+6=0$ to the intercept form and find the x and y intercepts.

LEVEL-2

8. The perpendicular distance of a line from the origin is 5 units and its slope is -1. Find the equation of the line.

ANSWERS

1. (i) Slope $=-\sqrt{3}$, y-intercept $=-2$ (ii) x-intercept $=-\frac{2}{\sqrt{3}}$, y-intercept $=-2$
   (iii) $p=1, \alpha=210°$
2. (i) $p=2, \alpha=\frac{\pi}{3}$ (ii) $p=1, \alpha=225°$ (iii) $p=2, \alpha=135°$ (iv) $p=3, \alpha=0$
   (v) $p=2, \alpha=\frac{\pi}{2}$
3. Slope $=-\frac{b}{a}$, y-intercept $=b$
4. $3x-4y+4=0$
6. $\alpha=210°, p=1$
7. x-intercept $=-2$, y-intercept $=3$
8. $x+y-5\sqrt{2}=0$

23.8 POINT OF INTERSECTION OF TWO LINES

Let the equations of two lines be

$$a_1x+b_1y+c_1=0 \quad \text{...(i)}$$

and, $$a_2x+b_2y+c_2=0 \quad \text{...(ii)}$$

Suppose these two lines intersect at a point $P(x_1, y_1)$. Then, (x_1, y_1) satisfies each of the given equations.

$\therefore \quad a_1x_1+b_1y_1+c_1=0$ and, $a_2x_1+b_2y_1+c_2=0$

Solving these two equations by cross-multiplication, we get

$$\frac{x_1}{b_1c_2-b_2c_1}=\frac{y_1}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1} \Rightarrow x_1=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, y_1=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$$

Hence, the coordinates of the point of intersection of lines (i) and (ii) are:

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right)$$

NOTE *To find the coordinates of the point of intersection of two non-parallel lines, we solve the given equations simultaneously and the values of x and y so obtained determine the coordinates of the point of intersection.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the coordinates of the point of intersection of the lines* $2x - y + 3 = 0$ *and* $x + 2y - 4 = 0$.

SOLUTION Solving the equations $2x - y + 3 = 0$ and $x + 2y - 4 = 0$, simultaneously we obtain

$$\frac{x}{4-6} = \frac{y}{3+8} = \frac{1}{4+1} \Rightarrow \frac{x}{-2} = \frac{y}{11} = \frac{1}{5} \Rightarrow x = -\frac{2}{5}, \; y = \frac{11}{5}$$

Hence, $(-2/5, 11/5$ is the required point of intersection.

EXAMPLE 2 *Find the area of the triangle formed by the lines* $y = x$, $y = 2x$ *and* $y = 3x + 4$.

SOLUTION The given equations are

$y = x$ ...(i) $y = 2x$ ...(ii) and $y = 3x + 4$ ...(iii)

Suppose the equations (i), (ii) and (iii) represent the sides AB, BC and CA respectively of a triangle ABC.

Solving (i) and (ii), we get: $x = 0$ and $y = 0$. Thus, AB and BC intersect at $B(0, 0)$.

Solving (ii) and (iii), we obtain: $x = -4$, $y = -8$. Thus, BC and CA intersect at $C(-4, -8)$.

Solving (iii) and (i), we get: $x = -2$ and $y = -2$. So, CA and AB intersect at $A(-2, -2)$.

Thus, the coordinates of the vertices of the triangle ABC are : $A(-2, -2)$, $B(0, 0)$ and $C(-4, -8)$.

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} -2 & -2 & 1 \\ 0 & 0 & 1 \\ -4 & -8 & 1 \end{vmatrix} = 4 \text{ sq. units.}$$

EXAMPLE 3 *Find the equations of the medians of a triangle formed by the lines* $x + y - 6 = 0$, $x - 3y - 2 = 0$ *and* $5x - 3y + 2 = 0$.

SOLUTION The given equations are :

$x + y - 6 = 0$ ...(i) $x - 3y - 2 = 0$ ...(ii) and $5x - 3y + 2 = 0$ ...(iii)

Suppose equations (i), (ii) and (iii) represent the sides, AB, BC and CA respectively of triangle ABC.

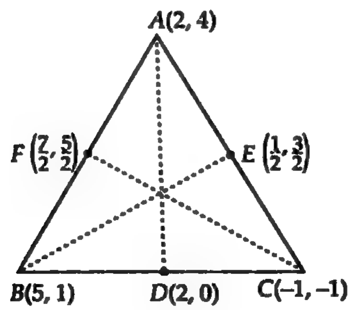


Fig. 23.62

Solving (i) and (ii), we get: $x = 5$ and $y = 1$. Thus, AB and BC intersect at $B(5, 1)$.

Solving (ii) and (iii), we get: $x = -1$ and $y = -1$. Thus, BC and CA intersect at $C(-1, -1)$.

Solving (i) and (iii), we get: $x = 2$ and $y = 4$. Thus, AB and CA intersect at $A(2, 4)$.

Thus, the coordinates of the vertices A, B and C of triangle ABC are $(2, 4)$, $(5, 1)$ and $(-1, -1)$ respectively. Let D, E and F be the mid-points of sides BC, CA and AB respectively. Then, the coordinates of D, E and F are

$$D\left(\frac{5-1}{2}, \frac{1-1}{2}\right)=(2, 0); E\left(\frac{2-1}{2}, \frac{4-1}{2}\right)=\left(\frac{1}{2}, \frac{3}{2}\right) \text{and } F\left(\frac{2+5}{2}, \frac{4+1}{2}\right)=\left(\frac{7}{2}, \frac{5}{2}\right)$$

respectively.

The median AD passes through $A(2, 4)$ and $D(2, 0)$. So, its equation is

$$y-4 = \frac{0-4}{2-2}(x-2)$$

$$\Rightarrow \quad x-2 = \frac{2-2}{0-4}(y-4) \Rightarrow x-2 = 0 \Rightarrow x = 2$$

The median BE passes through points $B(5, 1)$ and $E(1/2, 3/2)$. So, its equation is

$$y-1 = \frac{\frac{3}{2}-1}{\frac{1}{2}-5}(x-5)$$

$$\Rightarrow \quad y-1 = -\frac{1}{9}(x-5) \Rightarrow x+9y-14=0$$

The median CF passes through points $C(-1, -1)$ and $F(7/2, 5/2)$. So, its equation is

$$y+1 = \frac{\frac{5}{2}+1}{\frac{7}{2}+1}(x+1) \Rightarrow y+1 = \frac{7}{9}(x+1) \Rightarrow 7x-9y-2=0$$

Hence, the equations of the medians of the triangle are $x=2$, $x+9y-14=0$ and $7x-9y-2=0$.

EXAMPLE 4 *Find the value of m for which the lines $mx+(2m+3)y+m+6=0$ and $(2m+1)x+(m-1)y+m-9=0$ intersect at a point on y-axis.*

SOLUTION The equations of the lines are

$$mx+(2m+3)y+m+6=0 \qquad \text{...(i)}$$

$$(2m+1)x+(m-1)y+m-9=0 \qquad \text{...(ii)}$$

Solving these two equations by cross-multiplication, we obtain

$$\frac{x}{(2m+3)(m-9)-(m-1)(m+6)} = \frac{y}{(2m+1)(m+6)-m(m-9)} = \frac{1}{m(m-1)-(2m+1)(2m+3)}$$

$$\Rightarrow \quad \frac{x}{m^2-20m-21} = \frac{y}{m^2+22m+6} = \frac{1}{-3(m^2+3m+1)}$$

$$\Rightarrow \quad x = \frac{m^2-20m-21}{-3(m^2+3m+1)} \text{ and } y = \frac{m^2+22m+6}{-3(m^2+3m+1)}$$

So, given lines intersect at the point $\left(\frac{m^2-20m-21}{-3(m^2+3m+1)}, \frac{m^2+22m+6}{-3(m^2+3m+1)}\right)$.

If it lies on y-axis, then its x-coordinate is zero.

$$\therefore \quad \frac{m^2-20m-21}{-3(m^2+3m+1)} = 0 \Rightarrow m^2-20m-21=0 \Rightarrow (m-21)(m+1)=0 \Rightarrow m=-1, 21$$

EXAMPLE 5 *Find the area of the triangle formed by the lines $y=m_1x+c_1, y=m_2x+c_2$ and $x=0$.* [NCERT]

SOLUTION Let $y=m_1x+c_1, y=m_2x+c_2$ and $x=0$ be the sides AB, BC and CA respectively of a triangle ABC. Solving $y=m_1x+c_1$ and $y=m_2x+c_2$ as linear equations in x, y, we get

$$x = \frac{c_2-c_1}{m_1-m_2}, \quad y = \frac{m_1c_2-m_2c_1}{m_1-m_2}$$

So, the coordinates of B are $\left(\frac{c_2-c_1}{m_1-m_2}, \frac{m_1c_2-m_2c_1}{m_1-m_2}\right)$.

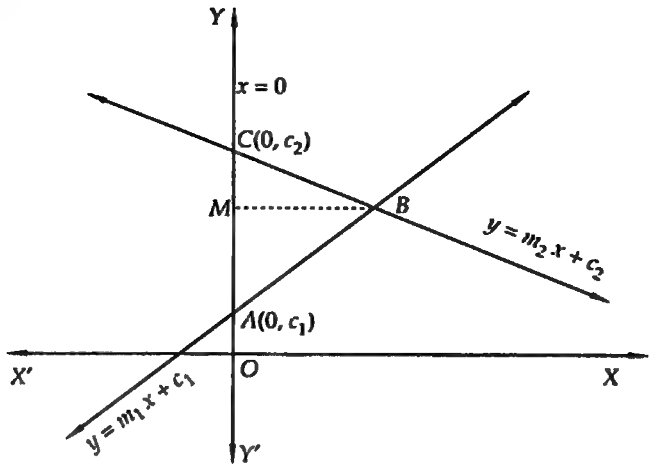


Fig. 23.63

Solving $y = m_2 x + c_2$ and $x = 0$, we get: $x = 0, y = c_2$. So, coordinates of C are $(0, c_2)$.

Similarly, by solving $x = 0$ and $y = m_1 x + c_1$, we get the coordinates of A as $(0, c_1)$.

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 0 & c_1 & 1 \\ 0 & c_2 & 1 \\ \dfrac{c_2 - c_1}{m_1 - m_2} & \dfrac{m_1 c_2 - m_2 c_1}{m_1 - m_2} & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left(\frac{c_2 - c_1}{m_1 - m_2}\right)(c_1 - c_2) = \frac{1}{2}\frac{(c_1 - c_2)^2}{m_1 - m_2} \text{ in magnitude.}$$

<u>ALITER</u> Given lines are

$y = m_1 x + c_1$ ... (i)

$y = m_2 x + c_2$ ... (ii)

$x = 0$ ... (iii)

Lines (i) and (ii) intersect line (iii) at $A\ (0, c_1)$ and $C\ (0, c_2)$ respectively.

Solving (i) and (ii), we obtain the coordinates of B as $\left(\dfrac{c_2 - c_1}{m_1 - m_2}, \dfrac{m_1 c_2 - m_2 c_1}{m_1 - m_2}\right)$.

$$\therefore \quad BM = x\text{-coordinate of } B = \frac{c_2 - c_1}{m_1 - m_2}$$

Clearly, $AC = |c_2 - c_1|$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}\text{ Base} \times \text{Height} = \frac{1}{2}(AC \times BM)$$

$$= \frac{1}{2}|c_2 - c_1| \times \left|\frac{c_2 - c_1}{m_1 - m_2}\right| = \frac{1}{2}\frac{(c_2 - c_1)^2}{|m_1 - m_2|}$$

EXAMPLE 6 *Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y = 0$.*

SOLUTION On solving the equations $x - 7y + 5 = 0$ and $3x + y = 0$, we get:

$$x = -\frac{5}{22} \text{ and } y = \frac{15}{22}.$$

So, the given lines intersect at the point whose coordinates are : $(-5/22, 15/22)$.

We know that, the equation of a line parallel to y-axis is of the form x=constant. So, let the equation of the required line be

$x = \lambda$ ...(i)

It passes through $(-5/22, 15/22)$.

$\therefore \quad \dfrac{-5}{22} = \lambda$

Substituting the value of λ in (i), we get:

$x = -5/22$ or, $22x + 5 = 0$ as the equation of the required line.

LEVEL-2

EXAMPLE 7 *Show that the lines $4x + y - 9 = 0, x - 2y + 3 = 0, 5x - y - 6 = 0$ make equal intercepts on any line of gradient 2.*

SOLUTION The equation of any line of gradient 2 is

$$y = 2x + c \quad \text{...(i)}$$

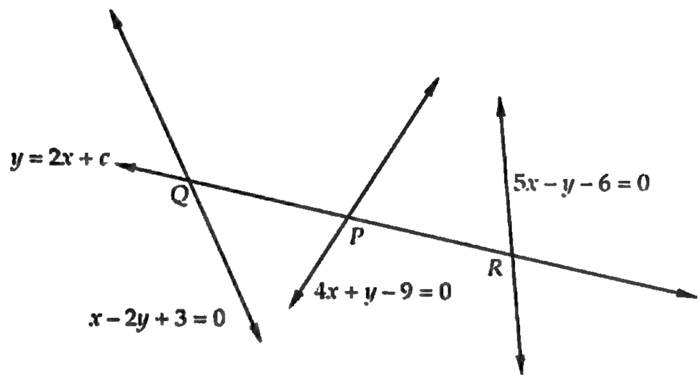


Fig. 23.64

The equations of given lines are

$$4x + y - 9 = 0 \quad \text{...(ii)}$$
$$x - 2y + 3 = 0 \quad \text{...(iii)}$$
$$5x - y - 6 = 0 \quad \text{...(iv)}$$

Solving (i) with (ii), (iii) and (iv) respectively, we obtain the coordinates of P, Q and R as

$$P\left(\frac{3}{2} - \frac{c}{6}, 3 + \frac{2c}{3}\right), Q\left(1 - \frac{2c}{3}, 2 - \frac{c}{3}\right) \text{and } R\left(2 + \frac{c}{3}, 4 + \frac{5c}{3}\right)$$

Clearly, P is the mid-point of QR. Therefore $PQ = PR$.

Hence, lines (ii), (iii) and (iv) make equal intercepts on any line of gradient 2.

EXAMPLE 8 *Two vertices of a triangle are $(3, -1)$ and $(-2, 3)$ and its orthocentre is at the origin. Find the coordinates of the third vertex.*

SOLUTION Clearly, A is the intersection of sides AB and AC of ΔABC. Side AB passes through $B(3, -1)$ and is perpendicular to OC whose slope is $-3/2$. So, equation of side AB is

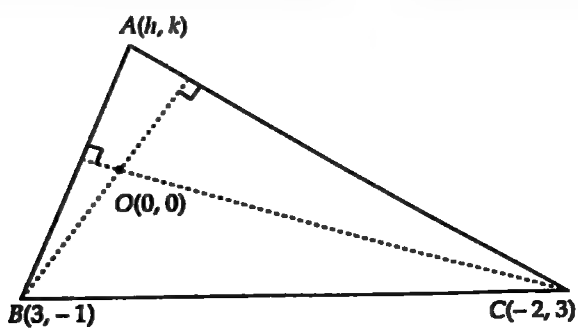


Fig. 23.65

$$y+1 = \frac{2}{3}(x-3) \text{ or, } 2x-3y-9 = 0 \quad \text{...(i)}$$

Similarly, side AC passes through $C(-2, 3)$ and is perpendicular to OB whose slope is $-1/3$. So, equation of side AC is

$$y-3 = 3(x+2) \text{ or, } 3x-y+9=0 \quad \text{...(ii)}$$

Solving (i) and (ii), we get

$$x = -36/7,\ y = -45/7$$

Hence, coordinates of A are $(-36/7, -45/7)$

EXAMPLE 9 *Two consecutive sides of a parallelogram are $4x+5y=0$ and $7x+2y=0$. If the equation of one diagonal is $11x+7y=9$, find the equation of the other diagonal.*

SOLUTION Let AB and AD be consecutive sides of parallelogram $ABCD$. Let the equations of AB and AD be $4x+5y=0$ and $7x+2y=0$ respectively. Clearly, these two lines intersect at $A(0, 0)$.

Solving $11x+7y = 9$ and $4x+5y = 0$, we get: $x=5/3$ and $y=-4/3$

So, the coordinates of B are $(5/3, -4/3)$.

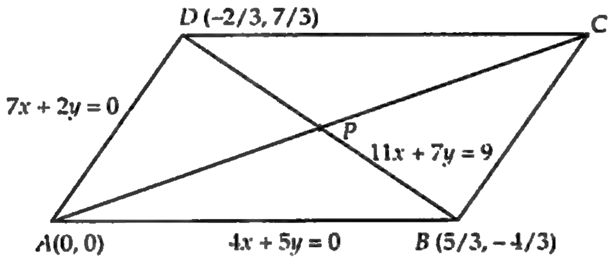


Fig. 23.66

Similarly, by solving $11x+7y = 9$ and $7x+2y = 0$, we obtain that the coordinates of D are $(-2/3, 7/3)$.

We know that the diagonals of a parallelogram bisect each other. So, P is the mid-point of BD and hence its coordinates are $\left(\frac{\frac{5}{3}-\frac{2}{3}}{2}, \frac{-\frac{4}{3}+\frac{7}{3}}{2}\right)$ or, $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Clearly, AC passes through $A(0, 0)$ and $C(1/2, 1/2)$.

Hence, equation of AC is $y-0=\frac{\frac{1}{2}-0}{\frac{1}{2}-0}(x-0)$ or, $y=x$.

EXERCISE 23.10

LEVEL-1

1. Find the point of intersection of the following pairs of lines :
   (i) $2x-y+3=0$ and $x+y-5=0$ (ii) $bx+ay=ab$ and $ax+by=ab$.
   (iii) $y=m_1x+\frac{a}{m_1}$ and $y=m_2x+\frac{a}{m_2}$.
2. Find the coordinates of the vertices of a triangle, the equations of whose sides are:
   (i) $x+y-4=0, 2x-y+3=0$ and $x-3y+2=0$
   (ii) $y(t_1+t_2)=2x+2at_1t_2$, $y(t_2+t_3)=2x+2at_2t_3$ and, $y(t_3+t_1)=2x+2at_1t_3$.

3. Find the area of the triangle formed by the lines
   (i) $y = m_1 x + c_1$, $y = m_2 x + c_2$ and $x = 0$
   (ii) $y = 0$, $x = 2$ and $x + 2y = 3$.
   (iii) $x + y - 6 = 0$, $x - 3y - 2 = 0$ and $5x - 3y + 2 = 0$
4. Find the equations of the medians of a triangle, the equations of whose sides are:
$$3x + 2y + 6 = 0, 2x - 5y + 4 = 0 \text{ and } x - 3y - 6 = 0$$
5. Prove that the lines $y = \sqrt{3}x + 1$, $y = 4$ and $y = -\sqrt{3}x + 2$ form an equilateral triangle.
6. Classify the following pairs of lines as coincident, parallel or intersecting:
   (i) $2x + y - 1 = 0$ and $3x + 2y + 5 = 0$ (ii) $x - y = 0$ and $3x - 3y + 5 = 0$
   (iii) $3x + 2y - 4 = 0$ and $6x + 4y - 8 = 0$.
7. Find the equation of the line joining the point (3, 5) to the point of intersection of the lines $4x + y - 1 = 0$ and $7x - 3y - 35 = 0$.
8. Find the equation of the line passing through the point of intersection of the lines $4x - 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes. [NCERT]

LEVEL-2

9. Show that the area of the triangle formed by the lines $y = m_1 x$, $y = m_2 x$ and $y = c$ is equal to $\frac{c^2}{4}(\sqrt{33} + \sqrt{11})$, where m_1, m_2 are the roots of the equation $x^2 + (\sqrt{3} + 2)x + \sqrt{3} - 1 = 0$.
10. If the straight line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the point of intersection of the lines $x + y = 3$ and $2x - 3y = 1$ and is parallel to $x - y - 6 = 0$, find a and b.
11. Find the orthocentre of the triangle the equations of whose sides are $x + y = 1$, $2x + 3y = 6$ and $4x - y + 4 = 0$.
12. Three sides AB, BC and CA of a triangle ABC are $5x - 3y + 2 = 0$, $x - 3y - 2 = 0$ and $x + y - 6 = 0$ respectively. Find the equation of the altitude through the vertex A.
13. Find the coordinates of the orthocentre of the triangle whose vertices are $(-1, 3)$, $(2, -1)$ and $(0, 0)$.
14. Find the coordinates of the incentre and centroid of the triangle whose sides have the equations $3x - 4y = 0$, $12y + 5x = 0$ and $y - 15 = 0$.
15. Prove that the lines $\sqrt{3}x + y = 0$, $\sqrt{3}y + x = 0$, $\sqrt{3}x + y = 1$ and $\sqrt{3}y + x = 1$ form a rhombus.
16. Find the equation of the line passing through the intersection of the lines $2x + y = 5$ and $x + 3y + 8 = 0$ and parallel to the line $3x + 4y = 7$.
17. Find the equation of the straight line passing through the point of intersection of the lines $5x - 6y - 1 = 0$ and $3x + 2y + 5 = 0$ and perpendicular to the line $3x - 5y + 11 = 0$.
[NCERT EXEMPLAR]

ANSWERS

1. (i) $\left(\frac{2}{3}, \frac{13}{3}\right)$ (ii) $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$ (iii) $\left(\frac{a}{m_1 m_2}, a\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right)$
2. (i) $\left(\frac{1}{3}, \frac{11}{3}\right), \left(-\frac{7}{5}, \frac{1}{5}\right), \left(\frac{5}{2}, \frac{3}{2}\right)$ (ii) $(at_1^2, 2at_1), (at_2^2, 2at_2), (at_3^2, 2at_3)$
3. (i) $\frac{(c_1 - c_2)^2}{2(m_1 - m_2)}$ (ii) 0 (iii) 12. sq. units
4. $41x - 112y - 70 = 0$, $16x - 59y - 120 = 0$ and $25x - 53y + 50 = 0$

6. (i) intersecting (ii) parallel (iii) coincident

7. $12x - y - 31 = 0$ 8. $x + y + 13 = 0$ 10. $a = 1, b = -1$

11. $\left(\frac{3}{7}, \frac{22}{7}\right)$ 12. $3x + y - 10 = 0$ 13. $(-4, -3)$ 14. $(-1, 8), \left(-\frac{16}{3}, 10\right)$

16. $3x + 4y + 3 = 0$ 17. $5x + 3y + 8 = 0$

HINTS TO NCERT & SELECTED PROBLEM

8. Given lines intersect at $(-8, -5)$. The equation of a line making equal intercepts on the coordinates axes is $\frac{x}{a} + \frac{y}{a} = 1$ or, $x + y = a$. It passes through $(-8, -5)$.

$\therefore \quad -8 - 5 + a = 0 \Rightarrow a = 13$

Hence, the equation of the line is $x + y = 13$.

17. The lines $5x - 6y - 1 = 0$ and $3x + 2y + 5 = 0$ intersect at the point $(-1, -1)$. The slope of the line $3x - 5y + 11 = 0$ is $3/5$. So, the slope of a line perpendicular to it is $-5/3$. Hence, the equation of the required line is

$y + 1 = -5/3\,(x + 1)$ or, $5x + 3y + 8 = 0$.

23.9 CONDITION OF CONCURRENCY OF THREE LINES

Three lines are said to be concurrent if they pass through a common point i.e. they meet at a point.

Thus, if three lines are concurrent the point of intersection of two lines lies on the third line. Let

$$a_1 x_1 + b_1 y + c_1 = 0 \quad ...(i) \qquad a_2 x + b_2 y + c_2 = 0 \quad ...(ii) \qquad a_3 x + b_3 y + c_3 = 0 \quad ...(iii)$$

be three concurrent lines. Then the point of intersection of (i) and (ii) must lie on the third.

The coordinates of the point of intersection of (i) and (ii) are:

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right)$$

[See section 22.8]

This point must lies on line (iii).

$$\therefore \qquad a_3\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}\right) + b_3\left(\frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right) + c_3 = 0$$

$$\Rightarrow \qquad a_3 (b_1 c_2 - b_2 c_1) + b_3 (c_1 a_2 - c_2 a_1) + c_3 (a_1 b_2 - a_2 b_1) = 0$$

$$\Rightarrow \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrency of three lines.

ANOTHER CONDITION OF CONCURRENCY OF THREE LINES

Three lines

$$L_1 \equiv a_1 x + b_1 y + c_1 = 0; \quad L_2 \equiv a_2 x + b_2 y + c_2 = 0; \quad L_3 \equiv a_3 x + b_3 y + c_3 = 0$$

are concurrent iff there exist constants $\lambda_1, \lambda_2, \lambda_3$ not all zero such that

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$$

i.e. $$\lambda_1 (a_1 x + b_1 y + c_1) + \lambda_2 (a_2 x + b_2 y + c_2) + \lambda_3 (a_3 x + b_3 y + c_3) = 0.$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove that the lines $3x + y - 14 = 0$, $x - 2y = 0$ and $3x - 8y + 4 = 0$ are concurrent.*

SOLUTION Given lines are $3x + y - 14 = 0$, $x - 2y + 0 = 0$ and $3x - 8y + 4 = 0$.

We have, $\begin{vmatrix} 3 & 1 & -14 \\ 1 & -2 & 0 \\ 3 & -8 & 4 \end{vmatrix} = 3(-8+0) - 1(4-0) - 14(-8+6) = -24 - 4 + 28 = 0.$

So, the given lines are concurrent.

EXAMPLE 2 *Show that the lines $x - y - 6 = 0$, $4x - 3y - 20 = 0$ and $6x + 5y + 8 = 0$ are concurrent. Also, find their common point of intersection.*

SOLUTION The given lines are

$$x - y - 6 = 0 \quad ...(i) \qquad 4x - 3y - 20 = 0 \quad ...(ii) \qquad 6x + 5y + 8 = 0 \quad ...(iii)$$

Solving (i) and (ii) by cross-multiplication, we get

$$\frac{x}{20-18} = \frac{y}{-24+20} = \frac{1}{-3+4} \Rightarrow x = 2 \text{ and } y = -4.$$

Thus, the first two lines intersect at the point $(2, -4)$. Putting $x = 2$ and $y = -4$ in (iii), we get

$$6 \times 2 + 5 \times -4 + 8 = 0$$

Thus, the point $(2, -4)$ lies on line (iii).

Hence, the given lines are concurrent and their common point of intersection is $(2, -4)$.

EXAMPLE 3 *Find the value of λ, if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33 = 0$ and $2x - 3y + \lambda = 0$ are concurrent.*

SOLUTION The given lines are concurrent, if

$$\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 3(-11\lambda - 99) + 4(8\lambda + 66) - 13(-24 + 22) = 0 \Rightarrow -\lambda - 7 = 0 \Rightarrow \lambda = -7$

<u>ALITER</u> The given equations are

$$3x - 4y - 13 = 0 \quad ...(i) \qquad 8x - 11y - 33 = 0 \quad ...(ii) \text{ and, } 2x - 3y + \lambda = 0 \quad ...(iii)$$

Solving equations (i) and (ii), we get $x = 11$ and $y = 5$. Thus, $(11, 5)$ is the point of intersection of lines (i) and (ii). The given lines will be concurrent if they pass through the common point i.e. the point of intersection of any two lies on the third. Therefore, the point $(11, 5)$ must lie on the line (iii).

$\therefore \quad 2 \times 11 - 3 \times 5 + \lambda = 0 \Rightarrow \lambda = -7.$

EXAMPLE 4 *If the lines $a_1 x + b_1 y + 1 = 0$, $a_2 x + b_2 y + 1 = 0$ and $a_3 x + b_3 y + 1 = 0$ are concurrent, show that the points (a_1, b_1), (a_2, b_2) and (a_3, b_3) are collinear.*

SOLUTION The given lines are

$$a_1 x + b_1 y + 1 = 0 \quad ...(i) \quad a_2 x + b_2 y + 1 = 0 \quad ...(ii) \text{ and } a_3 x + b_3 y + 1 = 0 \quad ...(iii).$$

If these lines are concurrent, we must have

$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$, which is the condition of collinearity of three points (a_1, b_1), (a_2, b_2) and (a_3, b_3).

Hence, if the given lines are concurrent, the given points are collinear.

LEVEL-2

EXAMPLE 5 *If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ are concurrent $(a \neq b \neq c \neq 1)$, prove that* $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$.

SOLUTION The equations of the given lines are

$$ax + y + 1 = 0$$
$$x + by + 1 = 0$$
$$x + y + c = 0$$

If these lines are concurrent, then

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$$\Rightarrow \quad a\begin{vmatrix} b & 1 \\ 1 & c \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & c \end{vmatrix} + 1\begin{vmatrix} 1 & b \\ 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad a(bc-1) - (c-1) + (1-b) = 0 \Rightarrow abc - a - c + 1 + 1 - b = 0 \Rightarrow abc = a + b + c - 2 \qquad \text{...(i)}$$

Now, $$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = \frac{(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b)}{(1-a)(1-b)(1-c)}$$

$$= \frac{3 - 2(a+b+c) + ab + bc + ca}{1 - (a+b+c) + ab + bc + ca - abc}$$

$$= \frac{3 - 2(a+b+c) + ab + bc + ca}{1 - (a+b+c) + (ab+bc+ca) - (a+b+c-2)} \qquad \text{[Using (i)]}$$

$$= \frac{3 - 2(a+b+c) + ab + bc + ca}{3 - 2(a+b+c) + ab + bc + ca} = 1$$

EXAMPLE 6 *Show that the following lines are concurrent:*

$$L_1 = (a-b)x + (b-c)y + (c-a) = 0$$
$$L_2 = (b-c)x + (c-a)y + (a-b) = 0$$

and, $$L_3 = (c-a)x + (a-b)y + (b-c) = 0.$$

SOLUTION Clearly,

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0, \text{ where } \lambda_1 = \lambda_2 = \lambda_3 = 1$$

Hence, the given lines are concurrent.

EXAMPLE 7 *Show that the altitudes of a triangle are concurrent.*

SOLUTION Let ABC be a triangle such that the coordinates of its vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Let AD, BE and CF be the altitudes drawn from the vertices A, B, C respectively to the opposite sides BC, CA and, AB respectively. Then,

Slope of $BC = \frac{y_2 - y_3}{x_2 - x_3}$, Slope of $CA = \frac{y_3 - y_1}{x_3 - x_1}$ and, Slope of $AB = \frac{y_2 - y_1}{x_2 - x_1}$

Since $AD \perp BC$, $BE \perp CA$ and $CF \perp AB$. Therefore,

Slope of $AD = -\frac{x_2 - x_3}{y_2 - y_3}$, Slope of $BE = -\frac{x_3 - x_1}{y_3 - y_1}$ and, Slope of $CF = -\frac{x_2 - x_1}{y_2 - y_1}$.

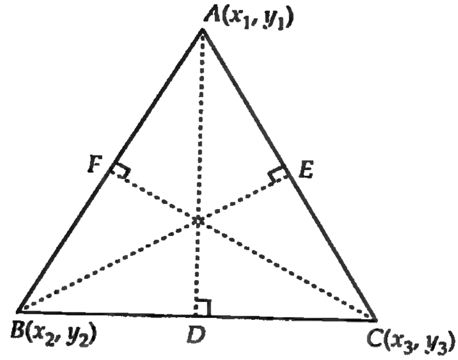


Fig. 23.67

The equation of altitude AD is

$$y - y_1 = -\frac{x_2 - x_3}{y_2 - y_3}(x - x_1)$$

i.e. $\quad L_1 \equiv x(x_2 - x_3) + y(y_2 - y_3) - x_1(x_2 - x_3) - y_1(y_2 - y_3) = 0 \quad$ ...(i)

Similarly, equations of altitudes BE and CF are

$$L_2 \equiv x(x_3 - x_1) + y(y_3 - y_1) - x_2(x_3 - x_1) - y_2(y_3 - y_1) = 0 \quad \text{...(ii)}$$

and, $\quad L_3 \equiv x(x_1 - x_2) + y(y_1 - y_2) - x_3(x_1 - x_2) - y_3(y_1 - y_2) = 0 \quad$ ...(iii)

Clearly, $1 \cdot L_1 + 1 \cdot L_2 + 1 \cdot L_3 = 0$.

Hence, the altitudes AD, BE and CF are concurrent.

EXAMPLE 8 *Prove analytically that the medians of a triangle are concurrent.*

SOLUTION Let ABC be a triangle the coordinates of whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Let D, E and F be the mid-points of sides BC, CA and AB respectively. The coordinates of D, E and F are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$, $E\left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2}\right)$ and $F\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ respectively.

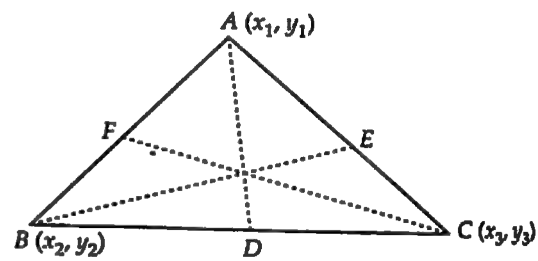


Fig. 23.68

Equation of median AD is

$$y - y_1 = \frac{y_1 - \frac{y_2 + y_3}{2}}{x_1 - \frac{x_2 + x_3}{2}}(x - x_1)$$

or, $$y - y_1 = \frac{2y_1 - y_2 - y_3}{2x_1 - x_2 - x_3}(x - x_1)$$

or, $\quad (2y_1 - y_2 - y_3)x - (2x_1 - x_2 - x_3)y - x_1(2y_1 - y_2 - y_3) + y_1(2x_1 - x_2 - x_3) = 0$

or, $\quad L_1 \equiv (2y_1 - y_2 - y_3)x - (2x_1 - x_2 - x_3)y + x_1(y_2 + y_3) - y_1(x_2 + x_3) = 0 \quad$ ...(i)

Similarly, equations of medians BE and CF are respectively

$$L_2 \equiv (2y_2 - y_1 - y_3)x - (2x_2 - x_1 - x_3)y + x_2(y_1 + y_3) - y_2(x_1 + x_3) = 0 \quad \text{...(ii)}$$

$$L_3 \equiv (2y_3 - y_1 - y_2)x - (2x_3 - x_1 - x_2)y + x_3(y_1 + y_2) - y_3(x_1 + x_2) = 0 \quad \text{...(iii)}$$

We observe that

$$1 \cdot L_1 + 1 \cdot L_2 + 1 \cdot L_3 = 0 \quad \text{(identically)}$$

Hence, medians AD, BE and CF are concurrent.

EXERCISE 23.11

LEVEL-1

1. Prove that the following sets of three lines are concurrent:
   (i) $15x - 18y + 1 = 0, 12x + 10y - 3 = 0$ and $6x + 66y - 11 = 0$
   (ii) $3x - 5y - 11 = 0, 5x + 3y - 7 = 0$ and $x + 2y = 0$
   (iii) $\frac{x}{a} + \frac{y}{b} = 1, \frac{x}{b} + \frac{y}{a} = 1$ and $y = x$.
2. For what value of λ are the three lines $2x - 5y + 3 = 0, 5x - 9y + \lambda = 0$ and $x - 2y + 1 = 0$ concurrent?
3. Find the conditions that the straight lines $y = m_1 x + c_1, y = m_2 x + c_2$ and $y = m_3 x + c_3$ may meet in a point. [NCERT]
4. If the lines $p_1 x + q_1 y = 1, p_2 x + q_2 y = 1$ and $p_3 x + q_3 y = 1$ be concurrent, show that the points $(p_1, q_1), (p_2, q_2)$ and (p_3, q_3) are collinear.

LEVEL-2

5. Show that the straight lines $L_1 = (b + c)x + ay + 1 = 0$, $L_2 = (c + a)x + by + 1 = 0$ and $L_3 = (a + b)x + cy + 1 = 0$ are concurrent.
6. If the three lines $ax + a^2 y + 1 = 0, bx + b^2 y + 1 = 0$ and $cx + c^2 y + 1 = 0$ are concurrent, show that at least two of three constants a, b, c are equal.
7. If a, b, c are in A.P., prove that the straight lines $ax + 2y + 1 = 0, bx + 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent.
8. Show that the perpendicular bisectors of the sides of a triangle are concurrent.

ANSWERS

2. $\lambda = 4$ 3. $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$

HINTS TO NCERT & SELECTED PROBLEM

3. If the lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ and $y = m_3 x + c_3$ are concurrent, then

$$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} m_1 & c_1 & 1 \\ m_2 & c_2 & 1 \\ m_3 & c_3 & 1 \end{vmatrix} = 0 \qquad \text{[Interchanging second and third column]}$$

$$\Rightarrow \quad m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$$

23.10 LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE

LINE PARALLEL TO A GIVEN LINE

THEOREM 1 *Prove that the equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$,* where λ is a constant.

PROOF Let m be the slope of the line $ax + by + c = 0$. Then,

$$m = -\frac{a}{b} \qquad \left[\text{Using : } m = -\frac{\text{Coeff. of } x}{\text{Coeff. of } y}\right]$$

The required line is parallel to the given line. So, the slope of the required line is also m. Let c_1 be the y-intercept of the required line. Then, its equation is

$$y = mx + c_1$$

$$\Rightarrow \quad y = -\frac{a}{b}x + c_1$$

$$\Rightarrow \quad ax + by - bc_1 = 0$$

$$\Rightarrow \quad ax + by + \lambda = 0, \quad \text{where } \lambda = -bc_1 = \text{constant.}$$

Q.E.D.

NOTE *To write a line parallel to a given line we keep the expression containing x and y same and simply replace the given constant by an unknown constant λ. The value of λ can be determined by some given condition.*

LINE PERPENDICULAR TO A GIVEN LINE

THEOREM 2 *Prove that the equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is a constant.*

PROOF Let m_1 be the slope of the given line and m_2 be the slope of a line perpendicular to the given line. Then, $m_1 = -\frac{a}{b}$. As the lines are perpendicular.

$$\therefore \quad m_1 m_2 = -1 \Rightarrow m_2 = -\frac{1}{m_1} = \frac{b}{a}$$

Let c_2 be the y-intercept of the required line. Then, its equation is

$$y = m_2 x + c_2$$

$$\Rightarrow \quad y = \frac{b}{a}x + c_2$$

$$\Rightarrow \quad bx - ay + ac_2 = 0$$

$$\Rightarrow \quad bx - ay + \lambda = 0, \quad \text{where } \lambda = ac_2 = \text{constant.}$$

Q.E.D.

To write a line perpendicular to a given line we may use the following algorithm.

ALGORITHM

STEP I *Interchange x and y.*

STEP II *If the coefficients of x and y in the given equation are of the same sign make them of opposite signs and if the coefficients are of opposite signs make them of the same sign.*

STEP III *Replace the given constant by a new constant λ which is determined by a given condition.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the line which is parallel to $3x - 2y + 5 = 0$ and passes through the point $(5, -6)$.*

SOLUTION The equation of any line parallel to the line $3x - 2y + 5 = 0$ is

$$3x - 2y + \lambda = 0 \qquad \text{...(i)}$$

This passes through $(5, -6)$.

$$\therefore \quad 3 \times 5 - 2 \times -6 + \lambda = 0 \Rightarrow \lambda = -27.$$

Putting $\lambda = -27$ in (i), we obtain $3x - 2y - 27 = 0$ as the required equation.

ALITER The slope of the given line is $3/2$. Therefore, the slope of the required line is also $3/2$. Since the required line passes through $(5, -6)$, so its equation is

$$y + 6 = \frac{3}{2}(x - 5) \quad \text{or,} \quad 3x - 2y - 27 = 0 \qquad [\text{Using: } y - y_1 = m(x - x_1)]$$

EXAMPLE 2 *Find the equation of the straight line that passes through the point (3, 4) and perpendicular to the line $3x + 2y + 5 = 0$.*

SOLUTION The equation of a line perpendicular to $3x + 2y + 5 = 0$ is

$$2x - 3y + \lambda = 0 \quad \text{...(i)}$$

This passes through the point (3, 4).

$\therefore \quad 3 \times 2 - 3 \times 4 + \lambda = 0 \Rightarrow \lambda = 6$

Putting $\lambda = 6$ in (i), we obtain $2x - 3y + 6 = 0$ as the required equation.

ALITER The slope of the given line is $-3/2$. Since the required line is perpendicular to the given line. So, the slope of the required line is $2/3$. As it passes through (3, 4). So, its equation is

$$y - 4 = \frac{2}{3}(x - 3) \text{ or, } 2x - 3y + 6 = 0 \qquad [\text{Using: } y - y_1 = m(x - x_1)].$$

EXAMPLE 3 *Find the equation of the line perpendicular to $x - 7y + 5 = 0$ and having x- intercept 3.*

SOLUTION The equation of a line perpendicular to $x - 7y + 5 = 0$ is

$$7x + y + \lambda = 0 \quad \text{...(i)}$$

Its x-intercept is 3. This means that the line cuts x-axis at a distance of 3 units from the origin. Consequently, it passes through the point (3, 0) on x-axis.

$\therefore \quad 21 + 0 + \lambda = 0 \Rightarrow \lambda = -21$

Putting $\lambda = -21$ in (i), we obtain $7x + y - 21 = 0$ as the equation of the required line.

EXAMPLE 4 *Find the coordinates of the foot of the perpendicular drawn from the point $(1, -2)$ on the line $y = 2x + 1$.*

SOLUTION Let M be the foot of the perpendicular drawn from $P(1, -2)$ on the line $y = 2x + 1$. Then, M is the point of intersection of $y = 2x + 1$ and a line passing through $P(1, -2)$ and perpendicular to $y = 2x + 1$. The equation of a line perpendicular to $y = 2x + 1$ or, $2x - y + 1 = 0$ is

$$x + 2y + \lambda = 0 \quad \text{...(i)}$$

This passes through $P(1, -2)$.

$\therefore \quad 1 - 4 + \lambda = 0 \Rightarrow \lambda = 3$

Putting $\lambda = 3$ in (i), we get

$$x + 2y + 3 = 0$$

Point M is the point of intersection of the lines

$$2x - y + 1 = 0 \text{ and } x + 2y + 3 = 0.$$

Solving these equations by cross-multiplication, we get

$$\frac{x}{-5} = \frac{y}{-5} = \frac{1}{5} \Rightarrow x = -1, \ y = -1.$$

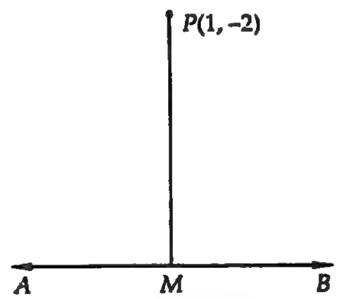


Fig. 23.69

Hence, the coordinates of the foot of the perpendicular are $(-1, -1)$.

EXAMPLE 5 *Find the equation of a straight line parallel to $2x + 3y + 11 = 0$ and which is such that the sum of its intercepts on the axes is 15.*

SOLUTION The equation of a line parallel to $2x + 3y + 11 = 0$ is

$$2x + 3y + \lambda = 0, \ \lambda \text{ is a constant} \quad \text{...(i)}$$

To find x-intercept of this line, we put $y = 0$ in its equation. Putting $y = 0$ in (i), we get

$\Rightarrow \quad 2x + \lambda = 0 \Rightarrow x = -\lambda/2$

So, x-intercept $= -\lambda/2$.

To find y-intercept of this line, we put $x = 0$ in its equation. Putting $x = 0$ in (i), we get

$$3y + \lambda = 0 \Rightarrow y = -\lambda/3$$

So, y-intercept $= -\lambda/3$.

It is given that the sum of the intercepts of the line (i) on the coordinate axes is 15.

$$\therefore \quad \left(-\frac{\lambda}{2}\right) + \left(-\frac{\lambda}{3}\right) = 15 \Rightarrow -\frac{5\lambda}{6} = 15 \Rightarrow \lambda = -18$$

Putting $\lambda = -18$ in (i), we get: $2x + 3y - 18 = 0$.

Hence, the equation of the required line is $2x + 3y - 18 = 0$.

EXAMPLE 6 *Show that the equation of a line passing through $(a \cos^3 \theta, a \sin^3 \theta)$ and perpendicular to the line $x \sec \theta + y \operatorname{cosec} \theta = a$ is $x \cos \theta - y \sin \theta = a \cos 2\theta$.*

SOLUTION The equation of a line perpendicular to the line $x \sec \theta + y \operatorname{cosec} \theta = a$ is

$$x \operatorname{cosec} \theta - y \sec \theta + \lambda = 0, \; \lambda \text{ is a constant} \quad \text{...(i)}$$

This line passes through $(a \cos^3 \theta, a \sin^3 \theta)$.

$$\therefore \quad a \cos^3 \theta \operatorname{cosec} \theta - a \sin^3 \theta \sec \theta + \lambda = 0$$

$$\Rightarrow \quad \lambda = a (\sin^3 \theta \sec \theta - \cos^3 \theta \operatorname{cosec} \theta)$$

Putting the value of λ in (i), we get

$$x \operatorname{cosec} \theta - y \sec \theta + a (\sin^3 \theta \sec \theta - \cos^3 \theta \operatorname{cosec} \theta) = 0$$

$$\Rightarrow \quad \frac{x}{\sin \theta} - \frac{y}{\cos \theta} + a\left(\frac{\sin^3 \theta}{\cos \theta} - \frac{\cos^3 \theta}{\sin \theta}\right) = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta + a (\sin^4 \theta - \cos^4 \theta) = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta + a (\sin^2 \theta + \cos^2 \theta)(\sin^2 \theta - \cos^2 \theta) = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta - a (\cos^2 \theta - \sin^2 \theta) = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta - a \cos 2\theta = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta = a \cos 2\theta.$$

EXAMPLE 7 *Find the image of the point $(-8, 12)$ with respect to the line mirror $4x + 7y + 13 = 0$.*

SOLUTION Let the image of the point $P(-8, 12)$ in the line mirror AB be $Q(\alpha, \beta)$. Then, the line segement PQ is perpendicularly bisected at R. So, the coordinates of R are

$$\left(\frac{\alpha - 8}{2}, \frac{\beta + 12}{2}\right).$$

As it lies on $4x + 7y + 13 = 0$.

$$\therefore \quad 2\alpha - 16 + \frac{7\beta + 84}{2} + 13 = 0 \Rightarrow 4\alpha + 7\beta + 78 = 0 \quad \text{...(i)}$$

The line segment PQ is perpendicular to AB.

$$\therefore \quad (\text{Slope of } AB) \times (\text{Slope of } PQ) = -1$$

$$\Rightarrow \quad -\frac{4}{7} \times \frac{\beta - 12}{\alpha + 8} = -1$$

$$\Rightarrow \quad 7\alpha - 4\beta + 104 = 0 \quad \text{...(ii)}$$

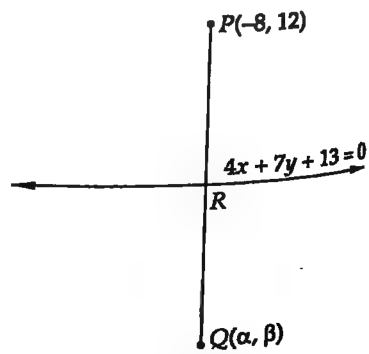


Fig. 23.70

Solving (i) and (ii), we get: $\alpha = -16, \beta = -2$.

Hence, the image of $(-8, 12)$ in the line mirror $4x + 7y + 13 = 0$ is $(-16, -2)$.

LEVEL-2

EXAMPLE 8 *A person stranding at a junction (crossing) of two straight paths represented by the equations $2x-3y-4=0$ and $3x-4y-5=0$, wants to reach the path whose equation is $6x-7y+8=0$ in the least time. Find the equation of the path that he should follow.* **[NCERT]**

SOLUTION The lines $2x-3y-4=0$ and $3x-4y-5=0$ intersect at $(-1,-2)$. In order to reach the path, represented by the equation $6x-7y+8=0$, in the least time, the person should move along the line passing through A and perpendicular to $6x-7y+8=0$.

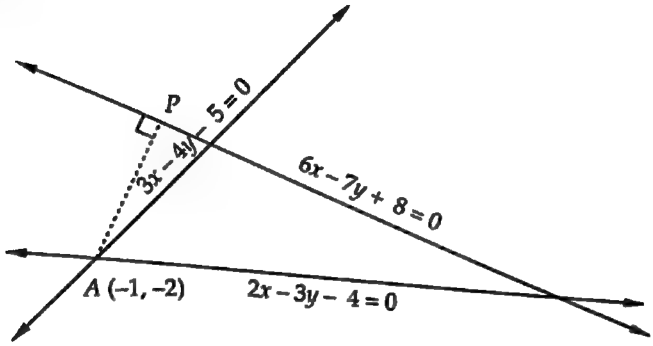


Fig. 23.71

Clearly, slope of the line $6x-7y+8=0$ is $\frac{6}{7}$. Therefore, slope of a line perpendicular to it is $-\frac{7}{6}$.

Hence, the equation of the required path is

$$y+2 = -\frac{7}{6}(x+1) \text{ or, } 7x+6y+19 = 0$$

EXAMPLE 9 *The equations of two sides of a triangle are $3x-2y+6 = 0$ and $4x+5y-20 = 0$ and the orthocentre is $(1,1)$. Find the equation of the third side.* **[NCERT]**

SOLUTION Let the equations of sides AB and AC of triangle ABC be respectively

$$3x-2y+6 = 0 \quad ...(i)$$

and, $$4x+5y-20 = 0 \quad ...(ii)$$

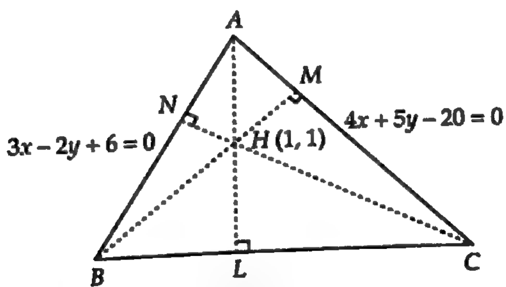


Fig. 23.72

Let $H(1,1)$ be the orthocentre of triangle ABC where the altitudes AL, BM and CN intersect each other. Clearly, BM passes through $H(1,1)$ and is perpendicular to AC. The equation of a line perpendicular to AC is

$$5x-4y+\lambda = 0 \quad ...(iii)$$

If it passes through the orthocentre $H(1,1)$, then

$$5-4+\lambda = 0 \Rightarrow \lambda = -1$$

Substituting $\lambda = -1$ in (iii), we get

$$5x-4y-1 = 0 \quad ...(iv)$$

This is the equation of altitude BM.

The vertex B of ΔABC is the intersection point of side AB and altitude BM. Solving their equations given by (i) and (iii), we get $x = -13$ and $y = -33/2$.

So, coordinates of B are $(-13, -33/2)$.

The altitude CN is perpendicular to AB. So, let its equation be

$$2x + 3y + \mu = 0 \quad \text{...(v)}$$

If it passes through the orthocentre $H(1, 1)$, then

$$2 + 3 + \mu = 0 \Rightarrow \mu = -5$$

Substituting $\mu = -5$ in (v), we get

$$2x + 3y - 5 = 0 \quad \text{...(vi)}$$

This is the equation of altitude CN.

The vertex C of ΔABC is the point of intersection of side AC and altitude CN.

Solving (ii) and (vi), we obtain that the coordinates of C are $(35/2, -10)$.

Thus, the coordinates of B and C are $(-13, -33/2)$ and $(35/2, -10)$ respectively. Hence, equation of side BC is

$$y + \frac{33}{2} = \frac{-10 + \frac{33}{2}}{\frac{35}{2} + 13}(x + 13) \text{ or, } \frac{2y + 33}{2} = \frac{13}{61}(x + 13) \text{ or, } 26x - 122y - 1675 = 0$$

EXAMPLE 10 *A straight line L is perpendicular to the line $5x - y = 1$. The area of the triangle formed by the line L and the coordinate axes is 5 square units. Find the equation of the line L.*

SOLUTION The equation of a line L perpendicular to the line $5x - y = 1$ is

$$x + 5y + \lambda = 0 \quad \text{...(i)}$$

This line meets x-axis at $y = 0$. Putting $y = 0$, we get $x = -\lambda$. So, the line L meets x-axis at $A(-\lambda, 0)$.

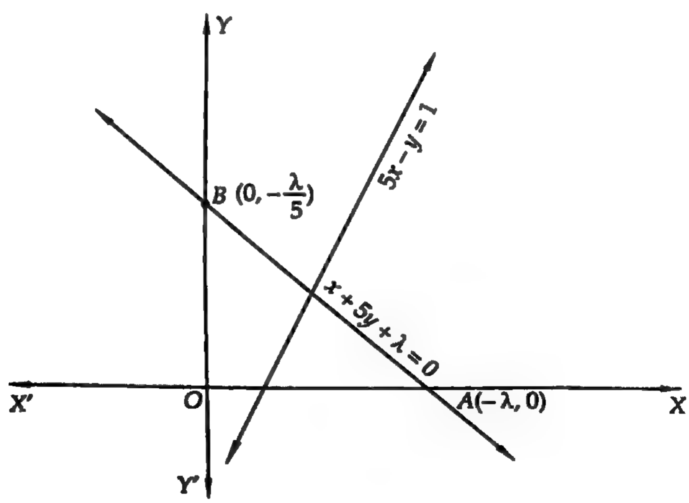


Fig. 23.73

The line L meets y-axis at $x = 0$. Putting $x = 0$ in the equation $x + 5y + \lambda = 0$, we obtain $y = -\lambda/5$.

So, the line L meets y-axis at $B(0, -\lambda/5)$.

Thus, the line L meets the coordinate axes at A and B such that $OA = -\lambda$ and $OB = -\lambda/5$.

It is given that

$$\text{Area of } \Delta OAB = 5$$

$$\Rightarrow \quad \frac{1}{2}(OA)(OB) = 5$$

$$\Rightarrow \quad \frac{1}{2}(-\lambda)\left(-\frac{\lambda}{5}\right) = 5 \Rightarrow \lambda^2 = 50 \Rightarrow \lambda = \pm 5\sqrt{2}$$

Substituting the value of λ in (i), we obtain $x + 5y \pm 5\sqrt{2} = 0$ as the equations of the required line L.

EXAMPLE 11 *Let $P(x_1, y_1)$ be a point and let $ax + by + c = 0$ be a line. If $L(h, k)$ is the foot of perpendicular drawn from P on this line and $Q(\alpha, \beta)$ is the image of P in the given line, then prove that*

$$\text{(i)} \quad \frac{h - x_1}{a} = \frac{k - y_1}{b} = -\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right) \quad \text{(ii)} \quad \frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

SOLUTION Suppose PQ makes an angle θ with x-axis. Since PQ is perpendicular to $ax + by + c = 0$.

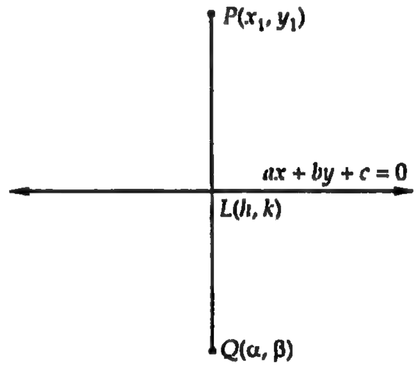


Fig. 23.74

$$\therefore \quad \text{Slope of } PQ \times (\text{Slope of } ax + by + c = 0) = -1$$

$$\Rightarrow \quad \tan\theta \times -\frac{a}{b} = -1$$

$$\Rightarrow \quad \tan\theta = \frac{b}{a}$$

$$\Rightarrow \quad \sin\theta = \frac{b}{\sqrt{a^2 + b^2}} \text{ and } \cos\theta = \frac{a}{\sqrt{a^2 + b^2}}$$

Since PQ passes through $P(x_1, y_1)$ and makes an angle θ with x-axis. Therefore, equation of PQ (in distance form) is $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta}$.

Let $PL = LQ = r$. Then, coordinates of L and Q are given by

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \text{ and } \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = 2r \text{ respectively.}$$

$$\therefore \quad \frac{h - x_1}{\cos\theta} = \frac{k - y_1}{\sin\theta} = r \qquad \text{...(i)}$$

$$\text{and,} \quad \frac{\alpha - x_1}{\cos\theta} = \frac{\beta - y_1}{\sin\theta} = 2r \qquad \text{...(ii)}$$

Now, $\dfrac{h - x_1}{\cos\theta} = \dfrac{k - y_1}{\sin\theta} = r \Rightarrow h = x_1 + r\cos\theta,\ k = y_1 + r\sin\theta$

So, the coordinates of L are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$.

Point L lies on $ax + by + c = 0$.

$$\therefore \quad a(x_1 + r\cos\theta) + b(y_1 + r\sin\theta) + c = 0$$

$$\Rightarrow \quad r = -\frac{ax_1 + by_1 + c}{a\cos\theta + b\sin\theta}$$

$$\Rightarrow \quad r = -\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \qquad \left[\because \cos\theta = \frac{a}{\sqrt{a^2 + b^2}}, \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}\right]$$

$$3y + \lambda = 0 \Rightarrow y = -\lambda/3$$

So, y-intercept $= -\lambda/3$.

It is given that the sum of the intercepts of the line (i) on the coordinate axes is 15.

$$\therefore \quad \left(-\frac{\lambda}{2}\right) + \left(-\frac{\lambda}{3}\right) = 15 \Rightarrow -\frac{5\lambda}{6} = 15 \Rightarrow \lambda = -18$$

Putting $\lambda = -18$ in (i), we get: $2x + 3y - 18 = 0$.

Hence, the equation of the required line is $2x + 3y - 18 = 0$.

EXAMPLE 6 *Show that the equation of a line passing through* $(a \cos^3 \theta, a \sin^3 \theta)$ *and perpendicular to the line* $x \sec \theta + y \operatorname{cosec} \theta = a$ *is* $x \cos \theta - y \sin \theta = a \cos 2\theta$.

SOLUTION The equation of a line perpendicular to the line $x \sec \theta + y \operatorname{cosec} \theta = a$ is

$$x \operatorname{cosec} \theta - y \sec \theta + \lambda = 0, \ \lambda \text{ is a constant} \quad ...(i)$$

This line passes through $(a \cos^3 \theta, a \sin^3 \theta)$.

$$\therefore \quad a \cos^3 \theta \operatorname{cosec} \theta - a \sin^3 \theta \sec \theta + \lambda = 0$$

$$\lambda = a(\sin^3 \theta \sec \theta - \cos^3 \theta \operatorname{cosec} \theta)$$

'ng the value of λ in (i), we get

$$x \operatorname{cosec} \theta - y \sec \theta + a(\sin^3 \theta \sec \theta - \cos^3 \theta \operatorname{cosec} \theta) = 0$$

$$\frac{x}{\sin \theta} - \frac{y}{\cos \theta} + a\left(\frac{\sin^3 \theta}{\cos \theta} - \frac{\cos^3 \theta}{\sin \theta}\right) = 0$$

$$x \cos \theta - y \sin \theta + a(\sin^4 \theta - \cos^4 \theta) = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta + a(\sin^2 \theta + \cos^2 \theta)(\sin^2 \theta - \cos^2 \theta) = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta - a(\cos^2 \theta - \sin^2 \theta) = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta - a \cos 2\theta = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta = a \cos 2\theta.$$

EXAMPLE 7 *Find the image of the point (– 8, 12) with respect to the line mirror* $4x + 7y + 13 = 0$.

SOLUTION Let the image of the point $P(-8, 12)$ in the line mirror AB be $Q(\alpha, \beta)$. Then, the line segement PQ is perpendicularly bisected at R. So, the coordinates of R are

$$\left(\frac{\alpha - 8}{2}, \frac{\beta + 12}{2}\right).$$

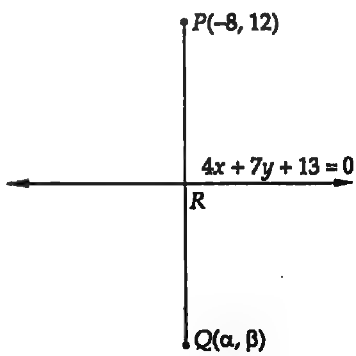


Fig. 23.70

As it lies on $4x + 7y + 13 = 0$.

$$\therefore \quad 2\alpha - 16 + \frac{7\beta + 84}{2} + 13 = 0 \Rightarrow 4\alpha + 7\beta + 78 = 0 \quad ...(i)$$

The line segment PQ is perpendicular to AB.

$$\therefore \quad (\text{Slope of } AB) \times (\text{Slope of } PQ) = -1$$

$$\Rightarrow \quad -\frac{4}{7} \times \frac{\beta - 12}{\alpha + 8} = -1$$

$$\Rightarrow \quad 7\alpha - 4\beta + 104 = 0 \quad ...(ii)$$

Solving (i) and (ii), we get: $\alpha = -16, \beta = -2$.

Hence, the image of $(-8, 12)$ in the line mirror $4x + 7y + 13 = 0$ is $(-16, -2)$.

LEVEL-2

EXAMPLE 8 *A person stranding at a junction (crossing) of two straight paths represented by the equations* $2x-3y-4=0$ *and* $3x-4y-5=0$, *wants to reach the path whose equation is* $6x-7y+8=0$ *in the least time. Find the equation of the path that he should follow.* **[NCERT]**

SOLUTION The lines $2x-3y-4=0$ and $3x-4y-5=0$ intersect at $(-1,-2)$. In order to reach the path, represented by the equation $6x-7y+8=0$, in the least time, the person should move along the line passing through A and perpendicular to $6x-7y+8=0$.

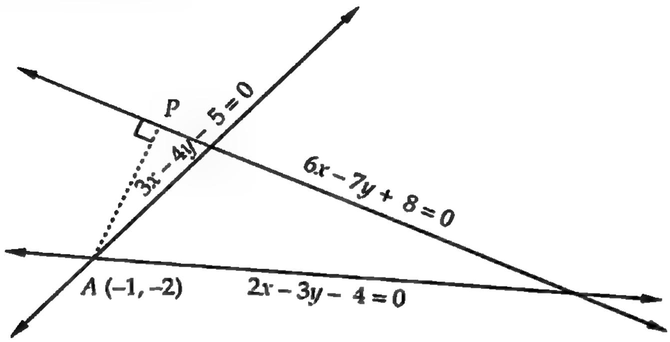


Fig. 23.71

Clearly, slope of the line $6x-7y+8=0$ is $\frac{6}{7}$. Therefore, slope of a line perpendicular to it is $-\frac{7}{6}$.

Hence, the equation of the required path is

$$y+2 = -\frac{7}{6}(x+1) \text{ or, } 7x+6y+19 = 0$$

EXAMPLE 9 *The equations of two sides of a triangle are* $3x-2y+6 = 0$ *and* $4x+5y-20 = 0$ *and the orthocentre is* $(1,1)$. *Find the equation of the third side.* **[NCERT]**

SOLUTION Let the equations of sides AB and AC of triangle ABC be respectively

$$3x-2y+6 = 0 \quad \text{...(i)}$$

and, $$4x+5y-20 = 0 \quad \text{...(ii)}$$

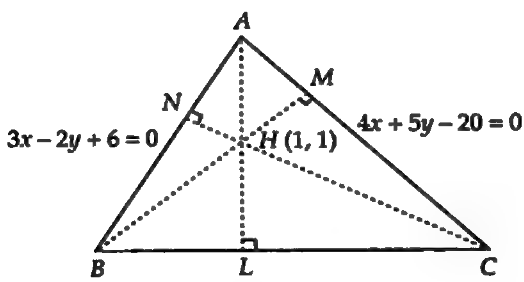


Fig. 23.72

Let $H(1,1)$ be the orthocentre of triangle ABC where the altitudes AL, BM and CN intersect each other. Clearly, BM passes through $H(1,1)$ and is perpendicular to AC. The equation of a line perpendicular to AC is

$$5x-4y+\lambda = 0 \quad \text{...(iii)}$$

If it passes through the orthocentre $H(1,1)$, then

$$5-4+\lambda = 0 \Rightarrow \lambda = -1$$

Substituting $\lambda = -1$ in (iii), we get

$$5x-4y-1 = 0 \quad \text{...(iv)}$$

This is the equation of altitude BM.

The vertex B of ΔABC is the intersection point of side AB and altitude BM. Solving their equations given by (i) and (iii), we get $x = -13$ and $y = -33/2$.

So, coordinates of B are $(-13, -33/2)$.

The altitude CN is perpendicular to AB. So, let its equation be

$$2x + 3y + \mu = 0 \quad \text{...(v)}$$

If it passes through the orthocentre $H(1, 1)$, then

$$2 + 3 + \mu = 0 \Rightarrow \mu = -5$$

Substituting $\mu = -5$ in (v), we get

$$2x + 3y - 5 = 0 \quad \text{...(vi)}$$

This is the equation of altitude CN.

The vertex C of $\Delta\, ABC$ is the point of intersection of side AC and altitude CN.

Solving (ii) and (vi), we obtain that the coordinates of C are $(35/2, -10)$.

Thus, the coordinates of B and C are $(-13, -33/2)$ and $(35/2, -10)$ respectively. Hence, equation of side BC is

$$y + \frac{33}{2} = \frac{-10 + \frac{33}{2}}{\frac{35}{2} + 13}(x + 13) \text{ or, } \frac{2y + 33}{2} = \frac{13}{61}(x + 13) \text{ or, } 26x - 122y - 1675 = 0$$

EXAMPLE 10 *A straight line L is perpendicular to the line $5x - y = 1$. The area of the triangle formed by the line L and the coordinate axes is 5 square units. Find the equation of the line L.*

SOLUTION The equation of a line L perpendicular to the line $5x - y = 1$ is

$$x + 5y + \lambda = 0 \quad \text{...(i)}$$

This line meets x-axis at $y = 0$. Putting $y = 0$, we get $x = -\lambda$. So, the line L meets x-axis at $A(-\lambda, 0)$.

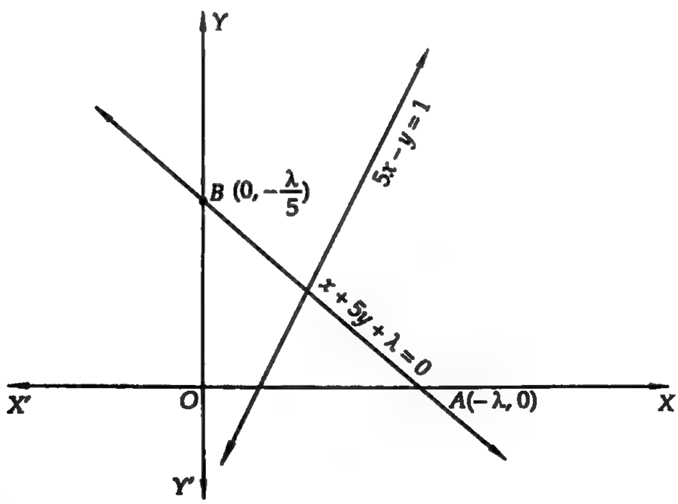


Fig. 23.73

The line L meets y-axis at $x = 0$. Putting $x = 0$ in the equation $x + 5y + \lambda = 0$, we obtain $y = -\lambda/5$.

So, the line L meets y-axis at $B(0, -\lambda/5)$.

Thus, the line L meets the coordinate axes at A and B such that $OA = -\lambda$ and $OB = -\lambda/5$.

It is given that

$$\text{Area of } \Delta\, OAB = 5$$

$\Rightarrow \quad \frac{1}{2}(OA)(OB) = 5$

$\Rightarrow \quad \frac{1}{2}(-\lambda)\left(-\frac{\lambda}{5}\right) = 5 \Rightarrow \lambda^2 = 50 \Rightarrow \lambda = \pm 5\sqrt{2}$

Substituting the value of λ in (i), we obtain $x + 5y \pm 5\sqrt{2} = 0$ as the equations of the required line L.

EXAMPLE 11 *Let $P(x_1, y_1)$ be a point and let $ax + by + c = 0$ be a line. If $L(h, k)$ is the foot of perpendicular drawn from P on this line and $Q(\alpha, \beta)$ is the image of P in the given line, then prove that*

(i) $\frac{h - x_1}{a} = \frac{k - y_1}{b} = -\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$ (ii) $\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$

SOLUTION Suppose PQ makes an angle θ with x-axis. Since PQ is perpendicular to $ax + by + c = 0$.

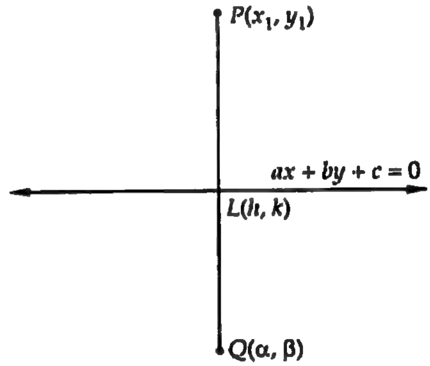


Fig. 23.74

$\therefore \quad$ Slope of $PQ \times$ (Slope of $ax + by + c = 0$) $= -1$

$\Rightarrow \quad \tan\theta \times -\frac{a}{b} = -1$

$\Rightarrow \quad \tan\theta = \frac{b}{a}$

$\Rightarrow \quad \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}$ and $\cos\theta = \frac{a}{\sqrt{a^2 + b^2}}$

Since PQ passes through $P(x_1, y_1)$ and makes an angle θ with x-axis. Therefore, equation of PQ (in distance form) is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}$.

Let $PL = LQ = r$. Then, coordinates of L and Q are given by

$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ and $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = 2r$ respectively.

$$\therefore \quad \frac{h - x_1}{\cos\theta} = \frac{k - y_1}{\sin\theta} = r \quad \text{...(i)}$$

$$\text{and,} \quad \frac{\alpha - x_1}{\cos\theta} = \frac{\beta - y_1}{\sin\theta} = 2r \quad \text{...(ii)}$$

Now, $\frac{h - x_1}{\cos\theta} = \frac{k - y_1}{\sin\theta} = r \Rightarrow h = x_1 + r\cos\theta,\ k = y_1 + r\sin\theta$

So, the coordinates of L are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$.

Point L lies on $ax + by + c = 0$.

$\therefore \quad a(x_1 + r\cos\theta) + b(y_1 + r\sin\theta) + c = 0$

$\Rightarrow \quad r = -\frac{ax_1 + by_1 + c}{a\cos\theta + b\sin\theta}$

$\Rightarrow \quad r = -\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \quad \left[\because \cos\theta = \frac{a}{\sqrt{a^2 + b^2}}, \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}\right]$

(i) Substituting the values of cos θ, sin θ and r in (i), we get

$$\frac{h-x_1}{\frac{a}{\sqrt{a^2+b^2}}}=\frac{k-y_1}{\frac{b}{\sqrt{a^2+b^2}}}=-\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \quad \frac{h-x_1}{a}=\frac{k-y_1}{b}=-\frac{ax_1+by_1+c}{a^2+b^2}$$

(ii) Substituting the values of cos θ, sin θ and r in (ii), we get

$$\frac{\alpha-x_1}{\frac{a}{\sqrt{a^2+b^2}}}=\frac{\beta-y_1}{\frac{b}{\sqrt{a^2+b^2}}}=-2\left(\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right)$$

$$\Rightarrow \quad \frac{\alpha-x_1}{a}=\frac{\beta-y_1}{b}=-2\left(\frac{ax_1+by_1+c}{a^2+b^2}\right)$$

EXAMPLE 12 *Find the centroid, incentre circum-centre and orthocentre of the triangle whose sides have the equations* $3x-4y=0, 12y+5x=0$ *and* $y-15=0$

SOLUTION Let ABC be the triangle whose sides BC, CA and AB have the equations $y-15=0$, $3x-4y=0$ and $5x+12y=0$ respectively. Solving these equations pair wise we can obtain the coordinates of the vertices A, B, C as $A(0,0)$, $B(-36,15)$ and $C(20,15)$ respectively.

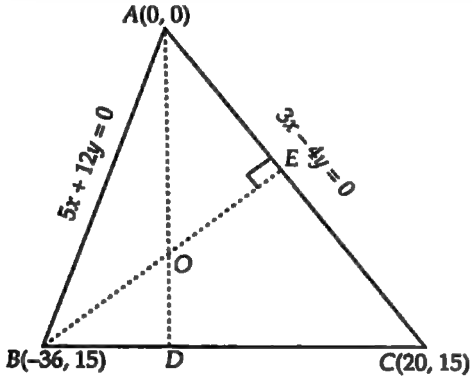


Fig. 23.75

Centroid: The coordinates of the centroid are

$$\left(\frac{0-36+20}{3},\frac{0+15+15}{3}\right)=\left(-\frac{16}{3},10\right) \qquad \left[\text{Using: }\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)\right]$$

In centre: We have,

$$a = BC=\sqrt{(-36-20)^2+(15-15)^2}=56,\ b=CA=\sqrt{20^2+15^2}=25,$$

and, $\quad c=AB=\sqrt{(-36-0)^2+(15-0)^2}=39.$

Using $\left(\dfrac{ax_1+bx_2+cx_3}{a+b+c},\dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$ the coordinates of the incentre are

$$\left(\frac{56\times 0+25\times -36+39\times 20}{56+25+39},\frac{56\times 0+25\times 15+39\times 15}{56+25+39}\right)=(-1,8)$$

Circum-centre: Let (x, y) be the coordinates of the circum-centre O (say). Then, $OA = OB = OC$.

Now, $OA = OB$

$\Rightarrow \quad OA^2 = OB^2$

$\Rightarrow \quad x^2 + y^2 = (x+36)^2 + (y-15)^2$

$\Rightarrow \quad 72x - 30y + 1521 = 0$ ...(i)

and, $OB = OC$

$\Rightarrow \quad OB^2 = OC^2$

$\Rightarrow \quad (x+36)^2 + (y-15)^2 = (x-20)^2 + (y-15)^2$

$\Rightarrow \quad 112x + 896 = 0 \Rightarrow x = -8$ ...(ii)

Solving (i) and (ii), we get : $x = -8$ and, $y = 63/2$.

So, the coordinates of circumcentre are $(-8, 63/2)$.

Orthocentre: AD is a line passing through A (0, 0) and perpendicular to $y - 15 = 0$. So, equation of AD is $x = 0$.

The equation of any line perpendicular to side AC having equation $3x - 4y = 0$ is $4x + 3y + \lambda = 0$. If it passes through $B(-36, 15)$, then

$$-144 + 45 + \lambda = 0 \Rightarrow \lambda = 99.$$

So, the equation of BE is $4x + 3y + 99 = 0$.

Solving the equations of AD and BE, we obtain $x = 0$ and $y = -33$.

Hence, the coordinates of the orthocentre are $(0, -33)$.

EXAMPLE 13 *Find the circumcentre of the triangle whose sides are $3x - y + 3 = 0$, $3x + 4y + 3 = 0$ and $x + 3y + 11 = 0$.*

SOLUTION Let ABC be the triangle whose sides AB, BC and CA have the equations $3x - y + 3 = 0$, $3x + 4y + 3 = 0$ and $x + 3y + 11 = 0$ respectively.

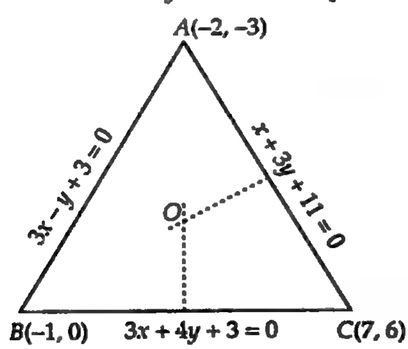


Fig. 23.76

The circumcentre of ΔABC is the point of concurrence of its perpendicular bisectors. So, let us first find the perpendicular bisector of sides BC and AC.

Solving the equations of AB and AC; BC and AB; AC and BC in pairs we obtain that the coordinates of the vertices A, B and C are $(-2, -3)$, $(-1, 0)$ and $(7, -6)$ respectively.

The equation of a line perpendicular to BC is $4x - 3y + \lambda = 0$.

This will pass through $(3, -3)$, the mid point of BC, if

$$12 + 9 + \lambda = 0 \Rightarrow \lambda = -21.$$

Putting $\lambda = -21$ in $4x - 3y + \lambda = 0$, we get

$$4x - 3y - 21 = 0. \qquad ...(i)$$

as the equation of the perpendicular bisector of BC.

The equation of a line perpendicular to AC is

$$3x - y + \lambda_1 = 0.$$

This will pass through $(5/2, -9/2)$ i.e. the mid point of AC, if

$$\frac{15}{2} + \frac{9}{2} + \lambda_1 = 0 \Rightarrow \lambda_1 = -12.$$

Putting $\lambda_1 = -12$ in $3x - y + \lambda_1 = 0$, we get

$$3x - y - 12 = 0. \qquad ...(ii)$$

as the perpendicular bisector of AC.

Solving (i) and (ii), we get: $x = 3$, $y = -3$.

Hence, the coordinates of the circumcentre of ΔABC are $(3, -3)$.

EXAMPLE 14 *Find the orthocentre of the triangle whose vertices are* $(at_1 t_2, a(t_1 + t_2))$, $(at_2 t_3, a(t_2 + t_3))$ *and* $(at_1 t_3, a(t_1 + t_3))$.

SOLUTION Let ABC be a triangle whose vertices are $A(at_1 t_2, a(t_1 + t_2))$, $B(at_2 t_3, a(t_2 + t_3))$ and $C(at_1 t_3, a(t_1 + t_3))$. The orthocentre of ΔABC is the point of concurrence of its altitudes. So, let us find their equations.

Clearly,

$$\text{Slope of } BC = \frac{a(t_2 + t_3) - a(t_1 + t_3)}{at_2 t_3 - at_1 t_3} = \frac{1}{t_3} \text{ and, Slope of } AC = \frac{a(t_1 + t_3) - a(t_1 + t_2)}{at_1 t_3 - at_1 t_2} = \frac{1}{t_1}.$$

The equation of the line through A perpendicular to BC i.e. the altitude through vertex A is

$$y - a(t_1 + t_2) = -t_3(x - at_1 t_2) \qquad ...(i)$$

The equation of the line through B perpendicular to AC i.e. the altitude through vertex B is

$$y - a(t_2 + t_3) = -t_1(x - at_2 t_3) \qquad ...(ii)$$

The point of intersection of (i) and (ii) is the orthocentre.

Subtracting (ii) from (i), we get $x = -a$.

Putting $x = -a$ in (i), we get $y = a(t_1 + t_2 + t_3 + t_1 t_2 t_3)$.

Hence, the coordinates of the orthocentre are $(-a, a(t_1 + t_2 + t_3 + t_1 t_2 t_3))$.

EXERCISE 23.12

LEVEL-1

1. Find the equation of a line passing through the point (2, 3) and parallel to the line $3x - 4y + 5 = 0$.
2. Find the equation of a line passing through $(3, -2)$ and perpendicular to the line $x - 3y + 5 = 0$
3. Find the equation of the perpendicular bisector of the line joining the points (1, 3) and (3, 1).
4. Find the equations of the altitudes of a ΔABC whose vertices are $A(1, 4)$, $B(-3, 2)$ and $C(-5, -3)$.
5. Find the equation of a line which is perpendicular to the line $\sqrt{3}x - y + 5 = 0$ and which cuts off an intercept of 4 units with the negative direction of y-axis.
6. If the image of the point (2, 1) with respect to a line mirror is (5, 2), find the equation of the mirror.
7. Find the equation of the straight line through the point (α, β) and perpendicular to the line $lx + my + n = 0$.

8. Find the equation of the straight line perpendicular to $2x - 3y = 5$ and cutting off an intercept 1 on the positive direction of the x-axis.
9. Find the equation of the straight line perpendicular to $5x - 2y = 8$ and which passes through the mid-point of the line segment joining (2, 3) and (4, 5).
10. Find the equation of the straight line which has y-intercept equal to $4/3$ and is perpendicular to $3x - 4y + 11 = 0$.
11. Find the equation of the right bisector of the line segment joining the points (a, b) and (a_1, b_1).
12. Find the image of the point (2, 1) with respect to the line mirror $x + y - 5 = 0$.
13. If the image of the point (2, 1) with respect to the line mirror be (5, 2), find the equation of the mirror.
14. Find the equation to the straight line parallel to $3x - 4y + 6 = 0$ and passing through the middle point of the join of points (2, 3) and (4, −1).
15. Prove that the lines $2x - 3y + 1 = 0$, $x + y = 3$, $2x - 3y = 2$ and $x + y = 4$ form a parallelogram.
16. Find the equation of a line drawn perpendicular to the line $\frac{x}{4} + \frac{y}{6} = 1$ through the point where it meets the y-axis. [NCERT]
17. The perpendicular from the origin to the line $y = mx + c$ meets it at the point $(-1, 2)$. Find the values of m and c. [NCERT]
18. Find the equation of the right bisector of the line segment joining the points (3, 4) and (−1, 2).
19. The line through $(h, 3)$ and (4, 1) intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h.
20. Find the image of the point (3, 8) with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror. [NCERT]
21. Find the coordinates of the foot of the perpendicular from the point (−1, 3) to the line $3x - 4y - 16 = 0$. [NCERT]
22. Find the projection of the point (1, 0) on the line joining the points (−1, 2) and (5, 4).
23. Find the equation of a line perpendicular to the line $\sqrt{3}\,x - y + 5 = 0$ and at a distance of 3 units from the origin.

LEVEL-2

24. The line $2x + 3y = 12$ meets the x-axis at A and y-axis at B. The line through (5, 5) perpendicular to AB meets the x-axis and the line AB at C and E respectively. If O is the origin of coordinates, find the area of figure $OCEB$.
25. Find the equation of the straight line which cuts off intercepts on x-axis twice that on y-axis and is at a unit distance from the origin.
26. The equations of perpendicular bisectors of the sides AB and AC of a triangle ABC are $x - y + 5 = 0$ and $x + 2y = 0$ respectively. If the point A is $(1, -2)$, find the equation of the line BC.

ANSWERS

1. $3x - 4y + 6 = 0$ 2. $3x + y - 7 = 0$ 3. $y = x$
4. $2x + 5y - 22 = 0, 6x + 7y + 4 = 0, 2x + y + 13 = 0$
5. $x + \sqrt{3}\,y + 4\sqrt{3} = 0$ 6. $3x + y - 12 = 0$
7. $m(x - \alpha) = l(y - \beta)$ 8. $3x + 2y - 3 = 0$ 9. $2x + 5y - 26 = 0$ 10. $4x + 3y - 4 = 0$
11. $2x(a_1 - a) + 2y(b_1 - b) + (a^2 + b^2) - (a_1^2 + b_1^2) = 0$ 12. (4, 3)

13. $3x + y = 12$ 14. $3x - 4y = 5$ 16. $2x - 3y + 18 = 0$ 17. $m = 1/2, c = 5/2$
18. $2x + y - 5 = 0$ 19. $22/9$ 20. $(-1, -4)$ 21. $\left(\frac{68}{25}, -\frac{49}{25}\right)$
22. $\left(\frac{1}{5}, \frac{12}{5}\right)$ 23. $x + \sqrt{3}\, y \pm 6 = 0$ 24. $\frac{23}{3}$ sq. units 25. $x + 2y \pm \sqrt{5} = 0$
26. $14x + 23y - 40 = 0$ 27. $5x + 3y + 8 = 0$

HINTS TO NCERT & SELECTED PROBLEMS

16. The line $\frac{x}{4} + \frac{y}{6} = 1$ cuts y-axis at (0, 6) and has slope is $-\frac{3}{2}$.

Hence, equation of the required line is $y - 6 = \frac{2}{3}(x - 0)$ or, $2x - 3y + 18 = 0$

17. Clearly, (–1, 2) lies on $y = mx + c$.

$\therefore \quad 2 = -m + c$ ...(i)

The line joining the origin to (– 1, 2) is perpendicular to $y = mx + c$.

$\therefore \quad \frac{2-0}{-1-0} \times m = -1 \Rightarrow m = \frac{1}{2}$

Putting $m = \frac{1}{2}$ in (i), we get $c = \frac{5}{2}$.

20. Let the image of the point $P(3, 8)$ in the line mirror $x + 3y = 7$ be $Q(\alpha\ \beta)$. Then, PQ is perpendicularly bisected at M. The coordinates of M are $\left(\frac{\alpha+3}{2}, \frac{\beta+8}{2}\right)$. Since M lies on $x + 3y = 7$.

$\therefore \quad \frac{\alpha+3}{2} + 3\left(\frac{\beta+8}{2}\right) = 7$

$\Rightarrow \quad \alpha + 3\beta + 13 = 0$ ...(i)

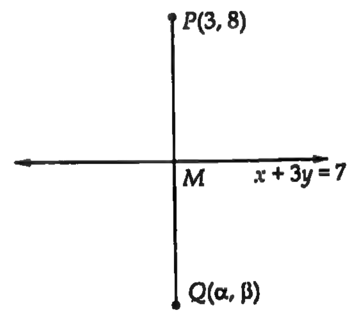


Fig. 23.77

Line segment PQ is perpendicular to $x + 3y = 7$.

$\therefore \quad \frac{\beta-8}{\alpha-3} \times -\frac{1}{3} = -1$

$\Rightarrow \quad 3\alpha - 9 = \beta - 8$

$\Rightarrow \quad 3\alpha - \beta - 1 = 0$ ...(ii)

Solving (i) and (ii), we get $\alpha = -1$ and $\beta = -4$.

Hence, the coordinates of Q are $(-1, -4)$.

21. Let $M(\alpha, \beta)$ be the foot of perpendicular from $P(-1, 3)$ on the line $3x - 4y - 16 = 0$. Then,

$$\frac{\beta-3}{\alpha+1} \times \frac{3}{4} = -1$$

$\Rightarrow \quad 4\alpha + 3\beta - 5 = 0$ ...(i)

Point $M(\alpha, \beta)$ lies on $3x - 4y - 16 = 0$.

$\therefore \quad 3\alpha - 4\beta - 16 = 0$ ...(ii)

Solving (i) and (ii), we get

$$\alpha = \frac{68}{25} \text{ and } \beta = -\frac{49}{25}.$$

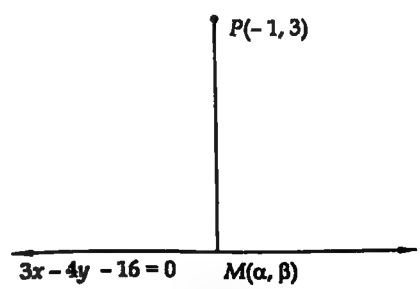


Fig. 23.78

Hence, the coordinates of M are $(68/25, -49/25)$.

23.11 ANGLE BETWEEN TWO STRAIGHT LINES WHEN THEIR EQUATIONS ARE GIVEN

THEOREM *Prove that the acute angle θ between the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ is given by*

$$\tan\theta = \left|\frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}\right|$$

PROOF Let m_1 and m_2 be the slopes of the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$. Then,

$$m_1 = -\frac{a_1}{b_1} \text{ and } m_2 = -\frac{a_2}{b_2}$$

Now, $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \quad \tan\theta = \left|\frac{-\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)}\right| \Rightarrow \tan\theta = \left|\frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}\right| \Rightarrow \theta = \tan^{-1}\left|\frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}\right|$$

Q.E.D.

CONDITION FOR THE LINES TO BE PARALLEL *If the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are parallel, then*

$$m_1 = m_2 \Rightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

CONDITION FOR THE LINES TO BE PERPENDICULAR *If the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are perpendicular, then*

$$m_1 m_2 = -1 \Rightarrow -\frac{a_1}{b_1} \times -\frac{a_2}{b_2} = -1 \Rightarrow a_1 a_2 + b_1 b_2 = 0$$

It follows from the above discussion that the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are:

(i) Coincident, if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ (ii) Parallel, if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$.

(iii) Intersecting, if $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ (iv) Perpendicular, if $a_1 a_2 + b_1 b_2 = 0$

To find the acute angle between two lines when their slopes are given, we may use the following algorithm.

ALGORITHM

STEP I *Obtain the equations of the lines.*

STEP II *Obtain the slopes m_1 and m_2 of two lines by using the formula: Slope $= -\dfrac{\text{Coeff. of } x}{\text{Coeff. of } y}$.*

STEP III *Use the formula : $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$ to find the acute angle θ between the lines.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the angles between the pairs of straight lines*

(i) $x-\sqrt{3}y-5=0$ and $\sqrt{3}\,x+y-7=0$ (ii) $y=(2-\sqrt{3})\,x+5$ and $y=(2+\sqrt{3})\,x-7$.

SOLUTION (i) The equations of two straight lines are:

$$x-\sqrt{3}y-5=0 \quad ...(i) \qquad \text{and} \qquad \sqrt{3}\,x+y-7=0 \quad ...(ii)$$

Let m_1 and m_2 be the slopes of these two lines. Then,

$$m_1=\frac{-1}{-\sqrt{3}}=\frac{1}{\sqrt{3}} \text{ and } m_2=-\frac{\sqrt{3}}{1}=-\sqrt{3}$$

We observe that $m_1m_2=-1$. Thus, the two lines are at right angle.

(ii) Let m_1 and m_2 be the slopes of the straight lines $y=(2-\sqrt{3})\,x+5$ and $y=(2+\sqrt{3})\,x-7$ respectively. Then, $m_1=2-\sqrt{3}$ and $m_2=2+\sqrt{3}$.

Let θ be the angle between the lines. Then,

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{(2-\sqrt{3})-(2+\sqrt{3})}{1+(2-\sqrt{3})(2+\sqrt{3})}\right|=\left|-\frac{2\sqrt{3}}{1+4-3}\right|=\sqrt{3}$$

$$\Rightarrow \quad \theta=\pi/3$$

Thus, the acute angle between the lines is of 60°.

EXAMPLE 2 *Find the tangent of the angle between the lines whose intercepts on the axes are respectively $a,-b$ and $b,-a$.*

SOLUTION The line which cuts off intercepts a and $-b$ on the coordinate axes passes through points $A\ (a, 0)$ and $B\ (0,-b)$.

$$\therefore \quad \text{Slope of line } AB=m_1=\frac{-b-0}{0-a}=\frac{b}{a}$$

The line which cuts off intercepts b and $-a$ on the coordinate axes passes through points $C\ (b, 0)$ and $D\ (0,-a)$.

$$\therefore \quad \text{Slope of line } CD=m_2=\frac{-a-0}{0-b}=\frac{a}{b}$$

Let θ be the angle between AB and CD. Then,

$$\tan\theta=\pm\frac{m_1-m_2}{1+m_1m_2}=\pm\frac{\frac{b}{a}-\frac{a}{b}}{1+\frac{b}{a}\cdot\frac{a}{b}}=\pm\frac{b^2-a^2}{2ab}$$

EXAMPLE 3 *Find the obtuse angle between the lines $x-2y+3=0$ and $3x+y-1=0$.*

SOLUTION Let m_1 and m_2 be the slopes of the straight lines $x-2y+3=0$ and $3x+y-1=0$. Then,

$$m_1=-\frac{1}{-2}=\frac{1}{2} \text{ and } m_2=-\frac{3}{1}=-3.$$

Let θ be the angle between the given lines. Then,

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\frac{1}{2}+3}{1-\frac{3}{2}}\right|=7 \Rightarrow \theta=\tan^{-1}(7).$$

Thus, the acute angle between the lines is $\tan^{-1}(7)$ and the obtuse angle is $\pi-\tan^{-1}(7)$.

EXAMPLE 4 *Find the value of k if the straight line* $2x + 3y + 4 + k(6x - y + 12) = 0$ *is perpendicular to the line* $7x + 5y - 4 = 0$.

SOLUTION The two lines are

$$x(2 + 6k) + y(3 - k) + 4 + 12k = 0 \quad ...(i) \quad \text{and} \quad 7x + 5y - 4 = 0 \quad ...(ii)$$

Let m_1 and m_2 be the slopes of (i) and (ii) respectively. Then,

$$m_1 = -\frac{2 + 6k}{3 - k}, \; m_2 = -\frac{7}{5}$$

If lines (i) and (ii) are perpendicular. Then,

$$m_1 m_2 = -1 \Rightarrow \left(-\frac{2 + 6k}{3 - k}\right)\left(-\frac{7}{5}\right) = -1 \Rightarrow 14 + 42k = -15 + 5k \Rightarrow k = -\frac{29}{37}.$$

EXAMPLE 5 *A line passing through the points $(a, 2a)$ and $(-2, 3)$ is perpendicular to the line $4x + 3y + 5 = 0$, find the value of a.*

SOLUTION Let m_1 be the slope of the line joining $A(a, 2a)$ and $B(-2, 3)$. Then, $m_1 = \dfrac{2a - 3}{a + 2}$

Let m_2 be the slope of the line $4x + 3y + 5 = 0$. Then, $m_2 = -\dfrac{4}{3}$.

Since given lines are perpendicular. Therefore,

$$m_1 m_2 = -1 \Rightarrow \frac{2a - 3}{a + 2} \times -\frac{4}{3} = -1 \Rightarrow 8a - 12 = 3a + 6 \Rightarrow a = 18/5.$$

EXAMPLE 6 *Classify the following pairs of lines as coincident, parallel or intersecting:*

(i) $x + 2y - 3 = 0$ and $-3x - 6y + 9 = 0$ (ii) $x + 2y + 1 = 0$ and $2x + 4y + 3 = 0$

(iii) $3x - 2y + 5 = 0$ and $2x + y - 9 = 0$

SOLUTION (i) The given lines are $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$, where $a_1 = 1, b_1 = 2, c_1 = -3, a_2 = -3, b_2 = -6$ and $c_2 = 9$.

Clearly, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = -\dfrac{1}{3}$. So, the given lines are coincident.

(ii) The given lines are $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$, where $a_1 = 1, b_1 = 2, c_1 = 1, a_2 = 2, b_2 = 4$, and $c_2 = 3$.

Clearly, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$. So, the given lines are parallel.

(iii) The given lines are $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$, where $a_1 = 3, b_1 = -2, c_1 = 5, a_2 = 2, b_2 = 1$, and $c_2 = -9$.

Clearly, $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$. So, the given lines are intersecting.

EXAMPLE 7 *The hypotenuse of a right isosceles triangle has its ends at the points* $(1, 3)$ *and* $(-4, 1)$. *Find the equations of the legs (perpendicular sides) of the triangle.* **[NCERT]**

SOLUTION Let ABC be the right triangle with diagonal AC. Let m be the slope of a line making 45° angle with AC.

Clearly, Slope of $AC = \dfrac{1 - 3}{-4 - 1} = \dfrac{2}{5}$. We observe that θ is the angle between AC and a line of slope m.

$\therefore \quad \tan 45° = \left|\dfrac{m - \frac{2}{5}}{1 + \frac{2m}{5}}\right|$

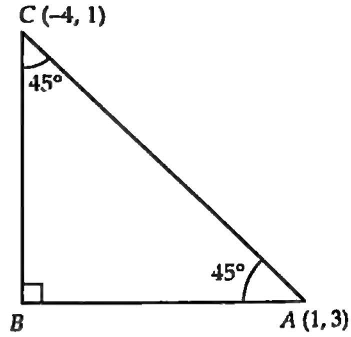


Fig. 23.79

$\Rightarrow \quad 2m + 5 = \pm(5m - 2)$

$\Rightarrow \quad 2m + 5 = 5m - 2 \text{ or, } 2m + 5 = -(5m - 2)$

$\Rightarrow \quad m = \dfrac{7}{3} \text{ or, } m = -\dfrac{3}{7}$

Thus, the lines making 45° angle with AC having slopes $\frac{7}{3}$ or $-\frac{3}{7}$. So, the possible equations of ꓤ are

$$y - 3 = \frac{7}{3}(x - 1) \text{ and } y - 3 = -\frac{3}{7}(x - 1) \Rightarrow 7x - 3y + 2 = 0 \text{ and } 3x + 7y - 24 = 0$$

ssible equations of BC are

$$y - 1 = \frac{7}{3}(x + 4) \text{ and } y - 1 = -\frac{3}{7}(x + 4) \Rightarrow 7x - 3y + 31 = 0 \text{ and } 3x + 7y + 5 = 0$$

the equations of the sides are:

$$7x - 3y + 2 = 0 \text{ and } 3x + 7y + 5 = 0 \quad \text{or, } 7x - 3y + 31 = 0 \text{ and } 3x + 7y - 24 = 0$$

PLE 8 *If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$. Find alues of m.* [NCERT]

LUTION Let θ be the acute angle which the line $y = mx + 4$ makes with the lines $y = 3x + 1$ and $2y = x + 3$. Then,

$$\tan\theta = \left|\frac{m - 3}{1 + 3m}\right| \text{ and, } \tan\theta = \left|\frac{m - \frac{1}{2}}{1 + \frac{m}{2}}\right|$$

$\Rightarrow \quad \left|\dfrac{m - 3}{1 + 3m}\right| = \left|\dfrac{2m - 1}{m + 2}\right|$

$\Rightarrow \quad \dfrac{m - 3}{3m + 1} = \pm\dfrac{2m - 1}{m + 2}$

$\Rightarrow \quad m^2 - m - 6 = \pm(6m^2 - m - 1)$

$\Rightarrow \quad 5m^2 + 5 = 0 \text{ or } 7m^2 - 2m - 7 = 0 \Rightarrow 7m^2 - 2m - 7 = 0 \Rightarrow m = \dfrac{1 \pm 5\sqrt{2}}{7}$

EXAMPLE 9 *Find the slope of the lines which make an angle of 45° with the line $3x - y + 5 = 0$.*

SOLUTION Let m be the slope of the line which make an angle of 45° with the line $3x - y + 5 = 0$. Then,

$$\tan 45° = \left|\frac{m - 3}{1 + 3m}\right| \qquad [\because \text{ Slope of } 3x - y + 5 = 0 \text{ is } 3]$$

$\Rightarrow \quad 1 = \left|\dfrac{m - 3}{1 + 3m}\right|$

$\Rightarrow \quad |1 + 3m| = |m - 3|$

$\Rightarrow \quad 1 + 3m = \pm(m - 3) \Rightarrow 1 + 3m = m - 3,\ 1 + 3m = -m + 3 \Rightarrow m = -2, \dfrac{1}{2}$

EXERCISE 23.13

LEVEL-1

1. Find the angles between each of the following pairs of straight lines:
   (i) $3x + y + 12 = 0$ and $x + 2y - 1 = 0$ (ii) $3x - y + 5 = 0$ and $x - 3y + 1 = 0$
   (iii) $3x + 4y - 7 = 0$ and $4x - 3y + 5 = 0$ (iv) $x - 4y = 3$ and $6x - y = 11$
   (v) $(m^2 - mn)\,y = (mn + n^2)\,x + n^3$ and $(mn + m^2)\,y = (mn - n^2)\,x + m^3$.
2. Find the acute angle between the lines $2x - y + 3 = 0$ and $x + y + 2 = 0$.
3. Prove that the points $(2, -1)$, $(0, 2)$, $(2, 3)$ and $(4, 0)$ are the coordinates of the vertices of a parallelogram and find the angle between its diagonals.
4. Find the angle between the line joining the points $(2, 0)$, $(0, 3)$ and the line $x + y = 1$.
5. If θ is the angle which the straight line joining the points (x_1, y_1) and (x_2, y_2) subtends at the origin, prove that $\tan\theta = \dfrac{x_2 y_1 - x_1 y_2}{x_1 x_2 + y_1 y_2}$ and $\cos\theta = \dfrac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\,\sqrt{x_2^2 + y_2^2}}$.
6. Prove that the straight lines $(a + b)\,x + (a - b)\,y = 2ab$, $(a - b)\,x + (a + b)\,y = 2ab$ and $x + y = 0$ form an isosceles triangle whose vertical angle is $2\tan^{-1}\left(\frac{a}{b}\right)$.
7. Find the angle between the lines $x = a$ and $by + c = 0$.
8. Find the tangent of the angle between the lines which have intercepts 3, 4 and 1, 8 on the axes respectively.
9. Show that the line $a^2 x + ay + 1 = 0$ is perpendicular to the line $x - ay = 1$ for all non-zero real values of a.
10. Show that the tangent of an angle between the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{a} - \frac{y}{b} = 1$ is $\frac{2ab}{a^2 - b^2}$. [NCERT EXEMPLAR]

ANSWERS

1. (i) 45° (ii) $\tan^{-1}\left(\frac{4}{3}\right)$ (iii) 90° (iv) $\tan^{-1}\left(\frac{23}{10}\right)$ (v) $\tan^{-1}\left(\frac{4m^2n^2}{m^4 - n^4}\right)$

2. $\tan^{-1} 3$ 3. $-\tan^{-1}\frac{1}{2} - \frac{\pi}{2}$ 4. $\tan^{-1}\left(\frac{1}{5}\right)$ 7. 90° 8. $\frac{4}{7}$

23.12 POSITION OF TWO POINTS RELATIVE TO A LINE

In this section, we shall see how to check whether two given points are on the same side or opposite sides of a given line.

Let the equation of the given line be $ax + by + c = 0$ ...(i)

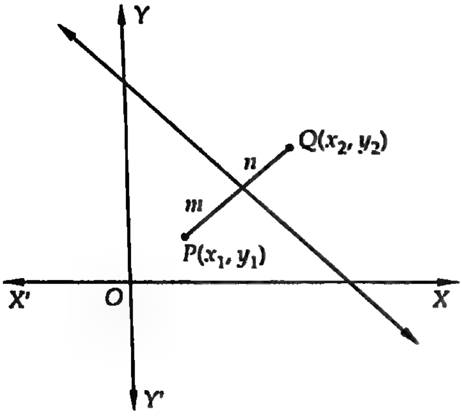
and let the coordinates of the two given points be $P(x_1, y_1)$ and $Q(x_2, y_2)$.

The coordinates of the point R which divides the line joining P and Q in the ratio $m : n$ are

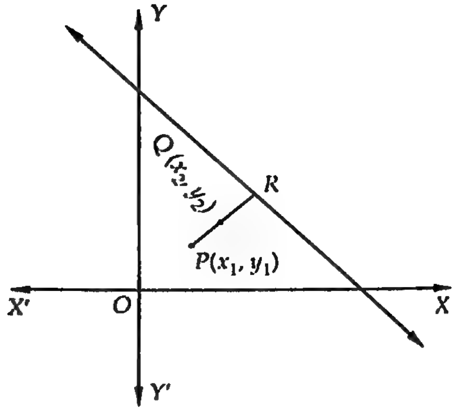
$$\left(\frac{m x_2 + n x_1}{m + n}, \frac{m y_2 + n y_1}{m + n}\right) \quad ...(ii)$$

If this point lies on (i), then

$$a\left(\frac{m x_2 + n x_1}{m + n}\right) + b\left(\frac{m y_2 + n y_1}{m + n}\right) + c = 0$$



(i)



(ii)

Fig. 23.80

$\Rightarrow \quad m(ax_2 + by_2 + c) + n(ax_1 + by_1 + c) = 0$

$$\Rightarrow \quad \frac{m}{n} = -\frac{a x_1 + by_1 + c}{ax_2 + by_2 + c} \qquad \text{...(iii)}$$

If the point R is between the points P and Q i.e. points P and Q are on the opposite sides of the given line, then the ratio $m : n$ is positive.

$$\therefore \quad -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) > 0$$

$$\Rightarrow \quad \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$$

$\Rightarrow \quad ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of opposite signs

If the point R is not between P and Q i.e. point P and Q are on the same side of the given line, then the ratio $m : n$ is negative.

$$\therefore \quad -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0 \qquad \text{[From (iii)]}$$

$$\Rightarrow \quad \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$$

$\Rightarrow \quad ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign.

Thus, the two points. (x_1, y_1) and (x_2, y_2) are on the same (or opposite) sides of the straight line $ax + by + c = 0$ according as the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same (or opposite) signs.

REMARK 1 *A point (x_1, y_1) will lie on the side of the origin relative to a line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and c have the same sign.*

REMARK 2 *A point (x_1, y_1) will lie on the opposite side of the origin relative to the line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and c have the opposite signs.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Are the points (3, 4) and (2, – 6) on the same or opposite sides of the line $3x - 4y = 8$?*

SOLUTION Let $Z = 3x - 4y - 8$. Then, the value of Z at (3, 4) is given by

$$Z_1 = 3 \times 3 - 4 \times 4 - 8 = 9 - 16 - 8 = -15 < 0$$

The value of Z at (2, – 6) is given by

$$Z_2 = 3 \times 2 - 4 \times -6 - 8 = 6 + 24 - 8 = 22 > 0$$

Since Z_1 and Z_2 are of opposite signs, therefore the two points are on the opposite sides of the given line.

EXAMPLE 2 *If the points* $(4, 7)$ *and* $(\cos\theta, \sin\theta)$, *where* $0 < \theta < \pi$, *lie on the same side of the line* $x + y - 1 = 0$, *then prove that* θ *lies in the first quadrant.*

SOLUTION If the points (4, 7) and $(\cos\theta, \sin\theta)$ lie on the same side of $x + y - 1 = 0$, then $4 + 7 - 1$ and $\cos\theta + \sin\theta - 1$ must be of the same sign.

$\therefore \quad \cos\theta + \sin\theta - 1 > 0$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta > \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos\theta \sin\frac{\pi}{4} + \sin\theta\cos\frac{\pi}{4} > \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \sin\left(\theta + \frac{\pi}{4}\right) > \frac{1}{\sqrt{2}} \Rightarrow \frac{\pi}{4} < \theta + \frac{\pi}{4} < \frac{3\pi}{4} \Rightarrow 0 < \theta < \frac{\pi}{2} \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$

Hence, θ lies in the first quadrant.

LEVEL-2

EXAMPLE 3 *Find the values of* β *so that the point* $(0, \beta)$ *lies on or inside the triangle having the sides* $3x + y + 2 = 0$, $2x - 3y + 5 = 0$ *and* $x + 4y - 14 = 0$.

SOLUTION Let ABC be the given triangle. The coordinates of the vertices of the triangle ABC are marked in Fig. 23.81. The point $P(0, \beta)$ will lie inside or on the triangle ABC, if the following three conditions hold simultaneously:

(i) A and P lie on the same side of BC (ii) B and P lie on the same side of AC,

(iii) C and P lie on the same side of AB.

Now,

A and P will lie on the same side of BC, if

$\quad (3 \times 2 + 3 + 2)(3 \times 0 + \beta + 2) > 0$

$\Rightarrow \quad 11(\beta + 2) \geq 0$

$\Rightarrow \quad \beta + 2 \geq 0$

$\Rightarrow \quad \beta \geq -2$ ...(i)

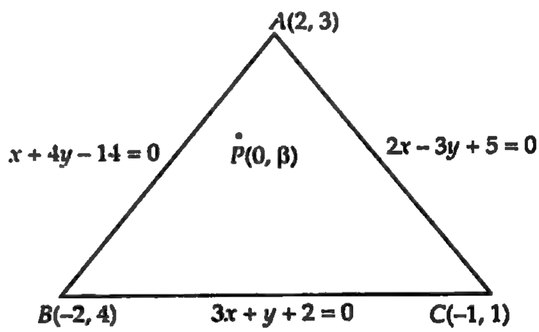


Fig. 23.81

B and P will lie on the same side of AC, if

$\quad (-2 \times 2 - 3 \times 4 + 5)(2 \times 0 - 3\beta + 5) \geq 0$

$\Rightarrow \quad -11(-3\beta + 5) \geq 0$

$\Rightarrow \quad 3\beta - 5 \geq 0$

$\Rightarrow \quad \beta \geq \frac{5}{3}$ ...(ii)

C and P will lie on the same side of AB, if

$\quad (-1 + 1 \times 4 - 14)(0 + 4\beta - 14) \geq 0$

$\Rightarrow \quad -22(2\beta - 7) \geq 0$

$\Rightarrow \quad 2\beta - 7 \leq 0$

$\Rightarrow \quad \beta \leq \frac{7}{2}$ ...(iii)

From (i), (ii) and (iii), we obtain that $\frac{5}{3} \leq \beta \leq \frac{7}{2}$ i.e., $\beta \in [5/3, 7/2]$

EXAMPLE 4 *Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$.*

SOLUTION Let ABC be the triangle, the equations of whose sides AB, BC and CA are respectively $2x + 3y - 1 = 0$ $x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$. The coordinates of the vertices are $A(1/3, 1/9)$, $B(-7, 5)$ and $C(5/4, 7/8)$. If the point $P(\alpha, \alpha^2)$ lies inside the ΔABC, then

(i) A and P must lie on the same side of BC

(ii) B and P must lie on the same side of AC

(iii) C and P must lie on the same side of AB.

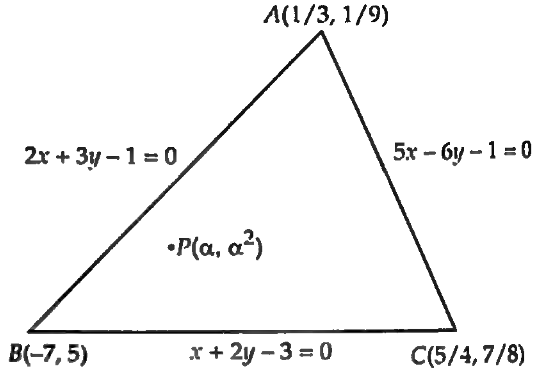


Fig. 23.82

Now,

A and P will lie on the same side of BC, if

$$\left(\frac{1}{3} + \frac{2}{9} - 3\right)\left(\alpha + 2\alpha^2 - 3\right) > 0$$

$\Rightarrow \quad \alpha + 2\alpha^2 - 3 < 0$

$\Rightarrow \quad 2\alpha^2 + \alpha - 3 < 0$

$\Rightarrow \quad (\alpha - 1)(2\alpha + 3) < 0$

$\Rightarrow \quad \alpha \in (-3/2, 1) \quad$ ...(i)

B and P will lie on the same side of CA, if

$$(-35 - 30 - 1)(5\alpha - 6\alpha^2 - 1) > 0$$

$\Rightarrow \quad 6\alpha^2 - 5\alpha + 1 > 0 \Rightarrow (3\alpha - 1)(2\alpha - 1) > 0 \Rightarrow \alpha \in (-\infty, 1/3) \cup (1/2, \infty) \quad$ ...(ii)

C and P will lie on the same side of AB, if

$$\left(\frac{5}{2} + \frac{21}{8} - 3\right)(2\alpha + 3\alpha^2 - 1) > 0$$

$\Rightarrow \quad 3\alpha^2 + 2\alpha - 1 > 0 \Rightarrow (\alpha + 1)(3\alpha - 1) > 0 \Rightarrow \alpha \in (-\infty, -1) \cup (1/3, \infty) \quad$ ...(iii)

From (i), (ii) and (iii), we obtain: $\alpha \in (-3/2, -1) \cup (1/2, 1)$

EXERCISE 23.14

LEVEL-2

1. Find the values of α so that the point $P(\alpha^2, \alpha)$ lies inside or on the triangle formed by the lines $x - 5y + 6 = 0$, $x - 3y + 2 = 0$ and $x - 2y - 3 = 0$.
2. Find the values of the parameter a so that the point $(a, 2)$ is an interior point of the triangle formed by the lines $x + y - 4 = 0$, $3x - 7y - 8 = 0$ and $4x - y - 31 = 0$.
3. Determine whether the point $(-3, 2)$ lies inside or outside the triangle whose sides are given by the equations $x + y - 4 = 0$, $3x - 7y + 8 = 0$, $4x - y - 31 = 0$.

ANSWERS

1. $\alpha \in [2, 3]$ 2. $a \in (22/3, 33/4)$ 3. Outside

23.13 DISTANCE OF A POINT FROM A LINE

THEOREM *Prove that the length of the perpendicular from a point (x_1, y_1) to a line $ax + by + c = 0$ is*

$$\left|\frac{a x_1 + b y_1 + c}{\sqrt{a^2 + b^2}}\right|$$

PROOF The line $ax + by + c = 0$ meets x-axis at $y = 0$. Therefore, putting $y = 0$ in $ax + by + c = 0$, we get $x = -\frac{c}{a}$. Thus, the coordinates of the point A where the line $ax + by + c = 0$ meets x-axis are $(-c/a, 0)$. Similarly, the coordinates of B where the line cuts y-axis are $(0, -c/b)$.

Let $P(x_1, y_1)$ be the point. Draw $PN \perp AB$.

Now,

$$\text{Area of } \Delta PAB = \frac{1}{2}\left| x_1\left(0 + \frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b} - y_1\right) + 0\,(y_1 - 0)\right|$$

$$= \frac{1}{2}\left| \frac{cx_1}{b} + \frac{cy_1}{a} + \frac{c^2}{ab}\right| = \left|(ax_1 + by_1 + c)\frac{c}{2\,ab}\right| \qquad \text{...(i)}$$

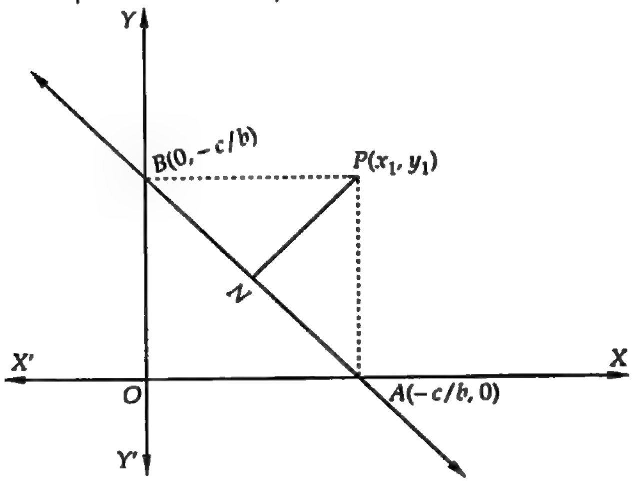


Fig. 23.83

Also,

$$\text{Area of } \Delta PAB = \frac{1}{2} AB \times PN = \frac{1}{2}\sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}} \times PN = \frac{c}{2\,ab}\sqrt{a^2 + b^2} \times PN \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\left|(ax_1 + by_1 + c)\frac{c}{2\,ab}\right| = \frac{c}{2\,ab}\sqrt{a^2 + b^2} \times PN$$

$$\Rightarrow \quad PN = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \qquad \text{Q.E.D.}$$

COROLLARY *The length of the perpendicular from the origin to the line $ax + by + c = 0$ is $\frac{|c|}{\sqrt{a^2 + b^2}}$.*

We may use the following algorithm for finding the length of the perpendicular from a point (x_1, y_1) to the line $ax + by + c = 0$.

ALGORITHM

STEP I *Write the equation of the line in the form $ax + by + c = 0$.*

STEP II *Substitute the coordinates x_1 and y_1 of the point in place of x and y respectively in the expression.*

STEP III *Divide the result obtained in step II by the square root of the sum of the squares of the coefficients of x and y.*

STEP IV *Take the modulus of the expression obtained in step III.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the distance between the line $12x - 5y + 9 = 0$ and the point (2, 1).*

SOLUTION Required distance $= \left|\dfrac{12 \times 2 - 5 \times 1 + 9}{\sqrt{12^2 + (-5)^2}}\right| = \dfrac{|24 - 5 + 9|}{13} = \dfrac{28}{13}$

EXAMPLE 2 *If p is the length of the perpendicular from the origin to the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$, then prove that*

(i) $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$ (ii) $a^4 + b^4 = 0$, if a^2, p^2, b^2 are in A.P. [NCERT]

SOLUTION (i) The equation given line is $bx + ay - ab = 0$ ...(i)

It is given that

p = Length of the perpendicular from the origin to line (i)

$$\Rightarrow \quad p = \frac{|b(0) + a(0) - ab|}{\sqrt{b^2 + a^2}} = \frac{ab}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \quad p^2 = \frac{a^2 b^2}{a^2 + b^2} \Rightarrow \frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2} \Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

(ii) If a^2, p^2, b^2 are in A.P, then $2p^2 = a^2 + b^2$.

Now, $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$ [from (i)]

$$\Rightarrow \quad p^2(a^2 + b^2) = a^2 b^2$$

$$\Rightarrow \quad \left(\frac{a^2 + b^2}{2}\right)(a^2 + b^2) = a^2 b^2 \qquad [\text{Using } 2p^2 = a^2 + b^2]$$

$$\Rightarrow \quad (a^2 + b^2)^2 = 2a^2 b^2 \Rightarrow a^4 + b^4 = 0.$$

EXAMPLE 3 *If p and p' be the perpendicular from the origin upon the straight lines $x \sec\theta + y \operatorname{cosec}\theta = a$ and $x \cos\theta - y \sin\theta = a \cos 2\theta$. Prove that: $4p^2 + p'^2 = a^2$.* [NCERT]

SOLUTION We have,

p = Length of the perpendicular from (0, 0) to $x \sec\theta + y \operatorname{cosec}\theta - a = 0$

$$\Rightarrow \quad p = \left|\frac{0 \sec\theta + 0 \operatorname{cosec}\theta - a}{\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}}\right| = \frac{a \cos\theta \sin\theta}{\sqrt{\sin^2\theta + \cos^2\theta}} = a \sin\theta \cos\theta$$

and, p' = Length of the perpendicular from (0, 0) to $x \cos\theta - y \sin\theta - a \cos 2\theta = 0$

$$\Rightarrow \quad p' = \left|\frac{0 \cos\theta - 0 \sin\theta - a \cos 2\theta}{\sqrt{\cos^2\theta + \sin^2\theta}}\right| = a \cos 2\theta.$$

Now, $4p^2 + p'^2 = 4a^2 \sin^2\theta \cos^2\theta + a^2 \cos^2 2\theta$

$$\Rightarrow \quad 4p^2 + p'^2 = a^2(2\sin\theta\cos\theta)^2 + a^2\cos^2 2\theta = a^2(\sin^2 2\theta + \cos^2 2\theta) = a^2.$$

EXAMPLE 4 *What are the points on x-axis whose perpendicular distance from the line $4x + 3y = 12$ is 4?*

SOLUTION Let the required point be $P(\alpha, 0)$. Then, Length of the perpendicular from $P(\alpha, 0)$ on $4x + 3y - 12 = 0$ is 4.

$\therefore \quad \left|\frac{4\alpha + 3 \times 0 - 12}{\sqrt{4^2 + 3^2}}\right| = 4$

$\Rightarrow \quad \left|\frac{4\alpha - 12}{5}\right| = 4 \Rightarrow |4\alpha - 12| = 20 \Rightarrow |\alpha - 3| = 5 \Rightarrow \alpha - 3 = \pm 5 \Rightarrow \alpha = 8, -2$

Hence, the required points are (8, 0) and (– 2, 0).

EXAMPLE 5 *Find the points on y-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3.*

SOLUTION Let te required point be $P(0, \alpha)$. It is given that the length of the perpendicular from $P(0, \alpha)$ on $4x - 3y - 12 = 0$ is 3.

$\therefore \quad \left|\frac{4 \times 0 - 3\alpha - 12}{\sqrt{4^2 + (-3)^2}}\right| = 3$

$\Rightarrow \quad |3\alpha + 12| = 15 \Rightarrow |\alpha + 4| = 15 \Rightarrow \alpha + 4 = \pm 5 \Rightarrow \alpha = 1, -9$

Hence, the required points are (0, 1) and (0, – 9).

EXAMPLE 6 *Find the equation of the straight line which cuts off intercept on X-axis which is twice that on Y-axis and is at a unit distance from the origin.* **[NCERT EXEMPLAR]**

SOLUTION Let the equation of the straight line be $\frac{x}{a} + \frac{y}{b} = 1$. It is given that $a = 2b$.

Putting $a = 2b$ in the above equation, we get

$x + 2y - 2b = 0$ ...(i)

This line is at a unit distance from the origin.

$\therefore \quad \left|\frac{0 + 2 \times 0 - 2b}{\sqrt{1^2 + 2^2}}\right| = 1 \Rightarrow \frac{|2b|}{\sqrt{5}} = 1 \Rightarrow |2b| = \sqrt{5} \Rightarrow 2b = \pm\sqrt{5} \Rightarrow b = \pm\frac{\sqrt{5}}{2}$

Substituting the value of b in (i), we obtain $x + 2y \pm \sqrt{5} = 0$ as the equations of the required line.

EXAMPLE 7 *The equation of the base of an equilateral triangle is $x + y - 2 = 0$ and the opposite vertex has coordinates $(2, -1)$. Find the area of the triangle.* **[NCERT EXEMPLAR]**

SOLUTION Let p be the altitude of the given triangle and 'a' be the length of each side. Then,

$p =$ Length of perpendicular from $(2, -1)$ on $x + y - 2 = 0$

$\Rightarrow p = \left|\frac{2 - 1 - 2}{\sqrt{1^2 + 1^2}}\right| = \frac{1}{\sqrt{2}}$

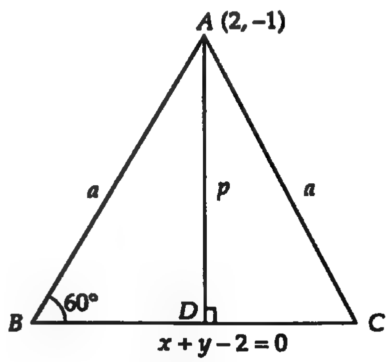


Fig. 23.84

In ΔABD, we have

$\sin 60° = \frac{p}{a} \Rightarrow \frac{\sqrt{3}}{2} = \frac{p}{a} \Rightarrow a = \frac{2p}{\sqrt{3}} \Rightarrow a = \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{2}} = \sqrt{\frac{2}{3}}$

$\therefore$ Area of the triangle $= \frac{\sqrt{3}a^2}{4} = \frac{\sqrt{3}}{4} \times \left(\sqrt{\frac{2}{3}}\right)^2 = \frac{1}{2\sqrt{3}}$ sq. units

EXAMPLE 8 *Prove that the length of perpendiculars from points $P(m^2, 2m)$, $Q(mn, m+n)$ and $R(n^2, 2n)$ to the line $x\cos^2\theta + y\sin\theta\cos\theta + \sin^2\theta = 0$ are in G.P.*

SOLUTION Let a, b and c denote the lengths of perpendiculars drawn from P, Q and R respectively on the line $x\cos^2\theta + y\sin\theta\cos\theta + \sin^2\theta = 0$. Then,

$$a = \left|\frac{m^2\cos^2\theta + 2m\sin\theta\cos\theta + \sin^2\theta}{\sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}}\right| = \left|\frac{(m\cos\theta + \sin\theta)^2}{\cos\theta}\right| \quad ...(i)$$

$$b = \left|\frac{mn\cos^2\theta + (m+n)\sin\theta\cos\theta + \sin^2\theta}{\sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}}\right| = \left|\frac{(m\cos\theta + \sin\theta)(n\cos\theta + \sin\theta)}{\cos\theta}\right| \quad ...(ii)$$

and, $$c = \left|\frac{n^2\cos^2\theta + 2n\sin\theta\cos\theta + \sin^2\theta}{\sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}}\right| = \left|\frac{(n\cos\theta + \sin\theta)^2}{\cos\theta}\right| \quad ...(iii)$$

$\therefore$ $$b^2 = \frac{(m\cos\theta + \sin\theta)^2 (n\cos\theta + \sin\theta)^2}{\cos^2\theta} = \frac{(m\cos\theta + \sin\theta)^2}{\cos\theta} \times \frac{(n\cos\theta + \sin\theta)^2}{\cos\theta}$$

$\Rightarrow$ $b^2 = ac$

$\Rightarrow$ a, b, c are in G.P.

LEVEL-2

EXAMPLE 9 *Find the coordinates of a point on $x + y + 3 = 0$, whose distance from $x + 2y + 2 = 0$ is $\sqrt{5}$.*

SOLUTION Let the required point be (x_1, y_1). Since it lies on $x + y + 3 = 0$.

$\therefore$ $x_1 + y_1 + 3 = 0$ ...(i)

Now,

Length of the perpendicular from (x_1, y_1) to $x + 2y + 2 = 0$ is $\sqrt{5}$.

$\Rightarrow$ $$\left|\frac{x_1 + 2y_1 + 2}{\sqrt{1^2 + 2^2}}\right| = \sqrt{5} \Rightarrow x_1 + 2y_1 + 2 = \pm 5 \quad ...(ii)$$

Solving equation (i) and (ii), we get: $x_1 = -9$, $y_1 = 6$ and $x_1 = 1$, $y_1 = -4$.

Hence, the required points are $(-9, 6)$ and $(1, -4)$.

ALITER Putting $x = t$ in $x + y + 3 = 0$, we get: $y = -3 - t$.

So, let the required point be $(t, -3 - t)$. This point is at a distance of $\sqrt{5}$ units from $x + 2y + 2 = 0$.

$\therefore$ $$\left|\frac{t - 6 - 2t + 2}{\sqrt{1^2 + 2^2}}\right| = \sqrt{5} \Rightarrow \left|\frac{-t-4}{\sqrt{5}}\right| = \sqrt{5} \Rightarrow t + 4 = \pm 5 \Rightarrow t = 1, -9$$

Hence, the required points are $(1, -4)$ and $(-9, 6)$.

EXAMPLE 10 *Find all points on $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$.*

SOLUTION Note that the coordinates of an arbitrary point on $x + y = 4$ can be obtained by putting $x = t$ (or $y = t$) and then obtaining y (or x) from the equation of the line, where t is a parameter. Putting $x = t$ in the equation $x + y = 4$ of the given line, we obtain $y = 4 - t$.

So, coordinates of an arbitrary point on the given line are $P(t, 4 - t)$.

Let $P(t, 4 - t)$ be the required point. Then, distance of P from the line $4x + 3y - 10 = 0$ is unity.

$\therefore$ $$\left|\frac{4t + 3(4 - t) - 10}{\sqrt{4^2 + 3^2}}\right| = 1 \Rightarrow |t + 2| = 5 \Rightarrow t + 2 = \pm 5 \Rightarrow t = -7 \text{ or, } t = 3$$

Hence, required points are $(-7, 11)$ and $(3, 1)$.

EXAMPLE 11 *Find the equations of lines passing through the point (1, 0) and at a distance $\frac{\sqrt{3}}{2}$ from the origin.* [NCERT EXEMPLAR]

SOLUTION Let m be the slope of a line passing through (1, 0). Then, its equation is

$$y-0=m(x-1) \text{ or, } mx-y-m=0 \quad \ldots(i)$$

It is given that line (i) is at a distance $\frac{\sqrt{3}}{2}$ from the origin.

$$\therefore \quad \left|\frac{m\times 0-0-m}{\sqrt{m^2+(-1)^2}}\right|=\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \frac{|m|}{\sqrt{m^2+1}}=\frac{\sqrt{3}}{2} \Rightarrow 4m^2=3m^2+3 \Rightarrow m^2=3 \Rightarrow m=\pm\sqrt{3}$$

Substituting the values of m in (i), we obtain

$\sqrt{3}x-y+\sqrt{3}=0$ and $\sqrt{3}x+y+\sqrt{3}=0$ as the equations of the line.

EXAMPLE 12 *Find the locus of a point which moves in such away that the square of its distnace from the point (3, −2) is numerically equal to its distance from the line $5x-12y=13$.* [NCERT EXEMPLAR]

SOLUTION Let $P(h, k)$ be a variable point moving in such a way that the square of its distance from $A(3, -2)$ is numerically equal to its distance from the line $5x-12y=13$.

$$\therefore \quad (h-3)^2+(k+2)^2=\frac{|5h-12k+13|}{\sqrt{5^2+(-12)^2}}$$

$$\Rightarrow \quad 13\left\{(h-3)^2+(k+2)^2\right\}=\pm(5h-12k+13)$$

$$\Rightarrow \quad 13(h^2+k^2)-83h+64k+182=0 \text{ or, } 13(h^2+k^2)-73h+40k+156=0$$

Hence, the locus of (h, k) is

$$13(x^2+y^2)-83x+64y+182=0 \text{ or, } 13(x^2+y^2)-73x+40y+156=0$$

EXAMPLE 13 *A point moves such that its distance from the point (4, 0) is half that of its distance from the line $x=16$, find its locus.* [NCERT EXEMPLAR]

SOLUTION Let $P(h, k)$ be the variable point.

By hypothesis

$$\sqrt{(h-4)^2+(k-0)^2}=\frac{1}{2}\left|\frac{h-16}{\sqrt{1^2+0^2}}\right|$$

$$\Rightarrow \quad 4\left\{(h-4)^2+k^2\right\}=(h-16)^2 \Rightarrow 3h^2+4k^2=192$$

Hence, the locus of (h, k) is $3x^2+4y^2=192$.

EXERCISE 23.15

LEVEL-1

1. Find the distance of the point (4, 5) from the straight line $3x-5y+7=0$.
2. Find the perpendicular distance of the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ from the origin.
3. Find the length of the perpendicular from the origin to the straight line joining the two points whose coordinates are $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$.

4. Show that the perpendiculars let fall from any point on the straight line $2x + 11y - 5 = 0$ upon the two straight lines $24x + 7y = 20$ and $4x - 3y - 2 = 0$ are equal to each other.
5. Find the distance of the point of intersection of the lines $2x + 3y = 21$ and $3x - 4y + 11 = 0$ from the line $8x + 6y + 5 = 0$.
6. Find the length of the perpendicular from the point $(4, -7)$ to the line joining the origin and the point of intersection of the lines $2x - 3y + 14 = 0$ and $5x + 4y - 7 = 0$.
7. What are the points on X-axis whose perpendicular distance from the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is a?
8. Show that the product of perpendiculars on the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ from the points $(\pm\sqrt{a^2 - b^2}, 0)$ is b^2. [NCERT]
9. Find the perpendicular distance from the origin of the perpendicular from the point $(1, 2)$ upon the straight line $x - \sqrt{3}y + 4 = 0$.
10. Find the distance of the point $(1, 2)$ from the straight line with slope 5 and passing through the point of intersection of $x + 2y = 5$ and $x - 3y = 7$. [NCERT]
11. What are the points on y-axis whose distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units?
12. In the triangle ABC with vertices $A(2, 3)$, $B(4, -1)$ and $C(1, 2)$, find the equation and the length of the altitude from the vertex A. [NCERT]
13. Show that the path of a moving point such that its distances from two lines $3x - 2y = 5$ and $3x + 2y = 5$ are equal is a straight line. [NCERT]
14. If sum of perpendicular distances of a variable point $P(x, y)$ from the lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$ is always 10. Show that P must move on a line. [NCERT]

LEVEL-2

15. If the length of the perpendicular from the point $(1, 1)$ to the line $ax - by + c = 0$ be unity, show that $\frac{1}{c} + \frac{1}{a} - \frac{1}{b} = \frac{c}{2ab}$.

ANSWERS

1. $\frac{6}{\sqrt{34}}$ 2. $\cos\left(\frac{\theta - \phi}{2}\right)$ 3. $a\cos\left(\frac{\alpha - \beta}{2}\right)$ 5. $\frac{59}{10}$

6. 1 7. $\left\{\frac{a}{b}\left(b \pm \sqrt{a^2 + b^2}\right), 0\right\}$ 9. $\frac{1}{2}(2 + \sqrt{3})$

10. $\frac{132}{\sqrt{650}}$ 11. $(0, 32/3), (0, -8/3)$ 12. $x - y + 1 = 0, \sqrt{2}$

HINTS TO NCERT & SELECTED PROBLEMS

2. The equation of the line joining $(\cos\theta, \sin\theta)$ and $(\cos\theta, \sin\phi)$ is

$$x\cos\left(\frac{\theta+\phi}{2}\right) + y\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

$\therefore$ Required distance $= \left|\dfrac{0\cos\left(\frac{\theta+\phi}{2}\right) + 0\sin\left(\frac{\theta+\phi}{2}\right) - \cos\left(\frac{\theta+\phi}{2}\right)}{\sqrt{\cos^2\left(\frac{\theta+\phi}{2}\right) + \sin^2\left(\frac{\theta+\phi}{2}\right)}}\right| = \cos\left(\frac{\theta-\phi}{2}\right)$

8. Let p_1 and p_2 be the lengths of perpendiculars drawn from points $P(\sqrt{a^2-b^2}, 0)$ and $Q(-\sqrt{a^2-b^2}, 0)$ on the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$. Then,

$$p_1 = \left|\frac{\frac{\sqrt{a^2-b^2}}{a}\cos\theta + \frac{0}{b}\sin\theta - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right|, \quad p_2 = \left|\frac{-\frac{\sqrt{a^2-b^2}}{a}\cos\theta + \frac{0}{b}\sin\theta - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right|$$

$$\therefore \quad p_1 p_2 = \left|\frac{\frac{\sqrt{a^2-b^2}}{a}\cos\theta - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right| \times \left|\frac{\frac{\sqrt{a^2-b^2}}{a}\cos\theta + 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right|$$

$$\Rightarrow \quad p_1 p_2 = \left|\frac{\left(\frac{\sqrt{a^2-b^2}}{a}\cos\theta - 1\right)\left(\frac{\sqrt{a^2-b^2}}{a}\cos\theta + 1\right)}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2 - \theta}{b^2}}\right|$$

$$\Rightarrow \quad p_1 p_2 = \left|\frac{\frac{a^2-b^2}{a^2}\cos^2\theta - 1}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}\right| = \left|\frac{(a^2-b^2)\cos^2\theta - a^2}{b^2\cos^2\theta + a^2\sin^2\theta}\right| b^2$$

$$\Rightarrow \quad p_1 p_2 = \left|\frac{-b^2\cos^2\theta - a^2\sin^2\theta}{b^2\cos^2\theta + a^2\sin^2\theta}\right| b^2 = b^2$$

10. Given lines $x + 2y = 5$ and $x - 3y = 7$ intersect at $\left(\frac{29}{5}, -\frac{2}{5}\right)$.

The equation of the line of slope 5 passing through this point is

$$y + \frac{2}{5} = 5\left(x - \frac{29}{5}\right) \text{ or, } 25x - 5y - 147 = 0$$

The distance d of the point (1, 2) from this line is

$$d = \left|\frac{25 - 10 - 147}{\sqrt{625 + 25}}\right| = \frac{132}{\sqrt{650}}$$

11. Let the required point on y-axis be $(0, a)$. Then,

$$\left|\frac{4 \times 0 + 3 \times a - 12}{\sqrt{4^2 + 3^2}}\right| = 4 \Rightarrow \left|\frac{3a - 12}{5}\right| = 4$$

$$\Rightarrow \quad 3a - 12 = \pm 20 \Rightarrow 3a = 32, -8 \Rightarrow a = \frac{32}{3}, -\frac{8}{3}$$

12. The equation of BC is

$$y+1 = \frac{2+1}{1-4}(x-4) \text{ or, } y+1 = -x+4$$

or, $$x+y-3 = 0$$

$\therefore$ $$AL = \left|\frac{2+3-3}{\sqrt{1+1}}\right| = \sqrt{2}$$

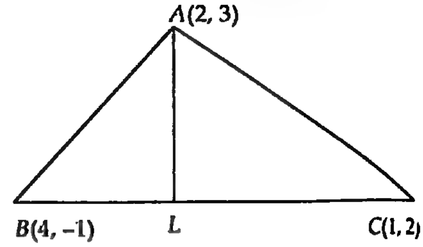


Fig. 23.85

Clearly, slope of BC having equation $x+y-3=0$ is -1. So, Slope of AL is 1. As it passes through A (2, 3). So, its equation is $y-3=1\,(x-2)$ or $x-y+1=0$

13. Let $P\,(h, k)$ be a moving point such that it is equidistant from the lines $3x-2y-5=0$ and $3x+2y-5=0$. Then,

$$\left|\frac{3h-2k-5}{\sqrt{9+4}}\right| = \left|\frac{3h+2k-5}{\sqrt{9+4}}\right|$$

$\Rightarrow$ $$|\,3h-2k-5\,| = |\,3h+2k-5\,|$$

$\Rightarrow$ $$3h-2k-5 = \pm(3h+2k-5)$$

$\Rightarrow$ $$4k = 0 \text{ or, } 6h-10 = 0 \Rightarrow k = 0 \text{ or, } 3h = 5$$

Hence, the locus of (h, k) is $y=0$ or $3x=5$, which are straight lines.

14. It is given that the sum of the perpendicular distances of a variable point $P\,(x, y)$ from the lines $x+y-5=0$ and $3x-2y+7=0$ is always 10.

$\therefore$ $$\frac{(x+y-5)}{\sqrt{2}} + \frac{(3x-2y+7)}{\sqrt{9+4}} = 10$$

$\Rightarrow$ $$(3\sqrt{2}+\sqrt{13})\,x + (\sqrt{13}-2\sqrt{2})\,y + (7\sqrt{2}-5\sqrt{13}-10\sqrt{26}) = 0$$

Clearly, it is a straight line.

23.14 DISTANCE BETWEEN PARALLEL LINES

If two lines are parallel, then they have the same distance between them throughout. Therefore to find the distance between two parallel lines choose an arbitrary point on one of them and find the length of the perpendicular on the other. In order to choose a point on a line, we give an arbitrary value to x or y and find the value of the other variable.

We may use the following algorithm to find the distance between two parallel lines.

ALGORITHM

Let the two parallel lines be $ax+by+c_1=0$ and $ax+by+c_2=0$. To find the distance between these two lines we proceed as follows:

STEP I *Choose a point on any one of the two lines by giving a particular value to x or y of your choice.*

STEP II *Find the length of the perpendicular from the chosen point in step I to the other line.*

STEP III *The length obtained in step II is the required distance between the parallel lines.*

THEOREM *Prove that the distance between two parallel lines* $ax+by+c_1 = 0$ *and* $ax+by+c_2 = 0$ *is given by* $\dfrac{|c_1-c_2|}{\sqrt{a^2+b^2}}$.

PROOF Given lines are

$$ax + by + c_1 = 0 \qquad ...(i)$$
$$ax + by + c_2 = 0 \qquad ...(ii)$$

Let $P(h, k)$ be a point on the line $ax + by + c_1 = 0$. Then,

$$ah + bk + c_1 = 0 \qquad ...(iii)$$

Clearly, distance 'd' between parallel lines (i) and (ii) is equal to the length of perpendicular from P on line (ii).

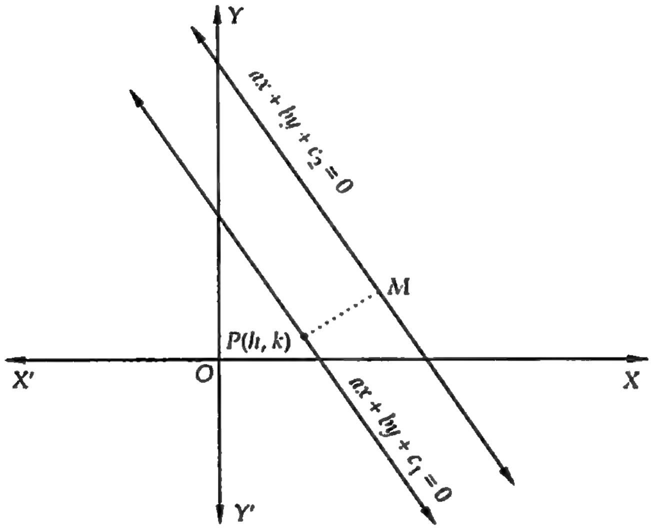


Fig. 23.86

$\therefore \quad d = PM$

$$\Rightarrow \quad d = \left|\frac{ah + bk + c_2}{\sqrt{a^2 + b^2}}\right| = \left|\frac{-c_1 + c_2}{\sqrt{a^2 + b^2}}\right| \qquad [\text{From (iii): } ah + bk = -c_1]$$

$$\Rightarrow \quad d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} \qquad \text{Q.E.D.}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the distance between the parallel lines* $3x - 4y + 9 = 0$ *and* $6x - 8y - 15 = 0$.

SOLUTION Putting $y = 0$ in $3x - 4y + 9 = 0$, we get $x = -3$. Thus, $(-3, 0)$ is a point on the line $3x - 4y + 9$.

Length of the perpendicular from $(-3, 0)$ to $6x - 8y - 15 = 0$ is given by

$$d = \frac{|-3 \times 6 - 8 \times 0 - 15|}{\sqrt{6^2 + (-8)^2}} = \frac{33}{10}$$

Hence, the distance between the given lines is $\frac{33}{10}$ units.

ALITER Given lines are

$$3x - 4y + 9 = 0 \quad ...(i) \qquad \text{and,} \quad 6x - 8y - 15 = 0 \;\text{ or, }\; 3x - 4y - \frac{15}{2} = 0 \quad ...(ii)$$

$$\therefore \quad \text{Required distance} = \frac{\left|9 - \left(\frac{-15}{2}\right)\right|}{\sqrt{3^2 + (-4)^2}} = \frac{9 + \frac{15}{2}}{5} = \frac{33}{10}$$

EXAMPLE 2 *Find the equations of lines parallel to $3x-4y-5=0$ at a unit distance from it.*

SOLUTION Equation of any line parallel to $3x-4y-5=0$ is

$$3x-4y+\lambda=0 \quad \text{...(i)}$$

Putting $x=-1$ in $3x-4y-5=0$, we get $y=-2$. Therefore, $(-1,-2)$ is a point on $3x-4y-5=0$.

Since the distance between the two lines is one unit. Therefore, the length of the perpendicular from $(-1,-2)$ to $3x-4y+\lambda=0$ is one unit.

i.e. $$\frac{|3\times-1-4\times-2+\lambda|}{\sqrt{3^2+(-4)^2}}=1$$

$$\Rightarrow \quad \frac{|5+\lambda|}{5}=1 \Rightarrow |5+\lambda|=5 \Rightarrow 5+\lambda=\pm 5 \Rightarrow \lambda=0 \text{ or } -10.$$

Substituting the values of λ in (i), we get

$3x-4y=0$ and, $3x-4y-10=0$ as the equations of the required lines.

ALITER Let the equation of a line parallel to $3x-4y-5=0$ be

$$3x-4y+\lambda=0 \quad \text{...(i)}$$

It is given that the distance between the line $3x-4y-5=0$ and line (i) is 1 unit.

$$\therefore \quad \frac{|\lambda-(-5)|}{\sqrt{3^2+(-4)^2}}=1 \Rightarrow \frac{|\lambda+5|}{5}=1 \Rightarrow |\lambda+5|=5 \Rightarrow \lambda+5=\pm 5 \Rightarrow \lambda=0,-10.$$

Substituting the values of λ in (i), we get $3x-4y=0$ and $3x-4y-10=0$ as the equations of required lines.

EXAMPLE 3 *Two sides of a square lie on the lines $x+y=1$ and $x+y+2=0$. What is its area?*

SOLUTION clearly, the length of the side of the square is equal to the distance between the parallel lines

$$x+y-1=0 \quad \text{...(i)} \qquad \text{and} \qquad x+y+2=0 \quad \text{...(ii)}$$

Putting $x=0$ in (i), we get $y=1$. So $(0,1)$ is a point on line (i).

$\therefore$ Distance between the parallel lines

$$= \{\text{Length of the perpendicular from } (0,1) \text{ to } x+y+2=0\} = \frac{|0+1+2|}{\sqrt{1^2+1^2}}=\frac{3}{\sqrt{2}}.$$

Thus, the length of the side of the square is $\frac{3}{\sqrt{2}}$ and hence its area is $\left(\frac{3}{\sqrt{2}}\right)^2=\frac{9}{2}$ square units

ALITER The equations of parallel sides of the square are $x+y-1=0$ and $x+y+2=0$.

$\therefore$ Length of the side of the square $=$ Distance between parallel side $=\frac{|2-(-1)|}{\sqrt{1^2+1^2}}=\frac{3}{\sqrt{2}}$

Hence, Area of the square $=(\text{Side})^2=\left(\frac{3}{\sqrt{2}}\right)^2=\frac{9}{2}$ sq. units.

EXAMPLE 4 *Prove that the line $5x-2y-1=0$ is mid-parallel to the lines $5x-2y-9=0$ and $5x-2y+7=0$.*

SOLUTION Clearly, the slope of each of the given lines is same equal to $5/2$. Hence, the line $5x-2y-1=0$ is parallel to each of the given lines.

In order to prove that the line $5x-2y-1=0$ is mid-parallel to the given lines it is sufficient to show that the line $5x-2y-1=0$ is equidistant from the given lines.

Putting $y = 0$ in $5x - 2y - 1 = 0$, we get $x = 1/5$. So, the coordinates of a point on $5x - 2y - 1 = 0$ are $(1/5, 0)$.

The distance d_1 between the lines $5x - 2y - 1 = 0$ and $5x - 2y - 9 = 0$ is given by

$$d_1 = \text{Length of the perpendicular from } (1/5, 0) \text{ to } 5x - 2y - 9 = 0$$

$$\Rightarrow \quad d_1 = \left|\frac{5 \times (1/5) - 2 \times 0 - 9}{\sqrt{5^2 + (-2)^2}}\right| = \frac{8}{\sqrt{29}}$$

The distance d_2 between the lines $5x - 2y - 1 = 0$ and $5x - 2y + 7 = 0$ is given by

$$d_2 = \text{Length of the perpendicular } (1/5, 0) \text{ to } 5x - 2y + 7 = 0$$

$$\Rightarrow \quad d_2 = \left|\frac{5 \times (1/5) - 2 \times 0 + 7}{\sqrt{5^2 + (-2)^2}}\right| = \frac{8}{\sqrt{29}}$$

Clearly, $d_1 = d_2$. Consequently the line $5x - 2y - 1 = 0$ is equidistant from the lines $5x - 2y - 9 = 0$ and $5x - 2y + 7 = 0$. Hence, the result follows.

LEVEL-2

EXAMPLE 5 *Prove that the parallelogram formed by the lines* $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{b} + \frac{y}{a} = 1$, $\frac{x}{a} + \frac{y}{b} = 2$ *and* $\frac{x}{b} + \frac{y}{a} = 2$ *is a rhombus.*

SOLUTION Let the given straight lines be AB, BC, CD and CA whose equations are respectively

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{...(i)} \qquad \frac{x}{b} + \frac{y}{a} = 1 \quad \text{...(ii)} \qquad \frac{x}{a} + \frac{y}{b} = 2 \quad \text{...(iii)} \quad \text{and} \quad \frac{x}{b} + \frac{y}{a} = 2 \quad \text{...(iv)}$$

Putting $y = 0$ in (i) and (ii), we get $x = a$ and $x = b$ respectively. So, the coordinate points on lines (i) and (ii) are (a, 0) and (b, 0) respectively.

Now, d_1 = Distance between the parallel lines (i) and (iii)

$\Rightarrow$ d_1 = Length of the perpendicular drawn from $(a, 0)$ upon the line (iii)

$$\Rightarrow \quad d_1 = \left|\frac{\frac{a}{a} + \frac{0}{b} - 2}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}\right| = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{ab}{\sqrt{a^2 + b^2}}$$

and, d_2 = Distance between the parallel lines (ii) and (iv)

$\Rightarrow$ d_2 = Length of the perpendicular drawn from $(b, 0)$ upon the line (iv)

$$\Rightarrow \quad d_2 = \left|\frac{\frac{b}{b} + \frac{0}{a} - 2}{\sqrt{\frac{1}{b^2} + \frac{1}{a^2}}}\right| = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{ab}{\sqrt{a^2 + b^2}}$$

Clearly, $d_1 = d_2$ i.e. the distances between the pairs of parallel lines are equal. Hence, $ABCD$ is a rhombus.

EXAMPLE 6 *Find the equation of the line mid-way between the parallel lines* $9x + 6y - 7 = 0$ *and* $3x + 2y + 6 = 0$. **[NCERT]**

SOLUTION The equations of the lines are

$$3x + 2y - \frac{7}{3} = 0 \quad \text{...(i)}$$

$$3x + 2y + 6 = 0 \quad \text{...(ii)}$$

Let the equation of the line mid-way between the parallel lines (i) and (ii) be

$$3x + 2y + \lambda = 0 \qquad \text{...(iii)}$$

Then,

Distance between lines (i) and (iii) = Distance between lines (ii) and (iii)

$$\Rightarrow \quad \frac{\left|\lambda + \frac{7}{3}\right|}{\sqrt{9+4}} = \frac{|\lambda - 6|}{\sqrt{9+4}}$$

$$\Rightarrow \quad \left|\lambda + \frac{7}{3}\right| = |\lambda - 6| \Rightarrow \lambda + \frac{7}{3} = -\lambda + 6 \Rightarrow 2\lambda = 6 - \frac{7}{3} \Rightarrow 2\lambda = \frac{11}{3} \Rightarrow \lambda = \frac{11}{6}$$

Hence, the equation of the required line is $3x + 2y + \frac{11}{6} = 0$.

EXERCISE 23.16

LEVEL-1

1. Determine the distance between the following pair of parallel lines :
   (i) $4x - 3y - 9 = 0$ and $4x - 3y - 24 = 0$
   (ii) $8x + 15y - 34 = 0$ and $8x + 15y + 31 = 0$
   (iii) $y = mx + c$ and $y = mx + d$
   (iv) $4x + 3y - 11 = 0$ and $8x + 6y = 15$
2. The equations of two sides of a square are $5x - 12y - 65 = 0$ and $5x - 12y + 26 = 0$. Find the area of the square.
3. Find the equation of two straight lines which are parallel to $x + 7y + 2 = 0$ and at unit distance from the point $(1, -1)$.
4. Prove that the lines $2x + 3y = 19$ and $2x + 3y + 7 = 0$ are equidistant from the line $2x + 3y = 6$.
5. Find the equation of the line mid-way between the parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.
6. Find the ratio in which the line $3x + 4y + 2 = 0$ divides the distance between the lines $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$. [NCERT EXEMPLAR]

ANSWERS

1. (i) 3 units (ii) $\frac{65}{17}$ units (iii) $\frac{|c-d|}{\sqrt{1+m^2}}$ (iv) $\frac{7}{10}$ units

2. 49 sq. units 3. $x + 7y + 6 \pm 5\sqrt{2} = 0$ 5. $18x + 12y + 11 = 0$ 6. 3 : 7

23.15 AREA OF A PARALLELOGRAM

Let $ABCD$ be a parallelogram the equations of whose sides AB, BC, CD and DA are $a_1 x + b_1 y + c_1 = 0$, $a_2 x + b_2 y + c_2 = 0$, $a_1 x + b_1 y + d_1 = 0$ and $a_2 x + b_2 y + d_2 = 0$. Let p_1 and p_2 be the distances between the pairs of parallel sides of the parallelogram. In Δs ALD and AMB, we obtain

$$\sin\theta = \frac{p_1}{AD} \text{ and, } \sin\theta = \frac{p_2}{AB} \text{ respectively.}$$

$$\Rightarrow \quad AD = \frac{p_1}{\sin\theta} \text{ and } AB = \frac{p_2}{\sin\theta}$$

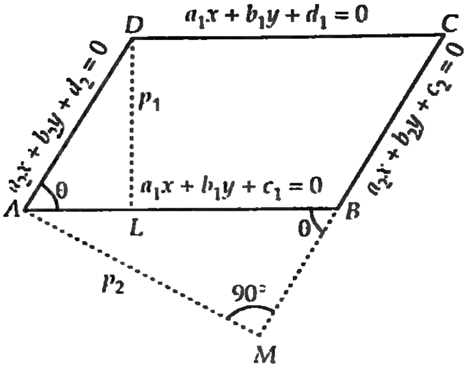


Fig. 23.87

Now,

$$\text{Area of parallelogram } ABCD = AB \times p_1 = \frac{p_1 p_2}{\sin\theta} \qquad \left[\because AB = \frac{p_2}{\sin\theta}\right]$$

Also, $$\text{Area of parallelogram } ABCD = AD \times p_2 = \frac{p_1 p_2}{\sin\theta} \qquad \left[\because AD = \frac{p_1}{\sin\theta}\right]$$

Thus, area of a parallelogram is $\frac{p_1 p_2}{\sin\theta}$, *where* p_1 *and* p_2 *are the distances between pairs of parallel sides and* θ *is the angle between two adjacent sides.*

Let m_1 and m_2 be the slopes of sides AB and AD respectively. Then,

$$m_1 = -\frac{a_1}{b_1} \text{ and, } m_2 = -\frac{a_2}{b_2}$$

The angle θ between AB and AD is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \quad \tan\theta = \frac{-\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 + \frac{a_1 a_2}{b_1 b_2}}$$

$$\Rightarrow \quad \tan\theta = \frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2} \Rightarrow \sin\theta = \frac{a_2 b_1 - a_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

We have,

$$p_1 = \text{Distance between parallel sides } AB \text{ and } AD = \frac{|c_1 - d_1|}{\sqrt{a_1^2 + b_1^2}}$$

and, $$p_2 = \text{Distance between parallel sides } AD \text{ and } BC = \frac{|c_2 - d_2|}{\sqrt{a_2^2 + b_2^2}}$$

$$\therefore \quad \text{Area of parallelogram } ABCD = \frac{p_1 p_2}{\sin\theta} = \frac{|c_1 - d_1||c_2 - d_2|}{|a_2 b_1 - a_1 b_2|} = \left|\frac{(c_1 - d_1)(c_2 - d_2)}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\right|$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Show that the area of the parallelogram formed by the lines* $2x-3y+a=0$, $3x-2y-a=0$, $2x-3y+3a=0$ *and* $3x-2y-2a=0$ *is* $\frac{2a^2}{5}$ *sq. units.*

SOLUTION Using the above formula, we obtain

$$\text{Area of given parallelogram} = \left|\frac{(3a-a)\{-2a-(-a)\}}{\begin{vmatrix}2 & -3\\ 3 & -2\end{vmatrix}}\right| = \frac{2a^2}{5} \text{ sq. units}$$

EXAMPLE 2 *Prove that the area of the parallelogram formed by the lines* $x\cos\alpha+y\sin\alpha=p$, $x\cos\alpha+y\sin\alpha=q$, $x\cos\beta+y\sin\beta=r$ *and* $x\cos\beta+y\sin\beta=s$ *is* $\pm(p-q)(r-s)\operatorname{cosec}(\alpha-\beta)$.

SOLUTION The equations of the sides of the parallelogram are:

$x\cos\alpha+y\sin\alpha-p=0$, $x\cos\alpha+y\sin\alpha-q=0$, $x\cos\beta+y\sin\beta-r=0$

and, $x\cos\beta+y\sin\beta-s=0$

$$\therefore \quad \text{Area of the parallelogram} = \left|\frac{\{(-p)-(-q)\}\{(-r)-(-s)\}}{\begin{vmatrix}\cos\alpha & \sin\alpha\\ \cos\beta & \sin\beta\end{vmatrix}}\right|$$

$$= \left|\frac{(q-p)(s-r)}{(\cos\alpha\sin\beta-\sin\alpha\cos\beta)}\right| = \pm\frac{(p-q)(r-s)}{\sin(\alpha-\beta)}$$

LEVEL-2

EXAMPLE 3 *Prove that the four straight lines* $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{b}+\frac{y}{a}=1$, $\frac{x}{a}+\frac{y}{b}=2$ *and* $\frac{x}{b}+\frac{y}{a}=2$ *form a rhombus. Find its area.*

SOLUTION The equations of the four sides are

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{...(i)} \qquad \frac{x}{b}+\frac{y}{a}=1 \quad \text{...(ii)}$$

$$\frac{x}{a}+\frac{y}{b}=2 \quad \text{...(iii)} \qquad \frac{x}{b}+\frac{y}{a}=2 \quad \text{...(iv)}$$

Clearly, (i), (iii) and (ii), (iv) form two sets of parallel lines. So, the four lines form a parallelogram.

Let p_1 be the distance between parallel lines (i) and (iii) and p_2 be the distance between (ii) and (iv). Then,

$$p_1 = \left|\frac{2-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right| = \frac{ab}{\sqrt{a^2+b^2}} \text{ and, } p_2 = \left|\frac{2-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right| = \frac{ab}{\sqrt{a^2+b^2}} \quad \left[\text{Using}: d = \left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|\right]$$

Clearly, $p_1=p_2$. So, the given lines form a rhombus.

$$\therefore \quad \text{Area of the rhombus} = \left|\frac{(2-1)(2-1)}{\begin{vmatrix}1/a & 1/b\\ 1/b & 1/a\end{vmatrix}}\right| = \left|\frac{1}{(1/a^2-1/b^2)}\right| = \frac{a^2b^2}{|b^2-a^2|}$$

EXAMPLE 4 *Show that the four lines $ax \pm by \pm c = 0$ enclose a rhombus whose area is $2c^2/ab$.*

SOLUTION The four lines are

$ax + by + c = 0$ ...(i) $\qquad ax + by - c = 0$ ...(ii)

$ax - by + c = 0$ ...(iii) $\qquad ax - by - c = 0$ ...(iv)

Clearly, (i), (ii) and (iii), (iv) are pairs of parallel lines. Solving (i) with (iii) and (ii) with (iv), we obtain the coordinates of C and A as $(-c/a, 0)$ and $(c/a, 0)$ respectively.

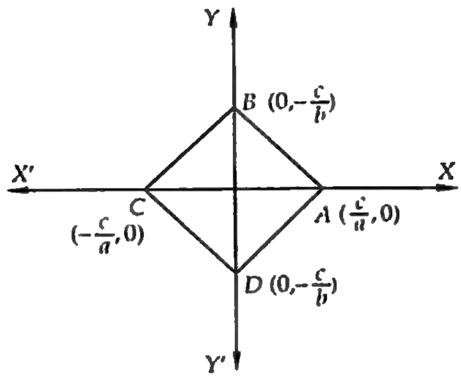


Fig. 23.88

Solving (ii) with (iii) and (i) with (iv), we obtain the coordinates of B and D as $(0, c/b)$ and $(0, -c/b)$ respectively.

Thus, the vertices of the parallelogram $ABCD$ are

$A\,(c/a, 0)$, $B\,(0, c/b)$, $C\,(-c/a, 0)$ and $D\,(0, -c/b)$

This shows that the vertices of the parallelogram are on the coordinate axes such that one diagonal is along x-axis and other along y-axis. Since, the diagonals are at right angles. Hence, $ABCD$ is a rhombus.

Area of the rhombus $= \dfrac{1}{2} AC \times BD = \dfrac{1}{2}\left(\dfrac{2c}{a} \times \dfrac{2c}{b}\right) = \dfrac{2c^2}{ab}$.

EXERCISE 23.17

LEVEL-1

1. Prove that the area of the parallelogram formed by the lines $a_1 x + b_1 y + c_1 = 0$, $a_1 x + b_1 y + d_1 = 0$, $a_2 x + b_2 y + c_2 = 0$, $a_2 x + b_2 y + d_2 = 0$ is $\left|\dfrac{(d_1 - c_1)(d_2 - c_2)}{a_1 b_2 - a_2 b_1}\right|$ sq. units.

   Deduce the condition for these lines to form a rhombus.

2. Prove that the area of the parallelogram formed by the lines $3x - 4y + a = 0$, $3x - 4y + 3a = 0$, $4x - 3y - a = 0$ and $4x - 3y - 2a = 0$ is $\dfrac{2a^2}{7}$ sq. units.

LEVEL-2

3. Show that the diagonals of the parallelogram whose sides are $lx + my + n = 0$, $lx + my + n' = 0$, $mx + ly + n = 0$ and $mx + ly + n' = 0$ include an angle $\pi/2$.

HINTS TO SELECTED PROBLEMS

1. Area $=\dfrac{p_1 p_2}{\sin\theta}$, where p_1, p_2 are the distance between the pairs of parallel lines and θ is he angle between two adjacent sides. For, rhombus use $p_1 = p_2$.
3. Use: $p_1 = p_2$.

23.16 EQUATIONS OF LINES PASSING THROUGH A GIVEN POINT AND MAKING A GIVEN ANGLE WITH A LINE

THEOREM *Prove the equations of the straight lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are*

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

PROOF Let $P(x_1, y_1)$ be the given point and let the given line be LMN, making an angle θ with the axis of x. Then, $m = \tan\theta$.

Let PMR and PNS be two required lines which make angle α with the given line. Let these lines meet the axis of X at R and S respectively. Suppose lines PMR and PNS make angles θ_1 and θ_2 with the positive direction of X-axis. Then, the equations of the two required lines are

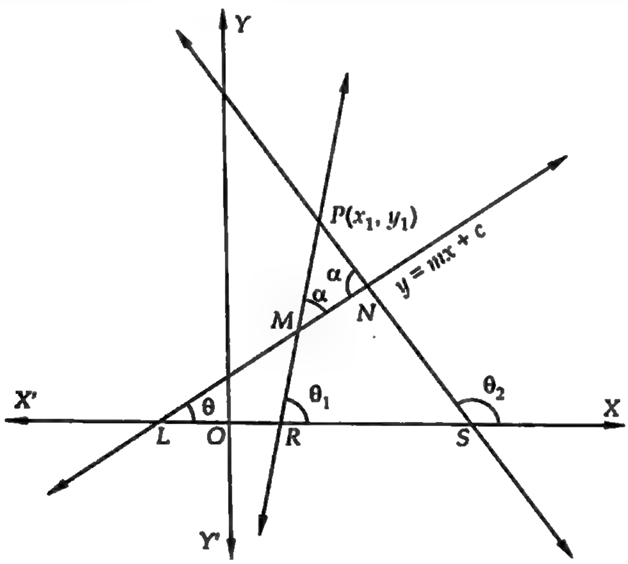


Fig. 23.89

$$y - y_1 = \tan\theta_1 (x - x_1) \quad \text{...(i)}$$

and, $$y - y_2 = \tan\theta_2 (x - x_2) \quad \text{...(ii)}$$

In ΔLMR, we have

$$\theta_1 = \theta + \alpha.$$

In ΔLNS, we have

$$\theta_2 = \theta + 180 - \alpha$$

Now, $\theta_1 = \theta + \alpha$

$\Rightarrow$ $\tan\theta_1 = \tan(\theta + \alpha) = \dfrac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \dfrac{m + \tan\alpha}{1 - m\tan\alpha}$

and, $\theta_2 = \theta + 180° - \alpha$

$\Rightarrow$ $\tan\theta_2 = \tan(180 + \theta - \alpha) = \tan(\theta - \alpha)$

$$\Rightarrow \qquad \tan\theta_2 = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha} = \frac{m - \tan\alpha}{1 + m\tan\alpha}$$

On substituting the values of $\tan\theta_1$ and, $\tan\theta_2$ in (i) and (ii), we get

$$y - y_1 = \frac{m + \tan\alpha}{1 - m\tan\alpha}(x - x_1) \quad \text{and} \quad y - y_1 = \frac{m - \tan\alpha}{1 + m\tan\alpha}(x - x_1)$$

These are the equations of the required lines.

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equations of the two straight lines through (7, 9) and making an angle of 60° with the line $x - \sqrt{3}\,y - 2\sqrt{3} = 0$.*

SOLUTION We know that the equations of two straight lines which pass through a point (x_1, y_1) and make a given angle α with the line $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Here, $x_1 = 7$, $y_1 = 9$, $\alpha = 60°$ and $m =$ (Slope of the line $x - \sqrt{3}\,y - 2\sqrt{3} = 0$) $= \dfrac{1}{\sqrt{3}}$

So, equations of required lines are

$$y - 9 = \frac{\frac{1}{\sqrt{3}} + \tan 60°}{1 - \frac{1}{\sqrt{3}}\tan 60°}(x - 7) \text{ and } y - 9 = \frac{\frac{1}{\sqrt{3}} - \tan 60°}{1 + \frac{1}{\sqrt{3}}\tan 60°}(x - 7)$$

or, $\quad (y - 9)\left(1 - \dfrac{1}{\sqrt{3}}\tan 60°\right) = \left(\dfrac{1}{\sqrt{3}} + \tan 60°\right)(x - 7)$

and, $\quad (y - 9)\left(1 + \dfrac{1}{\sqrt{3}}\tan 60°\right) = \left(\dfrac{1}{\sqrt{3}} - \tan 60°\right)(x - 7)$

or, $\quad 0 = \left(\dfrac{1}{\sqrt{3}} + \sqrt{3}\right)(x - 7)$, and $(y - 9)(2) = \left(\dfrac{1}{\sqrt{3}} - \sqrt{3}\right)(x - 7)$

or, $\quad x - 7 = 0$ and, $x + \sqrt{3}\,y = 7 + 9\sqrt{3}$.

Hence, the required lines are $x = 7$ and $x + \sqrt{3}\,y = 7 + 9\sqrt{3}$.

EXAMPLE 2 *Show that the equations of the straight lines passing through the point (3, – 2) and inclined at 60° to the line $\sqrt{3}\,x + y = 1$ are $y + 2 = 0$ and $y - \sqrt{3}\,x + 2 + 3\sqrt{3} = 0$.* [NCERT EXEMPLAR]

SOLUTION The equations of two straight lines passing through a point (x_1, y_1) and making an angle α with $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Here, $x_1 = 3$, $y_1 = -2$, $\alpha = 60°$ and $m =$ (Slope of $\sqrt{3}\,x + y = 1$) $= -\sqrt{3}$.

So, the equations of the required lines are

$$y + 2 = \frac{-\sqrt{3} - \tan 60°}{1 - \sqrt{3}\tan 60°}(x - 3) \text{ and } y + 2 = \frac{-\sqrt{3} + \tan 60°}{1 + \sqrt{3}\tan 60°}(x - 3)$$

or, $\quad y + 2 = \sqrt{3}(x - 3) \qquad$ and $y + 2 = 0$

or, $\quad y - \sqrt{3}\,x + 2 + 3\sqrt{3} = 0 \qquad$ and $y + 2 = 0$.

EXAMPLE 3 *Find the equations of the straight lines through (3, 2) which make acute angle of 45° with the line $x-2y-3=0$.* [NCERT]

SOLUTION Here, $x_1=3, y_1=2, \alpha=45°$ and, $m=$ (Slope of the line $x-2y-3=0$) $=\frac{1}{2}$

So, equations of required lines are

$$y-2=\frac{\frac{1}{2}-\tan 45°}{1+\frac{1}{2}\tan 45°}(x-3) \text{ and, } y-2=\frac{\frac{1}{2}+\tan 45°}{1-\frac{1}{2}\tan 45°}(x-3)$$

$$\Rightarrow \quad y-2=-\frac{1}{3}(x-3) \quad \text{and } y-2=3(x-3)$$

$$\Rightarrow \quad x+3y=9 \text{ and } 3x-y=7.$$

ALITER The equation of any line through (3, 2) is

$$y-2=m(x-3) \quad \text{...(i)}$$

where m is the slope of the line and is to be determined.

It is given that the line (i) makes an acute angle of 45° with the line $x-2y-3=0$. Therefore,

$$\tan 45° = \pm\frac{m-\frac{1}{2}}{1+\frac{1}{2}m} \qquad \left[\text{Using: } \tan\theta = \pm\frac{m_1-m_2}{1+m_1m_2}\right]$$

$$\Rightarrow \quad 1=\pm\left(\frac{2m-1}{2+m}\right) \Rightarrow 2+m=\pm(2m-1) \Rightarrow m=3, -\frac{1}{3}$$

Putting the values of m in (i), equations of required lines are

$$y-2=3(x-3) \text{ and } y-2=-\frac{1}{3}(x-3) \text{ or, } 3x-y=7 \text{ and } x+3y=9.$$

EXAMPLE 4 *A vertex of an equilateral triangle is (2, 3) and the opposite side is $x+y=2$. Find the equations of other sides.* [NCERT EXEMPLAR]

SOLUTION Let A (2, 3) be one vertex and $x+y=2$ be the opposite side of an equilateral triangle. Clearly, remaining two sides pass through the point A (2, 3) and make an angle of 60° with $x+y=2$.

Let m be the slope of $x+y=2$. Then, $m=-1$. Fig. 23.90 the equations of the other two sides are

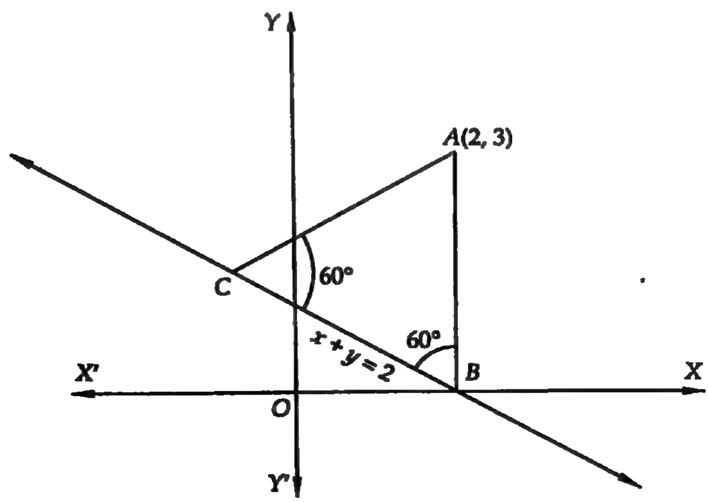


Fig. 23.90

$$y - 3 = \frac{-1 - \tan 60°}{1 - \tan 60°}(x-2) \text{ and } y - 3 = \frac{-1 + \tan 60°}{1 + \tan 60°}(x-2)$$

$$\Rightarrow \quad y - 3 = \frac{-(1+\sqrt{3})}{1-\sqrt{3}}(x-2) \quad \text{and} \quad y - 3 = \frac{\sqrt{3}-1}{\sqrt{3}+1}(x-2)$$

$$\Rightarrow \quad y - 3 = (2+\sqrt{3})(x-2) \quad \text{and,} \quad (y-3)(2+\sqrt{3}) = x - 2$$

$$\Rightarrow \quad (2+\sqrt{3})\,x - y = 1 + 2\sqrt{3} \quad \text{and,} \quad (2-\sqrt{3})\,x - y = 1 - 2\sqrt{3}.$$

EXAMPLE 5 *Show that the equation of the straight line through the origin making angle ϕ with the line $y = mx + b$ is* $\frac{y}{x} = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}$. [NCERT]

SOLUTION Let m_1 be the slope of the desired line. Then, its equation is

$$(y - 0) = m_1 (x - 0) \qquad \text{[Using : } y - y_1 = m(x - x_1)]$$

$$\Rightarrow \quad y = m_1 x \qquad \text{...(i)}$$

If $y = mx + b$ makes angle ϕ with line (i), then

$$\tan\phi = \pm \frac{m - m_1}{1 + mm_1} \quad \Rightarrow \quad m_1 = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}$$

Putting the values of m_1 in (i), we get: $y = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}\,x$ as the required equations of two lines.

EXAMPLE 6 *The opposite angular points of a square are* (3, 4) *and* (1, – 1). *Find the coordinates of the other two vertices.*

SOLUTION We have,

$$\text{Slope of } AC = \frac{-1-4}{1-3} = \frac{5}{2}$$

Clearly, AB and AD pass through A (3, 4) and make angle of 45° with AC whose slope is 5/2. Therefore, equations of AB and AD are given by

$$y - 4 = \frac{\frac{5}{2} \mp \tan 45°}{1 \pm \frac{5}{2}\tan 45°}(x - 3)$$

$$\Rightarrow \quad y - 4 = \frac{5 \mp 2}{2 \pm 5}(x - 3)$$

$$\Rightarrow \quad y - 4 = \frac{3}{7}(x-3) \quad \text{and} \quad y - 4 = -\frac{7}{3}(x-3)$$

$$\Rightarrow \quad 3x - 7y + 19 = 0 \text{ and } 7x + 3y - 33 = 0.$$

Thus, equations of AB and AD are $3x - 7y + 19 = 0$ and $7x + 3y - 33 = 0$ respectively. Since BC is a line parallel to AD. Therefore, equation of BC is $7x + 3y + \lambda = 0$. This passes through $(1, -1)$.

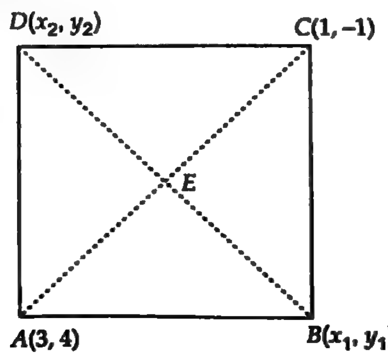


Fig. 23.91

$\therefore \quad 7-3+\lambda=0 \Rightarrow \lambda=-4.$

So, the equation of BC is $7x+3y-4=0$.

Since B is the point of intersection of AB and BC. Therefore, solving equations of AB and BC by cross-multiplication, we get

$$\frac{x}{-29}=\frac{y}{155}=\frac{1}{58} \Rightarrow x=-\frac{1}{2},\ y=\frac{5}{2}$$

So, the coordinates of B are $(-1/2, 5/2)$.

Let the coordinates of D be (x_2, y_2). Then, the coordinates of the mid-point of BD are

$$\left(\frac{x_2-\frac{1}{2}}{2}, \frac{y_2+\frac{5}{2}}{2}\right)$$

The coordinates of the mid-point of AC are $(2, 3/2)$.

Since the diagonals AC and BD bisect each other.

$$\therefore \quad \frac{x_2-\frac{1}{2}}{2}=2 \text{ and } \frac{y_2+\frac{5}{2}}{2}=\frac{3}{2} \Rightarrow x_2=\frac{9}{2} \text{ and } y_2=\frac{1}{2}$$

So, the coordinates of D are $(9/2, 1/2)$. Hence, the other two vertices are $(-1/2, 5/2)$ and $(9/2, 1/2)$.

LEVEL-2

EXAMPLE 7 *If one diagonal of a square is along the line $8x-15y=0$ and one of its vertex is at $(1, 2)$, then find the equations of sides of the square passing through this vertex.* [NCERT EXEMPLAR]

SOLUTION Let $ABCD$ be the given square whose one vertex is at $A\ (1, 2)$ and the diagonal BD is along the line $8x-15y=0$.

We observe that the sides AB and AD pass through the vertex $A\ (1, 2)$ and make $45°$ angle with the diagonal BD of slope $m=8/15$. Therefore, equations of AB and AD are given by

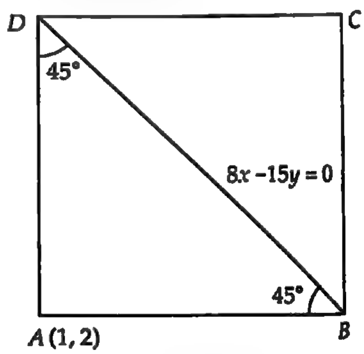


Fig. 23.92

$$y-2=\frac{\frac{8}{15}\pm\tan 45°}{1\mp\frac{8}{15}\tan 45°}(x-1)$$

or, $$y-2=\frac{8\pm 15}{15\mp 8}(x-1)$$

or, $$y-2=\frac{23}{7}(x-1) \text{ and } y-2=-\frac{7}{23}(x-1)$$

or, $$23x-7y-9=0 \text{ and } 7x+23y-53=0$$

EXAMPLE 8 *One side of a rectangle lies along the line $4x+7y+5=0$. Two of its vertices are $(-3, 1)$ and $(1, 1)$. Find the equations of the other three sides.*

SOLUTION Clearly, the point $(-3, 1)$ lies on the line $4x+7y+5=0$. So let $A\ (-3, 1)$ and $C\ (1, 1)$ be two vertices of the rectangle $ABCD$ and let the equation of the side AB be $4x+7y+5=0$.

Clearly, DC is a line parallel to AB passing through $C\ (1, 1)$.

Let the equation of DC be $4x+7y+\lambda=0$ ...(i)

This passes through $(1, 1)$.

$\therefore \quad 4+7+\lambda=0 \Rightarrow \lambda=-11.$

Putting the value of λ in (i), we get

$$4x + 7y - 11 = 0.$$

This is the equation of the side DC.

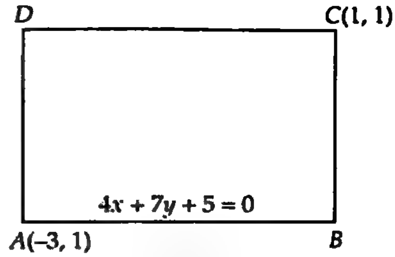


Fig. 23.93

Since AD is a line through A perpendicular to AB, therefore equation of AD is

$$7x - 4y + \lambda_1 = 0 \qquad \text{...(ii)}$$

This will pass through $(-3, 1)$, if

$$-21 - 4 + \lambda_1 = 0 \Rightarrow \lambda_1 = 25.$$

Putting the value of λ_1 in the equation of AD is $7x - 4y + 25 = 0$.

Clearly, BC is a line perpendicular to AB passing through $C(1, 1)$. Let the equation of BC be

$$7x - 4y + \mu = 0 \qquad \text{...(iii)}$$

This will pass through $(1, 1)$ if

$$7 - 4 + \mu = 0 \Rightarrow \mu = -3.$$

Putting the value of μ in (iii), we get $7x - 4y - 3 = 0$. This is the equation of BC.

EXAMPLE 9 *A line $4x + y = 1$ through the point $A(2, -7)$ meets the line BC whose equation is $3x - 4y + 1 = 0$ at the point B. Find the equation to the line AC so that $AB = AC$.*

SOLUTION The lines AB and BC meet at a point B. Let α be the angle between them. Then,

$$\tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2}, \text{ where, } m_1 = \text{Slope of } AB = -4, m_2 = \text{Slope of } BC = \frac{3}{4}$$

$$\Rightarrow \qquad \tan \alpha = \frac{-4 - 3/4}{1 + (-4) \times 3/4} = \frac{19}{8}$$

It is given that $AB = AC$. Therefore, triangle ABC is an isosceles triangle.

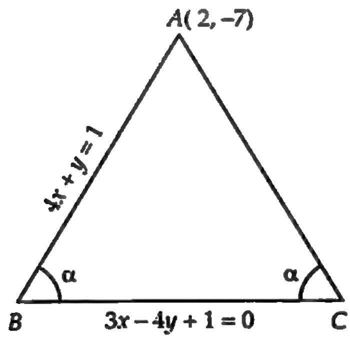


Fig. 23.94

Clearly, AB and AC both pass through $A(2, -7)$ and are equally inclined to $3x - 4y + 1 = 0$. So, their equations are given by

$$(y + 7) = \frac{m \pm \tan \alpha}{1 \mp m \tan \alpha}(x - 2), \text{ where, } m = \text{Slope of } BC = \frac{3}{4} \text{ and } \tan \alpha = \frac{19}{8}$$

$$\text{or,} \qquad (y + 7) = \frac{\frac{3}{4} + \frac{19}{8}}{1 - \frac{3}{4} \times \frac{19}{8}}(x - 2) \text{ and } y + 7 = \frac{\frac{3}{4} - \frac{19}{8}}{1 + \frac{3}{4} \times \frac{19}{8}}(x - 2)$$

or, $\quad y+7 = -4(x-2)$ and $y+7 = -\frac{52}{89}(x-2)$

$\Rightarrow \quad 4x+y = 1$ and $52x+89y+519 = 0$

Clearly, $4x+y=1$ is the equation of AB. So, equation of AC is $52x+89y+519=0$.

EXAMPLE 10 *The straight lines $3x+4y=5$ and $4x-3y=15$ intersect at the point A. On these lines, the points B and C are chosen such that $AB = AC$. Find the possible equations of the line BC passing through the point (1, 2).*

SOLUTION Let m_1 and m_2 be the slopes of the lines $3x+4y=5$ and $4x-3y=15$ respectively. Then, $m_1 = -\frac{3}{4}$ and $m_2 = \frac{4}{3}$. Clearly, $m_1 . m_2 = -1$. So, lines AB and AC are at right angle.

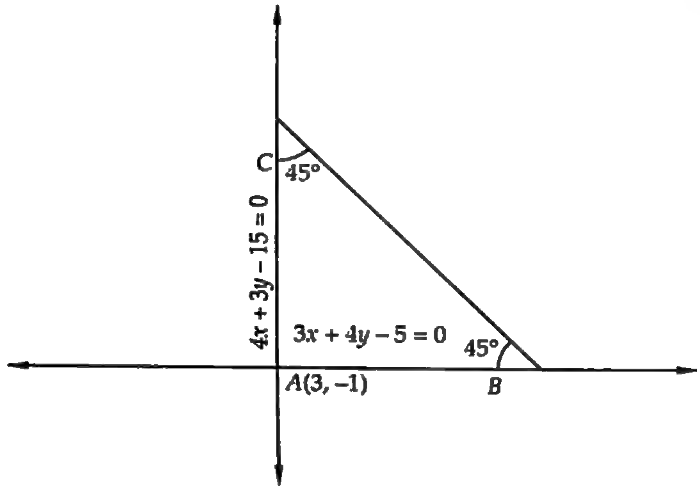


Fig. 23.95

Thus the triangle ABC is a right angled isosceles triangle. Hence, the line BC through (1, 2) will make an angle of 45° with the given lines. So, possible equations of BC are

$$(y-2) = \frac{m \pm \tan 45°}{1 \mp m \tan 45°}(x-1), \text{ where } m = \text{slope of } AB = -\frac{3}{4}$$

$$\Rightarrow \quad y-2 = \frac{-3/4 \pm 1}{1 \mp (-3/4)}(x-1)$$

$$\Rightarrow \quad (y-2) = \frac{-3 \pm 4}{4 \pm 3}(x-1)$$

$$\Rightarrow \quad y-2 = \frac{1}{7}(x-1) \text{ and } y-2 = -7(x-1)$$

$$\Rightarrow \quad x-7y+13 = 0 \text{ and } 7x+y-9 = 0$$

EXERCISE 23.18

LEVEL-1

1. Find the equation of the straight lines passing through the origin and making an angle of 45° with the straight line $\sqrt{3}\,x+y=11$.
2. Find the equations to the straight lines which pass through the origin and are inclined at an angle of 75° to the straight line $x+y+\sqrt{3}\,(y-x)=a$.
3. Find the equations of the straight lines passing through $(2,-1)$ and making an angle of 45° with the line $6x+5y-8=0$.
4. Find the equations to the straight lines which pass through the point (h, k) and are inclined at angle $\tan^{-1} m$ to the straight line $y=mx+c$.

5. Find the equations to the straight lines passing through the point (2, 3) and inclined at an angle of 45° to the line $3x + y - 5 = 0$.
6. Find the equations to the sides of an isosceles right angled triangle the equation of whose hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point (2, 2). **[NCERT EXEMPLAR]**

LEVEL-2

7. The equation of one side of an equilateral triangle is $x - y = 0$ and one vertex is $(2 + \sqrt{3}, 5)$. Prove that a second side is $y + (2 - \sqrt{3})x = 6$ and find the equation of the third side.
8. Find the equations of the two straight lines through (1, 2) forming two sides of a square of which $4x + 7y = 12$ is one diagonal.
9. Find the equations of two straight lines passing through (1, 2) and making an angle of 60° with the line $x + y = 0$. Find also the area of the triangle formed by the three lines.
10. Two sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point $(1, -10)$. Determine the equation of the third side.
11. Show that the point $(3, -5)$ lies between the parallel lines $2x + 3y - 7 = 0$ and $2x + 3y + 12 = 0$ and find the equation of lines through $(3, -5)$ cutting the above lines at an angle of 45°.
12. The equation of the base of an equilateral triangle is $x + y = 2$ and its vertex is $(2, -1)$. Find the length and equations of its sides. **[NCERT EXEMPLAR]**
13. If two opposite vertices of a square are (1, 2) and (5, 8), find the coordinates of its other two vertices and the equations of its sides.

ANSWERS

1. $y = (\sqrt{3} \pm 2)x$
2. $x = 0, y + \sqrt{3}x = 0$
3. $11x - y - 23 = 0$
4. $y = k, (1 - m^2)(y - k) = 2m(x - h)$
5. $x + 2y - 8 = 0$ and $2x - y - 1 = 0$
6. $7x + y - 16 = 0$ and $x - 7y + 12 = 0$
7. $y + (2 + \sqrt{3})x = 12 + 4\sqrt{3}$
8. $3x - 11y + 19 = 0$ and $11x + 3y - 17 = 0$
9. $y - 2 = (2 \pm \sqrt{3})(x - 1)$, Area $= \dfrac{3\sqrt{3}}{2}$ sq. units
10. $x - 3y - 31 = 0$ or $3x + y + 7 = 0$
11. $x - 5y - 28 = 0$ or $5x + y - 10 = 0$
12. $\sqrt{\dfrac{2}{3}}$, $(2 - \sqrt{3})x - y - 5 + 2\sqrt{3} = 0$, $(2 + \sqrt{3})x - y - 5 - 2\sqrt{3} = 0$
13. (6, 3), (0, 7), $x - 5y + 9 = 0$; $5x + y - 7 = 0$; $5x + y - 33 = 0$; $x - 5y + 35 = 0$

HINTS TO SELECTED PROBLEMS

6. The two sides pass through (2, 2) and make an angle of 45° with the line $3x + 4y = 4$.
8. The two sides pass through (1, 2) and make 45° angle with the diagonal having slope $m = -4/7$.
10. Any line through $(1, -10)$ is $y + 10 = m(x - 1)$. Since it makes equal angles, say θ, with the given lines. Therefore,

$$\tan\theta = \frac{m - 7}{1 + 7m} = -\frac{m - (-1)}{1 + m(-1)} \Rightarrow m = -3 \text{ or}, \frac{1}{3}.$$

23.17 FAMILY OF LINES THROUGH THE INTERSECTION OF TWO GIVEN LINES

THEOREM *Prove that the equation of the family of lines passing through the intersection of the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ is $(a_1 x + b_1 y + c_1) + \lambda (a_2 x + b_2 y + c_2) = 0$, where λ is a parameter.*

PROOF The equations of the lines are

$$a_1 x + b_1 y + c_1 = 0 \quad ...(i)$$

and, $$a_2 x + b_2 y + c_2 = 0 \quad ...(ii)$$

Let (α, β) be the point of intersection of the lines (i) and (ii). Then,

$$a_1 \alpha + b_1 \beta + c_1 = 0 \quad ...(iii)$$

and, $$a_2 \alpha + b_2 \beta + c_2 = 0 \quad ...(iv)$$

Now, consider the equation

$$(a_1 x + b_1 y + c_1) + \lambda (a_2 x + b_2 y + c_2) = 0 \quad ...(v)$$

Clearly, this is a first degree equation in x and y. So it represents a straight line.

We have,

$$(a_1 \alpha + b_1 \beta + c_1) + \lambda (a_2 \alpha + b_2 \beta + c_2) = 0 + \lambda\, 0 = 0 \quad \text{[Using (iii) and (iv)]}$$

So, (α, β) lies on the line given in equation (v).

Hence, equation (v) represents family lines through the point of intersection of lines (i) and (ii).

Thus, the family of straight lines through the intersection of lines $L_1 = a_1 x + b_1 y + c_1 = 0$ and $L_2 = a_2 x + b_2 y + c_2 = 0$ is

$$(a_1 x + b_1 y + c_1) + \lambda (a_2 x + b_2 y + c_2) = 0$$

i.e. $$L_1 + \lambda L_2 = 0$$

Q.E.D.

REMARK *The equation $L_1 + \lambda L_2 = 0$ represents a line passing through the intersection of the lines $L_1 = 0$ and $L_2 = 0$ which is a fixed point. Hence, $L_1 + \lambda L_2 = 0$ represents a family of straight lines, for different values of λ, which pass through a fixed-point.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the straight line which passes through the point $(2, -3)$ and the point of intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$.*

SOLUTION Any line through the intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$ has the equation

$$(x + y + 4) + \lambda (3x - y - 8) = 0 \quad ...(i)$$

This will pass through $(2, -3)$, if

$$(2 - 3 + 4) + \lambda (6 + 3 - 8) = 0 \Rightarrow 3 + \lambda = 0 \Rightarrow \lambda = -3.$$

Putting the value of λ in (i), the equation of the required line is $2x - y - 7 = 0$.

ALITER Solving the equations $x + y + 4 = 0$ and $3x - y - 8 = 0$ by cross-multiplication, we get $x = 1, y = -5$. So, the two lines intersect at the point $(1, -5)$. Hence, the required line passes through points $(2, -3)$ and $(1, -5)$ and so its equation is

$$y + 3 = -\frac{5+3}{1-2}(x - 2) \Rightarrow 2x - y - 7 = 0.$$

EXAMPLE 2 *Find the equation of the straight line which passes through the intersection of the lines $x - y - 1 = 0$ and $2x - 3y + 1 = 0$ and is parallel to (i) x-axis (ii) y-axis (iii) $3x + 4y = 14$.*

SOLUTION The equation of any line through the intersection of the lines $x-y-1=0$ and $2x-3y+1=0$ is

$$(x-y-1)+\lambda(2x-3y+1)=0$$

or, $$(2\lambda+1)x-y(3\lambda+1)+\lambda-1=0 \quad ...(i)$$

(i) The line in (i) will be parallel to x-axis if it is of the form $y=$constant.

$\therefore$ Coefficient of x in (i) $=0$

$\Rightarrow$ $2\lambda+1=0 \Rightarrow \lambda=-1/2$.

Putting $\lambda=-1/2$ in (i), we get $y=3$ as is the equation of the required line.

(ii) The line in (i) will be parallel to y-axis if it is of the form $x=\lambda$.

$\therefore$ Coefficient of y in (i) $=0$

$\Rightarrow$ $3\lambda+1=0 \Rightarrow \lambda=-1/3$.

Putting $\lambda=-1/3$ in (i), we get $x=4$ as the equation of the required line.

(iii) The line in (i) is parallel to the line $3x+4y-14=0$. Therefore, their slopes are equal

So, slope of line in equation (i) is same as that of the line $3x+4y-14=0$.

i.e. $$\frac{2\lambda+1}{3\lambda+1}=-\frac{3}{4} \Rightarrow \lambda=-\frac{7}{17}.$$

Putting this value of λ in (i) we get the equation of the required line as $3x+4y=24$.

EXAMPLE 3 *Find the equation of the straight line which passes through the point of intersection of the straight lines $x+2y=5$ and $3x+7y=17$ and is perpendicular to the straight line $3x+4y=10$.*

SOLUTION The equation of any line through the intersection of the lines $x+2y-5=0$ and $3x+7y-17=0$ is

$$(x+2y-5)+\lambda(3x+7y-17)=0$$

or, $$x(3\lambda+1)+y(7\lambda+2)-(17\lambda+5)=0 \quad ...(i)$$

This is perpendicular to the line $3x+4y=10$.

$\therefore$ Product of their slopes $=-1$.

$$\Rightarrow \quad -\left(\frac{3\lambda+1}{7\lambda+2}\right)\left(-\frac{3}{4}\right)=-1 \Rightarrow \lambda=-\frac{11}{37}$$

Putting this value of λ in (i), the equation of the required line is $4x-3y+2=0$.

EXAMPLE 4 *Obtain the equations of the lines passing through the intersection of lines $4x-3y-1=0$ and $2x-5y+3=0$ and equally inclined to the axes.*

SOLUTION The equation of any line through the intersection of the given lines is

$$(4x-3y-1)+\lambda(2x-5y+3)=0$$

or, $$x(2\lambda+4)-y(5\lambda+3)+3\lambda-1=0 \quad ...(i)$$

Let m be the slope of this line. Then, $m=\dfrac{2\lambda+4}{5\lambda+3}$.

As the line is equally inclined with the axes.

$$\therefore \quad m=\pm 1 \Rightarrow \frac{2\lambda+4}{5\lambda+3}=\pm 1 \Rightarrow \lambda=-1 \text{ or, } 1/3$$

Putting the values of λ in (i), we obtain

$x+y-2=0$ and $x=y$ as the equations of the required lines.

LEVEL-2

EXAMPLE 5 *If t_1 and t_2 are roots of the equation $t^2+\lambda t+1=0$, where λ is an arbitrary constant. Then prove that the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ always passes through a fixed point. Also, find that point.*

SOLUTION Since t_1 and t_2 are roots of the equation $t^2 + \lambda t + 1 = 0$.

$\therefore \quad t_1 + t_2 = -\lambda$ and, $t_1 t_2 = 1$ ...(i)

The equation of the line joining the points $(at_1^2, 2\,at_1)$ and $(at_2^2, 2\,at_2)$ is

$$y - 2\,at_1 = \frac{2\,at_2 - 2\,at_1}{at_2^2 - at_1^2}(x - at_1^2) \quad \text{or,} \quad y(t_2 + t_1) = 2x + 2\,at_1 t_2 \qquad \text{...(ii)}$$

Putting $t_1 + t_2 = -\lambda$ and $t_1 t_2 = 1$ in (ii), we get

$$2x + 2a = -\lambda y \Rightarrow (x + a) + \left(-\frac{\lambda}{2}\right) y = 0$$

This is a line passing through the intersection of the lines $x + a = 0$ and $y = 0$ which is a fixed point having coordinates $(-a, 0)$.

EXAMPLE 6 *Show that the straight lines given by $x(a + 2b) + y(a + 3b) = a + b$ for different values of a and b pass through a fixed point.*

SOLUTION The given equation can be written as

$$a(x + y - 1) + b(2x + 3y - 1) = 0$$

or, $\quad (x + y - 1) + \lambda(2x + 3y - 1) = 0$, where $\lambda = b/a$

This is of the form $L_1 + \lambda L_2 = 0$. So it represents a line passing through the intersection of $x + y - 1 = 0$ and $2x + 3y - 1 = 0$. Solving these two equations, we get the point $(2, -1)$, which is the fixed point.

EXAMPLE 7 *If a, b, c are variables such that $3a + 2b + 4c = 0$, then show that the family of lines given by $ax + by + c = 0$ pass through a fixed point. Also, find that point.*

SOLUTION We have,

$$3a + 2b + 4c = 0 \Rightarrow c = -\frac{3}{4}a - \frac{1}{2}b$$

Substituting this value of c in $ax + by + c = 0$, we get

$$ax + by - \frac{3}{4}a - \frac{1}{2}b = 0$$

$$\Rightarrow \quad a\left(x - \frac{3}{4}\right) + b\left(y - \frac{1}{2}\right) = 0 \Rightarrow \left(x - \frac{3}{4}\right) + \lambda\left(y - \frac{1}{2}\right) = 0, \quad \text{where } \lambda = \frac{b}{a}.$$

This equation is of the form $L_1 + \lambda L_2 = 0$ which represents a straight line through the intersection of the line $L_1 = 0$ and $L_2 = 0$ i.e. $x - \frac{3}{4} = 0$ and $y - \frac{1}{2} = 0$. Solving these two equations, we get the point $(3/4, 1/2)$, which is a fixed point.

ALITER We have

$$3a + 2b + 4c = 0$$

$$\Rightarrow \quad \left(\frac{3}{4}\right)a + \left(\frac{2}{4}\right)b + c = 0$$

$\Rightarrow \quad (3/4, 1/2)$ lies on the line $ax + by + c = 0$

Hence, the given family of lines pass through the point $(3/4, 1/2)$.

EXAMPLE 8 *Let a, b, c be parameters. Then, the equation $ax + by + c = 0$ will represent a family of straight lines passing through a fixed-point iff there exists a linear relation between a, b and c.*

SOLUTION First, let the equation $ax + by + c = 0$ represent a family of straight lines passing through a fixed-point (α, β) (say) for different values of a, b, c. Then, we have to prove that there is a linear relation between a, b and c. Since, $ax + by + c = 0$ represents a family of lines pasing through (α, β). Therefore,

$$a\alpha + b\beta + c = 0,$$ which is the required linear relation between a, b and c.

Conversely, let there be a linear relation between the parameters a, b, c. Then, we have to prove that the equation $ax + by + c = 0$ represents a family of lines passing through a fixed-point.

Let the linear relation be

$la + mb + nc = 0$, where l, m, n are constants.

$$\Rightarrow \quad \left(\frac{l}{n}\right) a + \left(\frac{m}{n}\right) b + c = 0$$

$\Rightarrow \quad ax + by + c = 0$ passes through the fixed-point $(l/n, m/n)$

EXAMPLE 9 *If the algebraic sum of the perpendiculars from the points* (2, 0), (0, 2), (1, 1) *to a variable line be zero, then prove that the line passes through a fixed-point whose coordinates are* (1, 1).

[NCERT EXEMPLAR]

SOLUTION Let the variable line be $ax + by = 1$. It is given that the algebraic sum of the perpendiculars from the points (2, 0), (0, 2) and (1, 1) to this line is zero. Therefore,

$$\frac{2a + ob - 1}{\sqrt{a^2 + b^2}} + \frac{oa + 2b - 1}{\sqrt{a^2 + b^2}} + \frac{a + b - 1}{\sqrt{a^2 + b^2}} = 0$$

$$\Rightarrow \quad 3a + 3b - 3 = 0 \Rightarrow a + b - 1 = 0 \Rightarrow a + b = 1.$$

This is a linear relation between the parameters a and b. So, the equation $ax + by = 1$ represents a family of straight lines passing through a fixed-point. Comparing $ax + by = 1$ and $a + b = 1$, we obtain that the coordinates of the fixed-point are (1, 1).

EXAMPLE 10 *A ray of light is sent along the line* $x - 2y - 3 = 0$ *upon reaching the line* $3x - 2y - 5 = 0$, *the ray is reflected from it. Find the equation of the line containing the reflected ray.*

SOLUTION The point of intersection of the lines $x - 2y - 3 = 0$ and $3x - 2y - 5 = 0$ is $B(1, -1)$. BP is the normal at P. Clearly, BP passes through $B(1, -1)$ and is perpendicular to $3x - 2y - 5 = 0$. So, equation of BP is

$$y + 1 = -(2/3)(x - 1) \text{ or, } 2x + 3y + 1 = 0$$

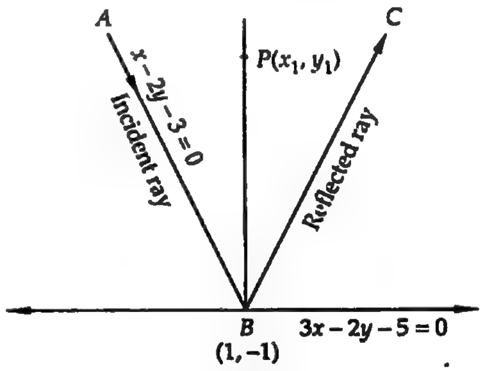


Fig. 23.96

Since the reflected ray passes through the intersection of $x - 2y - 3 = 0$ and the normal $2x + 3y + 1 = 0$. Therefore, equation of the reflected ray is

$$x - 2y - 3 + \lambda(2x + 3y + 1) = 0 \qquad \text{...(i)}$$

or, $$x(1 + 2\lambda) + y(3\lambda - 2) + (\lambda - 3) = 0 \qquad \text{...(ii)}$$

Let $P(x_1, y_1)$ be an arbitrary point on the normal at P. Then, P is equidistant from the incident ray and the reflected ray.

$$\therefore \quad \left|\frac{x_1 - 2y_1 - 3}{\sqrt{1 + 4}}\right| = \left|\frac{(x_1 - 2y_1 - 3) + \lambda(2x_1 + 3y_1 + 1)}{\sqrt{(1 + 2\lambda)^2 + (3\lambda - 2)^2}}\right|$$

$$\Rightarrow \quad \left|\frac{x_1 - 2y_1 - 3}{\sqrt{5}}\right| = \left|\frac{(x_1 - 2y_1 - 3) + \lambda \times 0}{\sqrt{(1 + 2\lambda)^2 + (3\lambda - 2)^2}}\right| \qquad \left[\begin{array}{l}\because (x_1, y_1) \text{ lies on } 2x + 3y + 1 = 0 \\ \therefore 2x_1 + 3y_1 + 1 = 0\end{array}\right]$$

$$\Rightarrow \quad \left|\frac{1}{\sqrt{5}}\right| = \left|\frac{1}{\sqrt{(2\lambda+1)^2+(3\lambda-2)^2}}\right|$$

$$\Rightarrow \quad 5 = (2\lambda+1)^2+(3\lambda-2)^2$$

$$\Rightarrow \quad 13\lambda^2 - 8\lambda = 0 \Rightarrow \lambda = 0 \text{ or, } \lambda = \frac{8}{13}$$

Since $\lambda = 0$ is not possible. So, $\lambda = \frac{8}{13}$ Putting the value of λ in (ii), we get $29x - 2y - 31 = 0$ as the equation of the line containing the reflected ray.

EXAMPLE 11 *Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv lx + my + n = 0$ intersect at a point P and make an angle θ with each other. Find the equation of the line L different from L_2 which passes through P and makes the same angle with L_1.*

SOLUTION Since the required line L passes through the intersection of $L_1 = 0$ and $L_2 = 0$. So, equation of the required line L is

$$L_1 + \lambda L_2 = 0 \text{ i.e. } (ax + by + c) + \lambda(lx + my + n) = 0 \quad \text{...(i)}$$

where λ is a parameter.

Since L_1 is the angle bisector of $L = 0$ and $L_2 = 0$. Therefore any point $A(x_1, y_1)$ on L_1 is equidistant from $L = 0$ and $L_2 = 0$.

$$\Rightarrow \quad \frac{|lx_1 + my_1 + n|}{\sqrt{l^2+m^2}} = \frac{|(ax_1 + by_1 + c) + \lambda(lx_1 + my_1 + n)|}{\sqrt{(a+\lambda l)^2 + (b+\lambda m)^2}} \quad \text{...(ii)}$$

But, $A(x_1, y_1)$ lies on L_1. So, it must satisfy the equation of L_1 i.e. $ax + by + c = 0$.

$$\therefore \quad ax_1 + by_1 + c = 0$$

Substituting $ax_1 + by_1 + c = 0$ in (ii), we get

$$\frac{|lx_1 + my_1 + n|}{\sqrt{l^2+m^2}} = \frac{|0 + \lambda(lx_1 + my_1 + n)|}{\sqrt{(a+\lambda l)^2 + (b+\lambda m)^2}}$$

$$\Rightarrow \quad \lambda^2(l^2+m^2) = (a+\lambda l)^2 + (b+\lambda m)^2$$

$$\Rightarrow \quad \lambda = -\frac{a^2+b^2}{2al+2bm}$$

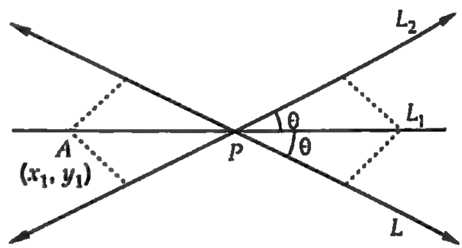


Fig. 23.97

Substituting the value of λ in (i), we get

$$(ax + by + c) - \frac{(a^2+b^2)}{2al+2bm}(lx + my + n) = 0$$

or, $2(al + bm)(ax + by + c) - (a^2 + b^2)(lx + my + n) = 0$ as the equation of the required line L.

EXERCISE 23.19

LEVEL-1

1. Find the equation of a straight line through the point of intersection of the lines $4x - 3y = 0$ and $2x - 5y + 3 = 0$ and parallel to $4x + 5y + 6 = 0$.
2. Find the equation of a straight line passing through the point of intersection of $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ and perpendicular to the straight line $x - y + 9 = 0$.
3. Find the equation of the line passing through the point of intersection of $2x - 7y + 11 = 0$ and $x + 3y - 8 = 0$ and is parallel to (i) x-axis (ii) y-axis.
4. Find the equation of the straight line passing through the point of intersection of $2x + 3y + 1 = 0$ and $3x - 5y - 5 = 0$ and equally inclined to the axes.

5. Find the equation of the straight line drawn through the point of intersection of the lines $x+y=4$ and $2x-3y=1$ and perpendicular to the line cutting off intercepts 5, 6 on the axes.

LEVEL-2

6. Prove that the family of lines represented by $x(1+\lambda)+y(2-\lambda)+5=0$, λ being arbitrary, pass through a fixed point. Also, find the fixed point.
7. Show that the straight lines given by $(2+k)x+(1+k)y=5+7k$ for different values of k pass through a fixed point. Also, find that point.
8. Find the equation of the straight line passing through the point of intersection of $2x+y-1=0$ and $x+3y-2=0$ and making with the coordinate axes a triangle of area 3/8 sq. units.
9. Find the equation of the straight line which passes through the point of intersection of the lines $3x-y=5$ and $x+3y=1$ and makes equal and positive intercepts on the axes.
10. Find the equations of the lines through the point of intersection of the lines $x-3y+1=0$ and $2x+5y-9=0$ and whose distance from the origin is $\sqrt{5}$.
11. Find the equations of the lines through the point of intersection of the lines $x-y+1=0$ and $2x-3y+5=0$ whose distance from the point (3, 2) is 7/5. [NCERT EXEMPLAR]

ANSWERS

1. $28x+35y-48=0$
2. $x+y+2=0$
3. $13y=27, 13x=23$
4. $19x+19y+3=0$ or, $19x-19y-23=0$
5. $25x-30y-23=0$
6. $(-5/3, -5/3)$
7. $(-2, 9)$
8. $3x+4y-3=0$ or, $12x+y-3=0$
9. $5x+5y=7$
10. $2x+y-5=0$
11. $3x-4y+6=0$, $4x-3y+1=0$

HINTS TO SELECTED PROBLEMS

11. The equation of the family of lines through the intersection of the lines $x-y+1=0$ and $2x-3y+5=0$ is

$$(x-y+1)+\lambda(2x-3y+5)=0 \text{ or, } x(2\lambda+1)+y(-3\lambda-1)+5\lambda+1=0 \quad \text{...(i)}$$

This is at a distance of $\frac{7}{5}$ units from the point (3, 2).

$$\therefore \quad \frac{|3(2\lambda+1)+2(-3\lambda-1)+5\lambda+1|}{\sqrt{(2\lambda+1)^2+(-3\lambda-1)^2}}=\frac{7}{5}$$

$$\Rightarrow \quad \frac{|5\lambda+2|}{\sqrt{13\lambda^2+10\lambda+2}}=\frac{7}{5}$$

$$\Rightarrow \quad 25(5\lambda+2)^2=49(13\lambda^2+10\lambda+2) \Rightarrow 6\lambda^2-5\lambda-1=0 \Rightarrow \lambda=1, -\frac{1}{6}$$

Substituting the values of λ in (i), we obtain

$3x-4y+6=0$ and $4x-3y+1=0$ as the required equations of the line.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write an equation representing a pair of lines through the point (a, b) and parallel to the coordinate axes.

2. Write the coordinates of the orthocentre of the triangle formed by the lines $x^2 - y^2 = 0$ and $x + 6y = 18$.

3. If the centroid of a triangle formed by the points (0, 0), ($\cos\theta$, $\sin\theta$) and ($\sin\theta$, $-\cos\theta$ lies on the line $y = 2x$, then write the value of $\tan\theta$.

4. Write the value of $\theta \in \left(0, \frac{\pi}{2}\right)$ for which area of the triangle formed by points $O(0, 0)$, $A(a\cos\theta, b\sin\theta)$ and $B(a\cos\theta, -b\sin\theta)$ is maximum.

5. Write the distance between the lines $4x + 3y - 11 = 0$ and $8x + 6y - 15 = 0$.

6. Write the coordinates of the orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$.

7. If the lines $x + ay + a = 0$, $bx + y + b = 0$ and $cx + cy + 1 = 0$ are concurrent, then write the value of $2abc - ab - bc - ca$.

8. Write the area of the triangle formed by the coordinate axes and the line $(\sec\theta - \tan\theta)x + (\sec\theta + \tan\theta)y = 2$.

9. If the diagonals of the quadrilateral formed by the lines $l_1 x + m_1 y + n_1 = 0$, $l_2 x + m_2 y + n_2 = 0$, $l_1 x + m_1 y + n_1' = 0$ and $l_2 x + m_2 y + n_2' = 0$ are perpendicular, then write the value of $l_1^2 - l_2^2 + m_1^2 - m_2^2$.

10. Write the coordinates of the image of the point (3, 8) in the line $x + 3y - 7 = 0$.

11. Write the integral values of m for which the x-coordinate of the point of intersection of the lines $y = mx + 1$ and $3x + 4y = 9$ is an integer.

12. If $a \neq b \neq c$, write the condition for which the equations $(b - c)x + (c - a)y + (a - b) = 0$ and $(b^3 - c^3)x + (c^3 - a^3)y + (a^3 - b^3) = 0$ represent the same line.

13. If a, b, c are in G.P. write the area of the triangle formed by the line $ax + by + c = 0$ with the coordinates axes.

14. Write the area of the figure formed by the lines $a|x| + b|y| + c = 0$.

15. Write the locus of a point the sum of whose distances from the coordinates axes is unity.

16. If a, b, c are in A.P., then the line $ax + by + c = 0$ passes through a fixed point. Write the coordinates of that point.

17. Write the equation of the line passing through the point $(1, -2)$ and cutting off equal intercepts from the axes.

18. Find the locus of the mid-points of the portion of the line $x\sin\theta + y\cos\theta = p$ intercepted between the axes.

ANSWER

1. $(x-a)(y-b) = 0$	2. (0, 0)	3. -3	4. $\pi/4$
5. 7/10 units	6. (0, 0)	7. -1	8. 2 sq. units
9. 0	10. $(-1, -4)$	11. $-1, -2$	12. $a + b + c = 0$
13. $\frac{1}{2}$ Sq. units	14. $\frac{2c^2}{\lvert ab\rvert}$ Sq. units	15. A square	16. $(1, -2)$
17. $x + y + 1 = 0$	18. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$		

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. L is a variable line such that the algebraic sum of the distances of the points (1, 1), (2, 0) and (0, 2) from the line is equal to zero. The line L will always pass through
   (a) (1, 1) (b) (2, 1) (c) (1, 2) (d) none of these
2. The acute angle between the medians drawn from the acute angles of a right angled isosceles triangle is
   (a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(\frac{3}{4}\right)$ (c) $\cos^{-1}\left(\frac{4}{5}\right)$ (d) $\cos^{-1}\left(\frac{5}{6}\right)$
3. The distance between the orthocentre and circumcentre of the triangle with vertices (1, 2). (2, 1) and $\left(\frac{3+\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}\right)$ is
   (a) 0 (b) $\sqrt{2}$ (c) $3 + \sqrt{3}$ (d) none of these
4. The equation of the straight line which passes through the point (– 4, 3) such that the portion of the line between the axes is divided internally by the point in the ratio 5 : 3 is
   (a) $9x - 20y + 96 = 0$ (b) $9x + 20y = 24$
   (c) $20x + 9y + 53 = 0$ (d) none of these
5. The point which divides the join of (1, 2) and (3, 4) externally in the ratio 1 : 1
   (a) lies in the III quadrant (b) lies in the II quadrant
   (c) lies in the I quadrant (d) cannot be found
6. A line passes through the point (2, 2) and is perpendicular to the line $3x + y = 3$. Its y-intercept is
   (a) 1/3 (b) 2/3 (c) 1 (d) 4/3
7. If the lines $ax + 12y + 1 = 0$, $bx + 13y + 1 = 0$ and $cx + 14y + 1 = 0$ are concurrent, then a, b, c are in
   (a) H.P. (b) G.P. (c) A.P. (d) none of these
8. The number of real values of λ for which the lines $x - 2y + 3 = 0$, $\lambda x + 3y + 1 = 0$ and $4x - \lambda y + 2 = 0$ are concurrent is
   (a) 0 (b) 1 (c) 2 (d) Infinite
9. The equations of the sides AB, BC and CA of ΔABC are $y - x = 2$, $x + 2y = 1$ and $3x + y + 5 = 0$ respectively. The equation of the altitude through B is
   (a) $x - 3y + 1 = 0$ (b) $x - 3y + 4 = 0$ (c) $3x - y + 2 = 0$ (d) none of these
10. If p_1 and p_2 are the lengths of the perpendiculars from the origin upon the lines $x \sec\theta + y \operatorname{cosec}\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$ respectively, then
    (a) $4p_1^2 + p_2^2 = a^2$ (b) $p_1^2 + 4p_2^2 = a^2$
    (c) $p_1^2 + p_2^2 = a^2$ (d) none of these
11. Area of the triangle formed by the points $((a+3)(a+4), a+3)$, $((a+2)(a+3), (a+2))$ and $((a+1)(a+2), (a+1))$ is
    (a) $25a^2$ (b) $5a^2$ (c) $24a^2$ (d) none of these

12. If $a+b+c=0$, then the family of lines $3ax+by+2c = 0$ pass through fixed point
(a) $(2, 2/3)$ (b) $(2/3, 2)$ (c) $(-2, 2/3)$ (d) none of these

13. The line segment joining the points $(-3, -4)$ and $(1, -2)$ is divided by y-axis in the ratio
(a) $1:3$ (b) $2:3$ (c) $3:1$ (d) $3:2$

14. The area of a triangle with vertices at $(-4, -1)$, $(1, 2)$ and $(4, -3)$ is
(a) 17 (b) 16 (c) 15 (d) none of these

15. The line segment joining the points $(1, 2)$ and $(-2, 1)$ is divided by the line $3x+4y=7$ in the ratio
(a) $3:4$ (b) $4:3$ (c) $9:4$ (d) $4:9$

16. If the point $(5, 2)$ bisects the intercept of a line between the axes, then its equation is
(a) $5x+2y=20$ (b) $2x+5y=20$ (c) $5x-2y=20$ (d) $2x-5y=20$

17. $A(6, 3)$, $B(-3, 5)$, $C(4, -2)$ and $D(x, 3x)$ are four points. If $\Delta DBC : \Delta ABC = 1:2$, then x is equal to
(a) $11/8$ (b) $8/11$ (c) 3 (d) none of these

18. If p be the length of the perpendicular from the origin on the line $x/a+y/b=1$, then
(a) $p^2=a^2+b^2$ (b) $p^2=\frac{1}{a^2}+\frac{1}{b^2}$ (c) $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$ (d) none of these

19. The equation of the line passing through $(1, 5)$ and perpendicular to the line $3x-5y+7=0$ is
(a) $5x+3y-20=0$ (b) $3x-5y+7=0$ (c) $3x-5y+6=0$ (d) $5x+3y+7=0$

20. The figure formed by the lines $ax \pm by \pm c = 0$ is
(a) a rectangle (b) a square (c) a rhombus (d) none of these

21. Two vertices of a triangle are $(-2, -1)$ and $(3, 2)$ and third vertex lies on the line $x+y=5$. If the area of the triangle is 4 square units, then the third vertex is
(a) $(0, 5)$ or, $(4, 1)$ (b) $(5, 0)$ or, $(1, 4)$ (c) $(5, 0)$ or, $(4, 1)$ (d) $(0, 5)$ or, $(1, 4)$

22. The inclination of the straight line passing through the point $(-3, 6)$ and the mid-point of the line joining the point $(4, -5)$ and $(-2, 9)$ is
(a) $\pi/4$ (b) $\pi/6$ (c) $\pi/3$ (d) $3\pi/4$

23. Distance between the lines $5x+3y-7=0$ and $15x+9y+14=0$ is
(a) $\frac{35}{\sqrt{34}}$ (b) $\frac{1}{3\sqrt{34}}$ (c) $\frac{35}{3\sqrt{34}}$ (d) $\frac{35}{2\sqrt{34}}$

24. The angle between the lines $2x-y+3=0$ and $x+2y+3=0$ is
(a) 90° (b) 60° (c) 45° (d) 30°

25. The value of λ for which the lines $3x+4y=5$, $5x+4y=4$ and $\lambda x+4y=6$ meet at a point is
(a) 2 (b) 1 (c) 4 (d) 3

26. Three vertices of a parallelogram taken in order are $(-1, -6)$, $(2, -5)$ and $(7, 2)$. The fourth vertex is
(a) $(1, 4)$ (b) $(4, 1)$ (c) $(1, 1)$ (d) $(4, 4)$

27. The centroid of a triangle is (2, 7) and two of its vertices are (4, 8) and (– 2, 6). The third vertex is

(a) (0, 0) (b) (4, 7) (c) (7, 4) (d) (7, 7)

28. If the lines $x + q = 0$, $y - 2 = 0$ and $3x + 2y + 5 = 0$ are concurrent, then the value of q will be

(a) 1 (b) 2 (c) 3 (d) 5

29. The medians AD and BE of a triangle with vertices $A(0, b)$, $B(0, 0)$ and $C(a, 0)$ are perpendicular to each other, if

(a) $a = \frac{b}{2}$ (b) $b = \frac{a}{2}$ (c) $ab = 1$ (d) $a = \pm\sqrt{2}\, b$

30. The equation of the line with slope $-3/2$ and which is concurrent with the lines $4x + 3y - 7 = 0$ and $8x + 5y - 1 = 0$ is

(a) $3x + 2y - 63 = 0$ (b) $3x + 2y - 2 = 0$

(c) $2y - 3x - 2 = 0$ (d) none of these

31. The vertices of a triangle are (6, 0), (0, 6) and (6, 6). The distance between its circumcentre and centroid is

(a) $2\sqrt{2}$ (b) 2 (c) $\sqrt{2}$ (d) 1

32. A point equidistant from the line $4x + 3y + 10 = 0$, $5x - 12y + 26 = 0$ and $7x + 24y - 50 = 0$ is

(a) (1, –1) (b) (1, 1) (c) (0, 0) (d) (0, 1)

33. The ratio in which the line $3x + 4y + 2 = 0$ divides the distance between the lines $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ is

(a) 1 : 2 (b) 3 : 7 (c) 2 : 3 (d) 2 : 5

34. The coordinates of the foot of the perpendicular from the point (2, 3) on the line $x + y - 11 = 0$ are

(a) (–6, 5) (b) (5, 6) (c) (–5, 6) (d) (6, 5)

35. The reflection of the point (4, –13) about the line $5x + y + 6 = 0$ is

(a) (–1, –14) (b) (3, 4) (c) (0, 0) (d) (1, 2)

ANSWERS

1. (a)	2. (c)	3. (a)	4. (a)	5. (d)	6. (d)	7. (c)	8. (a)
9. (b)	10. (a)	11. (d)	12. (b)	13. (c)	14. (a)	15. (d)	16. (b)
17. (a)	18. (c)	19. (a)	20. (c)	21. (b)	22. (d)	23. (c)	24. (a)
25. (b)	26. (b)	27. (b)	28. (c)	29. (d)	30. (b)	31. (c)	32. (c)
33. (b)	34. (b)	35. (a)					

SUMMARY

1. Every first degree equation in x, y represents a straight line.
2. The trigonometrical tangent of the angle that a non-vertical line makes with the positive direction of the x-axis in anticlockwise sense is called the slope or gradient of the line.
3. The slope m of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{m_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

4. Slope of a horizontal line is zero and slope of a vertical ine is undefined.
5. An acute angle θ between the lines having slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|, 1 + m_1 m_2 \neq 0$$

6. Two lines are parallel if and only if their slopes are equal.
7. Two lines are perpendicular if and only if the product of their slopes is -1.
8. Three points P, Q and R are collinear if and only if

   Slope of PQ = Slope of QR
9. If a straight line cuts x-axis at A and the y-axis at B, then OA and OB are known as the intercepts of the line on x-axis and y-axis respectively.
10. The equation of a line parallel to x-axis at a distance a from it is $y = a$ or $y = -a$ according as it is above or below x-axis.
11. The equation of a line parallel to y-axis at a distance b from it is $x = b$ or $x = -b$ according as it is on the right or on left side of y-axis.
12. The equation of x-axis is $y = 0$.
13. The equation of y-axis is $x = 0$.
14. The equation of a line with slope m and making an intercept c on y-axis is $y = mx + c$.
15. The equation of a line with slope m and passing through the origin is $y = mx$.
16. The equation of the line which passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

17. The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

18. The equation of the line making intercepts a and b on x and y-axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$.
19. The equation of the straight line upon which the length of the perpendicular from the origin is p and the angle between this perpendicular and positive x-axis is α is given by $x\cos\alpha + y\sin\alpha = p$.
20. The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$ where r is the distance of the point (x, y) on the line from the point (x_1, y_1).

   The coordinates of any point on the line at a distance r from the point (x_1, y_1) are

$$(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$$

21. The slope of the line $ax + by + c = 0$ is

$$-\frac{a}{b} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$$

22. Three lines $L_1 \equiv a_1 x + b_1 y + c_1 = 0$, $L_2 \equiv a_2 x + b_2 y + c_2 = 0$ and, $L_3 \equiv a_3 x + b_3 y + c_3 = 0$ are concurrent, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Also, these lines are concurrent iff there exist scalars $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$$

23. The equation of a line parallel to the line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is a constant.
24. The equation of a line perpendicular to the line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is a constant.
25. The perpendicular distance (d) of a line $ax + by + c = 0$ from a point (x_1, y_1) is given by

$$d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

26. The distance (d) between the parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is given by $d = \dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.

27. The equations of the lines passing through (x_1, y_1) and making an angle α with the line $y = mx + c$ are given by

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \pm m \tan\alpha}(x - x_1).$$

CHAPTER 24

THE CIRCLE

24.1 DEFINITION

A circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane is always constant.

The fixed point is called the centre of the *circle* and the constant distance is called the *radius* of the circle.

In Fig. 24.1, P is the moving point, C is the fixed point and CP is equal to the radius.

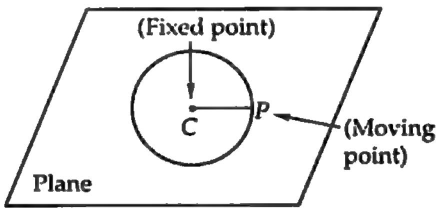


Fig. 24.1

EQUATION OF A CIRCLE By the equation of a circle is meant the equation of the circumference; it is a relation between the coordinates x, y of the moving point P, involving some constants depending upon the position of the centre and the length of the radius. In set theoretical notations it is the set of all points lying on the circumference of the circle.

24.2 STANDARD EQUATION OF A CIRCLE

In this section, we will find the equation of any circle whose centre and radius are given.

Let C be the centre of the circle and its coordinates be (h, k). Let the radius of the circle be a and let $P(x, y)$ be any point on the circumference. Then,

$$CP = a$$

$$\Rightarrow \quad CP^2 = a^2$$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = a^2$$

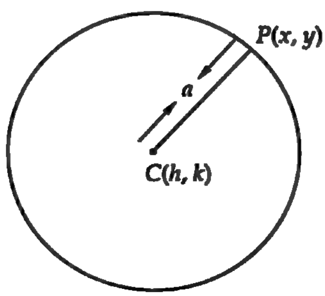


Fig. 24.2

This is the relation between the coordinates of any point on the circumference and hence it is the required equation of the circle having centre at (h, k) and radius equal to a.

<u>NOTE 1</u> *The above equation is known as the central form of the equation of a circle.*

NOTE 2 *If the centre of the circle is at the origin and radius is a, then from the above form the equation of the circle is* $x^2 + y^2 = a^2$.

ILLUSTRATION 1 *Find the equation of a circle whose centre is (2, –3) and radius 5.*

SOLUTION The equation of the required circle is

$$(x-2)^2 + (y+3)^2 = 5^2 \text{ or, } x^2 + y^2 - 4x + 6y - 12 = 0.$$

ILLUSTRATION 2 *Find the equation of a circle whose radius is 6 and the centre is at the origin.*

SOLUTION The equation of the required circle is

$$x^2 + y^2 = 6^2 \text{ or, } x^2 + y^2 = 36.$$

24.3 SOME PARTICULAR CASES

The equation of a circle with centre at (h, k) and radius equal to a, is

$$(x-h)^2 + (y-k)^2 = a^2 \quad \text{...(i)}$$

(i) *When the centre of the circle coincides with the origin (Fig. 24.3).*

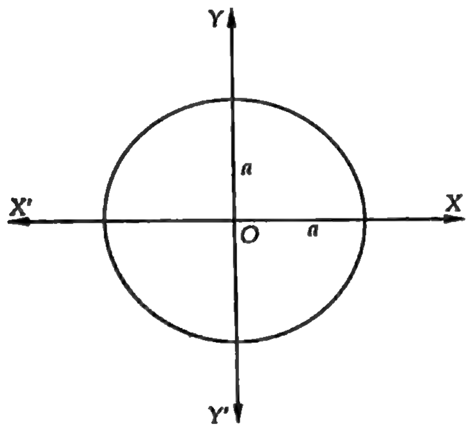


Fig. 24.3

In this case, $h = k = 0$. Putting $h = 0, k = 0$ in equation (i), we obtain $x^2 + y^2 = a^2$ as the equation of the circle having centre at the origin and radius equal to 'a'.

(ii) *When the circle passes through the origin (Fig. 24.4):*

Let O be the origin and $C(h, k)$ be the centre of the circle. Draw $CM \perp OX$.

Using Pythagoras Theorem in ΔOCM, we obtain

$$OC^2 = OM^2 + CM^2$$

$$\Rightarrow \quad a^2 = h^2 + k^2$$

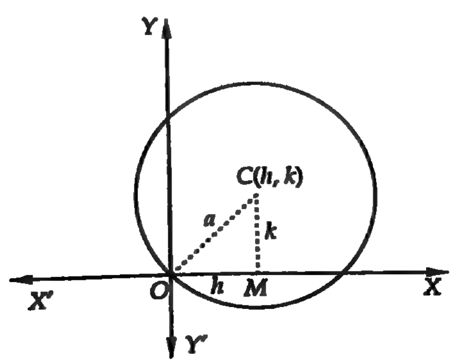


Fig. 24.4

The equation of the circle (i) then becomes

$$(x-h)^2 + (y-k)^2 = h^2 + k^2 \text{ or, } x^2 + y^2 - 2hx - 2ky = 0.$$

(iii) *When the circle touches x-axis (Fig. 24.5)*:

Let C (h, k) be the centre of the circle. Since the circle touches the x-axis. Therefore, $a = k$

Hence, the equation of the circle is

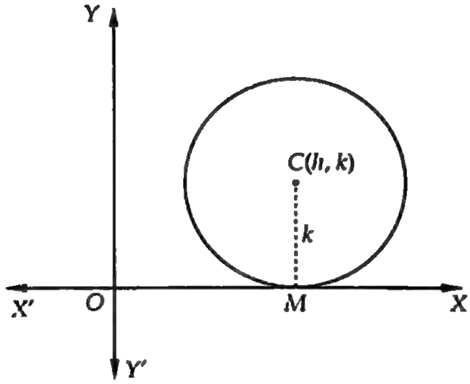


Fig. 24.5

$$(x-h)^2 + (y-a)^2 = a^2 \text{ or, } x^2 + y^2 - 2hx - 2ay + h^2 = 0$$

(iv) *When the circle touches y-axis (Fig. 24.6):*

Let C (h, k) be the centre of the circle. Since the circle touches the y-axis. Therefore, $h = a$

Hence, the equation of the circle is

$$(x-a)^2 + (y-k)^2 = a^2 \text{ or, } x^2 + y^2 - 2ax - 2ky + k^2 = 0.$$

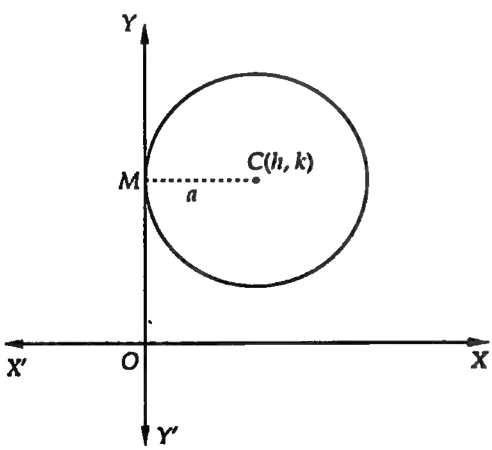


Fig. 24.6

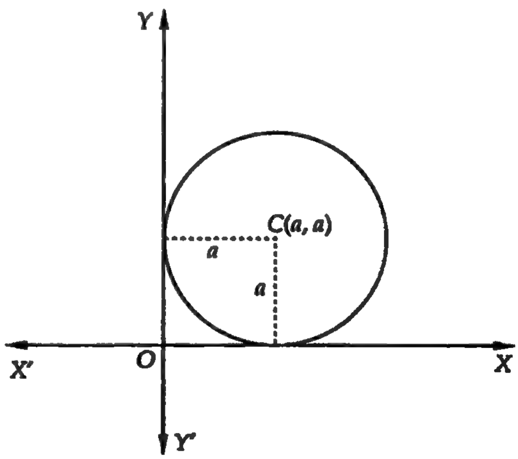


Fig. 24.7

(v) ***When the circle touches both the axes (Fig. 24.7):***

In this case we have, $h = k = a$

Hence, the equation of the circle is

$$(x-a)^2 + (y-a)^2 = a^2 \text{ or, } x^2 + y^2 - 2ax - 2ay + a^2 = 0.$$

(vi) *When the circle passes through the origin and centre lies on x-axis (Fig. 24.8):*
In this case, we have $k = 0$ and $h = a$.
Hence, the equation of the circle is

$$(x - a)^2 + (y - 0)^2 = a^2 \text{ or, } x^2 + y^2 - 2ax = 0.$$

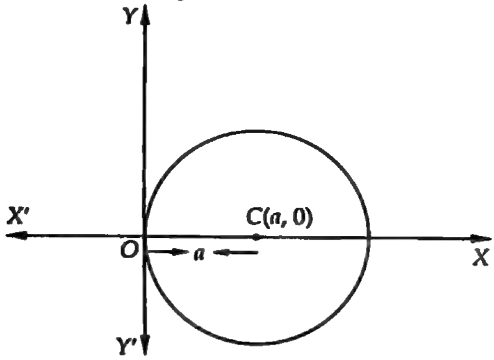


Fig. 24.8

(vii) *When the circle passes through the origin and centre lies on y-axis (Fig. 24.9):*
In this case, we have $h = 0$ and $k = a$.
Hence, the equation of the circle is

$$(x - 0)^2 + (y - a)^2 = a^2 \text{ or, } x^2 + y^2 - 2ay = 0.$$

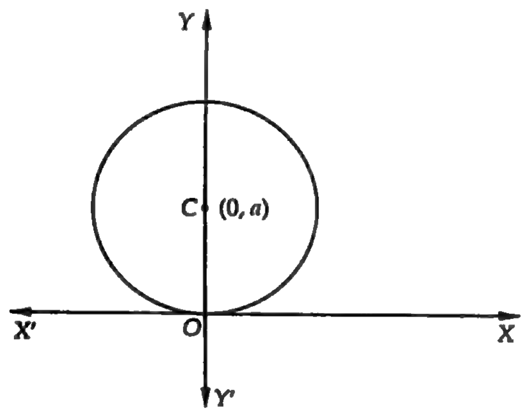


Fig. 24.9

IMPORTANT POINTS TO REMEMBER

(i) When a circle touches x-axis, then its radius is equal to the absolute value of the y-coordinates of the centre.

(ii) When a circle touches y-axis, the x-coordinates of its centre, in magnitude, is equal to the radius.

(iii) When a circle touches x-axis at the origin, then its centre lies on y-axis and absolute value of y-coordinates of the centre is equal to the radius.

(iv) When a circle touches y-axis at the origin, then its centre lies on x-axis at a distance equal to the radius of the circle.

(v) When a circle touches both the axis, then the coordinates of its centre are $(\pm a, \pm a)$, where a is the radius of the circle.

(vi) When a circle touches a line, then length of the perpendicular from its centre on the given line is equal to the radius of the circle.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE EQUATION OF A CIRCLE WHEN ITS CENTRE AND RADIUS ARE KNOWN**

EXAMPLE 1 *Find the equation of the circle whose centre is* (2, – 3) *and radius is 8.*

SOLUTION The equation of the circle is

$$(x-2)^2 + (y-(-3))^2 = 8^2 \qquad \text{[Using: } (x-h)^2+(y-k)^2=a^2\text{]}$$

$$\Rightarrow \quad (x-2)^2+(y+3)^2 = 8^2 \text{ or, } x^2+y^2-4x+6y-51 = 0.$$

EXAMPLE 2 *Find the equation of the circle which passes through the point of intersection of the lines* $3x-2y-1=0$ *and* $4x+y-27=0$ *and whose centre is* $(2,-3)$.

SOLUTION Let P be the point of intersection of the lines AB and LM whose equations are respectively

$$3x-2y-1=0 \quad \text{...(i)} \qquad \text{and} \qquad 4x+y-27=0 \quad \text{...(ii)}$$

Solving (i) and (ii), we get $x=5, y=7$. So, coordinates of P are (5, 7). Let C (2, – 3) be the centre of the circle. Since the circle passes through P.

$$\therefore \quad CP = \text{Radius}$$

$$\Rightarrow \quad \sqrt{(5-2)^2+(7+3)^2} = \text{Radius}$$

$$\Rightarrow \quad \text{Radius} = \sqrt{109}.$$

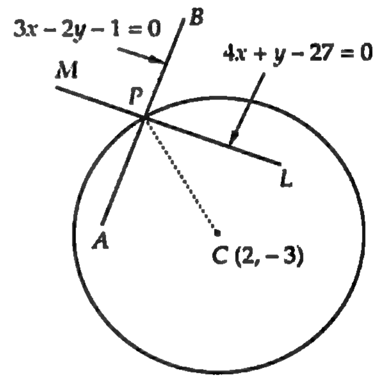


Fig. 24.10

Thus, the required circle has its centre at C (2, – 3)

and, radius $=\sqrt{109}$. So, its equation is

$$(x-2)^2+(y+3)^2 = (\sqrt{109})^2 \text{ or, } x^2+y^2-4x+6y-96=0$$

EXAMPLE 3 *Find the equation of the circle having centre at* $(3,-4)$ *and touching the line* $5x+12y-12=0$.

[NCERT EXEMPLAR]

SOLUTION Let C $(3,-4)$ be the centre of the circle. If the line $5x+12y-12=0$ touches the required circle at P. Then, CP is perpendicular to the line and is equal to the radius of the circle.

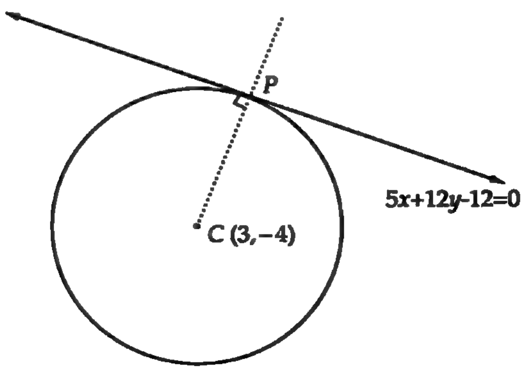


Fig. 24.11

$$\therefore \quad \text{Radius} = CP = \text{Length of perpendicular from } C(3,-4) \text{ on the line } 5x+12y-12=0$$

$$\Rightarrow \quad \text{Radius} = \left|\frac{5\times 3+12\times -4-12}{\sqrt{5^2+12^2}}\right| = \frac{45}{13}$$

Thus, the required circle has its centre at C $(3, -4)$ and radius $= \frac{45}{13}$.

Hence, its equation is $(x-3)^2+(y+4)^2 = \left(\frac{45}{13}\right)^2$

EXAMPLE 4 *Find the equation of a circle with origin as centre and which circumscribes an equilateral triangle whose median is of length 3a.*

SOLUTION Let the circle circumscribes an equilateral triangle ABC and let $AD = 3a$ be a median of ΔABC. It is given that the centre of the circle is at the origin O. Clearly, O lies on the median AD and coincides with the centroid of ΔABC.

$$\therefore \quad OA = \frac{2}{3}AD = \frac{2}{3}\times 3a = 2a$$

$$\Rightarrow \quad \text{Radius} = 2a$$

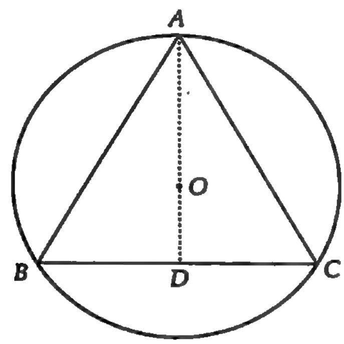


Fig. 24.12

Thus, the given circle has its centre at the origin O (0, 0) and radius $= 2a$.
Hence, the equation of the circle is $(x-0)^2+(y-0)^2 = (2a)^2$ or, $x^2+y^2 = 4a^2$.

EXAMPLE 5 *If the equations of the two diameters of a circle are $x-y=5$ and $2x+y=4$ and the radius of the circle is 5, find the equation of the* circle.

SOLUTION Let the diameters of the circle be AB and LM whose equations are respectively

$$x - y = 5 \qquad \text{...(i)} \qquad 2x + y = 4 \qquad \text{...(ii)}$$

Solving (i) and (ii), we get : $x = 3$ and $y = -2$.
Since the point of intersection of any two diameters of a circle is its centre. Therefore, coordinates of the centre of the required circle are $(3, -2)$ and its radius is 5 (given).

Hence, its equation is

$$(x-3)^2+(y+2)^2 = 5^2 \quad \text{or,} \quad x^2+y^2-6x+4y-12=0$$

EXAMPLE 6 *Find the equation of a circle whose diameters are $2x-3y+12=0$ and $x+4y-5=0$ and area is 154 square units.*

SOLUTION The centre is the point of intersection of diameters. Solving $2x-3y+12=0$ and $x+4y-5=0$, we get $x=-3$ and $y=2$. So, the coordinates of centre are $(-3, 2)$. Let r be the radius of the circle. Then,

$$\text{Area} = 154 \Rightarrow \pi r^2 = 154 \Rightarrow \frac{22}{7}\times r^2 = 154 \Rightarrow r = 7$$

Hence, the equation of the required circle is $(x+3)^2+(y-2)^2 = 49$.

EXAMPLE 7 *Find the equation of a circle of radius 5 whose centre lies on x-axis and passes through the point (2, 3).*

SOLUTION Let the coordinates of the centre of the required circle be C $(a, 0)$. Since it passes through P (2, 3).

$$\therefore \quad CP = \text{radius}$$
$$\Rightarrow \quad CP = 5$$
$$\Rightarrow \quad \sqrt{(a-2)^2 + (0-3)^2} = 5$$
$$\Rightarrow \quad (a-2)^2 + 9 = 25 \Rightarrow a - 2 = \pm 4 \Rightarrow a = 6 \text{ or, } a = -2$$

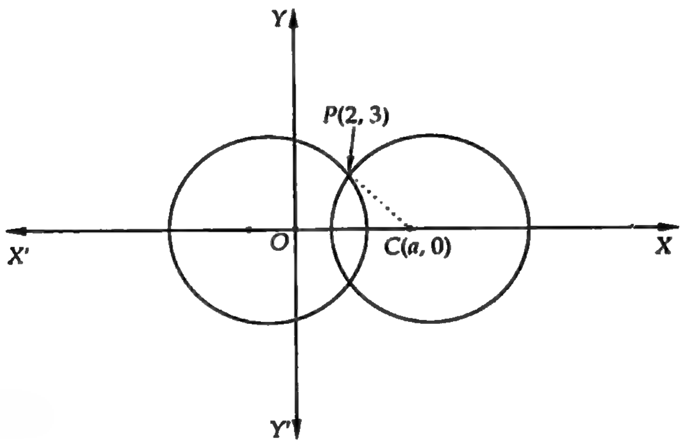


Fig. 24.13

Thus, the coordinates of the centre are (6, 0) or (–2, 0). Hence, the equations of the required circle are

$$(x-6)^2 + (y-0)^2 = 5^2 \quad \text{and} \quad (x+2)^2 + (y-0)^2 = 5^2$$
$$\Rightarrow \quad x^2 + y^2 - 12x + 11 = 0 \quad \text{and} \quad x^2 + y^2 + 4x - 21 = 0$$

Type II **ON FINDING THE CENTRE AND RADIUS OF A GIVEN CIRCLE**

EXAMPLE 8 *Find the centre and radius of each of the following circles:*

(i) $x^2 + (y+2)^2 = 9$ (ii) $x^2 + y^2 - 4x + 6y = 12$

(iii) $(x+1)^2 + (y-1)^2 = 4$ (iv) $x^2 + y^2 + 6x - 4y + 4 = 0$.

SOLUTION (i) We have,

$$x^2 + (y+2)^2 = 9 \Rightarrow (x-0)^2 + \{y-(-2)\}^2 = 3^2$$

Comparing this equation with $(x-a)^2\ (y-b)^2 = r^2$, we find that the given circle has its centre at (0, –2) and radius 3.

(ii) We have, $x^2 + y^2 - 4x + 6y = 12$

$$\Rightarrow \quad (x^2 - 4x) + (y^2 + 6y) = 12$$
$$\Rightarrow \quad (x^2 - 4x + 4) + (y^2 + 6y + 9) = 12 + 4 + 9$$
$$\Rightarrow \quad (x-2)^2 + (y+3)^2 = 5^2 \Rightarrow (x-2)^2 + \{y-(-3)\}^2 = 5^2$$

Comparing this equation with $(x-a)^2 + (y-b)^2 = r^2$, we find that the given circle has its centre at (2, –3) and radius 5.

(iii) We have, $(x+1)^2 + (y-1)^2 = 4$

$$\Rightarrow \quad \{x-(-1)\}^2 + (y-1)^2 = 2^2$$

Clearly, the given circle has its centre at (–1, 1) and radius 2.

(iv) We have, $x^2 + y^2 + 6x - 4y + 4 = 0$

$$\Rightarrow \quad (x^2 + 6x) + (y^2 - 4y) = -4$$

$$\Rightarrow \quad (x^2 + 6x + 9) + (y^2 - 4y + 4) = -4 + 9 + 4$$

$$\Rightarrow \quad (x + 3^2) + (y - 2)^2 = 3^2 \Rightarrow \{x - (-3)\}^2 + (y - 2)^2 = 3^2.$$

Clearly, this circle has its centre at (–3, 2) and radius 3.

Type III ON FINDING THE EQUATION OF A CIRCLE SATISFYING SOME GIVEN GEOMETRICAL CONDITIONS

EXAMPLE 9 *Find the equation of the circle which touches:*

(i) the x-axis and whose centre is (3, 4) *(ii) the x-axis at the origin and whose radius is 5*

(iii) both the axes and whose radius is 5 *(iv) the lines $x = 0$, $y = 0$ and $x = a$.*

SOLUTION (i) Clearly, radius $= CP = 4$ (Fig. 24.14) and the coordinates of the centre are (3, 4).

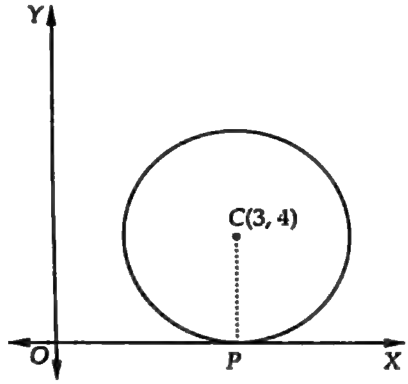


Fig. 24.14

Hence, the equation of the required circle is

$$(x - 3)^2 + (y - 4)^2 = 4^2 \text{ or, } x^2 + y^2 - 6x - 8y + 9 = 0$$

(ii) Since the circle touches the x-axis at the origin and has radius 5. So, the coordinates of the centre are (0, 5) as shown in Fig. 24.15. Hence, the equation of the circle is

$$(x - 0)^2 + (y - 5)^2 = 5^2 \text{ or, } x^2 + y^2 - 10y = 0.$$

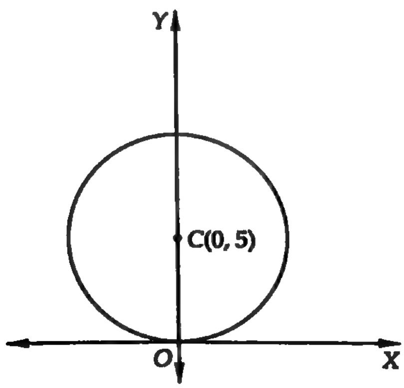


Fig. 24.15

(iii) The circle touches both the axes and has radius 5. So, the coordinates of the centre are (5, 5) and radius = 5 as shown in Fig. 24.16. So, the equation of the required circle is

$$(x - 5)^2 + (y - 5)^2 = 5^2 \text{ or, } x^2 + y^2 - 10x - 10y + 25 = 0.$$

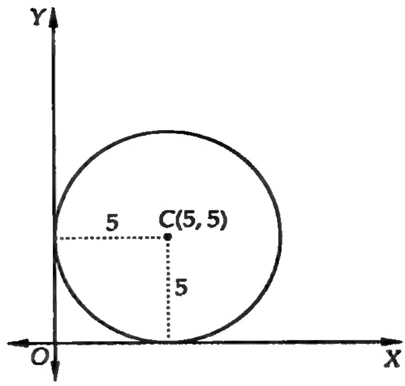


Fig. 24.16

Since, the circle may lie in any one of the four quadrants. So, there are four such circles. The equations of these circles are given by $x^2 + y^2 \pm 10x \pm 10y + 25 = 0$.

(iv) The circle touches the coordinate axes and the line $x = a$ as shown in Fig. 24.17. So, the centre of the required circle is at $(a/2, a/2)$ and radius $= a/2$.

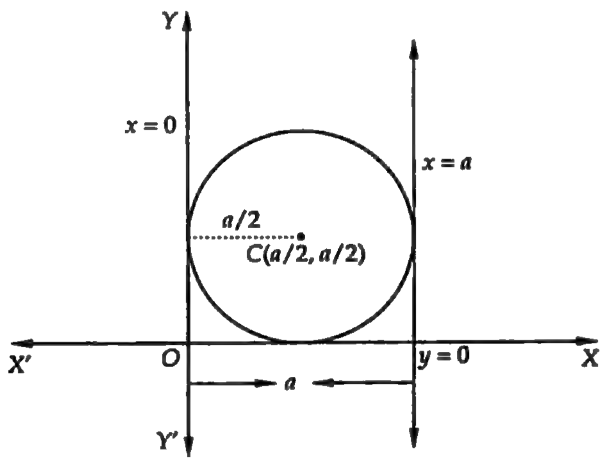


Fig. 24.17

Hence, its equation is $(x - a/2)^2 + (y - a/2)^2 = (a/2)^2$.

There may be two such circles, one lying above x-axis and other below x-axis. The circle lying below x-axis has its centre at $(a/2, -a/2)$ and radius $a/2$. The equation of this circles is

$$\left(x - \frac{a}{2}\right)^2 + \left(y + \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$$

Hence, the equations of the circles are $\left(x - \frac{a}{2}\right)^2 + \left(y \mp \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$.

EXAMPLE 10 *Find the equations of the circles which passes through two points on the x-axis which are at distances 4 from the origin and whose radius is 5.*

SOLUTION As is evident from Fig. 24.18 there are two circles which pass through two points A and A' on x-axis which are at a distance 4 from the origin. The centres of these circles lie on y-axis.

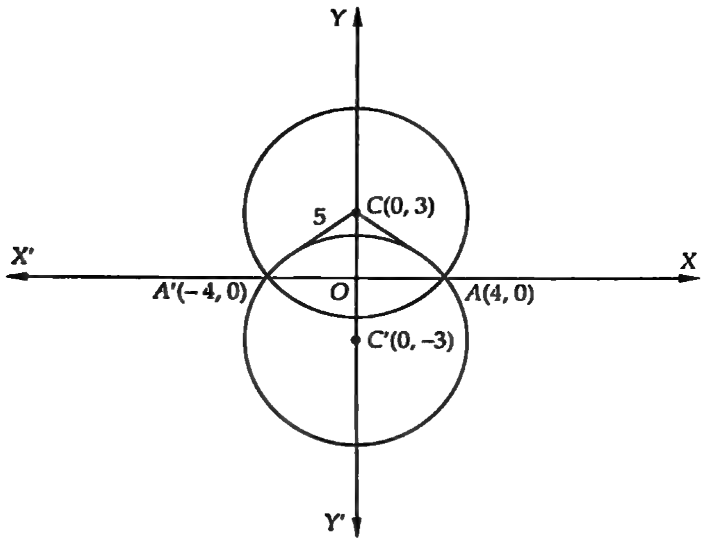


Fig. 24.18

Applying Pythagoras Theorem in $\Delta\ OAC$, we get

$$AC^2 = OA^2 + OC^2$$

$$\Rightarrow \quad 5^2 = 4^2 + OC^2 \Rightarrow OC = 3.$$

So, the coordinates of the centres of the required circles are C (0, 3) and C′ (0, − 3).

Hence, the equations of the required circles are

$$(x-0)^2 + (y \mp 3)^2 = 5^2 \text{ or, } x^2 + y^2 \mp 6y - 16 = 0.$$

EXAMPLE 11 *Find the equation of the circle which passes through the origin and cuts off intercepts 3 and 4 from the positive parts of the axes respectively.*

SOLUTION Let the circle cuts off intercepts OA and OB from OX and OY respectively. It is given that $OA = 3$ and $OB = 4$.

$$\therefore \quad OL = \frac{3}{2} \text{ and, } CL = 2$$

In $\Delta\ OLC$, we have

$$OC^2 = OL^2 + LC^2$$

$$\Rightarrow \quad OC^2 = \left(\frac{3}{2}\right)^2 + 2^2$$

$$\Rightarrow \quad OC = \frac{5}{2}.$$

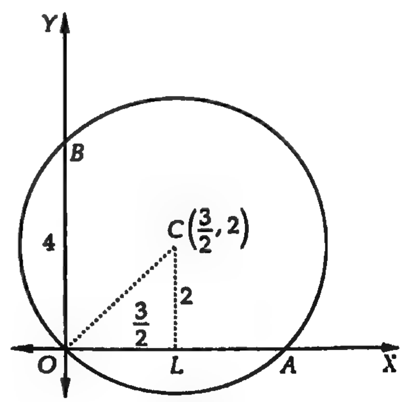


Fig. 24.19

Thus, the required circle has its centre at $(3/2, 2)$ and radius $5/2$.

Hence, its equation is $\left(x - \frac{3}{2}\right)^2 + (y-2)^2 = \left(\frac{5}{2}\right)^2$.

EXAMPLE 12 *Find the equation of a circle which touches y-axis at a distance of 4 units from the origin and cuts an intercept of 6 units along the positive direction of x-axis.*

SOLUTION The given circle touches y-axis at $L(0, 4)$ and cuts an intercept $AB = 6$ along the positive direction of x-axis. As shown in Fig. 24.20, there are two such circles.

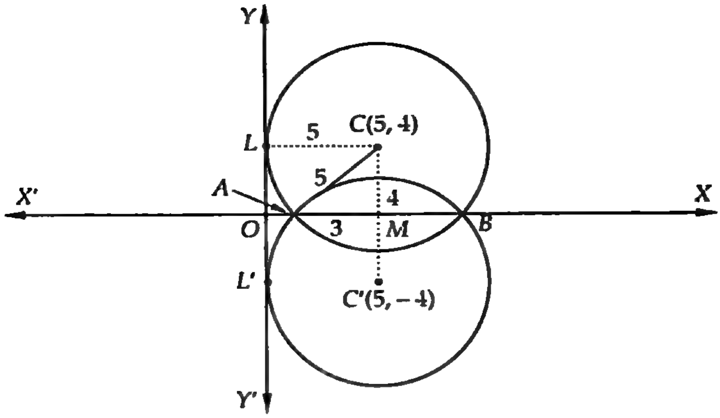


Fig. 24.20

In ΔCAM, we have

$$CA^2 = CM^2 + AM^2$$

$$\Rightarrow \quad CA^2 = 4^2 + 3^2$$

$$\Rightarrow \quad CA = 5$$

Also, $\quad CL = CA = 5.$

Thus, the coordinates of the centres are (5, 4) or (5, − 4) and radius = 5.

Hence, the equations of the required circles are

$$(x-5)^2 + (y \mp 4)^2 = 5 \Rightarrow x^2 + y^2 - 10x \mp 8y + 16 = 0.$$

EXAMPLE 13 *Find the equation of a circle which passes through the point (2, 0) and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)\, x + 5c^2 y = 1$ as $c \to 1$.*

SOLUTION We have,

$$3x + 5y = 1 \text{ and } (2 + c)\, x + 5c^2 y = 1.$$

Solving these two equations, we get

$$x = \frac{c^2 - 1}{3c^2 - c - 2} \text{ and } y = -\frac{c-1}{5\,(3c^2 - c - 2)}$$

Now, $\displaystyle \lim_{c \to 1} x = \lim_{c \to 1} \frac{c^2 - 1}{3c^2 - c - 2} = \lim_{c \to 1} \frac{(c-1)(c+1)}{(c-1)(3c+2)} = \lim_{c \to 1} \frac{c+1}{3c+2} = \frac{2}{5}$

and, $\displaystyle \lim_{c \to 1} y = \lim_{c \to 1} -\frac{c-1}{5\,(3c^2 - c - 2)} = -\lim_{c \to 1} \frac{(c-1)}{5\,(c-1)(3c+2)} = \lim_{c \to 1} \frac{1}{5\,(3c+2)} = -\frac{1}{25}$

Thus, the coordinates of the centre of the circle are C (2/5, −1/25). It passes through P (2, 0).

$$\therefore \quad \text{Radius} = CP = \sqrt{\left(2 - \frac{2}{5}\right)^2 + \left(0 + \frac{1}{25}\right)^2} = \sqrt{\frac{64}{25} + \frac{1}{625}} = \frac{\sqrt{1601}}{25}$$

Hence, the equation of the required circle is

$$\left(x - \frac{2}{5}\right)^2 + \left(y + \frac{1}{25}\right)^2 = \frac{1601}{625} \text{ or, } 25x^2 + 25y^2 - 20x + 2y - 60 = 0$$

LEVEL-2

EXAMPLE 14 *A circle of radius 5 units touches the coordinate axes in the first quadrant. If the circle makes one complete roll on x-axis along the positive direction of x-axis, find its equation in new position.*

SOLUTION Let C and C_1 be the centres of the circle in its initial and final positions. The coordinates of C are (5, 5). In making one complete roll on x-axis, the centre C moves through the distance $CC_1 = AB =$ Circumference of the circle $= 10\pi$.

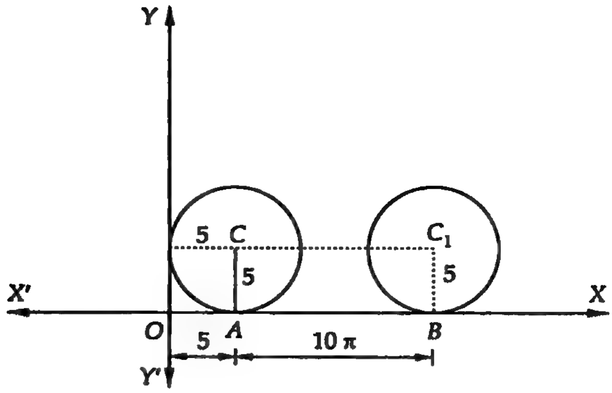


Fig. 24.21

So, the coordinates of the centre of the circle in the new position are $(5 + 10\pi, 5)$.
Radius of the circle in its new position is 5 units.
Hence, its equation is $\{x - (5 + 10\pi)\}^2 + (y - 5)^2 = 5^2$.

EXAMPLE 15 *A circle of radius 6 units touches the coordinate axes in the first quadrant. Find the equation of its image in the line mirror $y = 0$.*

SOLUTION The given circle has radius 6 and the co- ordinates of its centre C are (6, 6). The coordinates of its image C_1 in the line mirror $y = 0$ i.e. x-axis are (6, – 6).

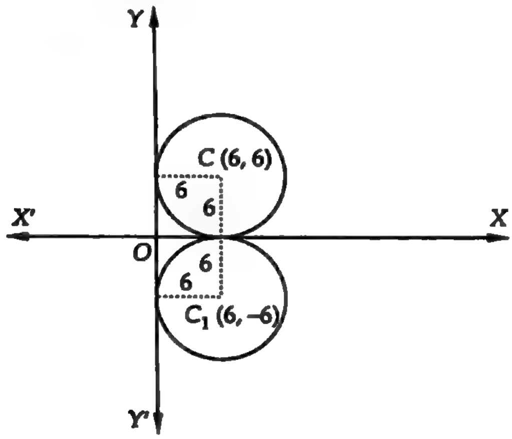


Fig. 24.22

So, the centre of the required circle is at C_1 (6, – 6) and its radius is 6.
Hence, its equation is

$$(x - 6)^2 + (y + 6)^2 = 6^2 \text{ or, } x^2 + y^2 - 12x + 12y + 36 = 0.$$

EXAMPLE 16 *Find the equation of the image of the circle $x^2 + y^2 + 8x - 16y + 64 = 0$ in the line mirror $x = 0$.*

SOLUTION The equation of the given circle is

$$x^2 + y^2 + 8x - 16y + 64 = 0$$

$$\Rightarrow \quad (x^2 + 8x + 16) + (y^2 - 16y + 64) = 16$$

$$\Rightarrow \quad (x + 4)^2 + (y - 8)^2 = 4^2$$

$$\Rightarrow \quad (x - (-4))^2 + (y - 8)^2 = 4^2$$

Clearly, its centre is at (– 4, 8) and radius = 4.

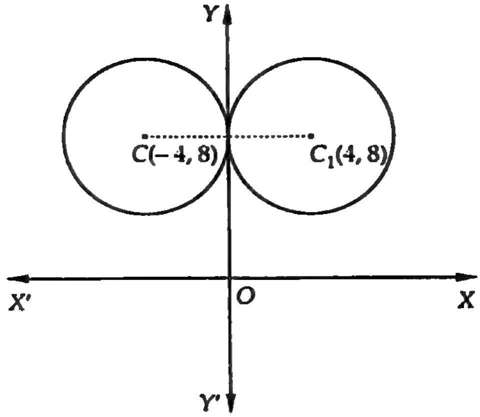


Fig. 24.23

The image of this circle in the line mirror has its centre C_1 (4, 8) and radius 4. So, its equation is

$$(x-4)^2 + (y-8)^2 = 4^2 \text{ or, } x^2 + y^2 - 8x - 16y + 64 = 0$$

EXAMPLE 17 *The circle $(x-a)^2 + (y-a)^2 = a^2$ is rolled on the y-axis in the positive direction through one complete revolution. Find the equation of the circle in its new-position.*

SOLUTION The given circle has its centre C (a, a) and radius $= a$. Clearly, it touches both the axes. When this circle rolls on y-axis and completes one revolution, its centre moves vertically through the distance equal to its circumference i.e. $2\pi a$. So, the coordinates of the centre of the new-circle are C_1 $(a, a + 2\pi a)$. Clearly, radius of the new circle is same as that of the given circle i.e. a.

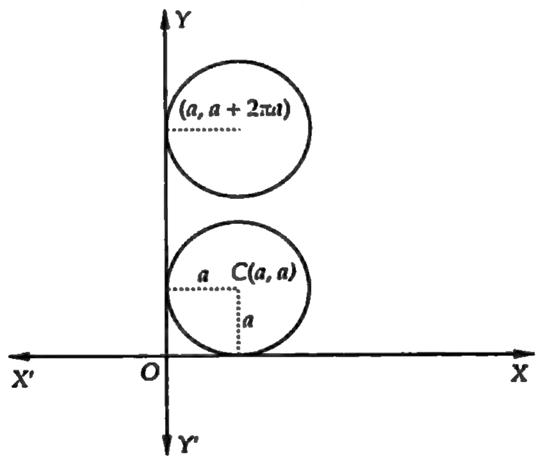


Fig. 24.24

Hence, the equation of the new circle is $(x-a)^2 + \{y-(a+2\pi a)\}^2 = a^2$.

EXAMPLE 18 *Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ externally at the point (5, 5).*

SOLUTION The equation of the given circle is

$$x^2 + y^2 - 2x - 4y - 20 = 0 \text{ or, } (x-1)^2 + (y-2)^2 = 5^2.$$

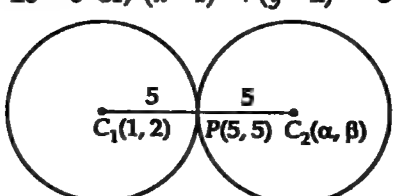


Fig. 24.25

Its centre is $C_1(1, 2)$ and radius = 5. This circle touches another circle of radius 5 externally at point $P(5, 5)$. Let its centre be $C_2(\alpha, \beta)$.

Clearly, $P(5, 5)$ is the mid-point of C_1C_2.

$$\therefore \quad \frac{\alpha+1}{2} = 5 \text{ and } \frac{\beta+2}{2} = 5 \Rightarrow \alpha = 9, \beta = 8$$

Thus, the required circle has its centre at (9, 8) and radius = 5.

Hence, the equation of the required circle is $(x-9)^2 + (y-8)^2 = 5^2$.

EXAMPLE 19 *Find the equation of a circle of radius 5 which lies within the circle $x^2 + y^2 + 14x + 10y - 26 = 0$ and which touches the given circle at the point $(-1, 3)$.*

SOLUTION The equation of the given circle is

$$x^2 + y^2 + 14x + 10y - 26 = 0 \text{ or, } (x-(-7))^2 + (y-(-5))^2 = 10^2$$

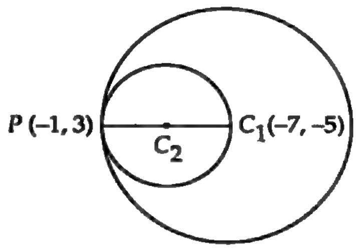


Fig. 24.26

Its centre is at $C_1(-7, -5)$ and, radius =10.

The required circle touches the above circle internally at $P(-1, 3)$ and has radius = 5 i.e. half of the radius of the given circle. So, its centre C_2 is the mid-point of C_1P. Therefore, coordinates of its centre C_2 are $\left(\frac{-1-7}{2}, \frac{3-5}{2}\right) = (-4, -1)$.

Hence, the equation of the required circle is $(x+4)^2 + (y+1)^2 = 5^2$.

EXAMPLE 20 *A circle of radius 2 lies in the first quadrant and touches both the axes. Find the equation of the circle with centre at (6, 5) and touching the above circle externally.*

SOLUTION The coordinates of the centre of the given circle are $C_1(2, 2)$ and the coordinates of the centre of the required circle are $C_2(6, 5)$.

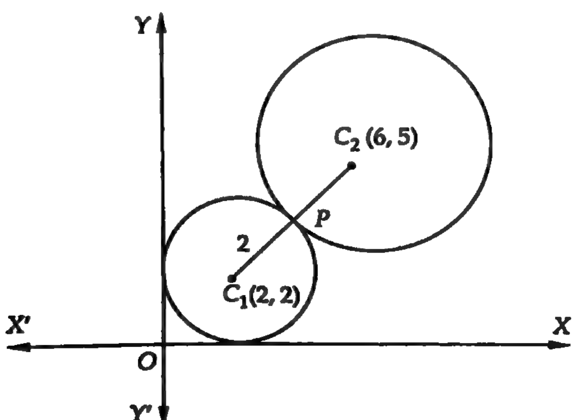


Fig. 24.27

Since it touches the given circle externally. Therefore,

$$C_1C_2 = C_1P + C_2P$$

$$\Rightarrow \quad \sqrt{(6-2)^2+(5-2)^2} = 2 + C_2P$$

$$\Rightarrow \quad 5 = 2 + C_2P$$

$$\Rightarrow \quad C_2P = 3$$

Thus, the required circle has its centre at $C_2(6, 5)$ and radius = 3.

Hence, its equation is $(x-6)^2 + (y-5)^2 = 3^2$ or, $x^2 + y^2 - 12x - 10y + 52 = 0$

EXAMPLE 21 *Show that the equation of the circle which touches the coordinate axes and whose centre lies on the line $lx + my + n = 0$ is $(l+m)^2(x^2+y^2) + 2n(l+m)(x+y) + n^2 = 0$.*

SOLUTION We know that the coordinates of the centre of a circle touching the coordinates axes in first quadrant are (a, a), where a is the radius of the circle. So, the equation of the circle is

$$(x-a)^2 + (y-a)^2 = a^2 \text{ or, } x^2 + y^2 - 2ax - 2ay + a^2 = 0 \quad \text{...(i)}$$

Since the centre (a, a) lies on $lx + my + n = 0$. Therefore,

$$la + ma + n = 0 \Rightarrow a = -\frac{n}{l+m}$$

Putting the value of a in (i), we obtain the equation of the circle as

$$x^2 + y^2 + \frac{2nx}{l+m} + \frac{2ny}{l+m} + \frac{n^2}{(1+m)^2} = 0 \text{ or, } (l+m)^2(x^2+y^2) + 2n(x+y)(l+m) + n^2 = 0.$$

EXAMPLE 22 *Find the equation of the circle which touches both the axes and the line $3x - 4y + 8 = 0$ and lies in the third quadrant.* **[NCERT EXEMPLAR]**

SOLUTION Let a be the radius of the circle. It is given that the circle touches both the axes and lies in the third quadrant. So, the coordinates of its centre are $(-a, -a)$ and the equation of the circle is

$$(x+a)^2 + (y+a)^2 = a^2 \text{ or, } x^2 + y^2 + 2ax + 2ay + a^2 = 0 \quad \text{...(i)}$$

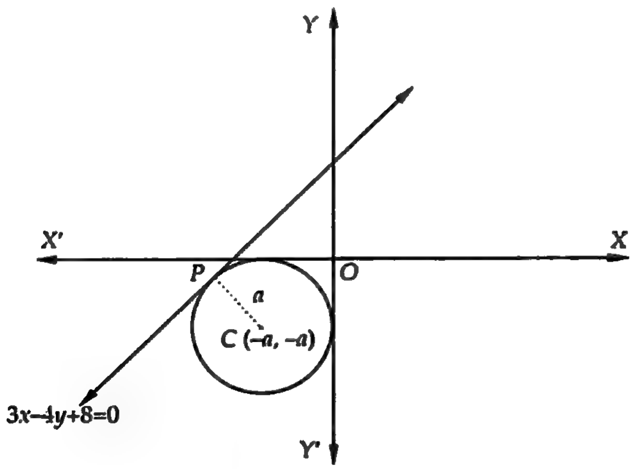


Fig. 24.28

The circle touches the line $3x - 4y + 8 = 0$. Therefore, length of the perpendicular from the centre $(-a, -a)$ to the line $3x - 4y + 8 = 0$ is equal to the radius of the circle.

i.e. $\quad CP = a$

$$\Rightarrow \quad \left|\frac{-3a+4a+8}{\sqrt{3^2+(-4)^2}}\right| = a \Rightarrow \frac{|a+8|}{5} = a \Rightarrow a + 8 = 5a \Rightarrow a = 2 \quad [\because a > 0 \therefore a + 8 > 0]$$

Substituting $a = 2$ in (i), we obtain

$x^2 + y^2 + 4x + 4y + 4 = 0$ as the required equation of the circle.

EXAMPLE 23 *Find the equation of the circle which touches the coordinate axes and whose centre lies on the line* $x - 2y = 3$.

SOLUTION Since the circle touches the coordinate axes and the line $x - 2y + 3 = 0$. So, its centre lies in third or in fourth quadrant. Let a be the radius of the circle.

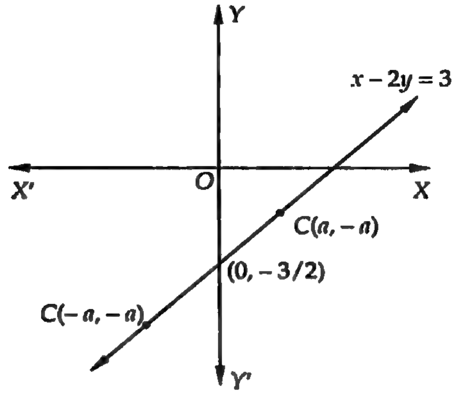


Fig. 24.29

<u>CASE I</u> *When centre is in third quadrant:*
In this case, the coordinates of the centre are $(-a, -a)$. As it lies on $x - 2y = 3$.

$$\therefore \quad -a + 2a = 3 \Rightarrow a = 3.$$

So, the equation of the circle is

$$(x + a)^2 + (y + a)^2 = a^2 \Rightarrow (x + 3)^2 + (y + 3)^2 = 3^2.$$

<u>CASE II</u> *When centre is in fourth quadrant:*
In this case, the coordinates of the centre are $(a, -a)$. As it lies on $x - 2y = 3$.

$$\therefore \quad a + 2a = 3 \Rightarrow a = 1$$

So, the equation of the circle is

$$(x - a)^2 + (y + a)^2 = a^2 \Rightarrow (x - 1)^2 + (y + 1)^2 = 1$$

<u>ALITER</u> We know that a circle touching both the axes has its centre either on $y = x$ or, $y = -x$.

<u>CASE I</u> *When centre of the circle is on* $y = x$:
It is also given that the centre of the circle lies on $x - 2y = 3$. Thus, centre of the required circle is the point of intersection of the lines $y = x$ and $x - 2y = 3$. Solving these two equations, we get $x = -3, y = -3$. Thus, the required circle has centre at $(-3, -3)$ and radius 3.
So, its equation is $(x + 3)^2 + (y + 3)^2 = 3^2$.

<u>CASE II</u> *When centre of the circle is on* $y = -x$:
In this case, centre is the point of intersection of the lines $y = -x$ and $x - 2y = 3$.
Solving these two equations, we obtain that the coordinates of the centre are $(1, -1)$. Radius of the required circle is 1 unit.
Thus, the equation of the required circle is $(x - 1)^2 + (y + 1)^2 = 1^2$.

EXAMPLE 24 *A circle has radius 3 units and its centre lies on the line* $y = x - 1$. *Find the equation of the circle, if it passes through* $(7, 3)$. **[NCERT EXEMPLAR]**

SOLUTION The coordinates of any point on the line $y = x - 1$ can be taken as $(t, t - 1)$. So, let $C\ (t, t - 1)$ be the centre of required circle. Its radius is 3. Therefore, equation of the required circle is

$$(x - t)^2 + \{y - (t - 1)\}^2 = 3^2 \qquad \text{...(i)}$$

It passes through $(7, 3)$.

$$\therefore \quad (7 - t)^2 + \{3 - (t - 1)\}^2 = 3^2$$

$$\Rightarrow \quad (7 - t)^2 + (4 - t)^2 = 9 \Rightarrow t^2 - 11t + 28 = 0 \Rightarrow (t - 4)(t - 7) = 0 \Rightarrow t = 4, 7$$

Substituting the values of t in (i), we obtain that the equations of the required circles are

$$(x-4)^2+(y-3)^2 = 3^2 \text{ and } (x-7)^2+(y-6)^2 = 3^2$$

EXAMPLE 25 *Find the equation of the circle whose centre is at* $(3,-1)$ *and which cuts off a chord of length 6 units on the line* $2x-5y+18=0$ **[NCERT EXEMPLAR]**

SOLUTION Let PQ be the chord cut off by the circle on the line $2x-5y+18=0$. Let $C(3,-1)$ be the centre of the circle and CL perpendicular drawn from C on the chord PQ. Then, L bisects PQ.

$$\therefore \quad PL = QL = \frac{1}{2}PQ \Rightarrow PL = QL = 3$$

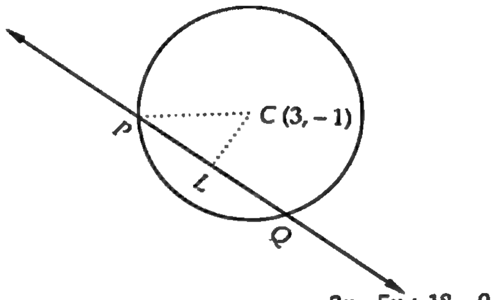


Fig. 24.30

Now,

$CL =$ Length of perpendicular from $C(3,-1)$ on $2x-5y+18=0$

$$\Rightarrow \quad CL = \left|\frac{2\times 3-5\times -1+18}{\sqrt{2^2+(-5)^2}}\right| = \sqrt{29}$$

Applying Pythagoras theorem in ΔCLP, we obtain

$$CP^2 = CL^2 + PL^2$$

$$\Rightarrow \quad CP^2 = (\sqrt{29})^2 + 3^2 = 38$$

$$\Rightarrow \quad CP = \sqrt{38}.$$

Thus, the coordinates of the centre of the circle are $(3,-1)$ and its radius is $\sqrt{38}$.

Hence, the equation of the circle is

$$(x-3)^2+(y+1)^2 = (\sqrt{38})^2 \text{ or, } x^2+y^2-6x+2y-28=0$$

EXAMPLE 26 *A rectangle ABCD is inscribed in a circle with a diameter lying along the line* $3y = x+10$. *If A and B are the points* $(-6,7)$ *and* $(4,7)$ *respectively, find the area of the rectangle and equation of the circle.*

SOLUTION Clearly, centre P of the desired circle lies on $3y = x+10$ and perpendicular bisector of AB. As AB is parallel to x-axis, therefore perpendicular bisector of AB passes through $(-1,7)$ and is parallel to y-axis. So, its equation is $x = -1$. Solving $3y = x+10$ and $x = -1$, we get $x=-1$ and $y=3$. Thus, the coordinates of the centre P are $(-1, 3)$.

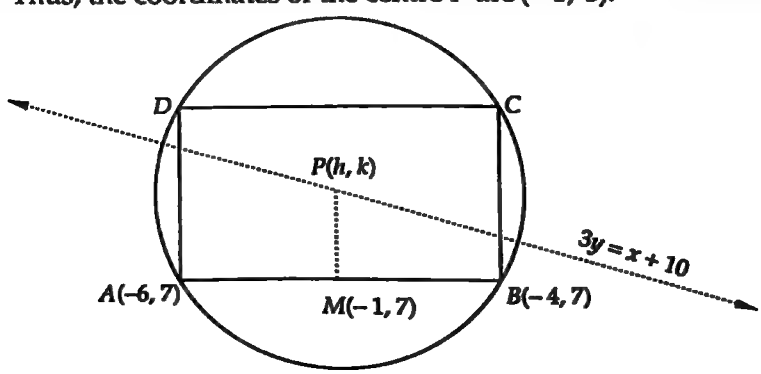


Fig. 24.31

Also, AP = Radius of the circle $= \sqrt{(-6+1)^2 + (7-3)^2} = \sqrt{41}$

Hence, the equation of the circle is

$$(x+1)^2 + (y-3)^2 = (\sqrt{41})^2 \text{ or, } x^2 + y^2 + 2x - 6y - 31 = 0$$

Now, $AD = 2\,PM = 2\sqrt{(-1+1)^2 + (3-7)^2} = 8$ and $AB = \sqrt{(-6-4)^2 + (7-7)^2} = 10$

$\therefore$ Area of rectangle $ABCD = AB \times AD = 10 \times 8 = 80$ square units.

EXAMPLE 27 *Find the equation of the circle passing through the points* $(1, -2)$ *and* $(4, -3)$ *and whose centre lies on the line* $3x + 4y = 7$.

SOLUTION Clearly, centre of the required circle lies on the perpendicular bisector of AB. Clearly,

$$\text{Slope of } AB = \frac{-3+2}{4-1} = \frac{-1}{3}$$

$\therefore$ Slope of $CP = 3$

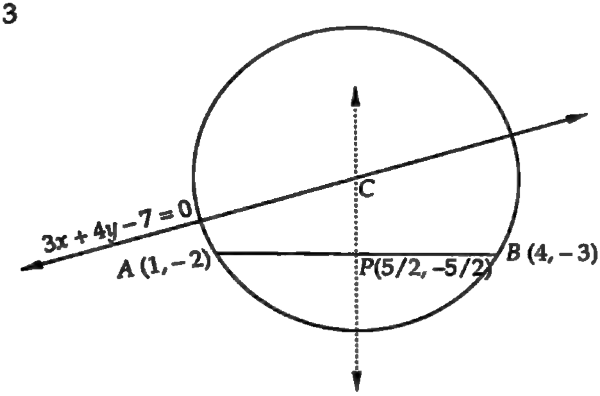


Fig. 24.32

The coordinates of the mid-point P of AB are $(5/2, -5/2)$. The perpendicular bisector of AB passes through $P(5/2, -5/2)$ and is perpendicular to AB. So, the equation of perpendicular bisector of AB is

$$y + \frac{5}{2} = 3\left(x - \frac{5}{2}\right) \text{ or, } 3x - y - 10 = 0$$

Solving $3x + 4y - 7 = 0$ and $3x - y - 10 = 0$, we get:

$$x = 47/15 \text{ and, } y = -3/5$$

So, the coordinates of C are $(47/15, -3/5)$.

Clearly, radius of the circle is AC.

$$\therefore \quad \text{Radius} = AC = \sqrt{\left(\frac{47}{15} - 1\right)^2 + \left(-\frac{3}{5} + 2\right)^2} = \sqrt{\left(\frac{32}{15}\right)^2 + \left(\frac{7}{5}\right)^2} = \frac{\sqrt{1465}}{15}$$

Hence, equation of the required circle is

$$\left(x - \frac{47}{15}\right)^2 + \left(y + \frac{3}{5}\right)^2 = \left(\frac{\sqrt{1465}}{15}\right)^2 \text{ or, } \left(x - \frac{47}{15}\right)^2 + \left(y + \frac{3}{5}\right)^2 = \frac{1465}{225}$$

EXAMPLE 28 *Find the equation of the circle which touches the lines* $4x - 3y + 10 = 0$ *and* $4x - 3y - 30 = 0$ *and whose centre lies on the line* $2x + y = 0$.

SOLUTION Clearly, the lines $4x - 3y + 10 = 0$ and $4x - 3y - 30 = 0$ are parallel and are touching the circle. It is given that the centre of the circle lies on the line $2x + y = 0$ which intersects the lines $4x - 3y + 10 = 0$ and $4x - 3y - 30 = 0$ at $A(-1, 2)$ and $B(3, -6)$ respectively. Therefore, centre of the circle is the mid-point of AB. So, the coordinates of the centre C are $(1, -2)$.

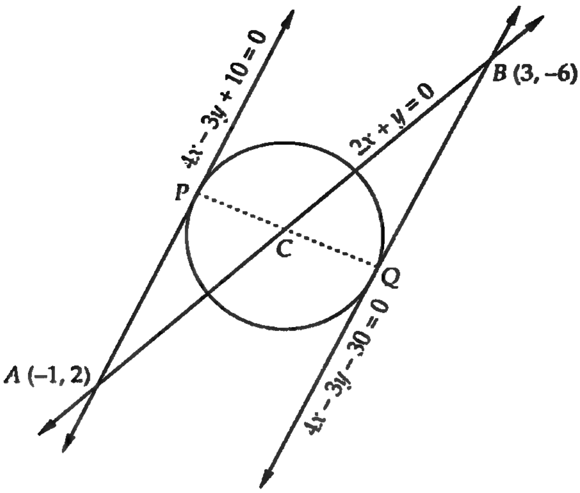


Fig. 24.33

Let d be the distance between parallel lines $4x - 3y + 10 = 0$ and $4x - 3y - 30 = 0$. Then,

$$d = \left| \frac{10 - (-30)}{\sqrt{4^2 + (-3)^2}} \right| = 8$$

$\therefore$ Radius $= \frac{1}{2}\,(PQ) = \frac{1}{2} \times d = 4$

Thus, the required circle has its centre at $C(1, -2)$ and radius $= 4$.
Hence, its equation is $(x-1)^2 + (y+2)^2 = 4^2$.

EXAMPLE 29 *Find the locus of the centre of the circle touching the line $x + 2y = 0$ and $x - 2y = 0$.*

SOLUTION Let (h, k) be the centre of the circle touching the lines $x + 2y = 0$ and $x - 2y = 0$. Let r be the radius of the circle. We know that the length of the perpendicular from the centre of a circle on the tangent line is equal to the radius of the circle.

$\therefore$ (Length of the perpendicular from (h, k) on $x + 2y = 0$) $= r$

and, (Length of the perpendicular from (h, k) on $x - 2y = 0$) $= r$.

$\Rightarrow \quad \frac{|h+2k|}{\sqrt{1+2^2}} = r$ and, $\left| \frac{h-2k}{\sqrt{1^2 + (-2)^2}} \right| = r$

$\Rightarrow \quad \frac{|h+2k|}{\sqrt{5}} = r$ and, $\frac{|h-2k|}{\sqrt{5}} = r$

$\Rightarrow \quad \frac{|h+2k|}{\sqrt{5}} = \frac{|h-2k|}{\sqrt{5}}$

$\Rightarrow \quad |h+2k| = |h-2k|$

$\Rightarrow \quad h + 2k = \pm (h - 2k)$

$\Rightarrow \quad h + 2k = h - 2k$ or, $h + 2k = -(h - 2k)$

$\Rightarrow \quad 4k = 0$ or, $2h = 0$

$\Rightarrow \quad h = 0$ or, $k = 0$

Hence, the locus of (h, k) is $x = 0$ or $y = 0$.

EXAMPLE 30 *Let C be any circle with centre $(0, \sqrt{2})$. Prove that at most two rational points can be there on C. (A rational point is a point both of whose coordinates are rational numbers)*

SOLUTION The equation of any circle C with centre $(0, \sqrt{2})$ is given by

$(x-0)^2 + (y - \sqrt{2})^2 = r^2$, where r is any positive real number.

or, $\quad x^2 + y^2 - 2\sqrt{2}y = r^2 - 2 \quad$ ...(i)

If possible, let $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ be three distinct rational points on circle C. Then,

$$x_1^2 + y_1^2 - 2\sqrt{2}\, y_1 = r^2 - 2 \quad \text{...(ii)}$$

$$x_2^2 + y_2^2 - 2\sqrt{2}\, y_2 = r^2 - 2 \quad \text{...(iii)}$$

$$x_3^2 + y_3^2 - 2\sqrt{2}\, y_3 = r^2 - 2 \quad \text{...(iv)}$$

We claim that at least two y_1, y_2 and y_3 are distinct. For if $y_1 = y_2 = y_3$, then P, Q and R lie on a line parallel to x-axis and a line parallel to x-axis does not cross the circle in more than two points. Thus, we have either $y_1 \neq y_2$ or, $y_1 \neq y_3$ or, $y_2 \neq y_3$.

Subtracting (ii) from (iii) and (iv), we get

$$(x_2^2 + y_2^2) - (x_1^2 + y_1^2) - 2\sqrt{2}\,(y_2 - y_1) = 0$$

and, $$(x_3^2 + y_3^2) - (x_1^2 + y_1^2) - 2\sqrt{2}\,(y_3 - y_1) = 0$$

$$\Rightarrow \quad a_1 - \sqrt{2} b_1 = 0 \text{ and } a_2 - \sqrt{2}\, b_2 = 0 \quad \text{...(v)}$$

where, $a_1 = (x_2^2 + y_2^2) - (x_1^2 + y_1^2)$, $b_1 = 2(y_2 - y_1)$

$a_2 = (x_3^2 + y_3^2) - (x_1^2 + y_1^2)$, $b_2 = 2(y_3 - y_1)$

Clearly, a_1, a_2, b_1, b_2 are rational numbers as $x_1, x_2, x_3, y_1, y_2, y_3$ are rational numbers.

Since either $y_1 \neq y_2$ or, $y_1 \neq y_3$. Therefore, either $b_1 \neq 0$ or, $b_2 \neq 0$.

If $b_1 \neq 0$, then

$$a_1 - \sqrt{2}\, b_1 = 0 \Rightarrow \frac{a_1}{b_1} = \sqrt{2}$$

This is not possible because $\frac{a_1}{b_1}$ is a rational number and $\sqrt{2}$ is an irrational number.

If $b_2 \neq 0$, then

$$a_2 - \sqrt{2}\, b_2 = 0 \Rightarrow \frac{a_2}{b_2} = \sqrt{2}$$

This is not possible because $\frac{a_2}{b_2}$ is a rational number and $\sqrt{2}$ is an irrational number.

Thus, in both the cases we arrive at a contradiction. This means that our supposition is wrong. Hence, there can be at most two rational points on circle C.

<u>ALITER</u> Let there be three points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ with rational coordinates on circle C having its equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Since P, Q, R lie on circle (i). Therefore,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0$$

These are three linear equations in g, f and c with rational coefficients. So, we get rational values of g, f, c. But, $f = \sqrt{2}$. Thus, we arrive at a contradiction. Hence, there can be at most two rational points on circle C.

EXERCISE 24.1

LEVEL-1

1. Find the equation of the circle with:
   (i) Centre (–2, 3) and radius 4.
   (ii) Centre (a, b) and radius $\sqrt{a^2+b^2}$.
   (iii) Centre (0, –1) and radius 1.
   (iv) Centre $(a \cos \alpha,\ a \sin \alpha)$ and radius a.
   (v) Centre (a, a) and radius $\sqrt{2}\, a$.
2. Find the centre and radius of each of the following circles:
   (i) $(x-1)^2+y^2 = 4$
   (ii) $(x+5)^2+(y+1)^2 = 9$
   (iii) $x^2+y^2-4x+6y = 5$
   (iv) $x^2+y^2-x+2y-3 = 0$.
3. Find the equation of the circle whose centre is (1, 2) and which passes through the point (4, 6).
4. Find the equation of the circle passing through the point of intersection of the lines $x+3y=0$ and $2x-7y=0$ and whose centre is the point of intersection of the lines $x+y+1=0$ and $x-2y+4=0$.
5. Find the equation of the circle whose centre lies on the positive direction of y-axis at a distance 6 from the origin and whose radius is 4.
6. If the equations of two diameters of a circle are $2x+y=6$ and $3x+2y=4$ and the radius is 10, find the equation of the circle.
7. Find the equation of a circle
   (i) which touches both the axes at a distance of 6 units from the origin.
   (ii) which touches x-axis at a distance 5 from the origin and radius 6 units
   (iii) which touches both the axes and passes through the point (2, 1).
   (iv) passing through the origin, radius 17 and ordinate of the centre is –15.
8. Find the equation of the circle which has its centre at the point (3, 4) and touches the straight line $5x+12y-1=0$.
9. Find the equation of the circle which touches the axes and whose centre lies on $x-2y=3$.
10. A circle whose centre is the point of intersection of the lines $2x-3y+4=0$ and $3x+4y-5=0$ passes through the origin. Find its equation.
11. A circle of radius 4 units touches the coordinate axes in the first quadrant. Find the equations of its images with respect to the line mirrors $x=0$ and $y=0$.
12. Find the equations of the circles touching y-axis at (0, 3) and making an intercept of 8 units on the x-axis.
13. Find the equations of the circles passing through two points on y-axis at distances 3 from the origin and having radius 5.
14. If the lines $2x-3y=5$ and $3x-4y=7$ are the diameters of a circle of area 154 square units, then obtain the equation of the circle.
15. If the line $y=\sqrt{3}x+k$ touches the circle $x^2+y^2=16$, then find the value of k. [NCERT EXEMPLAR]
16. Find the equation of the circle having $(1,-2)$ as its centre and passing through the intersection of the lines $3x+y=14$ and $2x+5y=18$. [NCERT EXEMPLAR]
17. If the lines $3x-4y+4=0$ and $6x-8y-7=0$ are tangents to a circle, then find the radius of the circle. [NCERT EXEMPLAR]

LEVEL-2

18. Show that the point (x, y) given by $x=\dfrac{2at}{1+t^2}$ and $y=a\left(\dfrac{1-t^2}{1+t^2}\right)$ lies on a circle for all real values of t such that $-1 \le t \le 1$, where a is any given real number. [NCERT EXEMPLAR]

19. The circle $x^2 + y^2 - 2x - 2y + 1 = 0$ is rolled along the positive direction of x-axis and makes one complete roll. Find its equation in new-position.

20. One diameter of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If the coordinates of A and B are $(-3, 4)$ and $(5, 4)$ respectively, find the equation of the circle.

21. If the line $2x - y + 1 = 0$ touches the circle at the point $(2, 5)$ and the centre of the circle lies on the line $x + y - 9 = 0$. Find the equation of the circle.

ANSWERS

1. (i) $(x+2)^2 + (y-3)^2 = 16$ (ii) $x^2 + y^2 - 2ax - 2by = 0$ (iii) $x^2 + y^2 + 2y = 0$
(iv) $x^2 + y^2 - (2a\cos\alpha).x - (2a\sin\alpha).y = 0$ (v) $x^2 + y^2 - 2ax - 2ay = 0$

2. (i) $(1, 0); 2$ (ii) $(-5, -1); 3$ (iii) $(2, -3); 3\sqrt{2}$ (iv) $\left(\frac{1}{2}, -1\right); \frac{\sqrt{17}}{2}$

3. $x^2 + y^2 - 2x - 4y - 20 = 0$ 4. $x^2 + y^2 + 4x - 2y = 0$ 5. $x^2 + y^2 - 12y + 20 = 0$

6. $x^2 + y^2 - 16x + 20y + 64 = 0$ 7. (i) $x^2 + y^2 - 12x - 12y + 36 = 0$
(ii) $x^2 + y^2 - 10x - 12y + 25 = 0$ (iii) $x^2 + y^2 - 2x - 2y + 1 = 0,\ x^2 + y^2 - 10x - 10y + 25 = 0$
(iv) $x^2 + y^2 \pm 16x + 30y = 0$ 8. $169(x^2 + y^2 - 6x - 8y) + 381 = 0$

9. $x^2 + y^2 + 6x + 6y + 9 = 0$ or $x^2 + y^2 - 2x + 2y + 1 = 0$

10. $\left(x + \frac{1}{17}\right)^2 + \left(y - \frac{22}{17}\right)^2 = \frac{485}{289}$

11. With respect to $x = 0;\ x^2 + y^2 + 8x - 8y + 16 = 0$
With respect to $y = 0\ x^2 + y^2 - 8x + 8y + 16 = 0$

12. $x^2 + y^2 \pm 10x - 6y + 9 = 0$ 13. $x^2 + y^2 \pm 8x - 9 = 0$

14. $x^2 + y^2 - 2x + 2y - 47 = 0$ 15. $k = \pm 8$

16. $x^2 + y^2 - 2x + 4y - 20 = 0$ 17. $3/2$

19. $(x - 1 - 2\pi)^2 + (y-1)^2 = 1$ 20. $x^2 + y^2 - 2x - 4y - 15 = 0$

21. $(x-6)^2 + (y-3)^2 = 20$

HINTS TO NCERT & SELECTED PROBLEMS

8. Radius = Length of the perpendicular from the centre $(3, 4)$ to the line $5x + 12y - 1 = 0$.

9. Let a be the radius of the circle. Clearly, the required circle lies either in third or in fourth quadrant. So, the coordinates of its centre are $(-a, -a)$ or $(a, -a)$. Since, centre lies on $x - 2y = 3$. Therefore, $a = 3$ or $a = 1$.

17. Clearly, Diameter = Distance between parallel tangents $3x - 4y + 4 = 0$ and $3x - 4y - 7/2 = 0$

18. $x = \frac{2at}{1+t^2}$ and, $y = a\left(\frac{1-t^2}{1+t^2}\right)$ are parametric equations of a curve. In order to obtain the cartesian equation, we will have to eliminate parameter t.

Clearly, $x^2 + y^2 = \frac{4at^2}{(1+t^2)^2} + \frac{a^2(1-t^2)^2}{(1+t^2)^2} = a^2\frac{(1-t^2)^2}{(1+t^2)^2} = a^2$, which is the cartesian equation of the curve representing a circle having centre at $(0, 0)$ and radius a.

24.4 GENERAL EQUATION OF A CIRCLE

THEOREM *Prove that the equation $x^2+y^2+2gx+2fy+c=0$ always represents a circle whose centre is $(-g,-f)$ and radius $=\sqrt{g^2+f^2-c}$.*

PROOF The given equation is $x^2+y^2+2gx+2fy+c=0$ ...(i)

$$\Rightarrow \quad (x^2+2gx+g^2)+(y^2+2fy+f^2) = g^2+f^2-c$$

$$\Rightarrow \quad (x+g)^2+(y+f)^2 = \left\{\sqrt{g^2+f^2-c}\right\}^2$$

$$\Rightarrow \quad \{x-(-g)\}^2+\{y-(-f)\}^2 = \left\{\sqrt{g^2+f^2-c}\right\}^2$$

This is of the form $(x-h)^2+(y-k)^2=a^2$ which represents a circle having centre at (h,k) and radius equal to a.

Hence, the given equation (i) represents a circle whose centre is at

$$(-g,-f) \text{ i.e. } \left(-\frac{1}{2}\text{ Coefficient of } x, -\frac{1}{2}\text{ Coefficient of } y\right)$$

and, $\quad \text{Radius} = \sqrt{g^2+f^2-c} = \sqrt{\left(\frac{1}{2}\text{ Coeff. of } x\right)^2+\left(\frac{1}{2}\text{ Coeff. of } y\right)^2-\text{Constant term}}$

Q.E.D.

NOTE 1 *The equation $x^2+y^2+2gx+2fy+c=0$ represents a circle of radius $\sqrt{g^2+f^2-c}$.*

If $g^2+f^2-c>0$, then the radius of the circle is real and hence the circle is also real.

If $g^2+f^2-c=0$, then the radius of the circle is zero. Such a circle is known as a point circle.

If $g^2+f^2-c<0$, then the radius $\sqrt{g^2+f^2-c}$ of the circle is imaginary but the centre is real. Such a circle is called an imaginary circle as it is not possible to draw such a circle.

NOTE 2 *Special features of the general equation $x^2+y^2+2gx+2fy+c=0$ of the circle are:*

(i) it is quadratic in both x and y.

(ii) Coefficient of x^2 = Coefficient of y^2.

(iii) there is no term containing xy i.e., the coefficient of xy is zero.

(iv) it contains three arbitrary constants viz. g, f and c.

NOTE 3 *The equation $ax^2+ay^2+2gx+2fy+c=0, a \neq 0$ also represents a circle. This equation can also be written as*

$$x^2+y^2+\frac{2g}{a}x+\frac{2f}{a}y+\frac{c}{a}=0.$$

The coordinates of the centre of the circle are $(-g/a, -f/a)$ and, radius $=\sqrt{\frac{g^2}{a^2}+\frac{f^2}{a^2}-\frac{c}{a}}$.

NOTE 4 ***On comparing the general equation $x^2+y^2+2gx+2fy+c=0$ of a circle with the general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$, we find that it represents a circle if $a=b$ i.e., coefficient of x^2 = coefficient of y^2 and $h=0$ i.e., coefficient of $xy=0$.***

NOTE 5 ***While solving problems it is advisable to keep the coefficient of x^2 and y^2 unity.***

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE CENTRE AND RADIUS OF A CIRCLE WHEN ITS EQUATION IS GIVEN**

RESULT *The coordinates of the centre of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are*

$$\left(-\frac{1}{2} \text{ Coefficient of } x,\ -\frac{1}{2} \text{ Coefficient of } y\right)$$

and, $\text{Radius} = \sqrt{\left(\frac{1}{2} \text{ Coefficient of } x\right)^2 + \left(\frac{1}{2} \text{ Coefficient of } y^2\right)^2 - \text{Constant term}}$

EXAMPLE 1 *Find the centre and radius of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$.*

SOLUTION The coordinates of the centre of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ are

$$\left(-\frac{1}{2} \text{ Coeff. of } x,\ -\frac{1}{2} \text{ Coeff. of } y\right) \text{i.e.} \left(-\frac{1}{2} \times -6, -\frac{1}{2} \times 4\right) = (3, -2)$$

and, $\text{Radius} = \sqrt{\left(-\frac{6}{2}\right)^2 + \left(\frac{4}{2}\right)^2 - (-12)} = \sqrt{9 + 4 + 12} = 5.$

EXAMPLE 2 *Find the centre and radius of the circle given by the equation*

$$2x^2 + 2y^2 + 3x + 4y + \frac{9}{8} = 0.$$

SOLUTION In the given equation the coefficients of x^2 and y^2 are not unity. So, we re-write the equation to make the coefficients of x^2 and y^2 unity.

We have, $2x^2 + 2y^2 + 3x + 4y + \frac{9}{8} = 0 \Rightarrow x^2 + y^2 + \frac{3}{2}x + 2y + \frac{9}{16} = 0.$

So, the coordinates of the centre are $(-3/4, -1)$ and, $\text{Radius} = \sqrt{\left(\frac{3}{4}\right)^2 + (1)^2 - \frac{9}{16}} = 1.$

Type II **ON FINDING THE EQUATION OF A CIRCLE SATISFYING GIVEN CONDITIONS**

EXAMPLE 3 *Find the equation of the circle whose centre is at the point (4, 5) and which passes through the centre of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$.*

SOLUTION The coordinates of the centre of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ are $C_1(3, -2)$. Therefore, the required circle passes through the point $C_1(3, -2)$ and has its centre at the point $C(4, 5)$. So, its radius is equal to

$$CC_1 = \sqrt{(4-3)^2 + (5+2)^2} = \sqrt{50}$$

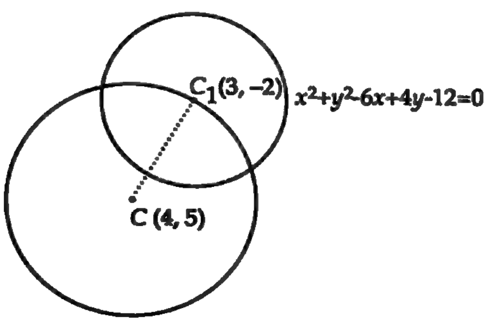


Fig. 24.34

Hence, the equation of the required circle is

$$(x-4)^2 + (y-5)^2 = (\sqrt{50})^2 \text{ or, } x^2 + y^2 - 8x - 10y - 9 = 0$$

EXAMPLE 4 *Find the equation of the circle concentric with the circle* $2x^2 + 2y^2 + 8x + 10y - 39 = 0$ *and having its area equal to* $16\,\pi$ *square units.*

SOLUTION The equation of the given circle is

$$2x^2 + 2y^2 + 8x + 10y - 39 = 0 \Rightarrow x^2 + y^2 + 4x + 5y - 39/2 = 0.$$

The coordinates of its centre are $(-2, -5/2)$. The required circle is concentric with the above circle, therefore the coordinates its centre are $(-2, -5/2)$.

Let r be the radius of the required circle. Then, its area is πr^2. But, it is given that its area is 16π sq. units.

$$\therefore \quad \pi r^2 = 16\pi \Rightarrow r = 4$$

Hence, the equation of the required circle is

$$(x+2)^2 + (y + 5/2)^2 = 4^2 \text{ or, } 4x^2 + 4y^2 + 16x + 20y - 23 = 0.$$

Type III **ON FINDING THE EQUATION OF A CIRCLE PASSING THROUGH THREE GIVEN POINTS**

EXAMPLE 5 *Find the equation of the circle that passes through the points (1, 0), (–1, 0) and (0, 1).*

SOLUTION Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

It passes through (1, 0), (–1, 0) and (0, 1). Therefore, on substituting the coordinates of three points successively in equation (i), we get

$$1 + 2g + c = 0 \text{ ...(ii)}, \quad 1 - 2g + c = 0 \text{ ...(iii)}, \quad 1 + 2f + c = 0 \text{ ...(iv)}$$

Subtracting (iii) from (ii), we get

$$4g = 0 \Rightarrow g = 0$$

Putting $g = 0$ in (ii), we obtain $c = -1$.

Now, putting $c = -1$ in (iv), we get $f = 0$.

Substituting the values of g, f and c in equation (i), we obtain the equation of the required circle as $x^2 + y^2 = 1$.

EXAMPLE 6 *Find the equation of the circle which passes through the points (5, – 8), (2, – 9) and (2, 1). Find also the coordinates of its centre and radius.*

SOLUTION Let the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

It passes through the points (5, – 8), (2, – 9) and (2, 1). Therefore,

$$89 + 10g - 16f + c = 0 \quad \text{...(ii)}$$
$$85 + 4g - 18f + c = 0 \quad \text{...(iii)}$$
$$5 + 4g + 2f + c = 0 \quad \text{...(iv)}$$

Subtracting (iii) from (ii), we obtain

$$4 + 6g + 2f = 0 \Rightarrow 2 + 3g + f = 0 \quad \text{...(v)}$$

Subtracting (iv) from (iii), we get

$$80 + 0g - 20f = 0 \Rightarrow f = 4$$

Putting $f = 4$ in (v), we get $g = -2$. Putting $f = 4, g = -2$ in (iv), we get

$$5 - 8 + 8 + c = 0 \Rightarrow c = -5$$

Substituting the values of g, f and c in equation (i), we obtain the equation of the required circle as

$$x^2 + y^2 - 4x + 8y - 5 = 0.$$

The coordinates of the centre are $(-g, -f)$ i.e $(2, -4)$.

and, Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{4 + 16 + 5} = 5.$

EXAMPLE 7 *The straight line $\frac{x}{a}+\frac{y}{b}=1$ cuts the coordinate axes at A and B. Find the equation of the circle passing through O (0, 0), A and B.*

SOLUTION The straight line $\frac{x}{a}+\frac{y}{b}=1$ cuts the coordinate axes at $A\ (a, 0)$ and $B\ (0, b)$.

Let $x^2+y^2+2gx+2fy+c = 0$ ...(i)

be the circle passing through O, A and B. Then,

$$0 + \ + c = 0 \quad \text{...(ii)}$$

$$a^2 + 2ga + c = 0 \quad \text{...(iii)}$$

$$b^2 + 2fb + c = 0 \quad \text{...(iv)}$$

Solving (ii), (iii) and (iv), we obtain

$$g = -\frac{a}{2}, f = -\frac{b}{2} \text{ and } c = 0.$$

Substituting these values in (i), we obtain the equation of the required circle as

$$x^2+y^2-ax-by = 0$$

<u>ALITER</u> The line represented by the equation $\frac{x}{a}+\frac{y}{b}=1$ meets the coordinate axes at $A\ (a, 0)$ and $B(0, b)$. Clearly, $\angle AOB = 90°$. So, AB is a diameter of the circle such that

$$AB=\sqrt{(a-0)^2+(0-b)^2}=\sqrt{a^2+b^2}$$

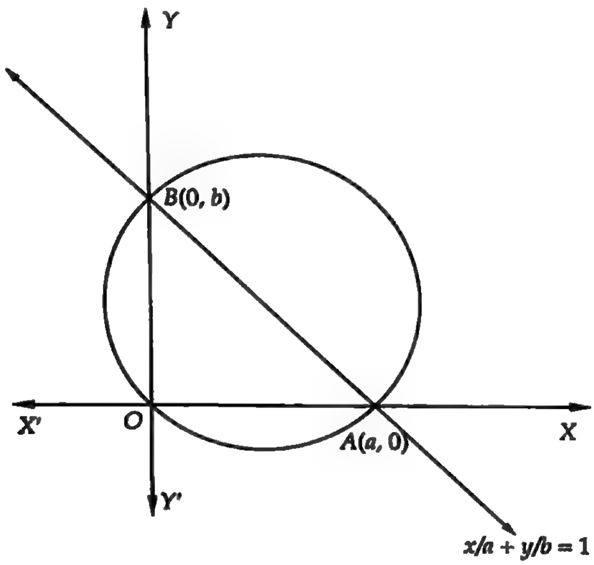


Fig. 24.35

$\therefore$ Radius $=\frac{1}{2}AB=\frac{1}{2}\sqrt{a^2+b^2}$.

The centre C of the circle is the mid-point of AB and so its coordinates are

$$\left(\frac{a+0}{2}, \frac{0+b}{2}\right)=\left(\frac{a}{2}, \frac{b}{2}\right).$$

Hence, the equation of the circle is

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\left(\frac{1}{2}\sqrt{a^2+b^2}\right)^2 \text{ or, } x^2+y^2-ax-by=0$$

EXAMPLE 8 *Find the equation of the circle passing through (1, 0) and (0, 1) and having the smallest possible radius.*

SOLUTION Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(i)$$

This passes through the points $A(1, 0)$ and $B(0, 1)$.

$$\therefore \quad 1 + 2g + c = 0 \text{ and, } 1 + 2f + c = 0 \Rightarrow g = -\left(\frac{c+1}{2}\right) \text{ and, } f = -\left(\frac{c+1}{2}\right)$$

Let r be the radius of circle (i). Then,

$$r = \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow \quad r = \sqrt{\left(\frac{c+1}{2}\right)^2 + \left(\frac{c+1}{2}\right)^2 - c} \Rightarrow r = \sqrt{\frac{c^2+1}{2}} \Rightarrow r^2 = \frac{1}{2}(c^2+1)$$

Now, $\frac{1}{2}c^2 \geq 0 \Rightarrow \frac{1}{2}c^2 + \frac{1}{2} \geq \frac{1}{2} \Rightarrow r^2 \geq \frac{1}{2}$

Thus, the minimum value of r^2 is $\frac{1}{2}$.

Also, $r^2 = \frac{1}{2} \Rightarrow \frac{1}{2}c^2 + \frac{1}{2} = \frac{1}{2} \Rightarrow c = 0$

So, r is minimum when $c = 0$ and in that case, the minimum value of r is $\frac{1}{\sqrt{2}}$.

Putting $c = 0$ in $g = -\frac{c+1}{2}$ and $f = -\frac{c+1}{2}$, we get $g = -\frac{1}{2}$ and $f = -\frac{1}{2}$.

Substituting the values of g, f and c in (i), we get $x^2 + y^2 - x - y = 0$ as the equation of the required circle.

NOTE *To prove that four given points are concyclic; find the equation of the circle passing through any of the three given points and show that the fourth point lies on it.*

Type IV **ON CONCYCLIC POINTS**

EXAMPLE 9 *Show that the points* (9, 1), (7, 9) (−2, 12) *and* (6, 10) *are concyclic.*

SOLUTION Let the equation of the circle passing through (9, 1), (7, 9) and (−2, 12) be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(i)$$

Then,

$$82 + 18g + 2f + c = 0 \quad ...(ii)$$

$$130 + 14g + 18f + c = 0 \quad ...(iii)$$

$$148 - 4g + 24f + c = 0 \quad ...(iv)$$

Subtracting (ii) from (iii), we get

$$48 - 4g + 16f = 0 \Rightarrow 12 - g + 4f = 0 \quad ...(v)$$

Subtracting (iii) from (iv), we get

$$18 - 18g + 6f = 0 \Rightarrow 3 - 3g + f = 0 \quad ...(vi)$$

Solving (v) and (vi) as simultaneous linear equations in g and f, we get: $f = -3, g = 0$.

Putting $f = -3, g = 0$ in (ii), we get

$$82 + 0 - 6 + c = 0 \Rightarrow c = -76$$

Substituting the values of g, f and c in (i), we get $x^2 + y^2 - 6y - 76 = 0$ as the equation of the circle passing through points (9, 1), (7, 9) and (−2, 12).

Clearly, point (6, 10) satisfies this equation. Hence, the given points are concyclic.

Type V **ON FINDING THE EQUATION OF A CIRCLE SATISFYING THREE GIVEN CONDITIONS**

EXAMPLE 10 *Find the equation of the circle which passes through the points $(1, -2)$ and $(4, -3)$ and has its centre on the line $3x + 4y = 7$.*

SOLUTION Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

It passes through $(1, -2)$ and $(4, -3)$.

$$\therefore \quad 5 + 2g - 4f + c = 0 \quad \text{...(ii)}$$

$$\text{and,} \quad 25 + 8g - 6f + c = 0 \quad \text{...(iii)}$$

The centre $(-g, -f)$ of (i) lies on $3x + 4y = 7$.

$$\therefore \quad -3g - 4f = 7 \quad \text{...(iv)}$$

Subtracting (ii) from (iii), we get

$$20 + 6g - 2f = 0 \Rightarrow 10 + 3g - f = 0 \quad \text{...(v)}$$

Solving (iv) and (v) as simultaneous equations, we get

$$g = -\frac{47}{15} \text{ and } f = \frac{3}{5}$$

Substituting the values of g and f in (ii), we get

$$5 - \frac{94}{15} - \frac{12}{5} + c = 0 \Rightarrow c = \frac{55}{15} = \frac{11}{3}$$

Substituting the values of g, f and c in (i) we obtain the required equation of the circle as

$$x^2 + y^2 - \frac{94}{15}x + \frac{6}{5}y + \frac{11}{3} = 0 \text{ or, } 15(x^2 + y^2) - 94x + 18y + 33 = 0$$

EXAMPLE 11 *Find the equation of the circle circumscribing the triangle formed by the lines $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$.*

SOLUTION Let the equations of sides AB, BC and CA of ΔABC are respectively

$$x + y = 6 \quad \text{...(i)} \qquad 2x + y = 4 \quad \text{...(ii)} \qquad \text{and} \qquad x + 2y = 5 \quad \text{...(iii)}$$

Solving (i) and (iii), (i) and (ii); (ii) and (iii) we get the coordinates of A, B and C. The coordinates A, B and C are $(7, -1)$, $(-2, 8)$ and $(1, 2)$ respectively.

Let the equation of the circumcircle of ΔABC be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(iv)}$$

It passes through the points $A(7, -1)$, $B(-2, 8)$ and $C(1, 2)$. Therefore,

$$50 + 14g - 2f + c = 0 \quad \text{...(v)}$$

$$68 - 4g + 16f + c = 0 \quad \text{...(vi)}$$

$$5 + 2g + 4f + c = 0 \quad \text{...(vii)}$$

Subtracting (v) from (vi), we get

$$18 - 18g + 18f = 0 \Rightarrow 1 - g + f = 0 \quad \text{...(viii)}$$

Subtracting (v) from (vii), we get: $-45 - 12g + 6f = 0$ ...(ix)

Solving (viii) and (ix), we get: $g = -17/2, f = -19/2$.

Putting the values of g and f in (v), we get $c = 50$.

Substituting the values of g, f and c in (iv), the equation of the required circumcircle is

$$x^2 + y^2 - 17x - 19y + 50 = 0$$

LEVEL-2

Type VI **MISCELLANEOUS EXAMPLES**

EXAMPLE 12 *Find the radius of the circle* $(x\cos\alpha + y\sin\alpha - a)^2 + (x\sin\alpha - y\cos\alpha - b)^2 = k^2$, *if* α *varies, the locus of its centre is again a circle. Also, find its centre and radius.*

SOLUTION The given equation is

$$(x\cos\alpha + y\sin\alpha - a)^2 + (x\sin\alpha - y\cos\alpha - b)^2 = k^2$$

$$\Rightarrow \quad x^2(\cos^2\alpha + \sin^2\alpha) + y^2(\sin^2\alpha + \cos^2\alpha) - 2(a\cos\alpha + b\sin\alpha)x - 2(a\sin\alpha - b\cos\alpha)y + a^2 + b^2 - k^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2x(a\cos\alpha + b\sin\alpha) - 2y(a\sin\alpha - b\cos\alpha) + a^2 + b^2 - k^2 = 0$$

The coordinates of the centre of this circle are $(a\cos\alpha + b\sin\alpha,\ a\sin\alpha - b\cos\alpha)$. Let its radius be r. Then,

$$r = \sqrt{(a\cos\alpha + b\sin\alpha)^2 + (a\sin\alpha - b\cos\alpha)^2 - (a^2 + b^2 - k^2)}$$

$$\Rightarrow \quad r = \sqrt{a^2(\cos^2\alpha + \sin^2\alpha) + b^2(\sin^2\alpha + \cos^2\alpha) - (a^2 + b^2 - k^2)}$$

$$\Rightarrow \quad r = \sqrt{a^2 + b^2 - a^2 - b^2 + k^2} = k$$

Let (p, q) be the coordinates of the centre of the given circle. Then,

$$p = a\cos\alpha + b\sin\alpha \quad \text{and} \quad q = a\sin\alpha - b\cos\alpha$$

To find the locus of (p, q) we have to eliminate α. Squaring and adding these two, we get

$$p^2 + q^2 = (a\cos\alpha + b\sin\alpha)^2 + (a\sin\alpha - b\cos\alpha)^2$$

$$\Rightarrow \quad p^2 + q^2 = a^2(\cos^2\alpha + \sin^2\alpha) + b^2(\sin^2\alpha + \cos^2\alpha)$$

$$\Rightarrow \quad p^2 + q^2 = a^2 + b^2$$

Hence, the locus of (p, q) is $x^2 + y^2 = a^2 + b^2$. This is a circle having centre at (0, 0) and radius equal to $\sqrt{a^2 + b^2}$.

EXAMPLE 13 *Find the area of an equilateral triangle inscribed in the circle*

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

SOLUTION Let ABC be an equilateral triangle inscribed in the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

Let $O(-g, -f)$ be the centre of the circle. Then,

$$OA = OB = OC = \sqrt{g^2 + f^2 - c}$$

In ΔOBD, we have

$$\sin 60^\circ = \frac{BD}{OB}$$

$$\Rightarrow \quad BD = \frac{\sqrt{3}}{2} OB$$

$$\Rightarrow \quad BD = \frac{\sqrt{3}}{2}\sqrt{g^2 + f^2 - c}$$

$$\therefore \quad BC = 2\,BD = \sqrt{3}\sqrt{g^2 + f^2 - c}$$

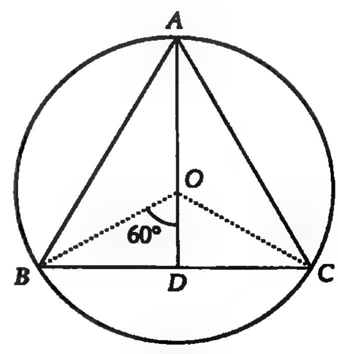


Fig. 24.36

$$\therefore \quad \text{Area of } \Delta ABC = \frac{\sqrt{3}}{4}(\text{Side})^2 = \frac{\sqrt{3}}{4}(BC)^2$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{\sqrt{3}}{4} \times 3(g^2 + f^2 - c) \text{ sq. units} = \frac{3\sqrt{3}}{4}(g^2 + f^2 - c) \text{ sq. units.}$$

EXAMPLE 14 *If the line $lx+my=1$ is a tangent to the circle $x^2+y^2=a^2$, then prove that (l, m) lies on a circle.* [NCERT EXEMPLAR]

SOLUTION If the line $lx+my-1=0$ touches the circle $x^2+y^2=a^2$, then length of the perpendicular from its centre $O\ (0, 0)$ is equal to the radius a.

$$\therefore \quad \left|\frac{m+m\times 0-1}{\sqrt{l^2+m^2}}\right|=a$$

$$\Rightarrow \quad \frac{1}{\sqrt{l^2+m^2}}=a$$

$$\Rightarrow \quad l^2+m^2=\frac{1}{a^2}$$

$\Rightarrow$ (l, m) satisfies the equation $x^2+y^2=\dfrac{1}{a^2}$

$\Rightarrow$ (l, m) lies on the circle $x^2+y^2=\dfrac{1}{a^2}$.

Hence, (l, m) lies on a circle.

EXAMPLE 15 *If the line $lx+my+n=0$ touches the circle $x^2+y^2=a^2$, then prove that $(l^2+m^2)\,a^2=n^2$.* [NCERT EXEMPLAR]

SOLUTION If the line $lx+my+n=0$ touches the circle $x^2+y^2=a^2$, then length of the perpendicular from its centre $O\ (0, 0)$ is equal to its radius a.

$$\therefore \quad \left|\frac{l\times 0+m\times 0+n}{\sqrt{l^2+m^2}}\right|=a$$

$\Rightarrow$ $(l^2+m^2)\,a^2=n^2$, which is the required condition.

EXAMPLE 16 *Prove that the locus of a point which moves such that the sum of the squares of its distances from the vertices of a triangle is constant is a circle having centre at the centroid of the triangle.* [NCERT EXEMPLAR]

SOLUTION Let $A\ (x_1, y_1)$, $B\ (x_2, y_2)$ and $C\ (x_3, y_3)$ be the vertices of ΔABC, and let $P(h, k)$ be a point which moves in such a way that

$$PA^2+PB^2+PC^2=c \quad \text{(constant)}$$

$$\Rightarrow (h-x_1)^2+(k-y_1)^2+(h-x_2)^2+(k-y_2)^2+(h-x_3)^2+(k-y_3)^2=c$$

$$\Rightarrow 3h^2+3k^2-2h\,(x_1+x_2+x_3)-2k\,(y_1+y_2+y_3)+x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2-c=0$$

$$\Rightarrow h^2+k^2-\frac{2}{3}(x_1+x_2+x_3)\,h-\frac{2}{3}(y_1+y_2+y_3)\,k+\frac{1}{3}(x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2-c)=0$$

$$\Rightarrow h^2+k^2-\frac{2}{3}(x_1+x_2+x_3)\,h-\frac{2}{3}(y_1+y_2+y_3)\,k+\lambda=0,$$

$$\text{where } \lambda=\frac{1}{3}(x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2-c)$$

Hence, the locus of (h, k) is

$$x^2+y^2-\frac{2}{3}(x_1+x_2+x_3)\,x-\frac{2}{3}(y_1+y_2+y_3)\,y+\lambda=0$$

Clearly, it represents a circle with centre at $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$, which is the centroid of ΔABC.

EXAMPLE 17 *If a circle of constant radius $3c$ passes through the origin and meets the axes at A and B, prove that the locus of the centroid of ΔABC is a circle of radius $2c$.* [NCERT EXEMPLAR]

SOLUTION Let the coordinates of A and B be $(a, 0)$ and $(0, b)$ respectively. Clearly, ΔOAB is a right triangle right-angled at O. Therefore, AB is a diameter of the circle.

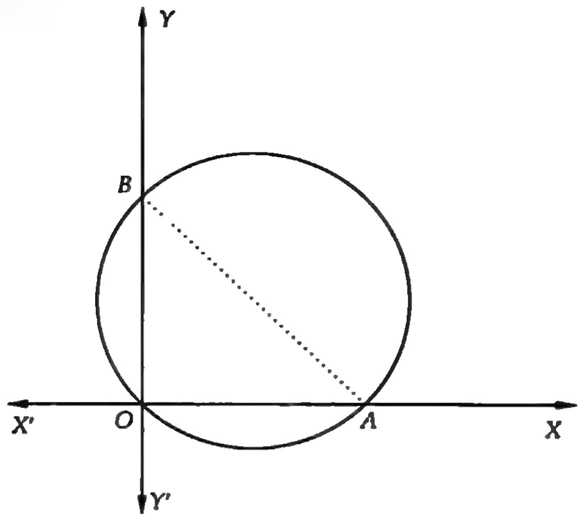


Fig. 24.37

$\therefore \quad AB = 2(3c) = 6c$

In ΔAOB,

$$OA^2 + OB^2 = AB^2 \qquad \text{[Using Pythagoras theorem]}$$

$$\Rightarrow \quad a^2 + b^2 = 36c^2 \qquad \text{...(i)}$$

Let (α, β) be the coordinates of the centroid of ΔOAB. Then,

$$\alpha = \frac{0+a+0}{3} = \frac{a}{3}, \quad \beta = \frac{0+0+b}{3} = \frac{b}{3} \Rightarrow a = 3\alpha \text{ and } b = 3\beta$$

Substituting the values of a and b in (i), we obtain

$$9\alpha^2 + 9\beta^2 = 36c^2 \text{ or, } \alpha^2 + \beta^2 = (2c)^2$$

Hence, the locus of (α, β) is $x^2 + y^2 = (2c)^2$, which is a circle of radius $2c$.

<u>ALITER</u> Let $OA = a$ and $OB = b$. Then, the coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively.

The equation of the circle passing through O, A and B is

$$x^2 + y^2 - ax - by = 0 \qquad \text{[See Example 7]} \qquad \text{...(i)}$$

Let (α, β) be the coordinates of the centroid of ΔOAB. Then,

$$\alpha = \frac{a}{3} \text{ and } \beta = \frac{b}{3} \Rightarrow a = 3\alpha \text{ and } b = 3\beta \qquad \text{...(ii)}$$

It is given that the radius of circle (i) is $3c$.

$$\therefore \quad \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - 0} = 3c \Rightarrow a^2 + b^2 = 36c^2 \qquad \text{...(iii)}$$

Eliminating a and b between (ii) and (iii), we obtain

$$9\alpha^2 + 9\beta^2 = 36c^2 \quad \text{or,} \quad \alpha^2 + \beta^2 = 4c^2$$

Hence, the locus of (α, β) is $x^2 + y^2 = (2c)^2$, which is a circle of radius $2c$.

EXERCISE 24.2

LEVEL-1

1. Find the coordinates of the centre and radius of each of the following circles :

(i) $x^2 + y^2 + 6x - 8y - 24 = 0$ (ii) $2x^2 + 2y^2 - 3x + 5y = 7$

(iii) $\frac{1}{2}(x^2 + y^2) + x\cos\theta + y\sin\theta - 4 = 0$ (iv) $x^2 + y^2 - ax - by = 0$

2. Find the equation of the circle passing through the points:
   (i) (5, 7), (8, 1) and (1, 3) (ii) (1, 2), (3, – 4) and (5, – 6)
   (iii) (5, – 8), (– 2, 9) and (2, 1) (iv) (0, 0), (– 2, 1) and (– 3, 2)
3. Find the equation of the circle which passes through (3, –2), (–2, 0) and has its centre on the line $2x - y = 3$.
4. Find the equation of the circle which passes through the points (3, 7), (5, 5) and has its centre on the line $x - 4y = 1$.
5. Show that the points (3, – 2), (1, 0), (–1, – 2) and (1, – 4) are concyclic.
6. Show that the points (5, 5), (6, 4), (–2, 4) and (7, 1) all lie on a circle, and find its equation, centre and radius.
7. Find the equation of the circle which circumscribes the triangle formed by the lines
   (i) $x + y + 3 = 0$, $x - y + 1 = 0$ and $x = 3$
   (ii) $2x + y - 3 = 0$, $x + y - 1 = 0$ and $3x + 2y - 5 = 0$
   (iii) $x + y = 2$, $3x - 4y = 6$ and $x - y = 0$.
   (iv) $y = x + 2$, $3y = 4x$ and $2y = 3x$. [NCERT EXEMPLAR]
8. Prove that the centres of the three circles $x^2 + y^2 - 4x - 6y - 12 = 0$, $x^2 + y^2 + 2x + 4y - 10 = 0$ and $x^2 + y^2 - 10x - 16y - 1 = 0$ are collinear.
9. Prove that the radii of the circles $x^2 + y^2 = 1$, $x^2 + y^2 - 2x - 6y - 6 = 0$ and $x^2 + y^2 - 4x - 12y - 9 = 0$ are in A.P.
10. Find the equation of the circle which passes through the origin and cuts off chords of lengths 4 and 6 on the positive side of the x-axis and y-axis respectively.
11. Find the equation of the circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and double of its area.
12. Find the equation to the circle which passes through the points (1, 1) (2, 2) and whose radius is 1. Show that there are two such circles.
13. Find the equation of the circle concentric with $x^2 + y^2 - 4x - 6y - 3 = 0$ and which touches the y-axis.
14. If a circle passes through the point (0, 0), (a, 0), (0, b), then find the coordinates of its centre.
15. Find the equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the straight line $y - 4x + 3 = 0$. [NCERT EXEMPLAR]

ANSWERS

1. (i) (– 3, 4); 7 (ii) $\left(\frac{3}{4}, -\frac{5}{4}\right)$; $\frac{3\sqrt{10}}{4}$
   (iii) $(-\cos\theta, -\sin\theta)$; 3 (iv) $\left(\frac{a}{2}, \frac{b}{2}\right)$; $\frac{1}{2}\sqrt{a^2 + b^2}$.
2. (i) $3(x^2 + y^2) - 29x - 19y + 56 = 0$ (ii) $x^2 + y^2 - 22x - 4y + 25 = 0$
   (iii) $x^2 + y^2 + 116x + 48y - 285 = 0$ (iv) $x^2 + y^2 - 3x - 11y = 0$
3. $x^2 + y^2 + 3x + 12y + 2 = 0$
4. $x^2 + y^2 + 6x + 2y - 90 = 0$
6. $x^2 + y^2 - 4x - 2y - 20 = 0$; (2, 1), 5
7. (i) $x^2 + y^2 - 6x + 2y - 15 = 0$ (ii) $x^2 + y^2 - 13x - 5y + 16 = 0$
   (iii) $x^2 + y^2 + 4x + 6y - 12 = 0$ (iv) $x^2 + y^2 - 46x + 22y = 0$

10. $x^2 + y^2 - 4x - 6y = 0$ 11. $x^2 + y^2 - 6x + 12y - 15 = 0$

12. $x^2 + y^2 - 4x - 2y + 4 = 0,\ x^2 + y^2 - 2x - 4y + 4 = 0$

13. $x^2 + y^2 - 4x - 6y + 9 = 0$ 14. $\left(\frac{a}{2}, \frac{b}{2}\right)$ 15. $x^2 + y^2 - 4x - 10y + 25 = 0$

HINTS TO SELECTED PROBLEMS

10. The circle passes through (0, 0), (4, 0) and (0, 6).
11. Centre of the given circle is (3, – 6), and radius = $\sqrt{9 + 36 - 15} = \sqrt{30}$. Let r be the radius of the required circle. Then, $\pi r^2 = 2(\pi(\sqrt{30})^2) \Rightarrow r = \sqrt{60}$.
13. The centre of the required circle is (2, 3). As it touches y-axis. So, its radius = x-coordinate of centre = 2.

24.5 DIAMETER FORM OF A CIRCLE

THEOREM *The equation of the circle drawn on the straight line joining two given points* (x_1, y_1) *and* (x_2, y_2) *as diameter is* $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

PROOF Let A and B be the extremities of the diameter AB having coordinates (x_1, y_1) and (x_2, y_2) respectively. Let $P(x, y)$ be any point on the circle. Join point P to points A and B. Then,

$$m_1 = \text{Slope of the line } AP = \frac{y - y_1}{x - x_1} \text{ and, } m_2 = \text{Slope of the line } BP = \frac{y - y_2}{x - x_2}$$

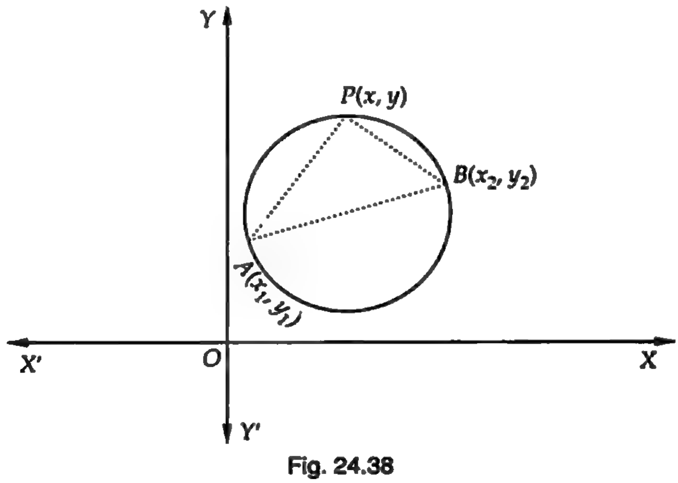


Fig. 24.38

The angle subtended at the point P in the semi-circle APB is a right angle.

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \quad \frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$$

$$\Rightarrow \quad (y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$$

$$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0 \qquad \text{...(i)}$$

This is the required equation of the circle having (x_1, y_1) and (x_2, y_2) as the coordinates of the end points of a diameter. Q.E.D.

REMARK 1 *If the coordinates of the end points of a diameter of a circle are given, we can also find the equation of the circle by finding the coordinates of the centre and radius. The centre is the mid-point of the diameter and radius is half of the length of the diameter.*

REMARK 2 *Equation (i) can also be written as*

$$x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1x_2 + y_1y_2 = 0$$

or, $x^2 + y^2 - x$ *(Sum of the abscissae)* $- y$ *(Sum of the ordinates)* + *Product of the abscissae* + *Product of the ordinates* = 0.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the circle, the coordinates of the end points of whose diameter are* (–1, 2) *and* (4, –3).

SOLUTION We know that the equation of the circle described on the line segment joining (x_1, y_1) and (x_2, y_2) as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Here, $x_1 = -1,\ x_2 = 4,\ y_1 = 2$ and $y_2 = -3$.

So, the equation of the required circle is

$$(x+1)(x-4) + (y-2)(y+3) = 0 \text{ or, } x^2 + y^2 - 3x + y - 10 = 0.$$

EXAMPLE 2 *Find the equation of the circle drawn on the intercept made by the line* $2x + 3y = 6$ *between the coordinate axes as diameter.*

SOLUTION The line $2x + 3y = 6$ meets x and y-axes at A (3, 0) and B (0, 2) respectively. Taking AB as a diameter, the equation of the required circle is

$$(x-3)(x-0) + (y-0)(y-2) = 0 \qquad [\text{Using: } (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0]$$

or, $x^2 + y^2 - 3x - 2y = 0$

EXAMPLE 3 *Find the equations of the circles drawn on the diagonals of the rectangle as its diameter whose sides are* $x = 6$, $x = -3$, $y = 3$ *and* $y = -1$.

SOLUTION Let the sides AB, BC, CD and DA of the rectangle $ABCD$ be represented by the equations $y = -1, x = 6, y = 3$ and $x = -3$ respectively. Then, the coordinates of the vertices are $A(-3, 1)$, $B(6, -1)$, $C(6, 3)$ and $D(-3, 3)$.

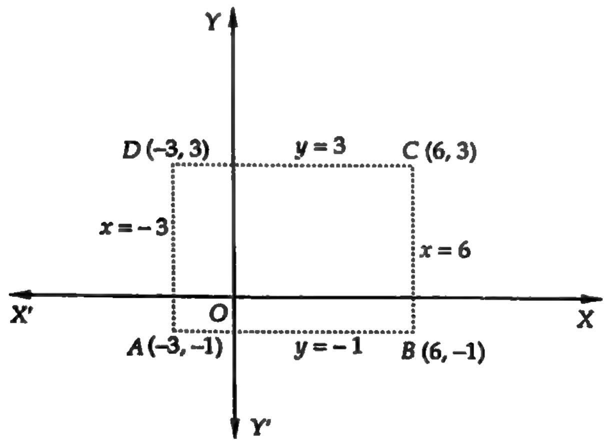


Fig. 24.39

The equation of the circle with diagonal AC as diameter is

$$(x+3)(x-6) + (y+1)(y-3) = 0 \text{ or, } x^2 + y^2 - 3x - 2y - 21 = 0$$

The equation of the circle with diagonal BD as diameter is

$$(x-6)(x+3) + (y+1)(y-3) = 0 \text{ or, } x^2 + y^2 - 3x - 2y - 21 = 0.$$

EXAMPLE 4 *If* $y = 2x$ *is a chord of the circle* $x^2 + y^2 - 10x = 0$, *find the equation of a circle with this chord as diameter.*

SOLUTION The points of intersection of the given chord and the given circle are obtained by solving $y = 2x$ and $x^2 + y^2 - 10x = 0$ simultaneously. Putting $y = 2x$ in $x^2 + y^2 - 10x = 0$, we get

$$5x^2 - 10x = 0 \Rightarrow 5x(x-2) = 0 \Rightarrow x = 0, 2.$$

Putting $x = 0$ and $x = 2$ respectively in $y = 2x$, we get $y = 0$ and $y = 4$.

Thus, the coordinates of the points of intersection of the given line and the given circle are $A(0, 0)$ and $B(2, 4)$. The equation of the circle with chord AB as diameter is

$$(x-0)(x-2) + (y-0)(y-4) = 0 \text{ or, } x^2 + y^2 - 2x - 4y = 0.$$

LEVEL-2

EXAMPLE 5 *If the abscissae and the ordinates of two points A and B be the roots of $ax^2 + bx + c = 0$ and $a'y^2 + b'y + c' = 0$ respectively, show that the equation of the circle described on AB as diameter is $aa'(x^2 + y^2) + a'bx + ab'y + (ca' + c'a) = 0$.*

SOLUTION Let (x_1, y_1) and (x_2, y_2) be the coordinates of points A and B respectively.
It is given that x_1, x_2 are roots of $ax^2 + bx + c = 0$ and y_1, y_2 are roots of $a'y^2 + b'y + c' = 0$.

$$\therefore \quad x_1 + x_2 = -\frac{b}{a}, \quad x_1 x_2 = \frac{c}{a}, \quad y_1 + y_2 = -\frac{b'}{a'} \text{ and } y_1 y_2 = \frac{c'}{a'} \quad \text{...(i)}$$

The equation of the circle with AB as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow \quad x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1 x_2 + y_1 y_2 = 0.$$

$$\Rightarrow \quad x^2 + y^2 - x\left(-\frac{b}{a}\right) - \left(-\frac{b'}{a'}\right)y + \frac{c}{a} + \frac{c'}{a'} = 0 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad aa'(x^2 + y^2) + a'bx + ab'y + (ca' + c'a) = 0$$

EXAMPLE 6 *Find the equation of the circle on the straight line joining the points of intersection of $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ as diameter.*

SOLUTION Suppose the line $lx + my = 1$ intersects the lines given by $ax^2 + 2hxy + by^2 = 0$ in A and B. Let the coordinates of A and B are (x_1, y_1) and (x_2, y_2) respectively. Eliminating y between $lx + my = 1$ and $ax^2 + 2hxy + by^2 = 0$, we obtain

$$x^2(am^2 - 2hlm + bl^2) - 2x(bl - hm) + b = 0$$

Clearly, x_1, x_2 are roots of this equation.

$$\therefore \quad x_1 + x_2 = \frac{2(bl - hm)}{am^2 - 2hlm + bl^2} \text{ and, } x_1 x_2 = \frac{b}{am^2 - 2hlm + bl^2}$$

Now, eliminating x between $lx + my = 1$ and $ax^2 + 2hxy + by^2 = 0$, we get

$$y^2(am^2 - 2h\,lm + bl^2) - 2y(am - hl) + a = 0.$$

Since y_1, y_2 are roots of this equation.

$$\therefore \quad y_1 + y_2 = \frac{2(am - hl)}{am^2 - 2hlm + bl^2} \text{ and, } y_1 y_2 = \frac{a}{am^2 - 2hlm + bl^2}$$

The equation of the circle with AB as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\text{or, } x^2 - x(x_1 + x_2) + x_1 x_2 + y^2 - y(y_1 + y_2) + y_1 y_2 = 0$$

$$\text{or, } x^2 + y^2 - \frac{2x(bl - hm)}{am^2 - 2hlm + bl^2} - \frac{2y(am - hl)}{am^2 - 2hlm + bl^2} + \frac{b}{am^2 - 2hlm + bl^2} + \frac{a}{am^2 - 2hlm + bl^2} = 0$$

or, $(x^2+y^2)(am^2-2hlm+bl^2)-2x(bl-hm)-2y(am-hl)+(a+b) = 0$

This is the required equation of the circle.

EXAMPLE 7 *On the line joining (1, 0) and (3, 0) an equilateral triangle is drawn, having its vertex in the first quadrant. Find the equation to the circles described on its sides as diameter.*

SOLUTION Let (1, 0) and (3, 0) be the coordinates of the points A and B respectively. Then,

$$AB = \sqrt{(1-3)^2+(0-0)^2} = 2.$$

Let $C(x_1, y_1)$ be the third vertex of the equilateral triangle ABC. Then, $AC = BC = 2$

Now, $AC = \sqrt{(x_1-1)^2+(y_1-0)^2}$, $BC = \sqrt{(x_1-3)^2+(y_1-0)^2}$

$\therefore \quad AC = BC$

$\Rightarrow \quad AC^2 = BC^2$

$\Rightarrow \quad (x_1-1)^2+{y_1}^2 = (x_1-3)^2+{y_1}^2 \Rightarrow 4x_1 = 8 \Rightarrow x_1 = 2$

Again, $AC = 2$

$\Rightarrow \quad \sqrt{(x_1-1)^2+y_1^2} = 2$

$\Rightarrow \quad (x_1-1)^2+y_1^2 = 4$

$\Rightarrow \quad (2-1)^2+y_1^2 = 4 \Rightarrow y_1 = \pm\sqrt{3} \Rightarrow y_1 = \sqrt{3}$ $\quad [\because x_1 = 2]$

So, the coordinates of C are $(2, \sqrt{3})$. $\quad [\because C(x_1, y_1)$ lies in first quadrant$]$

The equation of the circle on AC as diameter is

$$(x-1)(x-2)+(y-0)(y-\sqrt{3}) = 0 \text{ or, } x^2+y^2-3x-\sqrt{3}y+2 = 0.$$

Similarly, the equations of circles with AB and BC as diameters are

$$(x-1)(x-3)+(y-0)(y-0) = 0 \text{ and, } (x-3)(x-2)+(y-0)(y-\sqrt{3}) = 0$$

or, $\quad x^2+y^2-4x+3 = 0$ and, $x^2+y^2-5x-\sqrt{3}y+6 = 0$ respectively.

EXAMPLE 8 *Find the equations to the circles which pass through the origin and cut off equal chords of length 'a' from the straight lines $y = x$ and $y = -x$.*

SOLUTION From Fig. 24.40, we see that there will be four such circles which pass through the origin and cut off equal chords of length a from the straight lines $y = \pm x$.

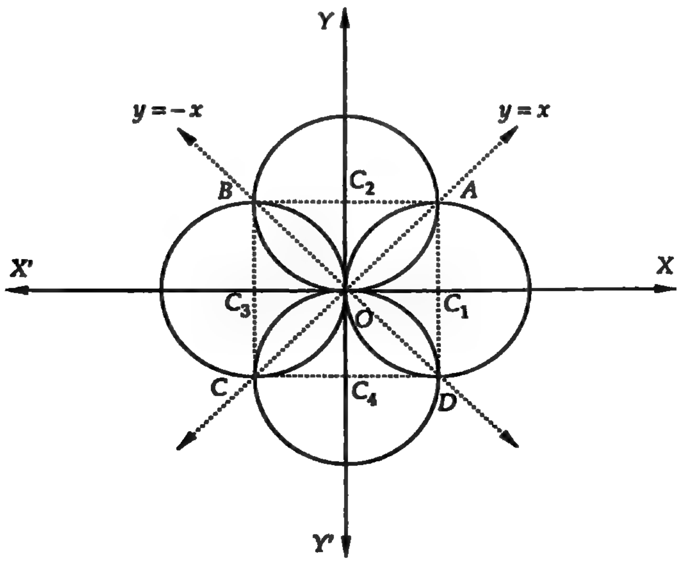


Fig. 24.40

Since $\angle AOB = \angle BOC = \angle COD = \angle DOA = \pi/2$. Therefore, AB, BC, CD and DA are diameters of the four circles.

Now, $\angle XOA = \pi/4$ and, $OA = a$

$\therefore \quad AC_1 = a \sin \frac{\pi}{4} = \frac{a}{\sqrt{2}}$ and, $OC_1 = a \cos \frac{\pi}{4} = \frac{a}{\sqrt{2}}$

So, the coordinates of A are $(a/\sqrt{2}, a/\sqrt{2})$.

Similarly, the coordinates of B, C and D are $(-a/\sqrt{2}, a/\sqrt{2}), (-a/\sqrt{2}, -a/\sqrt{2})$ and $(a/\sqrt{2}, -a/\sqrt{2})$ respectively.

The equation of the circle with AD as diameter is

$$\left(x - \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right) + \left(y - \frac{a}{\sqrt{2}}\right)\left(y + \frac{a}{\sqrt{2}}\right) = 0 \quad \text{or,} \quad x^2 + y^2 - \sqrt{2}\, ax = 0.$$

Similarly, the equations of the required circles with BC, CD and AB as diameters are

$$\left(x + \frac{a}{\sqrt{2}}\right)\left(x + \frac{a}{\sqrt{2}}\right) + \left(y - \frac{a}{\sqrt{2}}\right)\left(y + \frac{a}{\sqrt{2}}\right) = 0 \quad \text{or,} \quad x^2 + y^2 + \sqrt{2}\, ax = 0$$

$$\left(x + \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right) + \left(y + \frac{a}{\sqrt{2}}\right)\left(y + \frac{a}{\sqrt{2}}\right) = 0 \quad \text{or,} \quad x^2 + y^2 + \sqrt{2}\, ay = 0$$

and, $$\left(x - \frac{a}{\sqrt{2}}\right)\left(x + \frac{a}{\sqrt{2}}\right) + \left(y - \frac{a}{\sqrt{2}}\right)\left(y - \frac{a}{\sqrt{2}}\right) = 0 \quad \text{or,} \quad x^2 + y^2 - \sqrt{2}\, ay = 0$$

respectively.

EXERCISE 24.3

LEVEL-1

1. Find the equation of the circle, the end points of whose diameter are $(2, -3)$ and $(-2, 4)$. Find its centre and radius.
2. Find the equation of the circle the end points of whose diameter are the centres of the circles $x^2 + y^2 + 6x - 14y - 1 = 0$ and $x^2 + y^2 - 4x + 10y - 2 = 0$.
3. The sides of a square are $x = 6$, $x = 9$, $y = 3$ and $y = 6$. Find the equation of a circle drawn on the diagonal of the square as its diameter.
4. Find the equation of the circle circumscribing the rectangle whose sides are $x - 3y = 4$, $3x + y = 22$, $x - 3y = 14$ and $3x + y = 62$.
5. Find the equation of the circle passing through the origin and the points where the line $3x + 4y = 12$ meets the axes of coordinates.
6. Find the equation of the circle which passes through the origin and cuts off intercepts a and b respectively from x and y-axes.
7. Find the equation of the circle whose diameter is the line segment joining $(-4, 3)$ and $(12, -1)$. Find also the intercept made by it on y-axis.

LEVEL-2

8. The abscissae of the two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation of the circle with AB as diameter. Also, find its radius.
9. $ABCD$ is a square whose side is a; taking AB and AD as axes, prove that the equation of the circle circumscribing the square is $x^2 + y^2 - a(x + y) = 0$.
10. The line $2x - y + 6 = 0$ meets the circle $x^2 + y^2 - 2y - 9 = 0$ at A and B. Find the equation of the circle on AB as diameter.

11. Find the equation of the circle which circumscribes the triangle formed by the lines $x = 0$, $y = 0$ and $lx + my = 1$.
12. Find the equations of the circles which pass through the origin and cut off equal chords of $\sqrt{2}$ units from the lines $y = x$ and $y = -x$.

ANSWERS

1. $x^2 + y^2 - y - 16 = 0; \left(0, \frac{1}{2}\right), \frac{\sqrt{65}}{2}$
2. $x^2 + y^2 + x - 2y - 41 = 0$
3. $x^2 + y^2 - 15x - 9y + 72 = 0$
4. $x^2 + y^2 - 27x - 3y + 142 = 0$
5. $x^2 + y^2 - 4x - 3y = 0$
6. $x^2 + y^2 \pm ax \pm by = 0$
7. $x^2 + y^2 - 8x - 2y - 51 = 0, 4\sqrt{13}$
8. $x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0, \sqrt{a^2 + b^2 + p^2 + q^2}$
10. $x^2 + y^2 + 4x - 4y + 3 = 0$
11. $x^2 + y^2 - \frac{1}{l}x - \frac{1}{m}y = 0$
12. $x^2 + y^2 \pm 2y = 0,\ x^2 + y^2 \pm 2x = 0.$

HINTS TO SELECTED PROBLEMS

5. The line $3x + 4y = 12$ meets the coordinate axes at A (4, 0) and B (0, 3). We have to find the equation of the circle with AB as diameter.
6. The coordinates of the end points of a diameter are $(\pm a, 0)$ and $(0, \pm b)$.
9. The required circle has (0, 0) and (a, a) as the end points of a diameter.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the length of the intercept made by the circle $x^2 + y^2 + 2x - 4y - 5 = 0$ on y-axis.
2. Write the coordinates of the centre of the circle passing through (0, 0), (4, 0) and (0, – 6).
3. Write the area of the circle passing through (– 2, 6) and having its centre at (1, 2).
4. If the abscissae and ordinates of two points P and Q are roots of the equations $x^2 + 2ax - b^2 = 0$ and $x^2 + 2px - q^2 = 0$ respectively, then write the equation of the circle with PQ as diameter.
5. Write the equation of the unit circle concentric with $x^2 + y^2 - 8x + 4y - 8 = 0$.
6. If the radius of the circle $x^2 + y^2 + ax + (1 - a)y + 5 = 0$ does not exceed 5, write the number of integral values a.
7. Write the equation of the circle passing through (3, 4) and touching y-axis at the origin.
8. If the line $y = mx$ does not intersect the circle $(x + 10)^2 + (y + 10)^2 = 180$, then write the set of values taken by m.
9. Write the coordinates of the centre of the circle inscribed in the square formed by the lines $x = 2$, $x = 6$, $y = 5$ and $y = 9$.

ANSWERS

1. 6 units
2. (2, – 3)
3. 10π sq. units
4. $x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$
5. $x^2 + y^2 - 8x + 4y + 19 = 0$
6. 16
7. $3(x^2 + y^2) - 25x = 0$
8. $m \in \left(-2, -\frac{1}{2}\right)$
9. (4, 7)

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternatives in each of the following:

1. If the equation of a circle is $\lambda x^2 + (2\lambda - 3) y^2 - 4x + 6y - 1 = 0$, then the coordinates of centre are
   (a) $(4/3, -1)$ (b) $(2/3, -1)$ (c) $(-2/3, 1)$ (d) $(2/3, 1)$
2. If $2x^2 + \lambda xy + 2y^2 + (\lambda - 4)x + 6y - 5 = 0$ is the equation of a circle, then its radius is
   (a) $3\sqrt{2}$ (b) $2\sqrt{3}$ (c) $2\sqrt{2}$ (d) none of these
3. The equation $x^2 + y^2 + 2x - 4y + 5 = 0$ represents
   (a) a point (b) a pair of straight lines
   (c) a circle of non-zero radius (d) none of these
4. If the equation $(4a - 3)x^2 + ay^2 + 6x - 2y + 2 = 0$ represents a circle, then its centre is
   (a) $(3, -1)$ (b) $(3, 1)$ (c) $(-3, 1)$ (d) none of these
5. The radius of the circle represented by the equation $3x^2 + 3y^2 + \lambda xy + 9x + (\lambda - 6)y + 3 = 0$ is
   (a) $3/2$ (b) $\sqrt{17}/2$ (c) $2/3$ (d) none of these
6. The number of integral values of λ for which the equation $x^2 + y^2 + \lambda x + (1 - \lambda) y + 5 = 0$ is the equation of a circle whose radius cannot exceed 5, is
   (a) 14 (b) 18 (c) 16 (d) none of these
7. The equation of the circle passing through the point (1, 1) and having two diameters along the pair of lines $x^2 - y^2 - 2x + 4y - 3 = 0$, is
   (a) $x^2 + y^2 - 2x - 4y + 4 = 0$ (b) $x^2 + y^2 + 2x + 4y - 4 = 0$
   (c) $x^2 + y^2 - 2x + 4y + 4 = 0$ (d) none of these
8. If the centroid of an equilateral triangle is (1, 1) and its one vertex is (−1, 2), then the equation of its circumcircle is
   (a) $x^2 + y^2 - 2x - 2y - 3 = 0$ (b) $x^2 + y^2 + 2x - 2y - 3 = 0$
   (c) $x^2 + y^2 + 2x + 2y - 3 = 0$ (d) none of these
9. If the point $(2, k)$ lies outside the circles $x^2 + y^2 + x - 2y - 14 = 0$ and $x^2 + y^2 = 13$ then k lies in the interval
   (a) $(-3, -2) \cup (3, 4)$ (b) $-3, 4$
   (c) $(-\infty, -3) \cup (4, \infty)$ (d) $(-\infty, -2) \cup (3, \infty)$
10. If the point $(\lambda, \lambda + 1)$ lies inside the region bounded by the curve $x = \sqrt{25 - y^2}$ and y-axis, then λ belongs to the interval
    (a) $(-1, 3)$ (b) $(-4, 3)$
    (c) $(-\infty, -4) \cup (3, \infty)$ (d) none of these
11. The equation of the incircle formed by the coordinate axes and the line $4x + 3y = 6$ is
    (a) $x^2 + y^2 - 6x - 6y + 9 = 0$ (b) $4(x^2 + y^2 - x - y) + 1 = 0$
    (c) $4(x^2 + y^2 + x + y) + 1 = 0$ (d) none of these
12. If the circles $x^2 + y^2 = 9$ and $x^2 + y^2 + 8y + c = 0$ touch each other, then c is equal to
    (a) 15 (b) −15 (c) 16 (d) −16
13. If the circle $x^2 + y^2 + 2ax + 8y + 16 = 0$ touches x-axis, then the value of a is
    (a) ±16 (b) ±4 (c) ±8 (d) ±1

14. The equation of a circle with radius 5 and touching both the coordinate axes is

(a) $x^2 + y^2 \pm 10x \pm 10y + 5 = 0$ (b) $x^2 + y^2 \pm 10x \pm 10y = 0$

(c) $x^2 + y^2 \pm 10x \pm 10y + 25 = 0$ (d) $x^2 + y^2 \pm 10x \pm 10y + 51 = 0$

15. The equation of the circle passing through the origin which cuts off intercept of length 6 and 8 from the axes is

(a) $x^2 + y^2 - 12x - 16y = 0$ (b) $x^2 + y^2 + 12x + 16y = 0$

(c) $x^2 + y^2 + 6x + 8y = 0$ (d) $x^2 + y^2 - 6x - 8y = 0$

16. The equation of the circle concentric with $x^2 + y^2 - 3x + 4y - c = 0$ and passing through $(-1, -2)$ is

(a) $x^2 + y^2 - 3x + 4y - 1 = 0$ (b) $x^2 + y^2 - 3x + 4y = 0$

(c) $x^2 + y^2 - 3x + 4y + 2 = 0$ (d) none of these

17. The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ does not intersect x-axis, if

(a) $g^2 < c$ (b) $g^2 > c$ (c) $g^2 > 2c$ (d) none of these

18. The area of an equilateral triangle inscribed in the circle $x^2 + y^2 - 6x - 8y - 25 = 0$ is

(a) $\dfrac{225\sqrt{3}}{6}$ (b) 25π (c) $50\pi - 100$ (d) none of these

19. The equation of the circle which touches the axes of coordinates and the line $\dfrac{x}{3} + \dfrac{y}{4} = 1$ and whose centres lie in the first quadrant is $x^2 + y^2 - 2cx - 2cy + c^2 = 0$, where c is equal to

(a) 4 (b) 2 (c) 3 (d) 6

20. If the circles $x^2 + y^2 = a$ and $x^2 + y^2 - 6x - 8y + 9 = 0$, touch externally, then $a =$

(a) 1 (b) -1 (c) 21 (d) 16

21. If $(x, 3)$ and $(3, 5)$ are the extremities of a diameter of a circle with centre at $(2, y)$, then the values of x and y are

(a) $(3, 1)$ (b) $x = 4, y = 1$ (c) $x = 8, y = 2$ (d) none of these

22. If $(-3, 2)$ lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which is concentric with the circle $x^2 + y^2 + 6x + 8y - 5 = 0$, then $c =$

(a) 11 (b) -11 (c) 24 (d) none of these

23. Equation of the diameter of the circle $x^2 + y^2 - 2x + 4y = 0$ which passes through the origin is

(a) $x + 2y = 0$ (b) $x - 2y = 0$ (c) $2x + y = 0$ (d) $2x - y = 0$

24. Equation of the circle through origin which cuts intercepts of length a and b on axes is

(a) $x^2 + y^2 + ax + by = 0$ (b) $x^2 + y^2 - ax - by = 0$

(c) $x^2 + y^2 + bx + ay = 0$ (d) none of these

25. If the circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other, then

(a) $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c}$ (b) $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$ (c) $a + b = 2c$ (d) $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{2}{c}$

ANSWERS

1. (b)	2. (a)	3. (a)	4. (c)	5. (a)	6. (c)	7. (a)	8. (a)
9. (c)	10. (a)	11. (b)	12. (a)	13. (b)	14. (c)	15. (d)	16. (b)
17. (a)	18. (a)	19. (d)	20. (a)	21. (a)	22. (b)	23. (c)	24. (b)
25. (a)							

CHAPTER 25

PARABOLA

25.1 CONIC SECTIONS

A conic section, as the name implies, is a section cut-off from a circular (not necessary a right circular) cone by a plane in various ways. The shape of the section depends upon the position of the cutting plane.

Consider a double right circular cone of semi vertical angle α and let it be cut by a plane inclined at an angle θ to the axis of the cone. We will get different sections (curves) as follows:

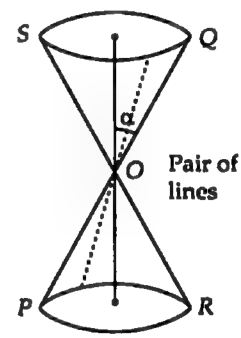


Fig. 25.1

CASE I *If the plane passes through the vertex O*

The curve of intersection is a pair of straight lines passing through the vertex which are

(i) real and distinct for $\theta < \alpha$.
(ii) coincident for $\theta = \alpha$ i.e. the plane touches the cone.
(iii) imaginary for $\theta > \alpha$.

CASE II *If the plane does not pass through the vertex O*

The curve of intersection is called

(i) a circle if $\theta = \dfrac{\pi}{2}$.
(ii) a parabola for $\theta = \alpha$ i.e. if the plane is parallel to the generator PQ.
(iii) an ellipse for $\theta > \alpha$ $(\theta \neq \pi / 2)$ i.e. if the plane cuts both the generating lines PQ and RS.
(iv) a hyperbola for $\theta < \alpha$ i.e. if the plane cuts both the cones.

Thus, we may get the section either as a pair of straight lines, a circle, a parabola, an ellipse or a hyperbola depending upon the different positions of the cutting plane. These curves of intersection are called the *conic sections*.

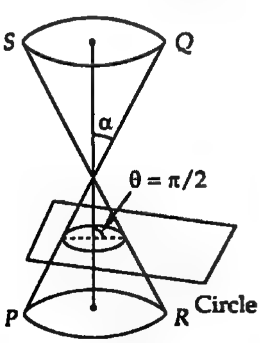


Fig. 25.2

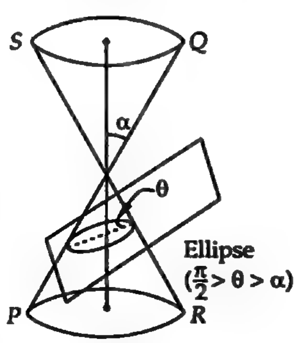


Fig. 25.3

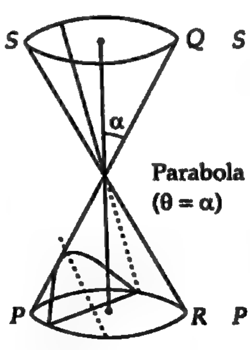


Fig. 25.4

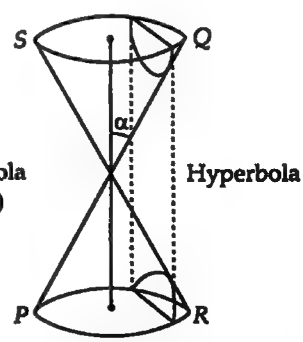


Fig. 25.5

25.2 ANALYTICAL DEFINITION OF CONIC SECTION

CONIC SECTION *A conic section or conic is the locus of a point P which moves in such a way that its distances from a fixed point S always bears a constant ratio to its distance from a fixed line, all being in the same plane.*

FOCUS *The fixed point is called the focus of the conic section.*

DIRECTRIX *The fixed straight line is called the directrix of the conic section.*

In general, every conic has four foci, two of them are real and the other two are imaginary. Due to two real foci, every conic has two directrices corresponding to each real focus.

ECCENTRICITY *The constant ratio is called the eccentricity of the conic section and is denoted by e.*

AXIS *The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.*

VERTEX *The points of intersection of the conic section and the axis are called vertices of the conic section.*

CENTRE *The point which bisects every chord of the conic passing through it, is called the centre of the conic.*

LATUS-RECTUM *The latus-rectum of a conic is the chord passing through the focus and perpendicular to the axis.*

NOTE *As mentioned above the eccentricity of a conic is generally denoted by e and*

(i) *for $e < 1$, the conic obtained is an ellipse;*
(ii) *for $e = 1$, the conic obtained is a parabola;*
(iii) *for $e > 1$, the conic is a hyperbola;*
(iv) *for $e = 0$, the conic is a circle.*

25.3 GENERAL EQUATION OF A CONIC SECTION WHEN ITS FOCUS, DIRECTRIX AND ECCENTRICITY ARE GIVEN

Let $S(\alpha, \beta)$ be the focus, $Ax + By + C = 0$ be the directrix and e be the eccentricity of a conic. Let $P(h, k)$ be any point on the conic. Let PM be the perpendicular from P, on the directrix. Then, by definition

$$SP = e \cdot PM$$

$$\Rightarrow \quad SP^2 = e^2 \, PM^2$$

$$\Rightarrow \quad (h-\alpha)^2 + (k-\beta)^2 = e^2 \left(\frac{Ah + Bk + C}{\sqrt{A^2 + B^2}} \right)^2$$

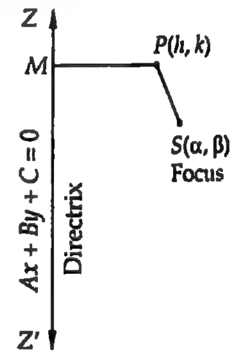


Fig. 25.6

Thus, the locus of (h, k) is

$$(x-\alpha)^2 + (y-\beta)^2 = e^2 \frac{(Ax + By + C)^2}{(A^2 + B^2)}$$

This is the cartesian equation of the conic section which, when simplified, can be written in the form

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, which is the general equation of second degree.

It can be easily shown that the general equation of second degree viz. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ always represents:

(i) a pair of straight lines, if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
(ii) a circle if $\Delta \neq 0$, $a = b$ and $h = 0$
(iii) a parabola if $\Delta \neq 0$ and $h^2 = ab$
(iv) an ellipse if $\Delta \neq 0$ and $h^2 < ab$
(v) a hyperbola if $\Delta \neq 0$ and $h^2 > ab$
(vi) a rectangular hyperbola if $\Delta \neq 0$, $h^2 > ab$ and $a + b = 0$.

25.4 THE PARABOLA

ANALYTICAL DEFINITION *A parabola is the locus of a point which moves in a plane such that its distance from a fixed point in the plane is always equal to its distance from a fixed straight line in the same plane.*

As defined in section 25.2, the fixed point is called the focus and the fixed straight line is called the directrix of the parabola. The line through the focus and perpendicular to the directrix is the axis of the parabola. The point on the axis midway between the focus and directrix is called the vertex of the parabola.

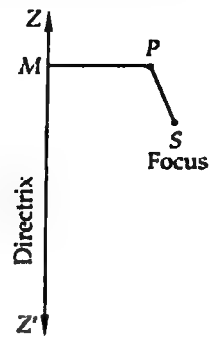


Fig. 25.7

Let S be the focus, ZZ' be the directrix and let P be any point on the parabola. Then, by definition

$$SP = PM$$

where PM is the length of the perpendicular from P on the directrix ZZ'.

ILLUSTRATION 1 *Find the equation of the parabola whose focus is* $(-3, 2)$ *and the directrix is* $x + y = 4$.

SOLUTION Let $P(x, y)$ be any point on the parabola whose focus is $S(-3, 2)$ and the directrix $x + y - 4 = 0$. Draw PM perpendicular to $x + y - 4 = 0$. Then,

$$SP = PM \quad \text{[By definition]}$$

$$\Rightarrow \quad SP^2 = PM^2$$

$$\Rightarrow \quad (x+3)^2 + (y-2)^2 = \left|\frac{x+y-4}{\sqrt{1+1}}\right|^2$$

$$\Rightarrow \quad 2\left(x^2 + y^2 + 6x - 4y + 13\right) = \left(x^2 + y^2 + 16 + 2xy - 8x - 8y\right)$$

$$\Rightarrow \quad x^2 + y^2 - 2xy + 20x + 10 = 0$$

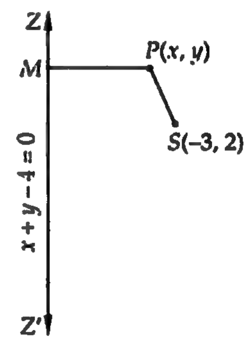


Fig. 25.8

Thus, the required equation of the parabola is $x^2 + y^2 - 2xy + 20x + 10 = 0$.

ILLUSTRATION 2 *Find the equation of the parabola whose focus is* $(-3, 0)$ *and the directrix is* $x + 5 = 0$.

SOLUTION Let $P(x, y)$ be any point on the parabola having its focus at $S(-3, 0)$ and directrix as the line $x + 5 = 0$. Then,

$SP = PM$, where PM is the length of the perpendicular from P on the directrix

$$\Rightarrow \quad SP^2 = PM^2$$

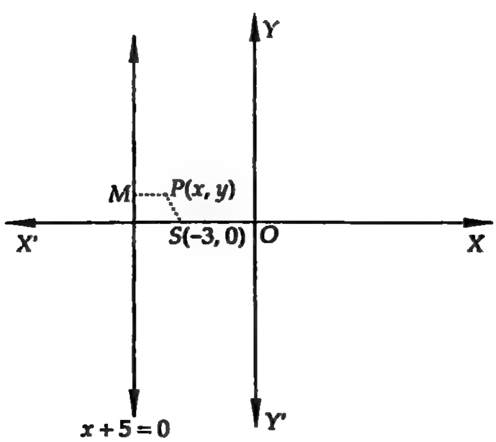


Fig. 25.9

$\Rightarrow \quad (x+3)^2 + (y-0)^2 = \left|\dfrac{x+0y+5}{\sqrt{1}+0}\right|^2$

$\Rightarrow \quad y^2 = 4x + 16$, which is the required equation of the parabola.

25.4.1 EQUATION OF THE PARABOLA IN ITS STANDARD FORM

Let S be the focus, $Z\,Z'$ be the directrix. Draw SK perpendicular from S on the directrix and bisect SK at A. Then,

$AS = AK$

$\Rightarrow$ Distance of A from the focus = Distance of A from the directrix

$\Rightarrow$ A lies on the parabola [By def.]

Let $SK = 2a$. Then, $AS = AK = a$.

Let us choose A as the origin, AS as x-axis and AY a line perpendicular to AS as y-axis. Then, the coordinates of S are $(a, 0)$ and the equation of the directrix ZZ' is $x = -a$.

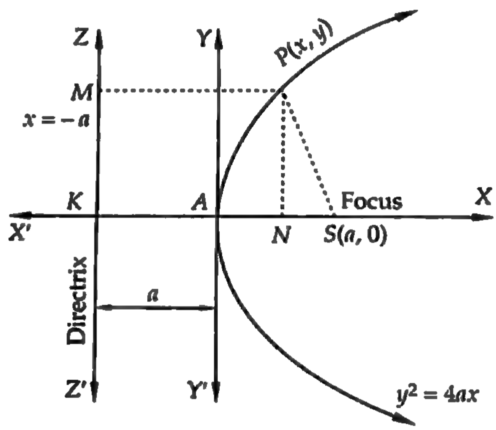


Fig. 25.10

Let $P(x, y)$ be any point on the parabola. Join SP and draw PM and PN perpendiculars on the directrix $Z\,Z'$ and X-axis. Then,

$PM = NK = AN + AK = x + a.$

Since P lies on the parabola. Therefore,

$SP = PM$ [By definition of parabola]

$\Rightarrow \quad SP^2 = PM^2$

$\Rightarrow \quad (x-a)^2 + (y-0)^2 = (x+a)^2$

$\Rightarrow \quad y^2 = 4ax$

This is the equation of the parabola in its standard form.

<u>NOTE</u> *The parabola has two real foci situated on its axis one of which is the focus S and the other lies at infinity. The corresponding directrix is also at infinity.*

25.4.2 TRACING OF THE PARABOLA $y^2 = 4ax,\ a > 0$

The given equation can be written as $y = \pm 2\sqrt{ax}$.

We observe the following:

(i) *Symmetry:* For every positive value of x, there are two equal and opposite values of y.

(ii) *Region:* For every negative value of x, the value of y is imaginary. Therefore, no part of the curve lies to the left of y-axis

(iii) *Origin:* The curve passes through the origin and the tangent at the origin is $x = 0$ i.e., y-axis

(iv) *Intersection with the axes:* The curve meets the coordinate axes only at the origin.

(v) *Portion Occupied:* As $x \to \infty$, $y \to \infty$. Therefore the curve extends to infinity to the right of axis of y.

With the help of the above facts and by joining some convenient points on the parabola the general shape of the parabola $y^2 = 4\,ax$ is as shown in Fig. 25.10.

25.4.3 VARIOUS RESULTS RELATED TO THE PARABOLA

As discussed in section 25.4, the focus of the parabola $y^2 = 4\,ax$ is at $(a, 0)$ and the directrix is $x = -a$. The axis is a line passing through the focus and perpendicular to the directrix. In Fig. 25.10 x-axis i.e., $y = 0$ is the axis of the parabola $y^2 = 4\,ax$. The axis meets the curve $y^2 = 4\,ax$ at A, the origin. So, the coordinates of the vertex are $(0, 0)$. Clearly, the vertex A is the midway between the focus and the directrix i.e., the vertex is equidistant from the focus and the directrix.

DOUBLE ORDINATE *Let P be any point on the parabola $y^2 = 4\,ax$. A chord passing through P perpendicular to the axis of the parabola is called the double ordinate through the point P.*

In Fig. 25.10, PP' is the double ordinate of point P.

LATUS-RECTUM *A double ordinate through the focus is called the latusrectum i.e. the latusrectum of a parabola is a chord passing through the focus perpendicular to the axis.*

In Fig. 25.10, LSL' is the latusrectum of the parabola $y^2 = 4\,ax$. By the symmetry of the curve $SL = SL' = \lambda$ (say). So, the coordinates of L are (a, λ). Since L lies on $y^2 = 4\,ax$. Therefore,

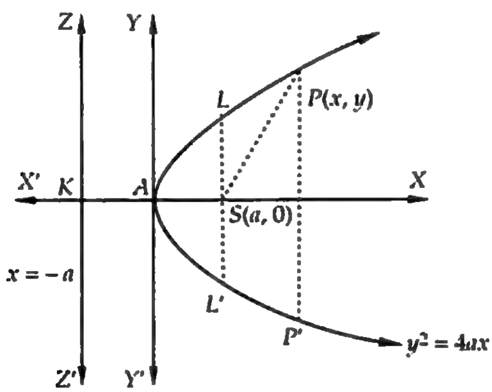


Fig. 25.11

$$\lambda^2 = 4\,a^2 \Rightarrow \lambda = 2a \Rightarrow LL' = 2\lambda = 4a.$$

$\therefore$ Latusrectum $= 4\,a$.

The coordinates of L and L', end points of the latusrectum, are $(a, 2\,a)$ and $(a, -2\,a)$ respectively.

FOCAL DISTANCE OF ANY POINT *The distance of $P\,(x, y)$ from the focus S is called the focal distance of the point P.*

Clearly, $SP = \sqrt{(x-a)^2 + (y-0)^2}$

$\Rightarrow \quad SP = \sqrt{(x-a)^2 + y^2}$

$\Rightarrow \quad SP = \sqrt{(x-a)^2 + 4\,ax}$ $\quad$ [$\because$ $P\,(x, y)$ lies on the parabola $\therefore\ y^2 = 4\,ax$]

$\Rightarrow \quad SP = \sqrt{(x+a)^2} = |x+a| = a + x$ $\quad$ [$\because\ x > 0, a > 0\ \therefore x + a > 0$]

Hence, $a + x$ is the focal distance of any point $P(x, y)$ on the parabola $y^2 = 4ax$.

FOCAL CHORD *A chord of the parabola is a focal chord, if it passes through the focus.*

25.4.4 SOME OTHER STANDARD FORMS OF PARABOLA

Proceeding as in section 25.4, we find that there are three other standard forms of parabola viz. $y^2 = -4ax$, $x^2 = 4ay$ and $x^2 = -4ay$ depending upon the choice of the axes. Thus, in all there are four standard forms. The shapes of the curves in these four standard forms and their corresponding results are as follows:

	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Coordinates of vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Coordinates of focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of the directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of the Latusrectum	$4a$	$4a$	$4a$	$4a$
Focal distance of a point $P(x, y)$	$a + x$	$a - x$	$a + y$	$a - y$

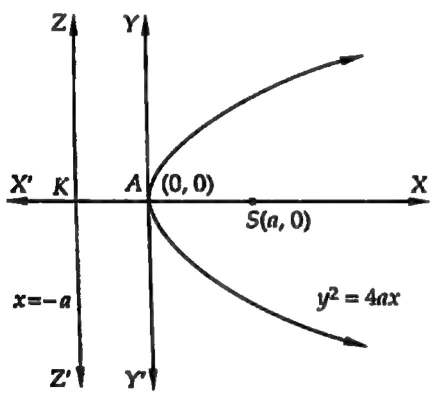


Fig. 25.12

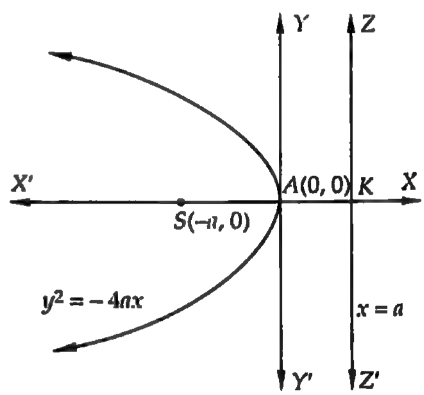


Fig. 25.13

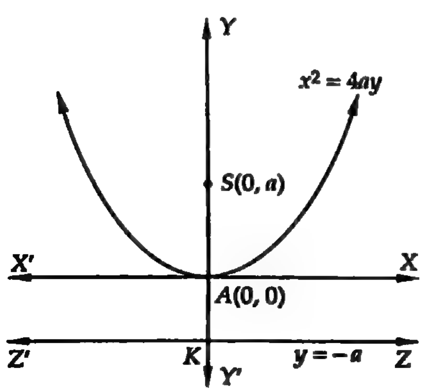


Fig. 25.14

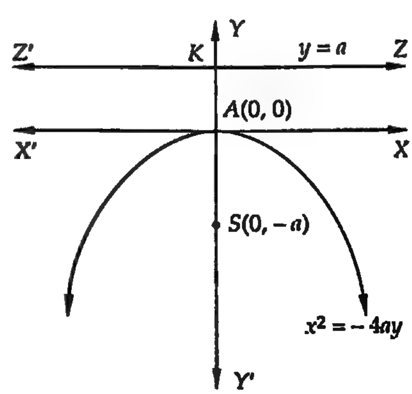


Fig. 25.15

REMARK *If the vertex of the parabola is at the point A (h, k) and its latusrectum is of length 4a, then its equation is*

(i) $(y-k)^2 = 4a(x-h)$ or, $(y-k)^2 = -4a(x-h)$ according as its axis is parallel to OX or OX'.

(ii) $(x-h)^2 = 4a(y-k)$ or, $(x-h)^2 = -4a(y-k)$ according as its axis is parallel to OY or OY'.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON FINDING THE EQUATION OF A PARABOLA WHEN ITS FOCUS AND DIRECTRIX ARE GIVEN

EXAMPLE 1 *Find the equation of the parabola whose focus is the point (0, 0) and whose directrix is the straight line $3x - 4y + 2 = 0$.*

SOLUTION Let $P(x, y)$ be any point on the parabola whose focus is $S(0, 0)$ and the directrix $3x - 4y + 2 = 0$. Draw PM perpendicular from P on the directrix. Then, by definition

$$SP = PM$$

$$\Rightarrow \quad SP^2 = PM^2$$

$$\Rightarrow \quad (x-0)^2 + (y-0)^2 = \left|\frac{3x-4y+2}{\sqrt{3^2+(-4)^2}}\right|^2$$

$$\Rightarrow \quad x^2 + y^2 = \frac{(3x-4y+2)^2}{25}$$

$$\Rightarrow \quad 25(x^2+y^2) = (3x-4y+2)^2$$

$$\Rightarrow \quad 25x^2 + 25y^2 = 9x^2 + 16y^2 + 4 - 24xy + 12x - 16y$$

$$\Rightarrow \quad 16x^2 + 9y^2 + 24xy - 12x + 16y - 4 = 0$$

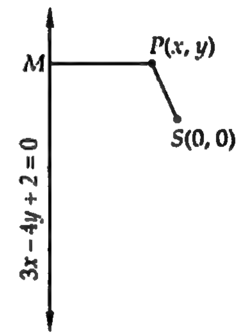


Fig. 25.16

This is the required equation of the parabola.

EXAMPLE 2 *Find the equation of the parabola whose focus is at (−1, −2) and the directrix the line $x - 2y + 3 = 0$.*

SOLUTION Let $P(x, y)$ be any point on the parabola whose focus is $S(-1, -2)$ and the directrix $x - 2y + 3 = 0$. Draw PM perpendicular from $P(x, y)$ on the directrix $x - 2y + 3 = 0$. Then, by definition

$$SP = PM$$

$$\Rightarrow \quad SP^2 = PM^2$$

$$\Rightarrow \quad (x+1)^2 + (y+2)^2 = \left|\frac{x-2y+3}{\sqrt{1+4}}\right|^2$$

$$\Rightarrow \quad 5\left\{(x+1)^2 + (y+2)^2\right\} = (x-2y+3)^2$$

$$\Rightarrow \quad 5(x^2 + y^2 + 2x + 4y + 5) = (x^2 + 4y^2 + 9 - 4xy + 6x - 12y)$$

$$\Rightarrow \quad 4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$$

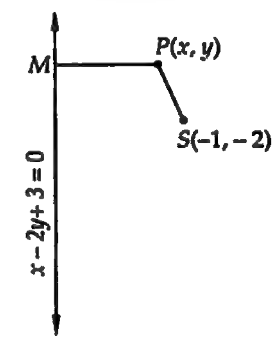


Fig. 25.17

This is the equation of the required parabola.

Type II ON FINDING THE FOCUS, DIRECTRIX, LATUS-RECTUM, AXIS ETC. FOR A GIVEN PARABOLA IN ONE OF THE STANDARD FORMS

EXAMPLE 3 *For the following parabolas find the coordinates of the foci, the equations of the directrices and the lengths of the latus-rectum:*

(i) $y^2 = 8x$ (ii) $x^2 = 6y$ (iii) $y^2 = -12x$ (iv) $x^2 = -16y$

SOLUTION (i) The given parabola $y^2 = 8x$ is of the form $y^2 = 4ax$, where $4a = 8$ i.e. $a = 2$.

The coordinates of the focus are $(a, 0)$ i.e. $(2, 0)$ and the equation of the directrix is $x = -a$ i.e. $x = -2$.

Length of the latus-rectum $= 4a = 8$.

(ii) The given parabola $x^2 = 6y$ is of the form $x^2 = 4ay$, where $4a = 6$ i.e. $a = 3/2$.

Clearly, the coordinates of the focus are $(0, a) = (0, 3/2)$ and the equation of the directrix is $y = -a$ i.e. $y = -3/2$.

Length of the latus-rectum $= 4a = 6$.

(iii) The given parabola $y^2 = -12x$ is of the form $y^2 = -4ax$, where $4a = 12$ i.e. $a = 3$.

Clearly, the coordinates of the focus are $(-a, 0) = (-3, 0)$ and the equation of the directrix is $x = a$ i.e. $x = 3$.

Length of the latus-rectum $= 4a = 12$.

(iv) The given parabola is of the form $x^2 = -4ay$, where $4a = 16$ i.e. $a = 4$.

Clearly, the coordinates of its focus are $(0, -a) = (0, -4)$ and the equation of the directrix is $y = a$ i.e. $y = 4$.

Length of the latus-rectum $= 4a = 16$.

Type III **ON FINDING THE VERTEX, FOCUS, AXIS, DIRECTRIX, LATUS-RECTUM ETC. OF THE PARABOLAS REDUCIBLE TO ONE OF THE FOUR STANDARD FORMS**

EXAMPLE 4 *Find the vertex, axis, focus, directrix, latus-rectum of the following parabolas. Also, draw their rough sketches:*

(i) $y^2 - 8y - x + 19 = 0$ (ii) $4y^2 + 12x - 20y + 67 = 0$

(iii) $y = x^2 - 2x + 3$ (iv) $x^2 + 2y - 3x + 5 = 0$

SOLUTION (i) The given equation is

$$y^2 - 8y - x + 19 = 0$$

$$\Rightarrow \quad y^2 - 8y = x - 19$$

$$\Rightarrow \quad y^2 - 8y + 16 = x - 19 + 16 \Rightarrow (y-4)^2 = (x-3) \qquad \text{...(i)}$$

Shifting the origin to the point (3, 4) without rotating the axes and denoting the new coordinates with respect to these new axes by X and Y, we have

$$x = X + 3, \quad y = Y + 4 \qquad \text{...(ii)}$$

Using these relations, equation (i) reduces to

$$Y^2 = X \qquad \text{...(iii)}$$

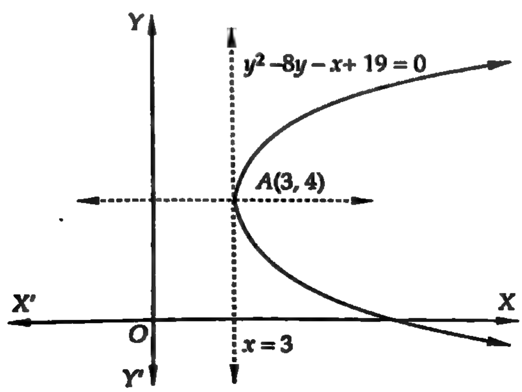


Fig. 25.18

This is of the form $Y^2 = 4aX$. On comparing, we get $4a = 1$ or, $a = \frac{1}{4}$.

Vertex: The coordinates of the vertex with respect to the new axes are $(X = 0, Y = 0)$.

So, the coordinates of the vertex with respect to the old axes are

$(3, 4)$ [Putting $X = 0, Y = 0$ in (ii)]

Axis: The equation of the axis of the parabola (iii) with respect to the new axes is $Y = 0$.

So, the equation of the axis with respect to the old axes is $y = 4$. [Putting $Y = 0$ in (ii)]

Focus: The coordinates of the focus with respect to the new axes are $(X = a, Y = 0)$

i.e., $(X = 1/4, Y = 0)$

So, the coordinates of the focus with respect to the old axes are

$(13/4, 4)$ [Putting $X = 1/4$ and $Y = 0$ in (ii)]

Directrix: The equation of the directrix with respect to the new axes is $X = -a$ i.e., $X = -1/4$.

So, the equation of the directrix with respect to the old axes is

$$x = -\frac{1}{4} + 3 \Rightarrow x = \frac{11}{4} \qquad \left[\text{Putting } X = -\frac{1}{4} \text{ in (ii)}\right]$$

Latus-rectum: The length of the latus-rectum of the parabola (iii) is equal to $4a = 1$.

(ii) The given equation is

$$4y^2 + 12x - 20y + 67 = 0$$

$$\Rightarrow \quad y^2 + 3x - 5y + \frac{67}{4} = 0$$

$$\Rightarrow \quad y^2 - 5y = -3x - \frac{67}{4}$$

$$\Rightarrow \quad y^2 - 5y + \left(\frac{5}{2}\right)^2 = -3x - \frac{67}{4} + \left(\frac{5}{2}\right)^2$$

$$\Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3x - \frac{42}{4}$$

$$\Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3\left(x + \frac{7}{2}\right) \qquad \text{...(i)}$$

Shifting the origin to the point (–7/2, 5/2) without rotating the axes and denoting the new coordinates with respect to these axes by X and Y, we have

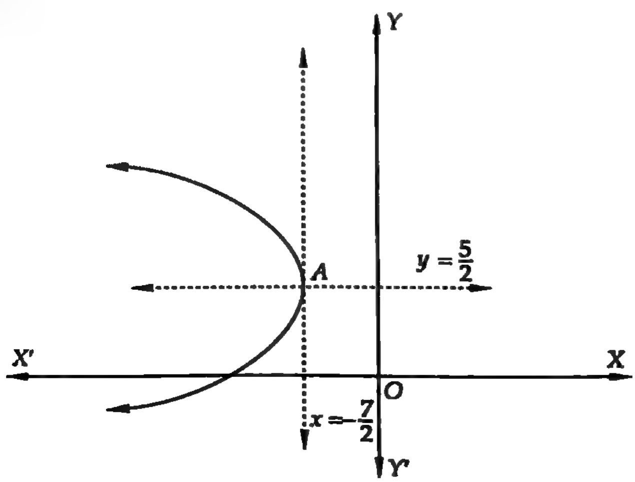


Fig. 25.19

$$x = X - \frac{7}{2}, \quad y = Y + \frac{5}{2} \qquad \text{...(ii)}$$

Using these relations, equation (i) reduces to

$$Y^2 = -3X \qquad \text{...(iii)}$$

This is of the form $Y^2 = -4aX$.

On comparing, we get: $4a = 3 \Rightarrow a = 3/4$.

Vertex: The coordinates of the vertex with respect to the new axes are $(X = 0, Y = 0)$.

So, the coordinates of the vertex with respect to the old axes are

$(-7/2, 5/2)$ [Putting $X = 0$, $Y = 0$ in (ii)]

Axis: The equation of the axis of the parabola with respect to the new axis is $Y = 0$. So, the equation of the axis with respect to the old axes is

$y = 5/2$ [Putting $Y = 0$ in (ii)]

Focus: The coordinates of the focus with respect to the new axes are $(X = -a, Y = 0)$ i.e. $(X = -3/4, Y = 0)$. So, the coordinates of the focus with respect to the old axes are

$(-17/4, 5/2)$ [Putting $X = -3/4$ and $Y = 0$ in (ii)]

Directrix: The equation of the directrix with respect to the new axes is $X = a$ *i.e.* $X = \frac{3}{4}$.

So, the equation of the directrix with respect to the old axes is $x = -\frac{11}{4}$ [Putting $X = 3/4$ in (ii)]

Latus-rectum: The length of the latus-rectum of the given parabola is $4a = 3$.

(iii) The given equation is

$$y = x^2 - 2x + 3$$

$$\Rightarrow \quad x^2 - 2x = y - 3$$

$$\Rightarrow \quad x^2 - 2x + 1 = y - 3 + 1$$

$$\Rightarrow \quad (x-1)^2 = y - 2 \qquad \text{...(i)}$$

Shifting the origin to the point (1, 2) without rotating the axes and denoting the new coordinates with respect to these axes by X and Y, we obtain

$$x = X + 1, \quad y = Y + 2 \qquad \text{...(ii)}$$

Using these relations, equation (i) reduces to

$$X^2 = Y \qquad \text{...(iii)}$$

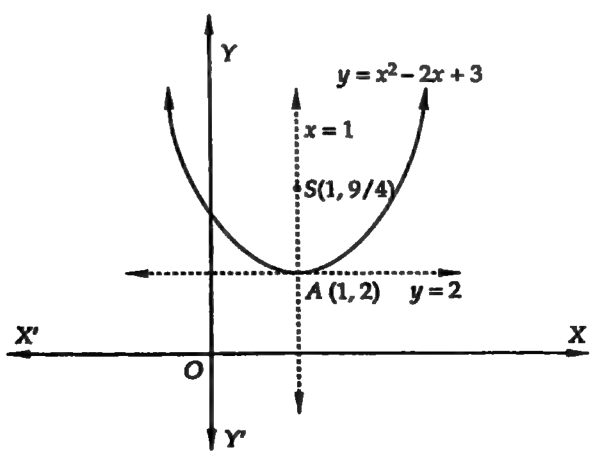


Fig. 25.20

This is of the form $X^2 = 4aY$. On comparing, we get

$4a = 1$ i.e. $a = 1/4$.

Vertex: The coordinates of the vertex with respect to the new axes are $(X = 0, Y = 0)$. So, the coordinates of the vertex with respect to the old axes are $(1, 2)$ [Putting $X = 0, Y = 0$ in (ii)]

Axis: The equation of the axis of the parabola with respect to the new axes is $X = 0$.

So, the equation of the axis with respect to the old axes is $x = 1$. [Putting $X = 0$ in (ii)]

Focus: The coordinates of the focus with respect to the new axes are $(X = 0, Y = a)$ i.e. $(X = 0, Y = 1/4)$

So, the coordinates of the focus S with respect to the old axes are

$(1, 9/4)$ [Putting $X = 0, Y = \frac{1}{4}$ in (ii)]

Directrix: The equation of the directrix with respect to the new axes is $Y = -a$ i.e. $Y = -1/4$.

So, the equation of the directrix with respect to the old axes is

$$y = -\frac{1}{4} + 2 \quad \text{or} \quad y = \frac{7}{4}$$ [Putting $Y = -\frac{1}{4}$ in (ii)]

Latus-rectum: The length of the latus-rectum of the given parabola is equal to $4a = 1$.

(iv) The given equation is

$$x^2 + 2y - 3x + 5 = 0$$

$$\Rightarrow \quad x^2 - 3x = -2y - 5$$

$$\Rightarrow \quad x^2 - 3x + \frac{9}{4} = -2y - 5 + \frac{9}{4}$$

$$\Rightarrow \quad \left(x - \frac{3}{2}\right)^2 = -2\left(y + \frac{11}{8}\right) \quad \text{...(i)}$$

Shifting the origin to the point $(3/2, -11/8)$ without rotating the coordinate axes and denoting the new coordinates with respect to the new axes by X and Y, we obtain

$$x = X + \frac{3}{2}, \quad y = Y - \frac{11}{8} \quad \text{...(ii)}$$

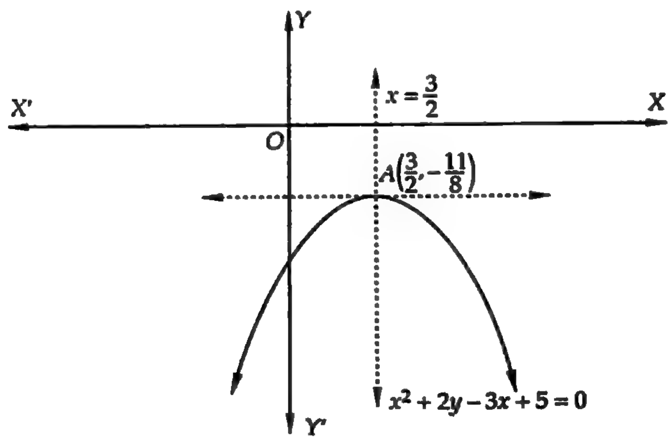


Fig. 25.21

Using these relations, equation (i) reduces to

$$X^2 = -2Y \quad \text{...(iii)}$$

This is of the form $X^2 = -4aY$. On comparing, we get : $4a = 2$ i.e. $a = 1/2$.

Vertex: The coordinates of the vertex with respect to the new axes are $(X = 0, Y = 0)$. So, the coordinates of the vertex with respect to the old axes are:

$(3/2, -11/8)$ [Putting $X = 0,\ Y = 0$ in (ii)]

Axis: The equation of the axis of the parabola with respect to the new axes is $X = 0$.

So, the equation of the axis with respect to the old axes is $x = \frac{3}{2}$ [Putting $X = 0$ in (ii)]

Focus: The coordinates of the focus with respect to the new axes are $(X = 0, Y = -a)$ i.e. $(X = 0, Y = -1/2)$. So, the coordinates of the focus with respect to the old axes are

$(3/2, -15/8)$ [Putting $X = 0, Y = -1/2$ in (ii)]

Directrix The equation of the directrix with respect to the new axes is $Y = a$ i.e. $Y = 1/2$. So, the equation of the directrix with respect to the old axes is $y = -\frac{7}{8}$ [Putting $Y = 1/2$ in (ii)]

Latus-rectum The length of the latus-rectum of the given parabola is equal to $4a = 2$.

EXAMPLE 5 *Find the vertex, focus, directrix, axis and latus-rectum of the parabola* $y^2 = 4x + 4y$.

SOLUTION The given equation is

$$y^2 = 4x + 4y$$

$$\Rightarrow \quad y^2 - 4y = 4x \Rightarrow y^2 - 4y + 4 = 4x + 4 \Rightarrow (y-2)^2 = 4(x+1) \quad \text{...(i)}$$

Shifting the origin to the point $(-1, 2)$ without rotating the axes and denoting the new coordinates with respect to these axes by X and Y, we have

$$x = X + (-1),\ y = Y + 2 \quad \text{...(ii)}$$

Using these relations equation (i), reduces to

$$Y^2 = 4X \quad \text{...(iii)}$$

This is of the form $Y^2 = 4aX$. On comparing, we get: $4a = 4 \Rightarrow a = 1$.

Vertex: The coordinates of the vertex with respect to new axes are $(X = 0, Y = 0)$. So, coordinates of the vertex with respect to old axes are $(-1, 2)$. [Putting $X = 0, Y = 0$ in (ii)]

Focus: The coordinates of the focus with respect to new axes are $(X = 1, Y = 0)$. So, coordinates of the focus with respect to old axes are $(0, 2)$. [Putting $X = 1, Y = 0$ in (ii)]

Directrix: Equation of the directrix of the parabola with respect to new axes is $X = -1$. So, equation of the directrix of the parabola with respect to old axes is

$x = -2$ [Putting $X = -1$ in (ii)]

Axis: Equation of the axis of the parabola with respect to new axes is $Y = 0$.

So, equation of axis with respect to old axes is $y = 2$. [Putting $Y = 0$ in (ii)]

Latus-rectum: The length of the latus-rectum = 4.

EXAMPLE 6 *Find the vertex, focus and directrix of the parabola* $4y^2 + 12x - 12y + 39 = 0$.

SOLUTION The given equation is

$$4y^2 + 12x - 12y + 39 = 0$$

$$\Rightarrow \quad 4y^2 - 12y = -12x - 39$$

$$\Rightarrow \quad 4(y^2 - 3y) = -12x - 39$$

$$\Rightarrow \quad 4\left(y^2 - 3y + \frac{9}{4}\right) = -12x - 39 + 9$$

$$\Rightarrow \quad 4\left(y-\frac{3}{2}\right)^2=-12\left(x+\frac{5}{2}\right)$$

$$\Rightarrow \quad \left(y-\frac{3}{2}\right)^2=-3\left(x+\frac{5}{2}\right) \quad \text{...(i)}$$

Shifting the origin to the point $(-5/2, 3/2)$ without rotating the axes and denoting the new coordinates with respect to these axes by X and Y, we obtain

$$x = X + \left(-\frac{5}{2}\right), \quad y = Y + \frac{3}{2} \quad \text{...(ii)}$$

Using these relations equation (i), reduces to

$$Y^2 = -3X \quad \text{...(iii)}$$

This is of the form $Y^2 = -4aX$. On comparing, we get: $a = \frac{3}{4}$.

Vertex: The coordinates of the vertex with respect to new axes are $(X = 0, Y = 0)$. So, coordinates of the vertex with respect to old axes are $(-5/2, 3/2)$. [Putting $X = 0$, $Y = 0$ in (ii)]

Focus: The coordinates of the focus of the parabola with respect to new axes are

$$\left(X=-\frac{3}{4}, Y=0\right).$$

So, coordinates of the focus with respect to old axes are

$$\left(-\frac{13}{4}, \frac{3}{2}\right) \qquad \left[\text{Putting } X=-\frac{3}{4}, Y=0 \text{ in (ii)}\right]$$

Directrix: The equation of the directrix of the parabola with respect to new axes is $X = \frac{3}{4}$. So, equation of the directrix of the parabola with respect to old axes is

$$x = -\frac{7}{4} \qquad [\text{Putting } X = 3/4 \text{ in (ii)}]$$

Type IV **ON FINDING THE EQUATION OF A PARABOLA WHEN ITS FOCUS AND VERTEX ARE GIVEN**

EXAMPLE 7 *Find the equation of the parabola with vertex (2, – 3) and focus (0, 5).*

SOLUTION In order to find the equation of a parabola, we need to know the coordinates of its focus and the equation of the directrix. We are given the coordinates of focus and vertex. So, we require the equation of the directrix. Let $Z(x_1, y_1)$ be the point of intersection of axis and the directrix. The vertex A is the mid-point of the line segment joining the focus S and the point Z of intersection of the axis and directrix. Therefor, $(2, -3)$ is the mid-point of the line segment joining $S(0, 5)$ and $Z(x_1, y_1)$.

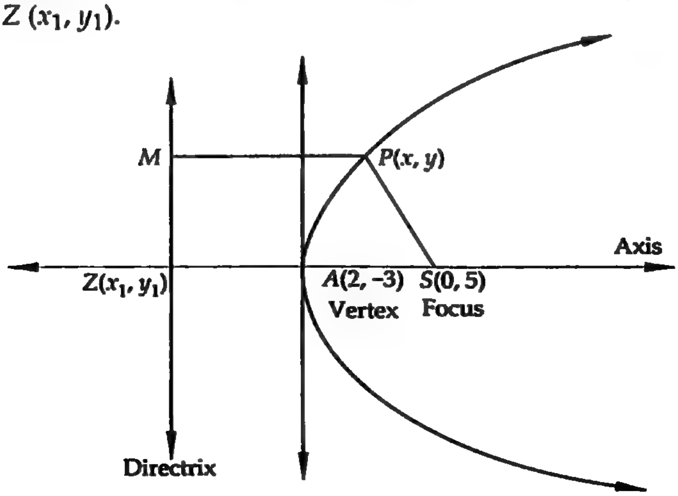


Fig. 25.22

$$\therefore \quad \frac{x_1+0}{2} = 2 \text{ and } \frac{y_1+5}{2} = -3 \Rightarrow x_1 = 4,\ y_1 = -11.$$

Thus, the directrix meets the axis at $Z(4, -11)$.

Let m_1 be the slope of AS. Then,

$$m_1 = \frac{5+3}{0-2} = -4$$

Let m_2 be the slope of the directrix. Since directrix is perpendicular to AS.

$$\therefore \quad m_1 m_2 = -1 \Rightarrow m_2 = -\frac{1}{m_1} = \frac{1}{4}.$$

Thus, the directrix passes through the point $Z(4, -11)$ and has slope $1/4$. Therefore, the equation of the directrix is

$$y + 11 = \frac{1}{4}(x-4) \text{ or, } x - 4y - 48 = 0.$$

Let $P(x, y)$ be any point on the required parabola, and let PM be the length of the perpendicular from P on the directrix. Then,

$$SP = PM \qquad \text{[By definition]}$$

$$\Rightarrow \quad SP^2 = PM^2$$

$$\Rightarrow \quad (x-0)^2 + (y-5)^2 = \left|\frac{x-4y-48}{\sqrt{1^2+(-4)^2}}\right|^2$$

$$\Rightarrow \quad 17x^2 + 17y^2 - 170y + 425 = x^2 + 16y^2 + 2304 - 8xy - 96x + 384y$$

$$\Rightarrow \quad 16x^2 + y^2 + 8xy + 96x - 554y - 1879 = 0$$ which is the required equation of the parabola.

EXAMPLE 8 *Find the equation of the parabola whose focus is $(1, -1)$ and whose vertex is $(2, 1)$. Also, find its axis and latus-rectum.*

SOLUTION In order to find the equation of a parabola, we require the coordinates of its focus and the equation of the directrix. Here, we are given the coordinates of the focus and vertex. So, we require the equation of the directrix. Let $Z(x_1, y_1)$ be the coordinates of the point of intersection of the axis and the directrix. Then, the vertex $A(2, 1)$ is the mid-point of the line segment joining $Z(x_1, y_1)$ and the focus $S(1, -1)$.

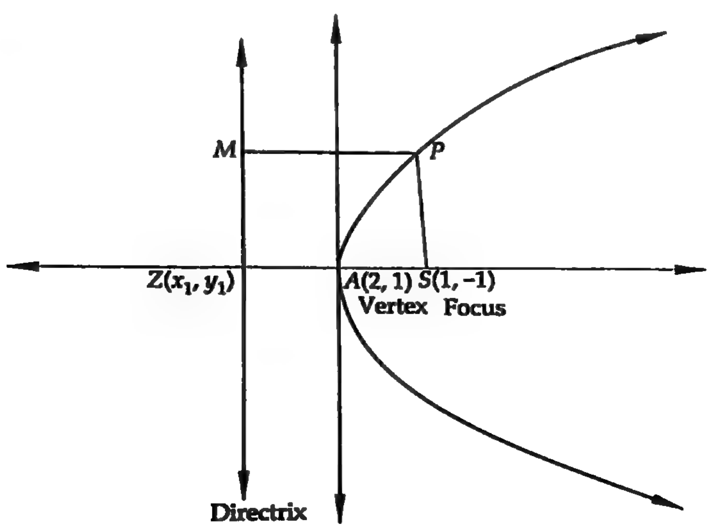


Fig. 25.23

$\therefore \quad \dfrac{x_1+1}{2} = 2$ and $\dfrac{y_1+(-1)}{2} = 1 \Rightarrow x_1 = 3,\ y_1 = 3.$

Thus, the directrix meets the axis at $Z(3, 3)$.

Let m_1 be the slope of the axis. Then,

$$m_1 = (\text{Slope of the line joining the focus } S \text{ and the vertex } A) = \frac{1+1}{2-1} = 2 \qquad \text{...(i)}$$

$\therefore \quad$ Slope of the directrix $= -\dfrac{1}{2}$ $\qquad$ [$\because$ Directrix is perpendicular to the axis]

Thus, the directrix passes through (3, 3) and has slope $-1/2$. So its equation is

$$y - 3 = -\frac{1}{2}(x-3) \text{ or, } x + 2y - 9 = 0$$

Let $P(x, y)$ be a point on the parabola. Then,

Distance of P from the focus = Distance of P from the directrix

$$\Rightarrow \quad \sqrt{(x-1)^2 + (y+1)^2} = \left| \frac{x + 2y - 9}{\sqrt{1^2 + 2^2}} \right|$$

$$\Rightarrow \quad (x-1)^2 + (y+1)^2 = \frac{(x+2y-9)^2}{5}$$

$$\Rightarrow \quad 5x^2 + 5y^2 - 10x + 10y + 10 = x^2 + 4y^2 + 81 + 4xy - 18x - 36y$$

$\Rightarrow \quad 4x^2 + y^2 - 4xy + 8x + 46y - 71 = 0$, which is the required equation of the parabola.

The axis passes through the focus $(1, -1)$, and its slope is $m_1 = 2$. Therefore, equation of the axis is

$$y + 1 = 2(x-1) \text{ or, } 2x - y - 3 = 0$$

Now,

Latus-rectum = 2 (Length of the perpendicular from the focus on the directrix)

$= 2$ [Length of the perpendicular from $(1, -1)$ on $x + 2y - 9 = 0$]

$$= 2\left| \frac{1-2-9}{\sqrt{1+4}} \right| = 2 \times \frac{10}{\sqrt{5}} = 4\sqrt{5}.$$

Type V **ON FINDING THE EQUATION OF A PARABOLA WHEN ITS VERTEX AND DIRECTRIX ARE GIVEN**

EXAMPLE 9 *Find the equation of the parabola whose vertex is at (2, 1) and the directrix is $x = y - 1$.*

SOLUTION In order to find the equation of a parabola, its focus and directrix are required. Here, we are given it directrix and vertex. So, we first find its focus which lies on the axis. The axis of the parabola is a line perpendicular to the directrix and passing through the vertex. The equation of a line perpendicular to $x - y + 1 = 0$ is $x + y + \lambda = 0$. This will pass through (2, 1), if

$$2 + 1 + \lambda = 0 \Rightarrow \lambda = -3.$$

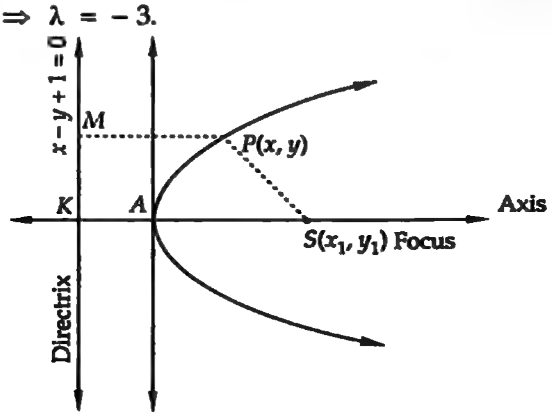


Fig. 25.24

Putting $\lambda = -3$ in $x + y + \lambda = 0$, we obtain

$$x + y - 3 = 0 \qquad ...(i)$$

as the axis of the parabola.

The equation of the directrix is

$$x - y + 1 = 0 \qquad ...(ii)$$

Solving (i) and (ii), we get $x = 1$, $y = 2$. So, the coordinates of K are $(1, 2)$.

Let (x_1, y_1) be the coordinates of the focus S. Then, A is the mid-point of KS.

$$\therefore \quad \frac{x_1 + 1}{2} = 2 \text{ and } \frac{y_1 + 2}{2} = 1 \Rightarrow x_1 = 3 \text{ and } y_1 = 0$$

So, the coordinates of the focus S are $(3, 0)$.

Let $P(x, y)$ be a point on the parabola. Then,

$$PS = PM$$

$$\Rightarrow \quad PS^2 = PM^2$$

$$\Rightarrow \quad (x-3)^2 + (y-0)^2 = \left| \frac{x - y + 1}{\sqrt{1^2 + (-1)^2}} \right|^2$$

$$\Rightarrow \quad 2(x^2 + y^2 - 6x + 9) = x^2 + y^2 + 1 - 2xy + 2x - 2y$$

$$\Rightarrow \quad x^2 + y^2 - 14x + 2y + 2xy + 17 = 0,$$ which is the required equation of the parabola.

EXAMPLE 10 *Find the equation of the parabola whose focus is (1, 1) and tangent at the vertex is* $x + y = 1$.

SOLUTION Here, we are given the coordinates of the focus and the equation of the tangent at the vertex. To find the equation of a parabola, we require the coordinates of its focus and the equation of the directrix. So, we first find the equation of the directrix of the parabola from the given components. Let S be the focus and A be the vertex of the parabola. Let K be the point of intersection of the axis and directrix. Since axis is a line passing through S (1, 1) and perpendicular to $x + y = 1$. So, let the equation of the axis be $x - y + \lambda = 0$.

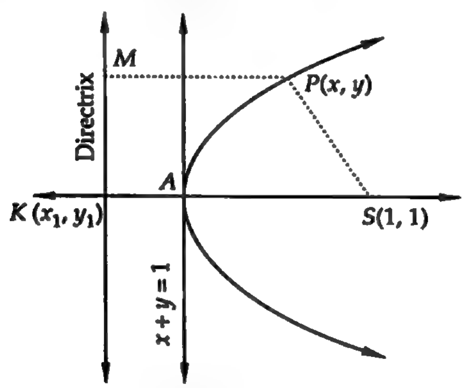


Fig. 25.25

This will pass through S (1, 1), if

$$1 - 1 + \lambda = 0 \Rightarrow \lambda = 0$$

So the equation of the axis is

$$x - y = 0 \qquad ...(i)$$

The vertex A is the point of intersection of $x - y = 0$ and $x + y = 1$. Solving these two equations, we get $x = 1/2$ and $y = 1/2$.

So, the coordinates of the vertex A are $(1/2, 1/2)$.

Let (x_1, y_1) be the coordinates of K. As A is the mid-point of SK.

$$\therefore \quad \frac{x_1+1}{2} = \frac{1}{2}, \frac{y_1+1}{2} = \frac{1}{2} \Rightarrow x_1 = 0, y_1 = 0$$

So, the coordinates of K are (0, 0). Since directrix is a line passing through K (0, 0) and parallel to $x + y = 1$. Therefore, equation of the directrix is

$$y - 0 = -1(x - 0) \text{ or, } x + y = 0. \quad \text{...(ii)}$$

Let $P(x, y)$ be any point on the parabola. Then,

Distance of P from the focus S = [Distance of P from the directrix $x + y = 0$]

$$\Rightarrow \quad \sqrt{(x-1)^2 + (y-1)^2} = \left|\frac{x+y}{\sqrt{1^2+1^2}}\right|$$

$$\Rightarrow \quad 2x^2 + 2y^2 - 4x - 4y + 4 = x^2 + y^2 + 2xy$$

$$\Rightarrow \quad x^2 + y^2 - 2xy - 4x - 4y + 4 = 0,$$ which is the required equation of the parabola.

EXAMPLE 11 *Find the equation of the parabola whose latus-rectum is 4 units, axis is the line* $3x + 4y - 4 = 0$ *and the tangent at the vertex is the line* $4x - 3y + 7 = 0$.

SOLUTION Let $P(x, y)$ be any point on the parabola and let PM and PN be perpendiculars from P on the axis and tangent at the vertex respectively. Then,

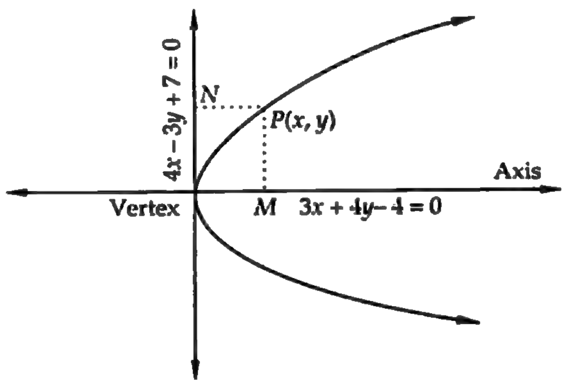


Fig. 25.26

$$PM^2 = (\text{Latusrectum})(PN)$$

$$\Rightarrow \quad \left|\frac{3x+4y-4}{\sqrt{3^2+4^2}}\right|^2 = 4\left|\frac{4x-3y+7}{\sqrt{4^2+(-3)^2}}\right|$$

$$\Rightarrow \quad (3x+4y-4)^2 = 20(4x-3y+7),$$ which is the required equation of the parabola.

LEVEL-2

Type VI **MISCELLANEOUS PROBLEMS ON PARABOLA**

EXAMPLE 12 *A double ordinate of the parabola* $y^2 = 4ax$ *is of length 8a. Prove that the lines from the vertex to its ends are at right angles.*

SOLUTION Let PQ be the double ordinate of length $8a$ of the parabola $y^2 = 4ax$. Then,

$PR = QR = 4a$. Let $AR = x_1$. Then, the coordinates of P and Q are $(x_1, 4a)$ and $(x_1, -4a)$ respectively. Since P lies on $y^2 = 4ax$.

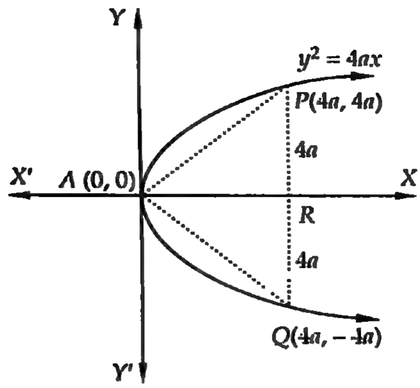


Fig. 25.27

$\therefore \quad (4a)^2 = 4ax_1 \Rightarrow x_1 = 4a.$

So, coordinates of P and Q are $(4a, 4a)$ and $(4a, -4a)$ respectively. Also, the coordinates of the vertex A are $(0, 0)$.

$\therefore \quad m_1 = \text{Slope of } AP = \dfrac{4a-0}{4a-0} = 1$, and, $m_2 = \text{Slope of } AQ = \dfrac{-4a-0}{4a-0} = -1$

Clearly, $m_1 m_2 = -1$. Hence, AP is perpendicular to AQ.

EXAMPLE 13 *The focal distance of a point on the parabola $y^2 = 12x$ is 4. Find the abscissa of this point.*

SOLUTION The given parabola is of the form $y^2 = 4ax$. On comparing, we obtain $4a = 12$ i.e. $a = 3$.

We know that the focal distance of any point (x, y) on $y^2 = 4\,ax$ is $x + a$.

Let the given point on the parabola $y^2 = 12x$ be (x, y). Then, its focal distance is $x + 3$.

$\therefore \quad x + 3 = 4 \Rightarrow x = 1.$

Hence, the abscissa of the given point is 1.

EXAMPLE 14 *Prove that the equation to the parabola whose vertex and focus are on the x-axis at a distance a and a' from the origin respectively is $y^2 = 4(a' - a)(x - a)$.*

SOLUTION Let O, A and S be respectively the origin, vertex and focus of the parabola. Then, $OA = a$, $OS = a'$. Therefore, the coordinates of S are $(a', 0)$. Let KK' be the directrix of the required parabola. Suppose SA produced meets the directrix at Z. Let the coordinates of Z be (x_1, y_1). Then,

$$\frac{x_1 + a'}{2} = a \text{ and } \frac{y_1 + 0}{2} = 0 \qquad [\because A \text{ is the mid-point of } SZ]$$

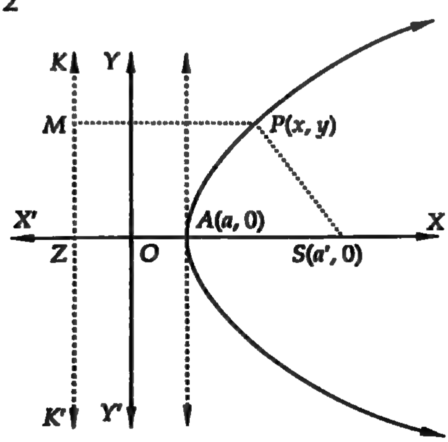


Fig. 25.28

$\Rightarrow \quad x_1 = 2a - a'$ and $y_1 = 0$

So, the equation of the directrix KK' is $x = x_1$ i.e. $x = 2a - a'$.

Let $P(x, y)$ be any point on the parabola. Then,

$SP = PM$ [By def.]

$$\Rightarrow \sqrt{(x-a')^2 + (y-0)^2} = \left|\frac{x - 2a + a'}{\sqrt{1+0}}\right|$$

$$\Rightarrow (x-a')^2 + y^2 = (x - 2a + a')^2$$

$$\Rightarrow (x-a')^2 + y^2 = [(x-a') - 2(a-a')]^2$$

$$\Rightarrow (x-a')^2 + y^2 = (x-a')^2 + 4(a-a')^2 - 4(x-a')(a-a')$$

$$\Rightarrow y^2 = 4(a-a')\{(a-a') - (x-a')\}$$

$$\Rightarrow y^2 = 4(a'-a)(x-a).$$

ALITER The parabola has its vertex at $(a, 0)$ and the length of its Latus-rectum = 4 (Distance between foucs and vertex) = $4(a'-a)$. The axis is along OX.

So, its equation is $(y-0)^2 = 4(a'-a)(x-a)$ or, $y^2 = 4(a'-a)(x-a)$

EXAMPLE 15 *Find the locus of the middle points of all chords of the parabola $y^2 = 4ax$ which are drawn through the vertex.*

SOLUTION Let OA be a chord, drawn through the vertex and $P(h, k)$ be its mid-point. Let the coordinates of A be (x_1, y_1). Then,

$$\frac{x_1 + 0}{2} = h, \ \frac{y_1 + 0}{2} = k \Rightarrow x_1 = 2h \ \text{ and } \ y_1 = 2k$$

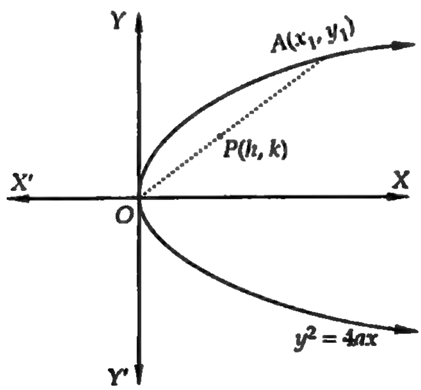


Fig. 25.29

So, the coordinates of A are $(2h, 2k)$. Since A lies on $y^2 = 4ax$.

$$\therefore \quad (2k)^2 = 4a(2h) \Rightarrow k^2 = 2ah$$

Hence, the locus of (h, k) is $y^2 = 2ax$.

EXAMPLE 16 *An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ whose vertex is at the vertex of the parabola. Find the length of its side.* **[NCERT EXEMPLAR]**

SOLUTION Let $AB = l$. Then,

$$AM = l\cos 30^\circ = \frac{l\sqrt{3}}{2} \text{ and, } BM = l\sin 30^\circ = \frac{l}{2}.$$

So, the coordinates of B are $\left(\frac{\sqrt{3}l}{2}, \frac{l}{2}\right)$. Since, B lies on $y^2 = 4ax$. So, coordinates of B satisfy $y^2 = 4ax$.

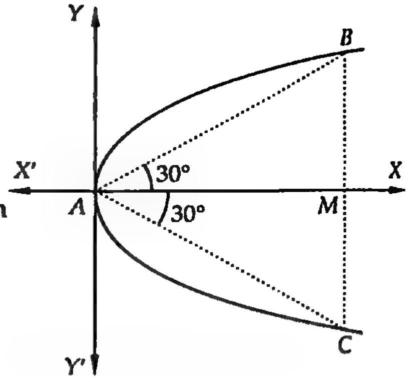


Fig. 25.30

$$\therefore \quad \frac{l^2}{4} = 4a\left(\frac{l\sqrt{3}}{2}\right) \Rightarrow l = 8a\sqrt{3}$$

EXAMPLE 17 *If y_1, y_2, y_3 be the ordinates of a vertices of the triangle inscribed in a parabola $y^2 = 4ax$, then show that the area of the triangle is $\frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$.*

SOLUTION Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC. Since (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie on the parabola. Therefore,

$$y_1^2 = 4ax_1,\ y_2^2 = 4ax_2 \text{ and } y_3^2 = 4ax_3 \Rightarrow x_1 = \frac{y_1^2}{4a},\ x_2 = \frac{y_2^2}{4a} \text{ and } x_3 = \frac{y_3^2}{4a}$$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$
$$= \frac{1}{2}\left[\frac{y_1^2}{4a}(y_2 - y_3) + \frac{y_2^2}{4a}(y_3 - y_1) + \frac{y_3^2}{4a}(y_1 - y_2)\right]$$
$$= \frac{1}{8a}[y_1^2(y_2 - y_3) + (y_2^2 y_3 - y_2 y_3^2) - y_1(y_2^2 - y_3^2)]$$
$$= \frac{1}{8a}[y_1^2(y_2 - y_3) + y_2 y_3(y_2 - y_3) - y_1(y_2^2 - y_3^2)]$$
$$= \frac{1}{8a}(y_2 - y_3)[y_1^2 + y_2 y_3 - y_1(y_2 + y_3)]$$
$$= \frac{1}{8a}(y_2 - y_3)[(y_1^2 - y_1 y_2) + (y_2 y_3 - y_1 y_3)]$$
$$= \frac{1}{8a}(y_2 - y_3)[y_1(y_1 - y_2) - y_3(y_1 - y_2)]$$
$$= \frac{1}{8a}(y_2 - y_3)(y_1 - y_2)(y_1 - y_3)$$
$$= -\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$$

$$\text{Hence, Area of } \Delta ABC = \frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$$

EXAMPLE 18 *PQ is a double ordinate of a parabola $y^2 = 4ax$. Find the locus of its points of trisection.*

SOLUTION Let R and S be the points of trisection of the double ordinates PQ. Let (h, k) be the coordinates of R. Then, $L = h$ and $RL = k$.

$$\therefore \quad RS = RL + LS = k + k = 2k.$$
$$\Rightarrow \quad PR = RS = SQ = 2k$$
$$\Rightarrow \quad LP = LR + RP = k + 2k = 3k$$

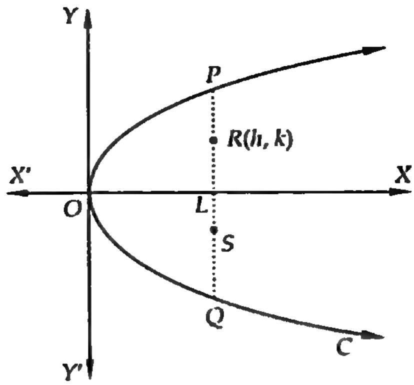


Fig. 25.31

Thus, the coordinates of P are $(h, 3k)$. Since $(h, 3k)$ lies on $y^2 = 4ax$.

$\therefore \quad 9k^2 = 4ah$

Hence, the locus of (h, k) is $9y^2 = 4ax$.

EXAMPLE 19 *If the line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$, prove that $ln = am^2$*

[NCERT EXEMPLAR]

SOLUTION The x-coordinates of the points of intersection of the line $lx + my + n = 0$ or $y = -\left(\frac{lx+n}{m}\right)$ and the parabola $y^2 = 4ax$ are roots of the equation

$$\left\{-\left(\frac{lx+n}{m}\right)\right\}^2 = 4ax \qquad \left[\text{On eliminating } y \text{ between } y = -\left(\frac{lx+n}{m}\right) \text{ and } y^2 = 4ax\right]$$

or, $\quad l^2x^2 + 2x(ln - 2am^2) + n^2 = 0$

If the line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$, then this equation has equal roots.

$$\therefore \quad 4\left(ln - 2am^2\right)^2 - 4l^2n^2 = 0 \qquad \text{[Putting discriminate equal to zero]}$$

$$\Rightarrow \quad -4a\,lm^2n + 4a^2m^4 = 0 \Rightarrow ln = am^2$$

25.9 SOME APPLICATIONS OF PARABOLA

Parabola has many applications in our day-to-day life. For example, if an object (projectile) is thrown in space, then the path of the projectle is a parabola. If we know the equation of the path of a projectile by using various properties of parabola studied in earlier sections, we can obtain many important results like greatest height attained by the projectile, its horizontal range reached etc.

Parabolic reflectors have the property that the light rays or sound waves coming parallel to its axis converge at the focus and then it reflects them parallel to the axis. Due to this property, parabolic reflectors are used in cars, automobiles, loudspeakers, solar cookers, telescopes etc.

If the roadway of a suspension bridge is loaded uniformly per horizontal metre, the suspension cable hangs in the form of arcs which closely approximate to parabolic arcs. Therefore, parabolic arcs are used in suspension cable bridge construction.

In this section, we shall discuss some examples on these applications

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If a parabolic reflector is 20 cm in diameter and 5 cm deep, find its focus.* [NCERT]

SOLUTION Let LAM be the parabolic reflector such that LM is its diameter and AN is its depth. It is given that $AN = 5$ cm and $LM = 20$ cm.

$\therefore \quad LN = 10$ cm

Taking A as the origin, AX along x-axis and a line through A perpendicular to AX as y-axis, let the equation of the reflector be

$$y^2 = 4ax \quad ...(i)$$

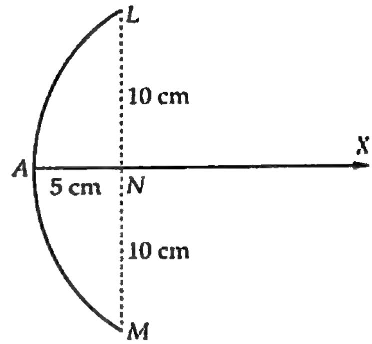


Fig. 25.32

The point L has coordinates (5, 10) and lies on (i). Therefore,

$$10^2 = 4a \times 5 \Rightarrow a = 5$$

So, the equation of the reflector is $y^2 = 20x$.

Its focus is at (5, 0) i.e. at point N.

Hence, the focus is at the mid-point of the given diameter.

EXAMPLE 2 *The focus of a parabolic mirror as shown in Fig. 25.33 is at a distance of 6 cm from its vertex. If the mirror is 20 cm deep, find the distance LM.* [NCERT]

SOLUTION Let the axis of the mirror be along the positive direction of x-axis and the vertex A be the origin.

Since the focus is at a distance of 6 cm from the vertex. Then, the coordinates of the focus are (6, 0). Therefore, the equation of the parabolic section is

$$y^2 = 24x \quad [\text{Putting } a = 6 \text{ in } y^2 = 4ax]$$

Since L (20, LN) lies on this parabola. Therefore,

$$LN^2 = 24 \times 20$$

$$\Rightarrow \quad LN = 4\sqrt{30}$$

$$\therefore \quad LM = 2LN = 8\sqrt{30} \text{ cm.}$$

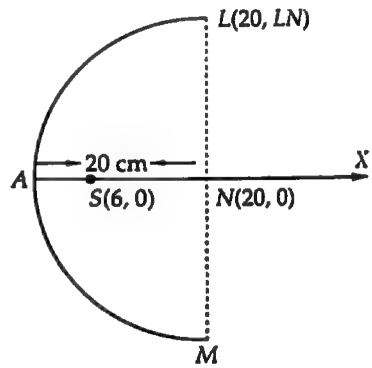


Fig. 25.33

EXAMPLE 3 *An arc is in the form of a parabola with its axis vertical. The arc is 10 m high and 5 m wide at the base. How wide is it 2m from the vertex of the parabola.* [NCERT]

SOLUTION Let the vertex of the parabola be at the origin and axis be along OY. Then, the equation of the parabola is

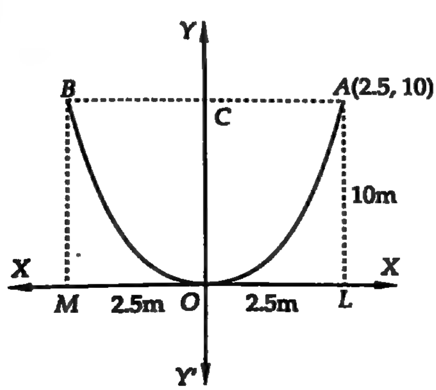


Fig. 25.34

$$x^2 = 4ay \quad \text{...(i)}$$

The coordinates of end A of the arc are (2.5, 10) and it lies on the parabola (i).

$\therefore \quad (2.5)^2 = 4a \times 10$

$\Rightarrow \quad a = \dfrac{6.25}{40} = \dfrac{625}{4000} = \dfrac{5}{32}$

Putting the value of a in (i), we obtain that the equation of the parabolic arc is $x^2 = \dfrac{5}{8}y$.

When $y = 2$, we obtain

$$x^2 = \frac{5}{8} \times 2 \Rightarrow x = \frac{\sqrt{5}}{2} \text{ m.}$$

Hence, the width of the arc at a height of 2 m from the vertex is $2 \times \dfrac{\sqrt{5}}{2}$ m $= \sqrt{5}$ m.

LEVEL-2

EXAMPLE 4 *The towers of a bridge, hung in the form of a parabola, have their tops 30 m above the roadway and are 200 metres apart. If the cable is 5 m above the roadway at the centre of the bridge, find the length of the vertical supporting cable 30 metres from the centre.* **[NCERT]**

SOLUTION Let CAB be the bridge and $X'OX$ be the roadway. Let A be the centre of the bridge. Taking $X'OX$ as x-axis and y-axis along OA, we find that the coordinates of A are (0, 5). Clearly, the bridge is in the shape of a parabola having its vertex at A (0, 5). Let its equation be

$$x^2 = 4a(y-5) \quad \text{...(i)}$$

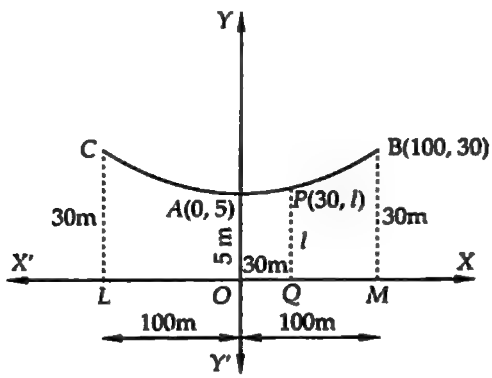


Fig. 25.35

It passes through B (100, 30).

$\therefore \quad (100)^2 = 4a(30-5) \Rightarrow a = 100.$

Putting the value of a in (i), we get

$$x^2 = 400(y-5) \quad \text{...(ii)}$$

Let l metres be the length of the vertical supporting cable 30 metres from the centre. Then, P (30, l) lies on (ii).

$\therefore \quad 900 = 400(l-5) \Rightarrow l = \dfrac{9}{4} + 5 = \dfrac{29}{4}$ m.

Hence, the length of the vertical supporting cable 30 metres from the centre of the bridge is $\dfrac{29}{4}$ m.

EXAMPLE 5 *A beam is supported at its ends by supports which are 12 metres apart. Since the load is connected at its centre, there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?* **[NCERT]**

SOLUTION Let O be the centre of the beam in deflected position. Taking O as the origin OX as x-axis and OY as y-axis. The equation representing the parabolic shape of the beam is $x^2 = 4ay$. This passes through $Q\left(6, \frac{3}{100}\right)$.

$$\therefore \quad 36 = 4a \times \frac{3}{100} \Rightarrow a = 300 \text{ m}$$

So, the equation of the curve representing deflected beam is $x^2 = 1200y$.

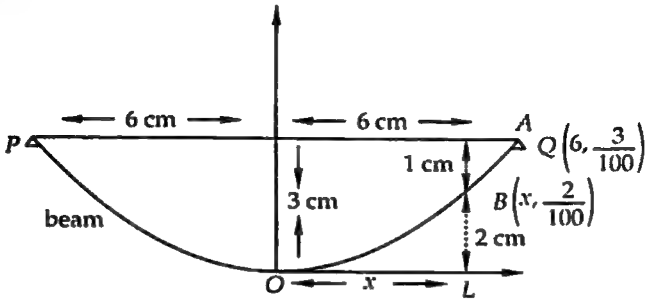


Fig. 25.36

Let the deflection of the beam be 1 cm $= \frac{1}{100}$ m at point B. Then, the coordinates of B are $\left(x, \frac{2}{100}\right)$, where $OL = x$. Since B lies on the parabola $x^2 = 1200\,y$.

$$\therefore \quad x^2 = 1200 \times \frac{2}{100} \Rightarrow x = \sqrt{24} = 2\sqrt{6} \text{ metres.}$$

Hence, the deflection of the beam is 1 cm at a distance of $2\sqrt{6}$ metres from the centre O.

EXERCISE 25.1

LEVEL-1

1. Find the equation of the parabola whose:
   (i) focus is $(3, 0)$ and the directrix is $3x + 4y = 1$
   (ii) focus is $(1, 1)$ and the directrix is $x + y + 1 = 0$
   (iii) focus is $(0, 0)$ and the directrix $2x - y - 1 = 0$
   (iv) focus is $(2, 3)$ and the directrix $x - 4y + 3 = 0$. [NCERT EXEMPLAR]
2. Find the equation of the parabola whose focus is the point $(2, 3)$ and directrix is the line $x - 4y + 3 = 0$. Also, find the length of its latus-rectum.
3. Find the equation of the parabola, if
   (i) the focus is at $(-6, -6)$ and the vertex is at $(-2, 2)$
   (ii) the focus is at $(0, -3)$ and the vertex is at $(0, 0)$
   (iii) the focus is at $(0, -3)$ and the vertex is at $(-1, -3)$
   (iv) the focus is at $(a, 0)$ and the vertex is at $(a', 0)$
   (v) the focus is at $(0, 0)$ and vertex is at the intersection of the lines $x + y = 1$ and $x - y = 3$.
4. Find the vertex, focus, axis, directrix and latus-rectum of the following parabolas
   (i) $y^2 = 8x$ (ii) $4x^2 + y = 0$ (iii) $y^2 - 4y - 3x + 1 = 0$
   (iv) $y^2 - 4y + 4x = 0$ (v) $y^2 + 4x + 4y - 3 = 0$ (vi) $y^2 = 8x + 8y$
   (vii) $4(y-1)^2 = -7(x-3)$ (viii) $y^2 = 5x - 4y - 9$ (ix) $x^2 + y = 6x - 14$
5. For the parabola $y^2 = 4px$ find the extremities of a double ordinate of length $8p$. Prove that the lines from the vertex to its extremities are at right angles.

6. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus-rectum. [NCERT EXEMPLAR]
7. Find the coordinates of the point of intersection of the axis and the directrix of the parabola whose focus is (3, 3) and directrix is $3x - 4y = 2$. Find also the length of the latus-rectum.
8. At what point of the parabola $x^2 = 9y$ is the abscissa three times that of ordinate ?
9. Find the equation of a parabola with vertex at the origin, the axis along x-axis and passing through (2, 3).
10. Find the equation of a parabola with vertex at the origin and the directrix, $y = 2$.
11. Find the equation of the parabola whose focus is (5, 2) and having vertex at (3, 2).
12. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest wire being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle. [NCERT]

LEVEL-2

13. Find the equations of the lines joining the vertex of the parabola $y^2 = 6x$ to the point on it which have abscissa 24. [NCERT EXEMPLAR]
14. Find the coordinates of points on the parabola $y^2 = 8x$ whose focal distance is 4. [NCERT EXEMPLAR]
15. Find the length of the line segment joining the vertex of the parabola $y^2 = 4ax$ and a point on the parabola where the line-segment makes an angle θ to the x-axis. [NCERT EXEMPLAR]
16. If the points (0, 4) and (0, 2) are respectively the vertex and focus of a parabola, then find the equation of the parabola. [NCERT EXEMPLAR]
17. If the line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$, then find the value of m. [NCERT EXEMPLAR]

ANSWERS

1. (i) $16x^2 + 9y^2 - 24xy - 144x + 8y + 224 = 0$
   (ii) $x^2 + y^2 - 2xy - 6x - 6y + 3 = 0$
   (iii) $x^2 + 4y^2 + 4xy + 4x - 2y - 1 = 0$
   (iv) $16x^2 + y^2 + 8xy - 74x - 78y + 212 = 0$.
2. $16x^2 + y^2 + 8xy - 74x - 78y + 212 = 0$, L.R. $= \frac{14}{\sqrt{17}}$.
3. (i) $(2x - y)^2 + 4(26x + 37y - 31) = 0$ (ii) $x^2 = -12y$
   (iii) $y^2 + 6y - 4x + 5 = 0$ (iv) $y^2 = -4(a' - a)(x - a')$
   (v) $(x + 2y)^2 + 40x - 20y - 100 = 0$.
4.

	vertex	*focus*	*axis*	*directrix*	*L.R.*
(i)	(0, 0)	(2, 0)	$y = 0$	$x = -2$	8
(ii)	(0, 0)	(0, −1/16)	x	$y = 1/16$	1/4
(iii)	(−1, 2)	(−1/4, 2)	$y = 2$	$x = -\frac{7}{4}$	3

(iv)	$(1, 2)$	$(0, 2)$	$y = 2$	$x = 2$	4
(v)	$(7/4, -2)$	$(3/4, -2)$	$y + 2 = 0$	$4x = 11$	4
(vi)	$(-2, 4)$;	$(0, 4)$	$y = 4$	$x + 4 = 0$	8
(vii)	$(3, 1)$	$(41/16, 1)$	$y = 1$	$x = 55/16$	7/4
(viii)	$(1, -2)$	$(9/4, -2)$	$y = -2$	$4x + 1 = 0.$	5
(ix)	$(3, -5)$	$(3, -21/4)$	$x = 3$	$4y + 19 = 0$	1

5. $(4p, 4\theta)$, $(4p, -4\theta)$ 6. 18 sq 7. $\left(\frac{18}{5}, \frac{11}{5}\right)$

8. $(3, 1)$ 9. $2y^2 = 9x$ 9. $2y^2 = 9x$ 10. $x^2 = -8y$

11. $y^2 - 4y - 8x + 28 = 0$ 12. 9.11 m (approx.) 13. $x \pm 2y = 0$

14. $(2, 4), (2, -4)$ 15. $4a \operatorname{cosec}\theta \cdot \cot\theta$ 16. $x^2 + 8y = 32$ 17. $m = 1$

HINTS TO SELECTED PROBLEMS

6. Required Area $= \frac{1}{2}(LL' \times OS) = \frac{1}{2} \times 12 \times 3 = 18$ sq. units

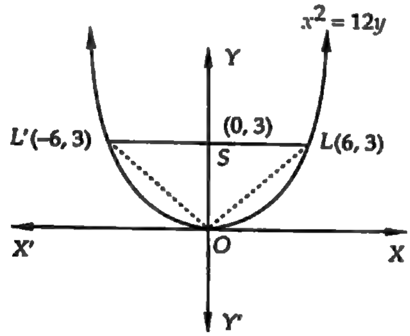


Fig. 25.37

12. Let $X'OX$ be the bridge and PAQ be the suspension cable. The suspension cable forms a parabola with vertex at $(0, 6)$. So, let the equation of the parabola formed by suspension cable be

$$(x - 0)^2 = 4a(y - 6) \qquad \text{...(i)}$$

It passes through $P(-50, 30)$ and $Q(50, 30)$.

$\therefore \quad 2500 = 4a(30 - 6) \Rightarrow 4a = \frac{2500}{24}$

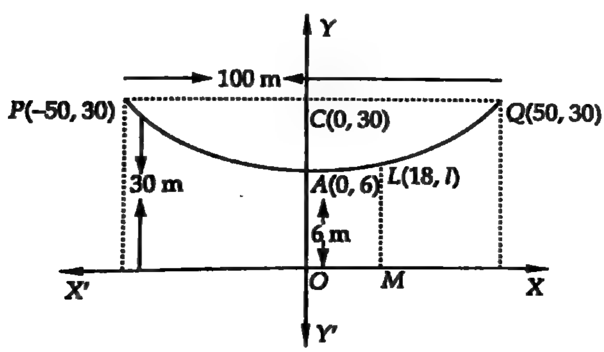


Fig. 25.38

Substituting this value of $4a$ in (i), we get

$$x^2 = \frac{2500}{24}(y-6) \qquad \text{...(ii)}$$

Let LM be the supporting wire attached at M which is 18 m from the middle O of the bridge. Let the coordinates of L be $(18, l)$. It lies on parabola (ii). Therefore,

$$18^2 = \frac{2500}{24}(l-6) \Rightarrow l-6 = 3.11 \Rightarrow l = 9.11 \text{ m}.$$

13. The parabola $y^2 = 6x$ is symmetric about x-axis. So, for a given abscissa there will be two points on the parabola as shown in Fig. 25.39. Let P and Q be two points on the parabola whose abscissa is 24. Let their coordinates be $(24, y_1)$ and $(24, -y_1)$ respectively.

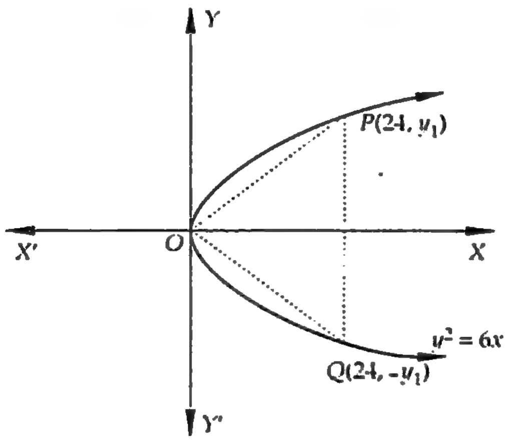


Fig. 25.39

Since $P(24, y_1)$ lies on $y^2 = 6x$.

$\therefore \quad y_1^2 = 6 \times 24 \Rightarrow y_1 = 12$

So, the coordinates of P and Q are $(24, 12)$ and $(24, -12)$ respectively.

The equations OP and OQ are

$$y - 0 = \frac{12-0}{24-0}(x-0) \text{ and } y - 0 = \frac{-12-0}{24-0}(x-0)$$

or, $\quad x = 2y$ and $x = -2y$ respectively.

14. Comparing $y^2 = 8x$ with $y^2 = 4ax$, we obtain $4a = 8$ or $a = 2$. The focal distance of any point $P(x, y)$ on $y^2 = 4ax$ is $a + x$. Therefore,

$$a + x = 4 \Rightarrow 2 + x = 4 \Rightarrow x = 2$$

Putting $x = 2$ in $y^2 = 8x$, we obtain $y = \pm 4$.

Hence, the coordinates of required points are $(2, 4)$ and $(2, -4)$.

15. Let $P(x, y)$ be a point on the parabola $y^2 = 4ax$ such that the segment OP makes an angle θ with x-axis. Then,

$$\tan\theta = \text{Slope of } OP \Rightarrow \tan\theta = \frac{y-0}{x-0} \Rightarrow y = x\tan\theta.$$

Since $P(x, y)$ lies on $y^2 = 4ax$. Therefore,

$$(x\tan\theta)^2 = 4ax \Rightarrow x = 4a\cot^2\theta \Rightarrow y = 4a\cot\theta$$

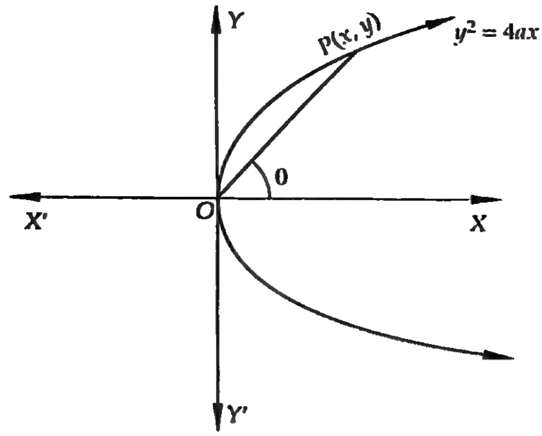


Fig. 25.40

Thus, the coordinates of P are $(4a\cot^2\theta, 4a\cot\theta)$.

Hence, $OP = \sqrt{x^2 + y^2} = \sqrt{16a^2\cot^4\theta + 16a^2\cot^2\theta} = 4a \operatorname{cosec}\theta \cot\theta$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the axis of symmetry of the parabola $y^2 = x$.
2. Write the distance between the vertex and focus of the parabola $y^2 + 6y + 2x + 5 = 0$.
3. Write the equation of the directrix of the parabola $x^2 - 4x - 8y + 12 = 0$.
4. Write the equation of the parabola with focus (0, 0) and directrix $x + y - 4 = 0$.
5. Write the length of the chord of the parabola $y^2 = 4ax$ which passes through the vertex and is inclined to the axis at $\frac{\pi}{4}$.
6. If b and c are lengths of the segments of any focal chord of the parabola $y^2 = 4ax$, then write the length of its latus-rectum.
7. PSQ is a focal chord of the parabola $y^2 = 8x$. If $SP = 6$, then write SQ.
8. Write the coordinates of the vertex of the parabola whose focus is at $(-2, 1)$ and directrix is the line $x + y - 3 = 0$.
9. If the coordinates of the vertex and focus of a parabola are $(-1, 1)$ and $(2, 3)$ respectively, then write the equation of its directrix.
10. If the parabola $y^2 = 4ax$ passes through the point (3, 2), then find the length of its latusrectum.
11. Write the equation of the parabola whose vertex is at $(-3, 0)$ and the directrix is $x + 5 = 0$.

ANSWERS

1. x-axis 2. 1/2 3. $y = -1$ 4. $x^2 + y^2 - 2xy + 8x + 8y - 16 = 0$ 5. $4\sqrt{2}a$
6. $\frac{4bc}{b+c}$ 7. 3 8. $(-1, 2)$ 9. $3x + 2y + 14 = 0$ 10. 4/3 11. $y^2 = 8(x+3)$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The coordinates of the focus of the parabola $y^2 - x - 2y + 2 = 0$ are

   (a) $(5/4, 1)$ (b) $(1/4, 0)$ (c) $(1, 1)$ (d) none of these

2. The vertex of the parabola $(y + a)^2 = 8a(x - a)$ is

   (a) $(-a, -a)$ (b) $(a, -a)$ (c) $(-a, a)$ (d) none of these

3. If the focus of a parabola is $(-2, 1)$ and the directrix has the equation $x + y = 3$, then its vertex is

   (a) $(0, 3)$ (b) $(-1, 1/2)$ (c) $(-1, 2)$ (d) $(2, -1)$

4. The equation of the parabola whose vertex is $(a, 0)$ and the directrix has the equation $x + y = 3a$, is

   (a) $x^2 + y^2 + 2xy + 6ax + 10ay + 7a^2 = 0$ (b) $x^2 - 2xy + y^2 + 6ax + 10ay - 7a^2 = 0$

   (c) $x^2 - 2xy + y^2 - 6ax + 10ay - 7a^2 = 0$ (d) none of these

5. The parametric equations of a parabola are $x = t^2 + 1$, $y = 2t + 1$. The cartesian equation of its directrix is

   (a) $x = 0$ (b) $x + 1 = 0$ (c) $y = 0$ (d) none of these

6. If the coordinates of the vertex and the focus of a parabola are $(-1, 1)$ and $(2, 3)$ respectively, then the equation of its directrix is

   (a) $3x + 2y + 14 = 0$ (b) $3x + 2y - 25 = 0$

   (c) $2x - 3y + 10 = 0$ (d) none of these.

7. The locus of the points of trisection of the double ordinates of a parabola is a

   (a) pair of lines (b) circle (c) parabola (d) straight line

8. The equation of the directrix of the parabola whose vertex and focus are $(1, 4)$ and $(2, 6)$ respectively is

   (a) $x + 2y = 4$ (b) $x - y = 3$ (c) $2x + y = 5$ (d) $x + 3y = 8$

9. If V and S are respectively the vertex and focus of the parabola $y^2 + 6y + 2x + 5 = 0$, then $SV =$

   (a) 2 (b) 1/2 (c) 1 (d) none of these

10. The directrix of the parabola $x^2 - 4x - 8y + 12 = 0$ is

    (a) $y = 0$ (b) $x = 1$ (c) $y = -1$ (d) $x = -1$

11. The equation of the parabola with focus $(0, 0)$ and directrix $x + y = 4$ is

    (a) $x^2 + y^2 - 2xy + 8x + 8y - 16 = 0$ (b) $x^2 + y^2 - 2xy + 8x + 8y = 0$

    (c) $x^2 + y^2 + 8x + 8y - 16 = 0$ (d) $x^2 - y^2 + 8x + 8y - 16 = 0$

12. The line $2x - y + 4 = 0$ cuts the parabola $y^2 = 8x$ in P and Q. The mid-point of PQ is

    (a) $(1, 2)$ (b) $(1, -2)$ (c) $(-1, 2)$ (d) $(-1, -2)$

13. In the parabola $y^2 = 4ax$, the length of the chord passing through the vertex and inclined to the axis at $\pi/4$ is

    (a) $4\sqrt{2}\,a$ (b) $2\sqrt{2}\,a$ (c) $\sqrt{2}\,a$ (d) none of these

14. The equation $16x^2 + y^2 + 8xy - 74x - 78y + 212 = 0$ represents

    (a) a circle (b) a parabola (c) an ellipse (d) a hyperbola

15. The length of the latus-rectum of the parabola $y^2 + 8x - 2y + 17 = 0$ is
(a) 2 (b) 4 (c) 8 (d) 16

16. The vertex of the parabola $x^2 + 8x + 12y + 4 = 0$ is
(a) $(-4, 1)$ (b) $(4, -1)$ (c) $(-4, -1)$ (d) $(4, 1)$

17. The vertex of the parabola $(y-2)^2 = 16(x-1)$ is
(a) $(1, 2)$ (b) $(-1, 2)$ (c) $(1, -2)$ (d) $(2, 1)$

18. The length of the latus-rectum of the parabola $4y^2 + 2x - 20y + 17 = 0$ is
(a) 3 (b) 6 (c) 1/2 (d) 9

19. The length of the latus-rectum of the parabola $x^2 - 4x - 8y + 12 = 0$ is
(a) 4 (b) 6 (c) 8 (d) 10

20. The focus of the parabola $y = 2x^2 + x$ is
(a) $(0, 0)$ (b) $(1/2, 1/4)$ (c) $(-1/4, 0)$ (d) $(-1/4, 1/8)$

21. Which of the following points lie on the parabola $x^2 = 4ay$?
(a) $x = at^2,\ y = 2at$ (b) $x = 2at,\ y = at^2$
(c) $x = 2at^2,\ y = at$ (d) $x = 2at,\ y = at^2$

22. The equation of the parabola whose focus is $(1, -1)$ and the directrix is $x + y + 7 = 0$ is
(a) $x^2 + y^2 - 2xy - 18x - 10y = 0$ (b) $x^2 - 18x - 10y - 45 = 0$
(c) $x^2 + y^2 - 18x - 10y - 45 = 0$ (d) $x^2 + y^2 - 2xy - 18x - 10y - 45 = 0$

ANSWERS

1. (a)	2. (b)	3. (c)	4. (b)	5. (a)	6. (a)	7. (c)	8. (a)
9. (b)	10. (c)	11. (a)	12. (c)	13. (a)	14. (b)	15. (c)	16. (b)
17. (a)	18. (c)	19. (c)	20. (c)	21. (d)	22. (d)		

SUMMARY

1. A parabola is the locus of a point which is equidistant from a fixed point (called focus) and a fixed line (called directrix).
   Thus, if (α, β) is the focus and $ax + by + c = 0$ is the equation of the directrix of a parabola, then its equation is
   $$(x-\alpha)^2 + (y-\beta)^2 = \frac{(ax+by+c)^2}{a^2+b^2}$$
   This equation is of the form
   $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ satisfying the conditions
   $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 - ab = 0$.
2. *Axis*: The straight line passing through the focus and perpendicular to the directrix is called the axis of the parabola.
3. *Vertex*: The point of intersection of the parabola and its axis is called the vertex of the parabola.

4. *Latus-rectum* : A chord passing through the focus and perpendicular to the axis is called the latus-rectum.
5. *Focal chord* : Any chord passing through the focus of a parabola is called its focal chord.
6. *Double ordinate* : Any chord perpendicular to the axis of a parabola is called double ordinate.
7. Following are four standard forms of parabola:

	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Coordinates of vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Coordinates of focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of the directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of the Latus-rectum	$4a$	$4a$	$4a$	$4a$
Focal distance of a point $P(x, y)$	$a + x$	$a - x$	$a + y$	$a - y$

CHAPTER 26

ELLIPSE

26.1 INTRODUCTION

In previous chapter, we have discussed that an ellipse is a particular case of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ when $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 < ab$. The analytical definition of an ellipse is as follows.

ELLIPSE *An ellipse is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed straight line (called directrix) is always constant which is always less than unity.*

The constant ratio is generally denoted by e and is known as the *eccentricity* of the ellipse.

If S is the focus, ZZ' is the directrix and P is any point on the ellipse, then by definition

$$\frac{SP}{PM} = e \Rightarrow SP = e.PM$$

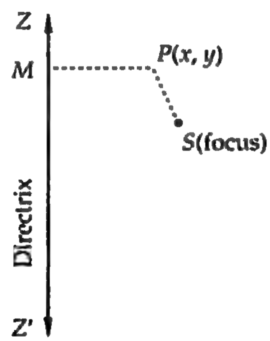


Fig. 26.1

ILLUSTRATION 1 *Find the equation of the ellipse whose focus is (1, 0), the directrix is $x + y + 1 = 0$ and eccentricity is equal to $1/\sqrt{2}$.*

SOLUTION Let S (1, 0) be the focus and ZZ' be the directrix. Let P (x, y) be any point on the ellipse and PM be perpendicular from P on the directrix. Then, by definition

$$SP = e \cdot PM, \quad \text{where } e = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad SP^2 = e^2 PM^2$$

$$\Rightarrow \quad (x-1)^2 + (y-0)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 \left|\frac{x+y+1}{\sqrt{1+1}}\right|^2$$

$$\Rightarrow \quad 4[(x-1)^2 + y^2] = (x+y+1)^2$$

$$\Rightarrow \quad 4x^2 + 4y^2 - 8x + 4 = x^2 + y^2 + 1 + 2xy + 2x + 2y$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 2xy - 10x - 2y + 3 = 0$$

This is the equation of the required ellipse.

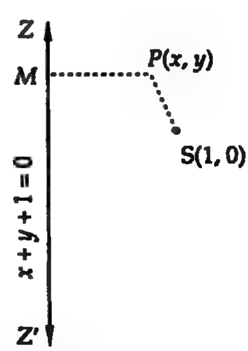


Fig. 26.2

26.2 EQUATION OF THE ELLIPSE IN STANDARD FORM $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let S be the focus, ZK the directrix and e the eccentricity of the ellipse whose equation is required. Draw SK perpendicular from S on the directrix. Divide SK internally and externally at A and A' (on KS produced) respectively in the ratio $e:1$.

$$\therefore \quad \frac{SA}{AK}=\frac{e}{1} \Rightarrow SA = e\cdot AK \qquad \text{...(i)}$$

$$\text{and,} \quad \frac{SA'}{A'K}=\frac{e}{1} \Rightarrow SA' = e\,A'K \qquad \text{...(ii)}$$

Since A and A' are such points that their distances from the focus bear constant ratio e (< 1) to their respective distances from the directrix. Therefore these points lie on the ellipse.

Let $AA' = 2a$ and C be the mid-point of AA'. Then, $CA = CA' = a$

Adding (i) and (ii), we get

$$SA + SA' = e(AK + A'K)$$

$$\Rightarrow \quad 2a = e(CK - CA + A'C + CK)$$

$$\Rightarrow \quad 2a = 2e\,CK \qquad [\because CA = A'C = a]$$

$$\Rightarrow \quad CK = \frac{a}{e} \qquad \text{...(iii)}$$

Subtracting (i) from (ii), we get

$$SA' - SA = e(A'K - AK)$$

$$\Rightarrow \quad (SC + CA') - (CA - CS) = e(AA')$$

$$\Rightarrow \quad 2CS = 2ae$$

$$\Rightarrow \quad CS = ae \qquad \text{...(iv)}$$

Now let us choose C as the origin. CAX as x-axis and a line CY perpendicular to AA' as y-axis. Therefore, coordinates of S are $(ae, 0)$ and equation of the directrix ZK is $x=\frac{a}{e}$.

Let $P(x, y)$ be any point on the ellipse. Join SP and draw $PM \perp ZK$. Then, by definition of the ellipse

$$SP = e\,PM$$

$$\Rightarrow \quad SP^2 = e^2\,PM^2$$

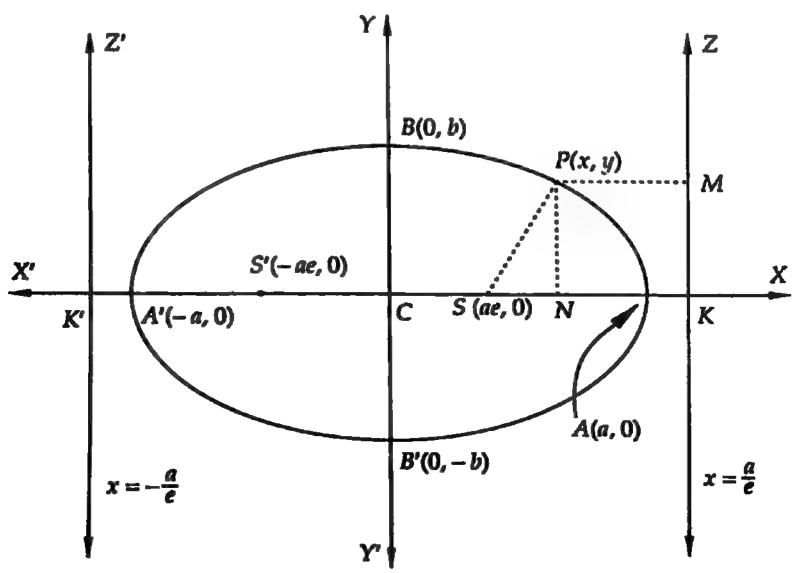


Fig. 26.3

$$\Rightarrow \quad SP^2 = e^2 (NK)^2$$

$$\Rightarrow \quad SP^2 = e^2 (CK - CN)^2$$

$$\Rightarrow \quad (x - ae)^2 + (y - 0)^2 = e^2 \left(\frac{a}{e} - x\right)^2$$

$$\Rightarrow \quad x^2 (1 - e^2) + y^2 = a^2 (1 - e^2)$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2 (1 - e^2)$$

This is the standard equation of the ellipse.

<u>NOTE</u> *We have, $e < 1$. Therefore, $1 - e^2 < 1 \Rightarrow a^2 (1 - e^2) < a^2 \Rightarrow b^2 < a^2$.*

26.2.1 TRACING OF THE ELLIPSE

We have, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ ...(i)

$\therefore \quad y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$ ...(ii) and, $x = \pm \frac{a}{b} \sqrt{b^2 - y^2}$ ...(iii)

In order to trace the ellipse (i), we observe the following points:

(a) *Symmetry*: For every value of x there are equal and opposite values of y [see (ii)]. Similarly, for every value of y there are equal and opposite values of x [see (iii)]. Thus, the curve is symmetric about both the axes.

(b) *Origin*: The curve does not pass through the origin.

(c) *Intersection with the axes*: The curve meets x axis at $y = 0$. Putting $y = 0$ in (iii), we get $x = \pm a$. So the curve meets x-axis at A $(a, 0)$ and A' $(-a, 0)$. Putting $x = 0$ in (ii), we get $y = \pm b$. So, the curve meets y-axis at B $(0, b)$ and B' $(0, -b)$.

(d) *Region*: If $x > a$ or $x < -a$, from (ii) we get imaginary values of y. Therefore, there is no part of the curve to the right of A or to the left of A'. If $y > b$ or $y < -b$, from (iii) we get imaginary values of x. Therefore, there is no part of the curve above B $(0, b)$ or below B' $(0, -b)$.
From (ii), we find that at $x = 0$, $y = \pm b$ and as x increases the values of y decrease and $y = 0$ at $x = a$. Therefore, the curve is a closed curve.
With the help of the above facts and by joining some convenient points on the ellipse, the general shape of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is as shown in Fig. 26.3.

26.2.2 SECOND FOCUS AND SECOND DIRECTRIX OF THE ELLIPSE

In Fig. 26.3 of an ellipse let P (x, y) be a point on the curve. Then as discussed above, we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1 \qquad ...(i)$$

where $CA = CA' = a$ and e is the eccentricity of the ellipse and the point S and the line ZK are the focus and directrix respectively.

Let S' and K' be points on the x-axis on the side of C which is opposite to the side of S such that $CS' = ae$ and $CK' = \frac{a}{e}$.

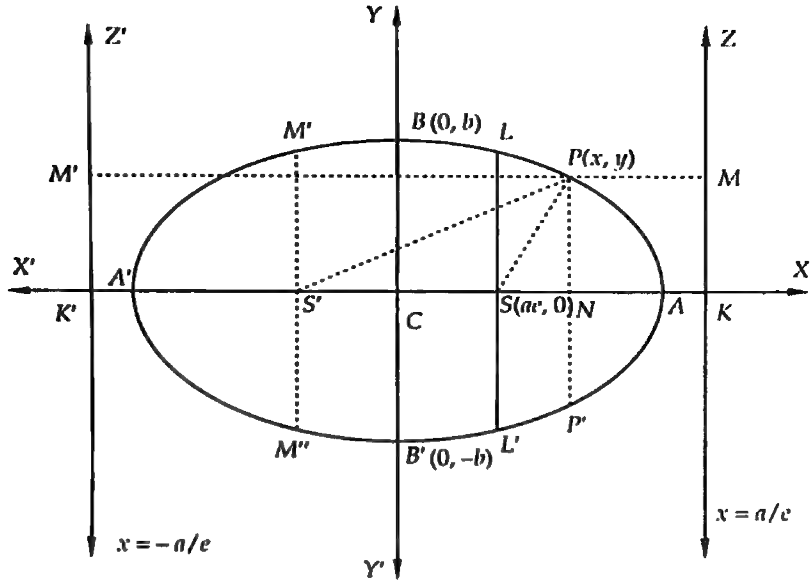


Fig. 26.4

Let $Z'K' \perp CK'$, $PM' \perp Z'K'$ as shown in Fig. 26.4. Join P and S'. Clearly $PM' = NK' = x + \frac{a}{e}$.

Now, equation (i) can be written as

$$x^2(1-e^2) + y^2 = a^2(1-e^2)$$

$$\Rightarrow \quad x^2 + y^2 + a^2e^2 = a^2 + e^2x^2$$

$$\Rightarrow \quad (x^2 + 2aex + a^2e^2) + y^2 = a^2 + 2aex + e^2x^2$$

$$\Rightarrow \quad (x + ae)^2 + y^2 = (a + ex)^2$$

$$\Rightarrow \quad (x + ae)^2 + (y - 0)^2 = e^2\left(x + \frac{a}{e}\right)^2$$

$$\Rightarrow \quad S'P^2 = e^2 PM'^2$$

$$\Rightarrow \quad S'P = e\,PM'$$

$\Rightarrow$ Distance of P from $S' = e$ (Distance of P from $Z'K'$)

Hence, we would have obtained the same curve had we started with S' as focus and $Z'K'$ as directrix. This shows that the ellipse has a second focus $S'(-ae, 0)$ and a second directrix $x = -\frac{a}{e}$.

26.2.3 VERTICES, MAJOR AND MINOR AXES, FOCI, DIRECTRICES AND CENTRE

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, we have the following definitions of some terms.

VERTICES *The points A and A' in Fig. 26.4 where the curve meets the line joining the foci S and S', are called the vertices of the ellipse. The coordinates of A and A' are (a, 0) and (– a, 0) respectively.*

MAJOR AND MINOR AXES *In Fig. 26.4 the distances* $AA' = 2a$ *and,* $BB' = 2b$ *are called the major and minor axes of the ellipse.*

Since $e < 1$ and $b^2 = a^2(1 - e^2)$. Therefore, $a > b \Rightarrow 2a > 2b \Rightarrow AA' > BB'$.

FOCI In Fig. 26.4, the points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the ellipse.

DIRECTRICES *ZK and Z' K' are two directrices of the ellipse and their equations are* $x = \frac{a}{e}$ *and* $x = -\frac{a}{e}$ *respectively.*

CENTRE *Since the centre of a conic section is a point which bisects every chord passing through it. In case of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *every chord, passing through C is bisected at C (0, 0). Therefore, C is the centre of the ellipse in Fig. 26.4 and it is the mid-point of AA'.*

ECCENTRICITY For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have

$$b^2 = a^2 (1 - e^2) \Rightarrow e^2 = 1 - \frac{b^2}{a^2} \Rightarrow e^2 = 1 - \frac{4b^2}{4a^2} = 1 - \left(\frac{2b}{2a}\right)^2$$

$$\Rightarrow \quad e = \sqrt{1 - \left(\frac{\text{Minor axis}}{\text{Major axis}}\right)^2}$$

26.2.4 ORDINATE, DOUBLE ORDINATE AND LATUS-RECTUM

We have the following terms associated to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$:

ORDINATE AND DOUBLE ORDINATE Let P be a point on the ellipse and let PN be perpendicular to the major axis AA' such that PN produced meets the ellipse at P'. Then, PN is called the ordinate of P and PNP' the double ordinate of P.

LATUS-RECTUM It is a double ordinate passing through the focus.

In Fig. 26.4, LSL' is the latus-rectum and LS is called the semi-latus-rectum. $MS'M'$ is also a latus-rectum. The coordinates of L are (ae, SL). As L lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the coordinates of L will satisfy the equation of the ellipse.

$$\therefore \quad \frac{(ae)^2}{a^2} + \frac{(SL)^2}{b^2} = 1$$

$$\Rightarrow \quad (SL)^2 = b^2 (1 - e^2)$$

$$\Rightarrow \quad (SL)^2 = b^2 \times \frac{b^2}{a^2} \qquad \left[\because b^2 = a^2 (1 - e^2) \Rightarrow 1 - e^2 = \frac{b^2}{a^2}\right]$$

$$\Rightarrow \quad SL = \frac{b^2}{a}$$

$$\therefore \quad SL = SL' = \frac{b^2}{a}$$

Hence, Length of the latus-rectum $LL' = 2\,(SL) = \frac{2b^2}{a} = 2a(1 - e^2)$

26.2.5 FOCAL DISTANCES OF A POINT ON THE ELLIPSE

The distances of any point on the ellipse from its foci are known as its focal distances.

THEOREM *The sum of the focal distances of any point on an ellipse is constant and equal to the length of the major axis of the ellipse.*

PROOF Let $P(x, y)$ be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (see Fig. 26.4). Then,

$$SP = e\ PM = e\,(NK) = e\,(CK - CN) = e\left(\frac{a}{e} - x\right) = a - ex \quad ...(i)$$

and, $$S'P = e\ PM' = e\,(NK') = e\,(CK' + CN) = e\left(\frac{a}{e} + x\right) = a + ex \quad ...(ii)$$

$\therefore$ $\quad SP + S'P = a - ex + a + ex = 2a =$ Major axis (= Constant)

Hence, the sum of the focal distances of a point on the ellipse is constant and is equal to the length of the major axis of the ellipse.

REMARK *On account of this property, a second definition of the ellipse may be given as follows:*

An ellipse is the locus of a point which moves in such a way that the sum of its distances from two fixed points (foci) is always constant.

26.3 EQUATION OF ELLIPSE IN OTHER FORMS

In the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a > b$ or $a^2 > b^2$ (denominator of x^2 is greater than that of y^2), then the major and minor axes lie along x-axis and y-axis respectively as shown in Fig. 26.4. But, if $a < b$ or $a^2 < b^2$ (denominator of x^2 is less than that of y^2), then the major axis of the ellipse lies along the y-axis and is of length $2b$ and the minor axis along the x-axis and is of length $2a$. The coordinates of foci S and S' are $(0, be)$ and $(0, -be)$ respectively. The equations of the directrices ZK and $Z'K'$ are $y = \frac{b}{e}$ and $y = -\frac{b}{e}$ respectively. The eccentricity e is given by the formula

$$a^2 = b^2(1 - e^2) \Rightarrow e = \sqrt{1 - \frac{a^2}{b^2}}$$

The shape of the ellipse is shown in Fig. 26.5.

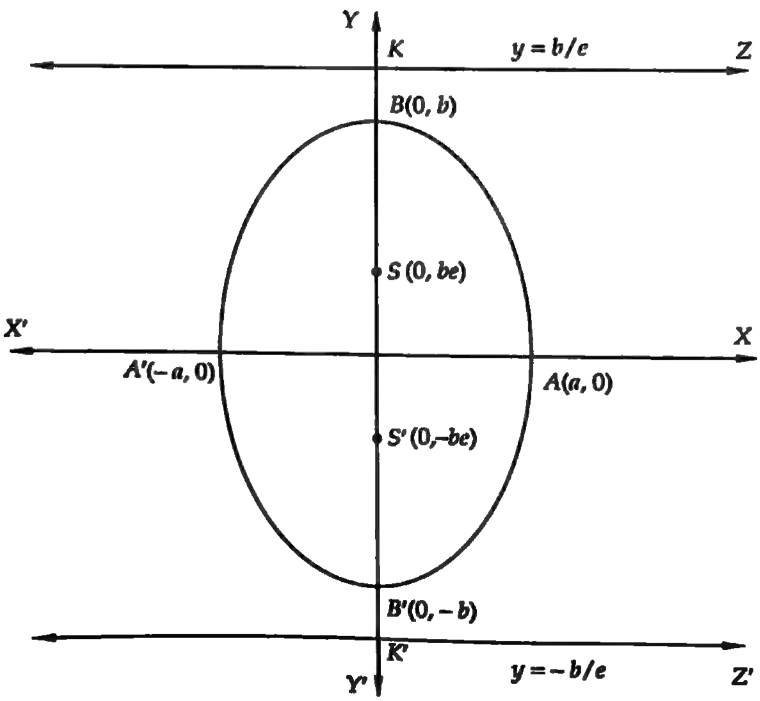


Fig. 26.5

Various results related to the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a<b)$ are given in the following table for ready reference.

	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a<b$
Coordinates of the centre	$(0, 0)$	$(0, 0)$
Coordinates of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, b)$ and $(0, -b)$
Coordinates of foci	$(ae, 0)$ and $(-ae, 0)$	$(0, be)$ and $(0, -be)$
Length of the major axis	$2a$	$2b$
Length of the minor axis	$2b$	$2a$
Equation of the major axis	$y=0$	$x=0$
Equation of the minor axis	$x=0$	$y=0$
Equations of the directrices	$x=\frac{a}{e}$ and $x=-\frac{a}{e}$	$y=\frac{b}{e}$ and $y=-\frac{b}{e}$
Eccentricity	$e=\sqrt{1-\frac{b^2}{a^2}}$	$e=\sqrt{1-\frac{a^2}{b^2}}$
Length of the latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Focal distances of a point (x, y)	$a \pm ex$	$b \pm ey$

SPECIAL FORM *If the centre of the ellipse is at point (h, k) and the directions of the axes are parallel to the coordinate axes, then its equation is* $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE EQUATION OF AN ELLIPSE WHEN ITS FOCUS, DIRECTRIX AND ECCENTRICITY ARE GIVEN**

EXAMPLE 1 *Find the equation of the ellipse with focus at (1, 1) and eccentricity $\frac{1}{2}$ and directrix $x-y+3=0$. Also, find the equation of its major axis.*

SOLUTION Let $P(x, y)$ be a point on the ellipse. Then, by definition

$$SP = e\,PM$$

Here $e=\frac{1}{2}$, coordinates of S are $(1, 1)$ and the equation of the directrix is $x-y+3=0$.

$$\therefore \quad SP = \frac{1}{2} PM$$

$$\Rightarrow \quad \sqrt{(x-1)^2+(y-1)^2} = \frac{1}{2}\left|\frac{x-y+3}{\sqrt{1^2+(-1)^2}}\right|$$

$$\Rightarrow \quad 8[(x-1)^2+(y-1)^2] = (x-y+3)^2$$

$$\Rightarrow \quad 7x^2+7y^2+2xy-22x-10y+7 = 0$$

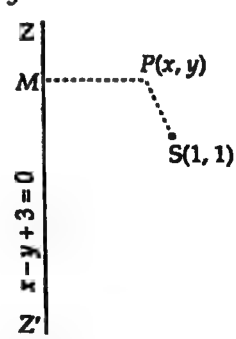


Fig. 26.6

This is the required equation of the ellipse.

The major axis is a line perpendicular to the directrix and passing through the focus. Therefore, the equation of the major axis is $y - 1 = -1\,(x - 1)$
or, $x + y - 2 = 0$.

EXAMPLE 2 *Find the equation of the ellipse whose eccentricity is 1/2, the focus is $(-1, 1)$ and the directrix is $x - y + 3 = 0$.*

SOLUTION Let $P\,(x, y)$ be any point on the ellipse whose focus is $S(-1, 1)$ and eccentricity $e = 1/2$. Let PM be perpendicular from P on the directrix. Then,

$$SP = e\,PM$$

$$\Rightarrow \quad SP = \frac{1}{2}(PM)$$

$$\Rightarrow \quad 4\,(SP)^2 = PM^2$$

$$\Rightarrow \quad 4\left\{(x+1)^2 + (y-1)^2\right\} = \left|\frac{x - y + 3}{\sqrt{1^2 + (-1)^2}}\right|^2$$

$$\Rightarrow \quad 8\,(x^2 + y^2 + 2x - 2y + 2) = (x - y + 3)^2$$

$$\Rightarrow \quad 7x^2 + 7y^2 + 10x - 10y + 2xy + 7 = 0$$

This is the required equation of the ellipse.

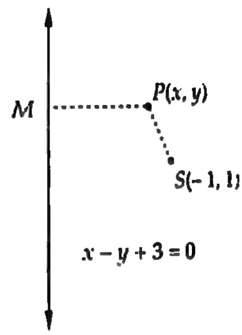


Fig. 26.7

Type II **ON FINDING THE VARIOUS ELEMENTS OF AN ELLIPSE WHEN ITS EQUATIONS IS GIVEN**

EXAMPLE 3 *For the following ellipses find the lengths of major and minor axes, coordinates of foci, vertices and the eccentricity:*

(i) $16x^2 + 25y^2 = 400$ (ii) $3x^2 + 2y^2 = 6$ (iii) $x^2 + 4y^2 - 2x = 0$

SOLUTION (i) We have,

$$16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1 \qquad \text{...(i)}$$

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 = 25$ and $b^2 = 16$ i.e. $a = 5$ and $b = 4$. Clearly, $a > b$, therefore the major and minor axes of the ellipse (i) are along x and y axes respectively.

$\therefore$ Length of major axis $= 2a = 10$, Length of minor axis $= 2b = 8$.

The coordinates of the vertices are $(a, 0)$ and $(-a, 0)$ i.e. $(5, 0)$ and $(-5, 0)$.

Let e be the eccentricity of the ellipse. Then,

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

The coordinates of the foci are $(ae, 0)$ and $(-ae, 0)$ i.e. $(3, 0)$ and $(-3, 0)$.

(ii) We have,

$$3x^2 + 2y^2 = 6 \Rightarrow \frac{x^2}{2} + \frac{y^2}{3} = 1 \qquad \text{...(i)}$$

This equation is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 = 2$ and $b^2 = 3$ i.e. $a = \sqrt{2}$ and $b = \sqrt{3}$. Clearly, $a < b$, so the major and minor axes of the given ellipse are along y and x-axes respectively.

$\therefore$ Length of the major axis $= 2b = 2\sqrt{3}$, Length of the minor axis $= 2 = 2\sqrt{2}$

The coordinates of the vertices are $(0, b)$ and $(0, -b)$ i.e. $(0, \sqrt{3})$ and $(0, -\sqrt{3})$.

The eccentricity e of the ellipse is given by

$$e = \sqrt{1-\frac{a^2}{b^2}} = \sqrt{1-\frac{2}{3}} = \frac{1}{\sqrt{3}}.$$

The coordinates of the foci are $(0, be)$ and $(0, -be)$ i.e. $(0, 1)$ and $(0, -1)$.

(iii) We have,

$$x^2 + 4y^2 - 2x = 0$$

$$\Rightarrow \quad (x^2 - 2x + 1) + 4y^2 = 0 + 1$$

$$\Rightarrow \quad (x-1)^2 + 4(y-0)^2 = 1$$

$$\Rightarrow \quad \frac{(x-1)^2}{1^2} + \frac{(y-0)^2}{(1/2)^2} = 1 \qquad \text{...(i)}$$

Shifting the origin at $(1, 0)$ without rotating the coordinate axes, we have

$$x = X + 1 \quad \text{and} \quad y = Y + 0 \qquad \text{...(ii)}$$

Using these relations in (i), it reduces to

$$\frac{X^2}{1^2} + \frac{Y^2}{(1/2)^2} = 1 \qquad \text{...(iii)}$$

Clearly, this equation is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $a^2 = 1$ and $b^2 = 1/4$ i.e. $a = 1$ and $b = 1/2$.

We find that $a > b$. So, the major and minor axes of the ellipse (iii) are along X and Y axes respectively.

$\therefore$ Length of the major axis $= 2a = 2$; Length of the minor axis $= 2b = 1$.

The eccentricity e is given by $e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$

The coordinates of the vertices with respect to the new axes are $(X = 1, Y = 0)$ and $(X = -1, Y = 0)$. So, the coordinates of the vertices with respect to the old axes are $(2, 0)$ and $(0, 0)$ [Putting $X = 1, Y = 0$ and $X = -1, Y = 0$ separately in (ii)]

The coordinates of the foci with respect to the new axes are

$$\left(X = \frac{\sqrt{3}}{\sqrt{2}}, Y = 0\right) \text{ and } \left(X = -\frac{\sqrt{3}}{\sqrt{2}}, Y = 0\right) \qquad [\text{Coordinates of foci are } (\pm ae, 0)]$$

So, the coordinates of the foci with respect to the old axes are

$$\left(\sqrt{\frac{3}{2}} + 1, 0\right) \text{ and } \left(1 - \sqrt{\frac{3}{2}}, 0\right) \qquad \left[\text{Putting } X = \pm\frac{\sqrt{3}}{\sqrt{2}}, Y = 0 \text{ in (ii)}\right]$$

EXAMPLE 4 *Show that $x^2 + 4y^2 + 2x + 16y + 13 = 0$ is the equation of an ellipse. Find its eccentricity, vertices, foci, directrices and, the length and the equation of the latus-rectum.*

SOLUTION We have,

$$x^2 + 4y^2 + 2x + 16y + 13 = 0$$

$$\Rightarrow \quad (x^2 + 2x + 1) + 4(y^2 + 4y + 4) = 4$$

$$\Rightarrow \quad (x+1)^2 + 4(y+2)^2 = 4$$

$$\Rightarrow \quad \frac{(x+1)^2}{2^2} + \frac{(y+2)^2}{1^2} = 1 \qquad \text{...(i)}$$

Shifting the origin at (–1, –2) without rotating the coordinate axes and denoting the new coordinates with respect to the new axes by X and Y, we have

$$x = X - 1 \quad \text{and} \quad y = Y - 2 \qquad \text{...(ii)}$$

Using these relations, equation (i) reduces to

$$\frac{X^2}{2^2} + \frac{Y^2}{1^2} = 1 \qquad \text{...(iii)}$$

This is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $a = 2$ and $b = 1$.

Thus, the given equation represents an ellipse. Clearly, $a > b$. So, the given equation represents an ellipse whose major and minor axes are along X and Y axes respectively.

Eccentricity: The eccentricity e is given by

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

Vertices: The vertices of the ellipse with respect to the new axes are $(X = \pm a,\ Y = 0)$ i.e. $(X = \pm 2,\ Y = 0)$. So, the vertices with respect to the old axes are given by

$$(\pm 2 - 1, -2) \text{ i.e. } (-3, -2) \text{ and } (1, -2) \qquad [\text{Putting } x = \pm 2,\ y = 0 \text{ in (ii)}]$$

Foci: The coordinates of the foci with respect to the new axes are given by $(X = \pm ae,\ Y = 0)$ i.e. $(X = \pm\sqrt{3},\ Y = 0)$. So, the coordinates of foci with respect to the old axes are given by

$$(\pm\sqrt{3} - 1,\ -2) \qquad [\text{Putting } X = \pm\sqrt{3},\ Y = 0 \text{ in (ii)}]$$

Directrices: The equations of the directrices with respect to the new axes are $X = \pm\frac{a}{e}$ i.e. $X = \pm\frac{4}{\sqrt{3}}$. So, the equations of the directrices with respect to the old axes are

$$x = \pm\frac{4}{\sqrt{3}} - 1 \text{ i.e. } x = \frac{4}{\sqrt{3}} - 1 \text{ and } x = -\frac{4}{\sqrt{3}} - 1 \qquad \left[\text{Putting } X = \pm\frac{4}{\sqrt{3}} \text{ in (ii)}\right]$$

Length of the latus-rectum: The length of the latusrectum $= \frac{2b^2}{a} = \frac{2}{2} = 1.$

Equations of Latus-recta: The equations of the latusrecta with respect to the new axes are $X = \pm ae$ i.e. $X = \pm\sqrt{3}$. So, the equations of the latus-recta with respect to the old axes are

$$x = \pm\sqrt{3} - 1 \text{ i.e. } x = \sqrt{3} - 1 \text{ and } x = -\sqrt{3} - 1. \qquad [\text{Putting } X = \pm\sqrt{3} \text{ in (ii)}]$$

EXAMPLE 5 *Find the eccentricity, centre, vertices, foci, minor axis, major axis, directrices and latus-rectum of the ellipse* $25x^2 + 9y^2 - 150x - 90y + 225 = 0$.

SOLUTION The equation of the ellipse is

$$25x^2 + 9y^2 - 150x - 90y + 225 = 0$$

$$\Rightarrow \quad 25x^2 - 150x + 9y^2 - 90y = -225$$

$$\Rightarrow \quad 25(x^2 - 6x) + 9(y^2 - 10y) = -225$$

$$\Rightarrow \quad 25(x^2 - 6x + 9) + 9(y^2 - 10y + 25) = -225 + 225 + 225$$

$$\Rightarrow \quad 25(x-3)^2 + 9(y-5)^2 = 225$$

$$\Rightarrow \quad \frac{(x-3)^2}{9} + \frac{(y-5)^2}{25} = 1 \qquad \text{...(i)}$$

Shifting the origin at (3, 5) without rotating the coordinate axes and denoting the new coordinates with respect to the new axes by X and Y, we have

$$x = X + 3 \quad \text{and} \quad y = Y + 5 \qquad \text{...(ii)}$$

Using these relations, equation (i) reduces to

$$\frac{X^2}{3^2} + \frac{Y^2}{5^2} = 1 \qquad \text{...(iii)}$$

This is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $a^2 = 3^2$ and $b^2 = 5^2$. Clearly, $a < b$. So, equation (iii) represents an ellipse whose major and minor axes along Y and X axes respectively.

Eccentricity: The eccentricity e is given by

$$e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

Centre: The coordinates of the centre with respect to new axes are $(X = 0,\ Y = 0)$. So, the coordinates of the centre with respect to old axes are $(3, 5)$.

Vertices: The vertices of the ellipse with respect to the new axes are $(X = 0, Y = \pm b)$ i.e. $(X = 0, Y = \pm 5)$. So, the vertices with respect to the old axes are

$$(3, 5 \pm 5) \text{ i.e. } (3, 0) \text{ and } (3, 10) \qquad [\text{Putting } X = 0, Y = \pm 5 \text{ in (ii)}]$$

Foci: The coordinates of the foci with respect to the old axes are $(X = 0, Y = \pm be)$ i.e. $(X = 0, Y = \pm 4)$. So, the coordinates of the foci with respect to the old axes are

$$(3, \pm 4 + 5) \text{ i.e. } (3, 1) \text{ and } (3, 9) \qquad [\text{Putting } X = 0, Y = \pm 4 \text{ in (ii)}]$$

Directrices: The equations of the directrices with respect to the new axes are $Y = \pm \frac{b}{e}$ i.e. $Y = \pm \frac{25}{4}$.

So, the equations of the directrices with respect to the old axes are

$$y = \pm \frac{25}{4} + 5 \text{ i.e. } y = -\frac{5}{4} \text{ and } y = \frac{45}{4}. \qquad \left[\text{Putting } Y = \pm \frac{25}{4} \text{ in (ii)}\right]$$

Axes: Lengths of the major and minor axes are: Major axis $= 2b = 10$, Minor axis $= 2a = 6$.

Equation of the major axis with respect to the new axes is $X = 0$. So, the equation of the major axis with respect to the old axes is $x = 3$. [Putting $X = 0$ in (ii)]

The equation of the minor axis with respect to the new axes is $Y = 0$. So, the equation of the minor axis with respect to the old axes is $y = 5$. [Putting $Y = 0$ in (ii)]

Latus-rectum: The length of the latus-rectum $= \frac{2a^2}{b} = \frac{2 \times 9}{5} = \frac{18}{5}$.

The equations of the latus-recta with respect to the new axes are $Y = \pm ae$ i.e. $Y = \pm 4$. So, the equations of the latus-recta with respect to the old axes are

$$y = \pm 4 + 5 \text{ i.e. } y = 1 \text{ and } y = 9. \qquad [\text{Putting } Y = \pm 4 \text{ in (ii)}]$$

EXAMPLE 6 *Find the eccentricity, foci and the length of the latusrectum of the ellipse* $x^2 + 4y^2 + 8y - 2x + 1 = 0$.

SOLUTION The given equation of the ellipse is

$$x^2 + 4y^2 + 8y - 2x + 1 = 0$$

$$\Rightarrow \quad x^2 - 2x + 4y^2 + 8y = -1$$

$$\Rightarrow \quad (x^2 - 2x + 1) + 4(y^2 + 2y + 1) = -1 + 1 + 4$$

$$\Rightarrow \quad (x - 1)^2 + 4(y + 1)^2 = 4$$

$$\Rightarrow \quad \frac{(x-1)^2}{2^2} + \frac{(y+1)^2}{1^2} = 1 \qquad ...(i)$$

Shifting the origin to $(1, -1)$ without rotating the axes and denoting the new coordinates with respect to these axes by X and Y, we obtain

$$x = X + 1, \ y = Y - 1 \qquad ...(ii)$$

Using these relations equation (i) reduces to

$$\frac{X^2}{2^2} + \frac{Y^2}{1^2} = 1$$

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$. On compairing, we get

$$a^2 = 2^2 \text{ and } b^2 = 1 \Rightarrow a = 2 \text{ and } b = 1.$$

Let e be the eccentricity of the ellipse. Then,

$$b^2 = a^2(1-e^2) \Rightarrow 1 = 4(1-e^2) \Rightarrow e = \frac{\sqrt{3}}{2}$$

The coordinates of foci with respect to new axes are $(X = \pm ae, \ Y = 0)$ i.e., $(X = \pm\sqrt{3}, \ Y = 0)$.
So, coordinates of foci with respect to old axes are $(1 \pm \sqrt{3}, -1)$ [Putting $X = \pm\sqrt{3}, Y = 0$ in (ii)]

$$\text{Length of the latus-rectum} = \frac{2b^2}{a} = \frac{2(1)^2}{2} = 1.$$

EXAMPLE 7 *Find the distance between the directrices the ellipse* $\frac{x^2}{36} + \frac{y^2}{20} = 1$.

[NCERT EXEMPLAR]

SOLUTION Comparing the given equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we obtain $a^2 = 36$ and $b^2 = 20$.

Let e be the eccentricity of the ellipse. Then,

$$b^2 = a^2(1-e^2) \Rightarrow 20 = 36(1-e^2) \Rightarrow 36e^2 = 16 \Rightarrow e = \frac{2}{3}$$

$$\therefore \quad \text{Distance between the directrices} = \frac{2a}{e} = \frac{2\times 6}{2/3} = 18.$$

Type III **ON FINDING SOME ELEMENTS OF AN ELLIPSE FROM GIVEN ELEMENTS**

EXAMPLE 8 *If the eccentricity of an ellipse is* $\frac{5}{8}$ *and the distance between its foci is 10, then find the latusrectum of the ellipse.*

[NCERT EXEMPLAR]

SOLUTION Let the equation of the required ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let e its eccentricity.

We have, $e = \frac{5}{8}$ and $2ae = 10$

$$\Rightarrow \quad e = \frac{5}{8} \text{ and } ae = 5$$

$$\Rightarrow \quad e = \frac{5}{8} \text{ and } a = 8$$

$$\therefore \quad b^2 = a^2(1-e^2) \Rightarrow b^2 = 64\left(1 - \frac{25}{64}\right) = 39$$

$$\text{Hence, length of the latusrectum} = \frac{2b^2}{a} = 2\times\frac{39}{8} = \frac{39}{4}$$

EXAMPLE 9 *If the latusrectum of an ellipse is equal to half of minor axis, find its eccentricity.*

[NCERT EXEMPLAR]

SOLUTION Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let e be its eccentricity.

It is given that

$$\text{Latusrectum} = \frac{1}{2}\ (\text{Minor axis})$$

$$\Rightarrow \quad \frac{2b^2}{a} = \frac{1}{2}(2b)$$

$$\Rightarrow \quad 2b = a$$

$$\Rightarrow \quad 4b^2 = a^2$$

$$\Rightarrow \quad 4a^2(1-e^2) = a^2 \Rightarrow 4-4e^2 = 1 \Rightarrow 4e^3 = 3 \Rightarrow e = \frac{\sqrt{3}}{2}$$

Hence, the eccentricity is $\frac{\sqrt{3}}{2}$.

Type IV **ON FINDING THE EQUATION OF AN ELLIPSE WHEN SOME OF ITS ELEMENTS ARE GIVEN**

EXAMPLE 10 *Find the equation of the ellipse whose axes are along the coordinate axes, vertices are* $(\pm 5, 0)$ *and foci at* $(\pm 4, 0)$.

SOLUTION Let the equation of the required ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$

The coordinates of its vertices and foci are $(\pm a, 0)$ and $(\pm ae, 0)$ respectively. But, the coordinates of vertices and foci are given as $(\pm 5, 0)$ and $(\pm 4, 0)$.

$$\therefore \quad a = 5 \ \text{ and } \ ae = 4 \Rightarrow e = \frac{4}{5}$$

Now, $b^2 = a^2(1-e^2) \Rightarrow b^2 = 25\left(1-\frac{16}{25}\right) = 9.$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{25}+\frac{y^2}{9}=1$, which is the equation of the required ellipse.

EXAMPLE 11 *Find the equation of the ellipse whose axes are along the coordinate axes, vertices are* $(0, \pm 10)$ *and eccentricity* $e = 4/5$.

SOLUTION Let the equation of the required ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$

Since the vertices of the ellipse are on y-axis. So, the coordinates of the vertices are $(0, \pm b)$. But, the coordinates of vertices are given to be $(0, \pm 10)$.

$$\therefore \quad b = 10.$$

Now, $a^2 = b^2(1-e^2) \Rightarrow a^2 = 100\left(1-\frac{16}{25}\right) = 36$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{36}+\frac{y^2}{100}=1$ as the equation of the required ellipse.

EXAMPLE 12 *Find the equation of the ellipse whose axes are along the coordinate axes, foci at* $(0, \pm 4)$ *and eccentricity* $4/5$.

SOLUTION Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$

The coordinates of the foci are $(0, \pm 4)$. This means that the major and minor axes of the ellipse are along y and x axes respectively and the coordinates of foci are $(0, \pm be)$.

$\therefore \quad be = 4$

$\Rightarrow \quad b\,(4/5) = 4 \qquad [\because e = 4/5]$

$\Rightarrow \quad b = 5.$

Now, $a^2 = b^2(1-e^2) \Rightarrow a^2 = 25\left(1-\frac{16}{25}\right) = 9$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{9}+\frac{y^2}{25}=1$ as the equation of the required ellipse.

EXAMPLE 13 *The foci of an ellipse are* $(\pm 2, 0)$ *and its eccentricity is* $1/2$, *find its equation if it is given that its centre is at the origin and axes are along the coordinates axes.*

SOLUTION Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The, coordinates of its foci are $(\pm ae, 0)$.

But, the coordinates of foci are given as $(\pm 2, 0)$.

$\therefore \quad ae = 2$

$\Rightarrow \quad a \times \frac{1}{2} = 2 \qquad [\because e = 1/2]$

$\Rightarrow \quad a = 4.$

Now, $b^2 = a^2(1-e^2) \Rightarrow b^2 = 16\left(1-\frac{1}{4}\right) = 12.$

Substituting $a = 4$ and $b^2 = 12$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we obtain $\frac{x^2}{16}+\frac{y^2}{12}=1$ as the equation of the ellipse.

EXAMPLE 14 *Find the equation of the ellipse with foci at* $(\pm 5, 0)$ *and* $x = \frac{36}{5}$ *as one of the directrices.*

[NCERT EXEMPALR]

SOLUTION Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let e be its eccentricity. The coordinates of its foci and the equations of the directrices are $(\pm ae, 0)$ and $x = \pm a/e$ respectively. But, it is given that the coordinates of foci are $(\pm 5, 0)$ and the equations of one of the directrices is $x = 36/5$.

$\therefore \quad ae = 5$ and $\frac{a}{e} = \frac{36}{5}$

$\Rightarrow \quad ae \times \frac{a}{e} = 5 \times \frac{36}{5}$

$\Rightarrow \quad a^2 = 36$

$\Rightarrow \quad a = 6$

Now, $b^2 = a^2(1-e^2)$

$\Rightarrow \quad b^2 = a^2 - (ae)^2 = 36 - 25 = 11$

$\Rightarrow \quad b = \sqrt{11}$

Substituting $a = 6$ and $b = \sqrt{11}$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we obtain $\frac{x^2}{36}+\frac{y^2}{11}=1$ as the required equation.

EXAMPLE 15 *Find the equation of the ellipse whose axes are parallel to the coordinate axes having its centre at the point (2, – 3) one focus at (3, – 3) and one vertex at (4, – 3).*

SOLUTION Let $2a$ and $2b$ be the major and minor axes of the ellipse. Then, its equation is

$$\frac{(x-2)^2}{a^2}+\frac{(y+3)^2}{b^2}=1 \quad ...(i)$$

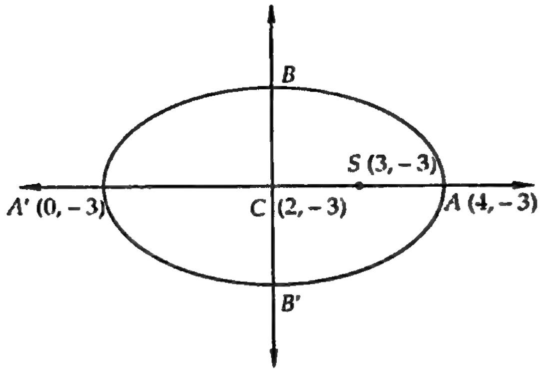


Fig. 26.8

Clearly,

$CA = a$ (= Semi-Major axis)

$$\Rightarrow \quad \sqrt{(4-2)^2+(-3+3)^2} = a$$

$$\Rightarrow \quad a = 2. \quad ...(ii)$$

Since the distance between the focus and centre of an ellipse is equal to ae, where e is the eccentricity.

$\therefore \quad CS = ae$

$$\Rightarrow \quad \sqrt{(2-3)^2+(-3+3)^2} = ae$$

$$\Rightarrow \quad ae = 1 \quad ...(iii)$$

From (ii) and (iii), we get : $e = \frac{1}{2}$.

Now, $b^2 = a^2(1-e^2) \Rightarrow b^2 = 4\left(1-\frac{1}{4}\right) = 3.$

Substituting the values of a and b in (i), we obtain

$$\frac{(x-2)^2}{4}+\frac{(y+3)^2}{3}=1,$$ as the required equation of the ellipse.

LEVEL-2

EXAMPLE 16 *Find the equation of the set of all points the sum of whose distances from the points* (3, 0) *and* (9, 0) *is* 12.

SOLUTION Let $P(x, y)$ be a point such that the sum of its distances from $S(3, 0)$ and $S'(9, 0)$ is 12.

i.e. $\quad PS + PS' = 12$

$$\Rightarrow \quad \sqrt{(x-3)^2+(y-0)^2}+\sqrt{(x-9)^2+(y-0)^2}=12$$

$$\Rightarrow \quad \sqrt{(x-3)^2+y^2}=12-\sqrt{(x-9)^2+y^2}$$

$$\Rightarrow \quad (x-3)^2+y^2=144-24\sqrt{(x-9)^2+y^2}+\{(x-9)^2+y^2\}$$

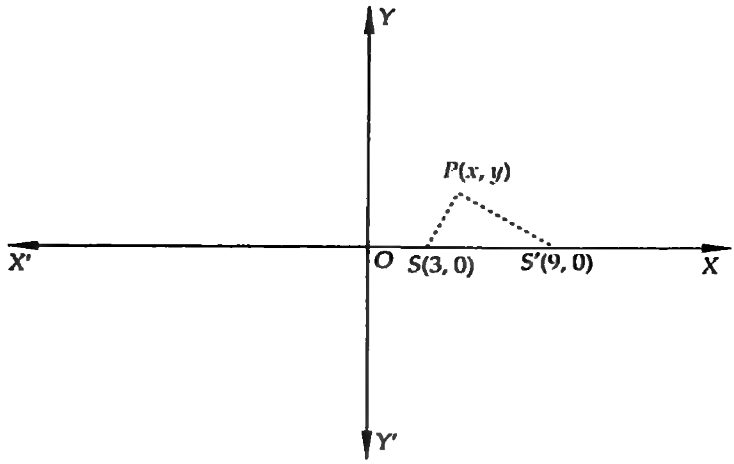


Fig. 26.9

$\Rightarrow \quad \{(x-3)^2+y^2\}-\{(x-9)^2+y^2\}=144-24\sqrt{(x-9)^2+y^2}$

$\Rightarrow \quad 12x-72=144-24\sqrt{(x-9)^2+y^2}$

$\Rightarrow \quad 12x-216=-24\sqrt{(x-9)^2+y^2}$

$\Rightarrow \quad x-18=-2\sqrt{(x-9)^2+y^2}$

$\Rightarrow \quad (x-18)^2=4\{(x-9)^2+y^2\}$

$\Rightarrow \quad x^2-36x+324=4x^2-72x+324+4y^2$

$\Rightarrow \quad 3x^2-36x+4y^2=0$, which is the required equation.

ALITER We know that the sum of the focal distances of a point on the ellipse is constant equal to major axis. Therefore, the curve is an ellipse having its foci at $S(3, 0)$ and $S'(9, 0)$ and major axis $2a=12$. The distance between the foci S and S' is 6.

$\therefore \quad 2ae=6 \Rightarrow 12e=6 \Rightarrow e=\frac{1}{2}$

Now, $\quad b^2=a^2(1-e^2) \Rightarrow b^2=36\left(1-\frac{1}{4}\right)=27$

The centre of the ellipse is the mid-point of segment SS'. So, the coordinates of centre are $(6, 0)$. Hence, the equation of the ellipse is

$$\frac{(x-6)^2}{6^2}+\frac{(y-0)^2}{27}=1 \text{ or, } 3x^2+4y^2-36x=0$$

EXAMPLE 17 *Find the equation of the ellipse whose centre is at the origin, foci are $(1, 0)$ and $(-1, 0)$ and eccentricity is $1/2$.*

SOLUTION Here coordinates of two foci S and S' are $(1, 0)$ and $(-1, 0)$ respectively. Therefore, $SS'=2$. Let $2a$ and $2b$ be the lengths of the major and minor axes of the required ellipse and e be the eccentricity. Then, $SS'=2ae \Rightarrow 2ae=2 \Rightarrow ae=1 \Rightarrow a\left(\frac{1}{2}\right)=1 \Rightarrow a=2$.

Let $P(x, y)$ be any point on the ellipse. Then,

$SP+S'P=2a$ [See section 26.2.5]

$\Rightarrow \quad SP+S'P=4$ [$\because a=2$]

$\Rightarrow \quad \sqrt{(x-1)^2+(y-0)^2}+\sqrt{(x+1)^2+(y-0)^2}=4$

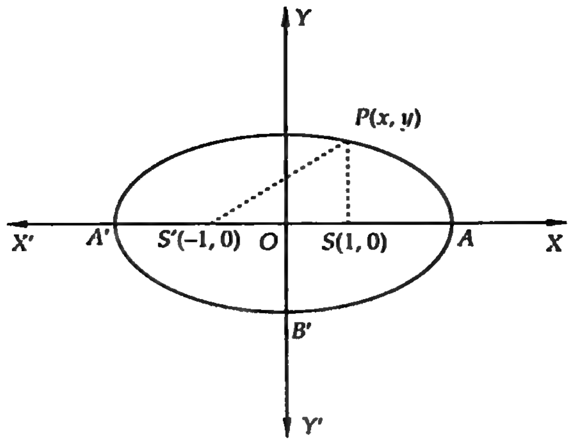


Fig. 26.10

$\Rightarrow \quad \sqrt{(x-1)^2+y^2} = 4-\sqrt{(x+1)^2+y^2}$

$\Rightarrow \quad \left\{\sqrt{(x-1)^2+y^2}\right\}^2 = \left\{4-\sqrt{(x+1)^2+y^2}\right\}^2$

$\Rightarrow \quad (x-1)^2+y^2 = 16-8\sqrt{(x+1)^2+y^2}+(x+1)^2+y^2$

$\Rightarrow \quad \left\{(x-1)^2+y^2\right\}-\left\{(x+1)^2+y^2\right\} = 16-8\left\{\sqrt{(x+1)^2+y^2}\right\}$

$\Rightarrow \quad -4x = 16-8\left\{\sqrt{(x+1)^2+y^2}\right\}$

$\Rightarrow \quad x+4 = 2\left\{\sqrt{(x+1)^2+y^2}\right\}$

$\Rightarrow \quad (x+4)^2 = 4\left\{(x+1)^2+y^2\right\}$

$\Rightarrow \quad 3x^2+4y^2-12 = 0$, which is the required equation of the ellipse.

ALITER Let $S(1, 0)$ and $S'(-1, 0)$ be the foci of the ellipse and e be its eccentricity. The centre of the ellipse is the mid-point of segment SS'. So, the coordinates of the centre are (0, 0). Let $2a$ and $2b$ be the lengths of major and minor axes of the ellipse.

Now, $\quad SS' = \sqrt{(-1-1)^2+(0-0)^2} = 2$

$\Rightarrow \quad 2ae = 2$

$\Rightarrow \quad ae = 1$

$\Rightarrow \quad a \times \frac{1}{2} = 1 \qquad \left[\because e = \frac{1}{2} \text{ (given)}\right]$

$\Rightarrow \quad a = 2.$

Now, $\quad b^2 = a^2(1-e^2) = 4\left(1-\frac{1}{4}\right) = 3$

Hence, the equation of the ellipse is $\frac{x^2}{4}+\frac{4^2}{3} = 1.$

EXAMPLE 18 *Find the equation of the ellipse whose foci are (2, 3), (−2, 3) and whose semi-minor axis is* $\sqrt{5}$.

SOLUTION Let S and S' be two foci of the required ellipse. Then, the coordinates of S and S' are (2, 3) and (−2, 3) respectively. Therefore, $SS' = 4$

Let $2a$ and $2b$ be the lengths of the axes of the ellipse and e be its eccentricity. Then,

$$SS' = 2ae. \Rightarrow 2ae = 4 \Rightarrow ae = 2.$$

Now, $b^2 = a^2(1-e^2) \Rightarrow 5 = a^2 - 2^2 \Rightarrow a = 3.$

Let $P(x, y)$ be any point on the ellipse. Then,

$$SP + S'P = 2a \qquad \text{[See section 26.2.5]}$$

$$\Rightarrow \sqrt{(x-2)^2+(y-3)^2} + \sqrt{(x+2)^2+(y-3)^2} = 6$$

$$\Rightarrow \left\{(x-2)^2+(y-3)^2\right\} - \left\{(x+2)^2+(y-3)^2\right\} = 36 - 12\left\{\sqrt{(x+2)^2+(y-3)^2}\right\}$$

$$\Rightarrow -8x = 36 - 12\left\{\sqrt{(x+2)^2+(y-3)^2}\right\}$$

$$\Rightarrow (2x+9)^2 = 9\{(x+2)^2+(y-3)^2\}$$

$\Rightarrow$ $5x^2 + 9y^2 - 54y + 36 = 0$, which is the required equation of the ellipse.

EXAMPLE 19 *A rod AB of length 15 cm rests in between two coordinate axes in such a way that the end point A lies on x-axis and end point B lies on y-axis. A point is taken on the rod in such a way that AP = 6 cm. Show that the locus of P is an ellipse. Also, find its eccentricity.* [NCERT]

SOLUTION Let the coordinates of A and B be $(a, 0)$ and $(0, b)$ respectively. Let the coordinates of P be (h, k).

We have, $AP = 6$ cm and $AB = 15$ cm.

$\therefore \quad BP = 9$ cm.

Since $P(h, k)$ divides AB in the ratio 6 : 9. Therefore,

$$h = \frac{9a}{15} \text{ and } k = \frac{6b}{15}$$

$$\Rightarrow a = \frac{15h}{9} \text{ and } b = \frac{15k}{6}$$

$$\Rightarrow a = \frac{5h}{3} \text{ and } b = \frac{5k}{2}$$

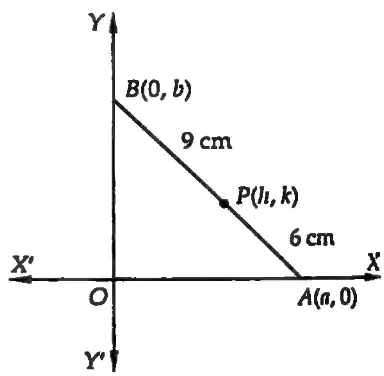


Fig. 26.11

In ΔOAB, we have,

$$OA^2 + OB^2 = AB^2$$

$$\Rightarrow a^2 + b^2 = 15^2 \Rightarrow \frac{25h^2}{9} + \frac{25k^2}{4} = 15^2 \Rightarrow 4h^2 + 9k^2 = 324$$

Hence, the locus of $P(h, k)$ is $4x^2 + 9y^2 = 324$. Clearly, it represents an ellipse.

Now, $4x^2 + 9y^2 = 324 \Rightarrow \dfrac{x^2}{81} + \dfrac{y^2}{36} = 1$

Comparing this equation with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we obtain

$$a^2 = 81 \text{ and } b^2 = 36 \Rightarrow a = 9 \text{ and } b = 6$$

Let e be the eccentricity of the ellipse. Then,

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{36}{81}} = \frac{\sqrt{5}}{3}.$$

EXAMPLE 20 *An arc is in the form of a semi-ellipse. It is 8 m wide and 2m high at the centre. Find the height of the arch at a point 1.5 m from one end.*

SOLUTION Let ABA' be the given arc such that $AA' = 8$ m and $OB = 2$ m. Let the arc be a part of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then,

$$AA' = 8 \text{ m} \Rightarrow 2a = 8 \Rightarrow a = 4$$

and, $OB = 2 \text{ m} \Rightarrow b = 2.$

So, the equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1 \quad \text{...(i)}$$

We have, to find the height of the arc at point P such that $AP = 1.5$ m. In other words, we have to find the y-coordinate at P.

$\because \quad OA = 4 \text{ m and } AP = 1.5 \text{ m}$

$\therefore \quad OP = OA - AP = (4 - 1.5) \text{ m} = 2.5 \text{ m}.$

Thus, the coordinates of M are $\left(\frac{5}{2}, PM\right)$.

Since M lies on the ellipse (i). Therefore,

$$\frac{25}{4 \times 16} + \frac{PM^2}{4} = 1$$

$$\Rightarrow \quad \frac{PM^2}{4} = 1 - \frac{25}{64}$$

$$\Rightarrow \quad \frac{PM^2}{4} = \frac{39}{64}$$

$$\Rightarrow \quad PM = \sqrt{\frac{39}{16}} \text{ m} = \frac{\sqrt{39}}{4} \text{ m}$$

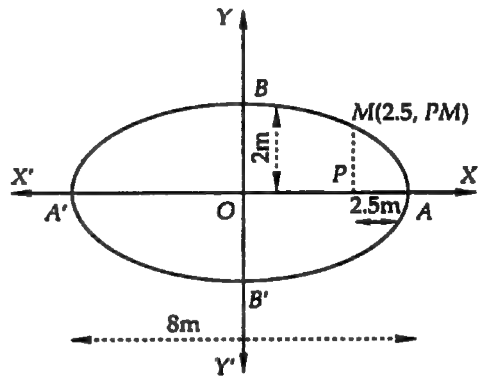


Fig. 26.12

Hence, the height of the arc at a point 1.5 m from one end is $\frac{\sqrt{39}}{4}$ m.

EXAMPLE 21 *A man running a race-course notes that the sum of the distances from the two flag posts from him is always 10 metres and the distance between the flag posts is 8 metres. Find the equation of the path traced by the man.* [NCERT]

SOLUTION Clearly, the path traced by the man is an ellipse having its foci at two flag posts. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(1 - e^2)$$

It is given that the sum of the distances of the man from the two flag posts is 10 metres. This means that the sum of the focal distnaces of a point on the ellipse is 10 m.

$\therefore \quad 2a = 10 \Rightarrow a = 5$ [$\because$ Sum of the focal distances of a point $= 2a$]

It is also given that the distance between the flag posts is 8 metres.

$\therefore \quad 2ae = 8 \Rightarrow ae = 4$ [$\because$ Distance between two foci $= 2ae$]

Now, $b^2 = a^2(1 - e^2)$

$$\Rightarrow \quad b^2 = a^2 - a^2e^2 = 25 - 16 = 9$$

Hence, the equation of the path is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

EXAMPLE 22 *A bar of given length moves with its extremities on two fixed straight lines at right angles. Show that any point on the bar describes an ellipse.* **[NCERT EXEMPLAR]**

SOLUTION Let AB be a bar of length l which slides between the coordinate axes and let $P(h, k)$ be a point on the bar such that $PA = a$ and $PB = b$.

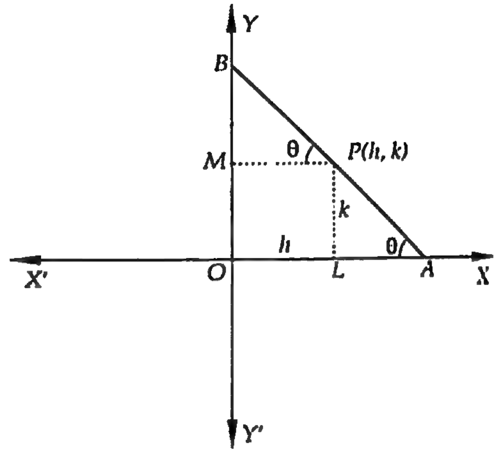


Fig. 26.13

Let $\angle OAB = \theta$. Then, $\angle MPB = \theta$.

In Δ's ALP and PMB, we have

$$\sin\theta = \frac{PL}{AP} \text{ and } \cos\theta = \frac{PM}{BP}$$

$$\Rightarrow \quad \sin\theta = \frac{k}{a} \text{ and } \cos\theta = \frac{h}{b}$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = \frac{k^2}{a^2} + \frac{h^2}{b^2}$$

$$\Rightarrow \quad \frac{h^2}{b^2} + \frac{k^2}{a^2} = 1$$

Hence, the locus of $P(h, k)$ is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, which is an ellipse.

EXAMPLE 23 *A straight rod of given length slides between two fixed bars which include an angle of 90°. Show that the locus of a point on the rod which divides it in a given ratio is an ellipse. If this ratio be 1/2, show that the eccentricity of the ellipse is $\sqrt{2}/3$.*

SOLUTION Let the two lines be along the coordinate axes. Let PQ be the rod of length a such that $\angle OPQ = \theta$. Then, the coordinates of P and Q are $(a\cos\theta, 0)$ and $(0, a\sin\theta)$ respectively. Let $R(h, k)$ be the point dividing PQ in the ratio $\lambda : 1$. Then,

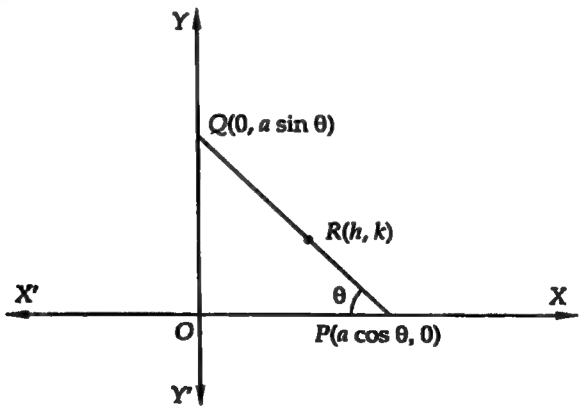


Fig. 26.14

$$h = \frac{a\cos\theta}{\lambda+1} \quad \text{and } k = \frac{\lambda a\sin\theta}{\lambda+1}$$

$$\Rightarrow \quad \cos\theta = \frac{h}{a}(\lambda+1) \text{ and } \sin\theta = \frac{k}{a\lambda}(\lambda+1)$$

$$\Rightarrow \quad \cos^2\theta + \sin^2\theta = \frac{h^2}{a^2}(\lambda+1)^2 + \frac{k^2}{a^2\lambda^2}(\lambda+1)^2$$

$$\Rightarrow \quad \frac{h^2}{\left(\frac{a}{\lambda+1}\right)^2} + \frac{k^2}{\left(\frac{a\lambda}{\lambda+1}\right)^2} = 1$$

Hence, the locus of (h, k) is $\dfrac{x^2}{\left(\frac{a}{\lambda+1}\right)^2} + \dfrac{y^2}{\left(\frac{a\lambda}{\lambda+1}\right)^2} = 1$, which is an ellipse.

Let e be the eccentricity of this ellipse. Then,

$$e = \sqrt{1 - \frac{\left(\frac{a\lambda}{\lambda+1}\right)^2}{\left(\frac{a}{\lambda+1}\right)^2}} \qquad \left[\because\ e = \sqrt{1 - \frac{b^2}{a^2}}\right]$$

$$\Rightarrow \quad e = \sqrt{1-\lambda^2}$$

When $\lambda = \frac{1}{2}$, we obtain

$$e = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}.$$

EXAMPLE 24 *A point moves so that the sum of the squares of its distances from two intersecting straight lines is constant. Prove that its locus is an ellipse.*

SOLUTION Let us assume that the two intersecting lines intersects at the origin and they are equally inclined with the positive direction of x-axis i.e. $\angle XOA = \angle XOC = \theta$.

The equations OA and OB are respectively

$$y = x \tan\theta \quad \text{and} \quad y = -x\tan\theta$$

or, $\quad x\sin\theta - y\cos\theta = 0$ and $x\sin\theta + y\cos\theta = 0$

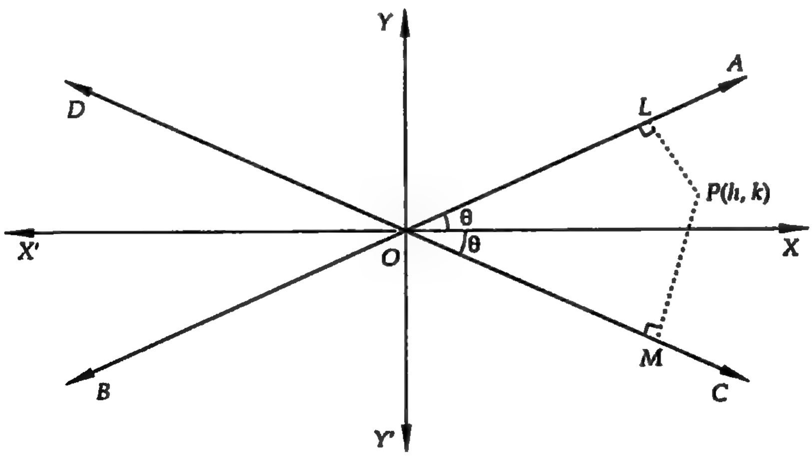


Fig. 26.15

Let $P(h, k)$ be a variable point such that the sum of the squares of its distances from OA and OB is constant.

i.e. $\quad PL^2 + PM^2 = \lambda^2$ (constant)

$$\Rightarrow \quad \left(\frac{h\sin\theta - k\cos\theta}{\sqrt{\sin^2\theta + \cos^2\theta}}\right)^2 + \left(\frac{h\sin\theta + k\cos\theta}{\sqrt{\sin^2\theta + \cos^2\theta}}\right)^2 = \lambda^2$$

$\Rightarrow \quad (h \sin \theta - k \cos \theta)^2 + (h \sin \theta + k \cos \theta)^2 = 2\lambda^2$

$\Rightarrow \quad h^2 \sin^2 \theta + k^2 \cos^2 \theta = \lambda^2$

$\Rightarrow \quad \dfrac{h^2}{\lambda^2 \operatorname{cosec}^2 \theta} + \dfrac{k^2}{\lambda^2 \sec^2 \theta} = 1$

Hence, the locus of (h, k) is

$\dfrac{x^2}{(\lambda \operatorname{cosec} \theta)^2} + \dfrac{y^2}{(\lambda \sec \theta)^2} = 1$, which is an ellipse having its centre at the intersection point of the given lines.

EXERCISE 26.1

LEVEL-1

1. Find the equation of the ellipse whose focus is $(1, -2)$, the directrix $3x - 2y + 5 = 0$ and eccentricity equal to $1/2$.
2. Find the equation of the ellipse in the following cases:
   (i) focus is $(0, 1)$, directrix is $x + y = 0$ and $e = \frac{1}{2}$.
   (ii) focus is $(-1, 1)$, directrix is $x - y + 3 = 0$ and $e = \frac{1}{2}$.
   (iii) focus is $(-2, 3)$, directrix is $2x + 3y + 4 = 0$ and $e = \frac{4}{5}$.
   (iv) focus is $(1, 2)$, directrix is $3x + 4y - 5 = 0$ and $e = \frac{1}{2}$.
3. Find the eccentricity, coordinates of foci, length of the latus-rectum of the following ellipse:
   (i) $4x^2 + 9y^2 = 1$ (ii) $5x^2 + 4y^2 = 1$ (iii) $4x^2 + 3y^2 = 1$
   (iv) $25x^2 + 16y^2 = 1600$. (v) $9x^2 + 25y^2 = 225$ [NCERT EXEMPLAR]
4. Find the equation to the ellipse (referred to its axes as the axes of x and y respectively) which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{\frac{2}{5}}$.
5. Find the equation of the ellipse in the following cases:
   (i) eccentricity $e = \frac{1}{2}$ and foci $(\pm 2, 0)$
   (ii) eccentricity $e = \frac{2}{3}$ and length of latus-rectum $= 5$
   (iii) eccentricity $e = \frac{1}{2}$ and semi-major axis $= 4$
   (iv) eccentricity $e = \frac{1}{2}$ and major axis $= 12$
   (v) The ellipse passes through $(1, 4)$ and $(-6, 1)$.
   (vi) Vertices $(\pm 5, 0)$, foci $(\pm 4, 0)$ [NCERT]
   (vii) Vertices $(0, \pm 13)$, foci $(0, \pm 5)$ [NCERT]
   (viii) Vertices $(\pm 6, 0)$, foci $(\pm 4, 0)$ [NCERT]
   (ix) Ends of major axis $(\pm 3, 0)$, ends of minor axis $(0, \pm 2)$ [NCERT]
   (x) Ends of major axis $(0, \pm \sqrt{5})$, ends of minor axis $(\pm 1, 0)$ [NCERT]
   (xi) Length of major axis 26, foci $(\pm 5, 0)$ [NCERT]
   (xii) Length of minor axis 16 foci $(0, \pm 6)$ [NCERT]

(xiii) Foci$(\pm 3, 0)$, $a = 4$ [NCERT]

6. Find the equation of the ellipse whose foci are (4, 0) and (– 4, 0), eccentricity = 1/3.
7. Find the equation of the ellipse in the standard form whose minor axis is equal to the distance between foci and whose latus-rectum is 10.
8. Find the equation of the ellipse whose centre is (– 2, 3) and whose semi-axis are 3 and 2 when major axis is (i) parallel to x-axis (ii) parallel to y-axis.
9. Find the eccentricity of an ellipse whose latus-rectum is
   (i) half of its minor axis (ii) half of its major axis.
10. Find the centre, the lengths of the axes, eccentricity, foci of the following ellipse:
    (i) $x^2 + 2y^2 - 2x + 12y + 10 = 0$ (ii) $x^2 + 4y^2 - 4x + 24y + 31 = 0$
    (iii) $4x^2 + y^2 - 8x + 2y + 1 = 0$ (iv) $3x^2 + 4y^2 - 12x - 8y + 4 = 0$
    (v) $4x^2 + 16y^2 - 24x - 32y - 12 = 0$ (vi) $x^2 + 4y^2 - 2x = 0$
11. Find the equation of an ellipse whose foci are at (± 3, 0) and which passes through (4, 1).
12. Find the equation of an ellipse whose eccentricity is 2/3, the latus-rectum is 5 and the centre is at the origin.
13. Find the equation of an ellipse with its foci on y-axis, eccentricity 3/4, centre at the origin and passing through (6, 4).
14. Find the equation of an ellipse whose axes lie along coordinate axes and which passes through (4, 3) and (–1, 4).
15. Find the equation of an ellipse whose axes lie along the coordinate axes, which passes through the point (–3, 1) and has eccentricity equal to $\sqrt{2/5}$.
16. Find the equation of an ellipse, the distance between the foci is 8 units and the distance between the directrices is 18 units.
17. Find the equation of an ellipse whose vertices are (0, ± 10) and eccentricity $e = \frac{4}{5}$.

LEVEL-2

18. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with x-axis.
19. Find the equation of the set of all points whose distances from (0, 4) are $\frac{2}{3}$ of their distances from the line $y = 9$. [NCERT EXEMPLAR]

ANSWERS

1. $43x^2 + 48y^2 + 12xy - 134x + 228y + 235 = 0$
2. (i) $7x^2 + 7y^2 - 2xy - 16y + 8 = 0$ (ii) $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$
   (iii) $325(x^2 + y^2 + 4x - 6y + 13) = 16(2x + 3y + 4)^2$
   (iv) $91x^2 + 84y^2 - 24xy - 170x - 360y + 475 = 0$
3. (i) $e = \frac{\sqrt{5}}{3}; \left(\pm\frac{\sqrt{5}}{6}, 0\right); \frac{4}{9}$ (ii) $e = \frac{1}{\sqrt{5}}; \left(0, \pm\frac{1}{2\sqrt{5}}\right); \frac{4}{5}$
   (iii) $e = \frac{1}{2}; \left(0, \pm\frac{1}{2\sqrt{3}}\right); \frac{\sqrt{3}}{2}$ (iv) $e = \frac{3}{5}, (0, \pm 6); \frac{64}{5}$
   (v) $e = \frac{4}{5}; (\pm 4, 0); \frac{18}{5}$
4. $3x^2 + 5y^2 = 32$

5. (i) $3x^2 + 4y^2 = 48$ (ii) $20x^2 + 36y^2 = 405$ (iii) $3x^2 + 4y^2 = 48$

(iv) $3x^2 + 4y^2 = 108$ (v) $3x^2 + 7y^2 = 115$ (vi) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

(vii) $\frac{x^2}{144} + \frac{y^2}{169} = 1$ (viii) $\frac{x^2}{36} + \frac{y^2}{20} = 1$ (ix) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

(x) $\frac{x^2}{1} + \frac{y^2}{5} = 1$ (xi) $\frac{x^2}{169} + \frac{y^2}{144} = 1$ (xii) $\frac{x^2}{64} + \frac{y^2}{100} = 1$

(xiii) $\frac{x^2}{16} + \frac{y^2}{7} = 1$ 6. $\frac{x^2}{9} + \frac{y^2}{8} = 16$ 7. $x^2 + 2y^2 = 100$

8. (i) $4x^2 + 9y^2 + 16x - 54y + 61 = 0$ (ii) $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

9. (i) $e = \frac{\sqrt{3}}{2}$ (ii) $e = \frac{1}{\sqrt{2}}$

10.

	Centre	Major axis	Minor axis	Eccentricity	Foci
(i)	$(1, -3)$	6	$3\sqrt{2}$	$\frac{1}{\sqrt{2}}$	$\left(1 \pm \frac{3}{\sqrt{2}}, -3\right)$
(ii)	$(2, -3)$	6	3	$\frac{\sqrt{3}}{2}$	$\left(2 \pm \frac{3\sqrt{3}}{2}, -3\right)$
(iii)	$(1, -1)$	4	2	$\frac{\sqrt{3}}{2}$	$(1, <196>1 \pm \sqrt{3})$
(iv)	$(2, 1)$	4	$2\sqrt{3}$	$\frac{1}{2}$	$(2 \pm 1, 1)$
(v)	$(3, 1)$	8	4	$\sqrt{3}/2$	$(3 \pm 2\sqrt{3}, 1)$
(vi)	$(1, 0)$	2	1	$\sqrt{3}/2$	$(1 \pm \sqrt{3}/2), 0)$

11. $\frac{x^2}{18} + \frac{y^2}{9} = 1$ 12. $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$ 13. $\frac{x^2}{43} + \frac{7y^2}{688} = 1$ 14. $\frac{7x^2}{247} + \frac{15y^2}{247} = 1$

15. $3x^2 + 5y^2 = 32$ 16. $\frac{x^2}{36} + \frac{y^2}{20} = 1$ 17. $100x^2 + 36y^2 = 3600$

18. $x^2 + 9y^2 = 81$ 19. $9x^2 + 5y^2 = 180$

HINTS TO NCERT & SELECTED PROBLEMS

5. (vi) Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(i)$$

It is given that vertices are at $(\pm 5, 0)$ and foci are at $(\pm 4, 0)$. Therefore, $a = 5$ and $ae = 4$. Now,

$$b^2 = a^2(1 - e^2) \Rightarrow b^2 = a^2 - (ae)^2 = 25 - 16 = 9$$

Substituting the values of a and b in (i), we obtain $\frac{x^2}{25} + \frac{y^2}{9} = 1$ as the equation of the ellipse.

(vii) Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(i)$$

Its vertices are at $(0, \pm 13)$ and foci are at $(0, \pm 5)$.

$\therefore \quad b = 13$ and $be = 5$

Now, $a^2 = b^2(1-e^2) \Rightarrow a^2 = b^2 - (be)^2 = 169 - 25 = 144$

Substituting the values of a and b in (i), we obtain $\frac{x^2}{144} + \frac{y^2}{169}$ as the equation of the ellipse.

(viii) Proceed as in 5 (vi).

(ix) Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

It is given that ends of major and minor axes are at $(\pm 3, 0)$ and $(0, \pm 2)$. But, coordinates of end points of major and minor axes are $(\pm a, 0)$ and $0, \pm b)$.

$\therefore \quad a = 3$ and $b = 2$.

Hence, the equation of the ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

(x) Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

where $b > a$.

The coordinates of end points of its major and minor axes are $(0, \pm b)$ and $(\pm a, 0)$.

$\therefore \quad b = \sqrt{5}$ and $a = 1$

Hence, the equation of the ellipse is $\frac{x^2}{5} + \frac{y^2}{1} = 1$.

(xi) Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is given that $2a = 26$ and $ae = 5$

$\Rightarrow \quad a = 13$ and $a^2e^2 = 25$

$\Rightarrow \quad a = 13$ and $a^2 - b^2 = 25 \qquad \left[\because\ b^2 = a^2(1-e^2)\right]$

$\Rightarrow \quad a = 13$ and $b^2 = 169 - 25 = 144$

Hence, the equation of the ellipse is $\frac{x^2}{169} + \frac{y^2}{144} = 1$.

(xii) Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, b > a$. It is given that

$2a = 16$ and $be = 6$. Therefore, $a = 8$ and $be = 6$.

Now,

$be = 6 \Rightarrow b^2e^2 = 36 \Rightarrow b^2 - a^2 = 36 \qquad \left[\because\ a^2 = b^2(1-e^2)\right]$

$\Rightarrow \quad b^2 = 64 + 36 = 100$

Hence, the equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{100} = 1$.

(xiii) Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is given that $a = 4$ and $ae = 3$.

$\therefore \quad b^2 = a^2(1-e^2) \Rightarrow b^2 = 16 - 9 = 7$

Hence, the equation of the ellipse is $\frac{x^2}{16}+\frac{y^2}{7}=1$.

11. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then, $ae = 3$ and $\frac{16}{a^2}+\frac{1}{b^2} = 1$

$\Rightarrow \quad \frac{16}{a^2}+\frac{1}{a^2-a^2e^2}=1 \Rightarrow \frac{16}{a^2}+\frac{1}{a^2-9}=1$

$\Rightarrow \quad a^4-26a^2+144=0 \Rightarrow (a^2-18)(a^2-8)=0 \Rightarrow a^2=18, a^2=8.$

$\therefore \quad b^2 = a^2(1-e^2)$ and $ae = 3 \Rightarrow b^2=a^2-9.$

Now, $a^2=18 \Rightarrow b^2=18-9 = 9$

$a^2=8 \Rightarrow b^2=8-9=-1$, which is not possible.

Hence, the equation of the ellipse is $\frac{x^2}{18}+\frac{y^2}{9}=1$.

12. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. We have, $e=\frac{2}{3}$, and $\frac{2b^2}{a}=5$.

Now, $\frac{2b^2}{a}=5 \Rightarrow 2b^2=5a \Rightarrow 2a^2(1-e^2)=5a \Rightarrow 2a\left(1-\frac{4}{9}\right)=5 \Rightarrow a=\frac{9}{2}.\backslash ff$

$\therefore \quad \frac{2b^2}{a} = 5 \Rightarrow b^2=\frac{45}{4}.$

Hence, the equation of the ellipse is $\frac{4x^2}{81}+\frac{4y^2}{45}=1$.

13. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where b > a. We have, $e=\frac{3}{4}$

$\therefore \quad a^2=b^2(1-e^2) \Rightarrow a^2=\frac{7}{16}b^2.$

So, the equation of the ellipse becomes $\frac{x^2}{a^2}+\frac{7y^2}{16a^2}=1$. It passes through (6, 4).

$\therefore \quad \frac{36}{a^2}+\frac{112}{16a^2}=1 \Rightarrow a^2=43.$

$\therefore \quad b^2 = \frac{16a^2}{7} \Rightarrow b^2=\frac{16\times 43}{7}=\frac{688}{7}$

Hence, the equation of the ellipse is $\frac{x^2}{43}+\frac{7y^2}{688}=1$.

14. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. It passes through (4, 3) and (–1, 4).

$\therefore \quad \frac{16}{a^2}+\frac{9}{b^2}=1$ and $\frac{1}{a^2}+\frac{16}{b^2}=1$

$\Rightarrow \quad 16\alpha+9\beta=1$ and $\alpha+16\beta=1$, where $\alpha=\frac{1}{a^2}$ and $\beta=\frac{1}{b^2}$.

Solving these two equations, we get $\alpha=\frac{7}{247}$ and $\beta=\frac{15}{247}$.

Therefore, $a^2=\frac{247}{7}$ and $b^2=\frac{247}{15}$.

Hence, the equation of the ellipse is $7x+15y^2=247$.

16. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

We have,

$$2ae = 8 \text{ and } 2a/e = 18 \Rightarrow 2ae \times \frac{2a}{e} = 8 \times 18 \Rightarrow 4a^2 = 8 \times 18 \Rightarrow a = 6.$$

Now, 2ae = 8 and $a = 6 \Rightarrow e = 2/3$.

$$\therefore \quad b^2 = a^2(1-e^2) = 36\left(1-\frac{4}{9}\right) = 20.$$

Hence, the equation of the ellipse is $\frac{x^2}{36}+\frac{y^2}{20}=1$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If the lengths of semi-major and semi-minor axes of an ellipse are 2 and $\sqrt{3}$ and their corresponding equations are $y-5=0$ and $x+3=0$, then write the equation of the ellipse.
2. Write the eccentricity of the ellipse $9x^2+5y^2-18x-2y-16=0$.
3. Write the centre and eccentricity of the ellipse $3x^2+4y^2-6x+8y-5=0$.
4. PSQ is a focal chord of the ellipse $4x^2+9y^2=36$ such that $SP=4$. If S' is the another focus, write the value of $S'Q$.
5. Write the eccentricity of an ellipse whose latus-rectum is one half of the minor axis.
6. If the distance between the foci of an ellipse is equal to the length of the latus-rectum, write the eccentricity of the ellipse.
7. If S and S' are two foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and B is an end of the minor axis such that $\Delta BSS'$ is equilateral, then write the eccentricity of the ellipse.
8. If the minor axis of an ellipse subtends an equilateral triangle with vertex at one end of major axis, then write the eccentricity of the ellipse.
9. If a latus-rectum of an ellipse subtends a right angle at the centre of the ellipse, then write the eccentricity of the ellipse.

ANSWERS

1. $3x^2+4y^2+18x-40y+115=0$ 2. $\frac{1}{2}$ 3. $(1,-1), \frac{1}{2}$ 4. $\frac{26}{5}$
5. $\frac{\sqrt{3}}{2}$ 6. $\frac{\sqrt{5}-1}{2}$ 7. $\frac{1}{2}$ 8. $\sqrt{\frac{2}{3}}$ 9. $\frac{\sqrt{5}-1}{2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. For the ellipse $12x^2+4y^2+24x-16y+25=0$
   (a) centreis $(-1, 2)$
   (b) lengths of the axes are $\sqrt{3}$ and 1
   (c) eccentricity $=\sqrt{\frac{2}{3}}$
   (d) all of these

2. The equation of the ellipse with focus $(-1, 1)$, directrix $x - y + 3 = 0$ and eccentricity $1/2$ is
   (a) $7x^2 + 2xy + 7y^2 + 10x + 10y + 7 = 0$ (b) $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$
   (c) $7x^2 + 2xy + 7y^2 + 10x - 10y - 7 = 0$ (d) none of these
3. The equation of the circle drawn with the two foci of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as the end-points of a diameter is
   (a) $x^2 + y^2 = a^2 + b^2$ (b) $x^2 + y^2 = a^2$
   (c) $x^2 + y^2 = 2a^2$ (d) $x^2 + y^2 = a^2 - b^2$
4. The eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if its latus-rectum is equal to one half of its minor axis, is
   (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) none of these
5. The eccentricity of the ellipse, if the distance between the foci is equal to the length of the latus-rectum, is
   (a) $\frac{\sqrt{5}-1}{2}$ (b) $\frac{\sqrt{5}+1}{2}$ (c) $\frac{\sqrt{5}-1}{4}$ (d) none of these
6. The eccentricity of the ellipse, if the minor axis is equal to the distance between the foci, is
   (a) $\frac{\sqrt{3}}{2}$ (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{2}}{3}$
7. The difference between the lengths of the major axis and the latus-rectum of an ellipse is
   (a) ae (b) $2ae$ (c) ae^2 (d) $2ae^2$
8. The eccentricity of the conic $9x^2 + 25y^2 = 225$ is
   (a) 2/5 (b) 4/5 (c) 1/3 (d) 1/5 (e) 3/5
9. The latus-rectum of the conic $3x^2 + 4y^2 - 6x + 8y - 5 = 0$ is
   (a) 3 (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{2}{\sqrt{3}}$ (d) none of these
10. The equations of the tangents to the ellipse $9x^2 + 16y^2 = 144$ from the point (2, 3) are
    (a) $y = 3, x = 5$ (b) $x = 2, y = 3$ (c) $x = 3, y = 2$ (d) $x + y = 5, y = 3$
11. The eccentricity of the ellipse $4x^2 + 9y^2 + 8x + 36y + 4 = 0$ is
    (a) $\frac{5}{6}$ (b) $\frac{3}{5}$ (c) $\frac{\sqrt{2}}{3}$ (d) $\frac{\sqrt{5}}{3}$
12. The eccentricity of the ellipse $4x^2 + 9y^2 = 36$ is
    (a) $\frac{1}{2\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{\sqrt{5}}{3}$ (d) $\frac{\sqrt{5}}{6}$
13. The eccentricity of the ellipse $5x^2 + 9y^2 = 1$ is
    (a) 2/3 (b) 3/4 (c) 4/5 (d) 1/2
14. For the ellipse $x^2 + 4y^2 = 9$
    (a) the eccentricity is 1/2 (b) the latus-rectum is 3/2
    (c) a focus is $(3\sqrt{3}, 0)$ (d) a directrix is $x = -2\sqrt{3}$

15. If the latus-rectum of an ellipse is one half of its minor axis, then its eccentricity is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}}{4}$

16. An ellipse has its centre at $(1, -1)$ and semi-major axis $= 8$ and it passes through the point $(1, 3)$. The equation of the ellipse is

(a) $\frac{(x+1)^2}{64}+\frac{(y+1)^2}{16}=1$ (b) $\frac{(x-1)^2}{64}+\frac{(y+1)^2}{16}=1$

(c) $\frac{(x-1)^2}{16}+\frac{(y+1)^2}{64}=1$ (d) $\frac{(x+1)^2}{64}+\frac{(y-1)^2}{16}=1$

17. The sum of the focal distances of any point on the ellipse $9x^2+16y^2=144$ is

(a) 32 (b) 18 (c) 16 (d) 8

18. If $(2, 4)$ and $(10, 10)$ are the ends of a latus-rectum of an ellipse with eccentricity $1/2$, then the length of semi-major axis is

(a) $20/3$ (b) $15/3$ (c) $40/3$ (d) none of these

19. The equation $\frac{x^2}{2-\lambda}+\frac{y^2}{\lambda-5}+1=0$ represents an ellipse, if

(a) $\lambda<5$ (b) $\lambda<2$ (c) $2<\lambda<5$ (d) $\lambda<2$ or $\lambda>5$

20. The eccentricity of the ellipse $9x^2+25y^2-18x-100y-116=0$, is

(a) $25/16$ (b) $4/5$ (c) $16/25$ (d) $5/4$

21. If the major axis of an ellipse is three times the minor axis, then its eccentricity is equal to

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{2\sqrt{2}}{3}$ (e) $\frac{2}{3\sqrt{2}}$

22. The eccentricity of the ellipse $25x^2+16y^2=400$ is

(a) $3/5$ (b) $1/3$ (c) $2/5$ (d) $1/5$

23. The eccentricity of the ellipse $5x^2+9y^2=1$ is

(a) $2/3$ (b) $3/4$ (c) $4/5$ (d) $1/2$

24. The eccentricity of the ellipse $4x^2+9y^2=36$ is

(a) $\frac{1}{2\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{\sqrt{5}}{3}$ (d) $\frac{\sqrt{5}}{6}$

ANSWERS

1. (d)	2. (b)	3. (d)	4. (b)	5. (a)	6. (c)	7. (d)	8. (b)
9. (a)	10. (d)	11. (d)	12. (c)	13. (a)	14. (b)	15. (c)	16. (b)
17. (d)	18. (a)	19. (c)	20. (b)	21. (d)	22. (a)	23. (a)	24. (c)

SUMMARY

1. **An ellipse is the locus of a point in a plane which moves in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed straight line (called directrix) is always constant which is always less than unity.**

The constant ratio of generally denoted by e and is known as the eccentricity of the ellipse.

If S is the focus, ZZ' is the directrix and P is any point on the ellipse, such that M is the foot of perpendicular from P on ZZ', then $SP = e \cdot PM$.

The equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents an ellipse, if

$$\Delta = abc+2fgh-af^2-bg^2-ch^2 \neq 0 \text{ and } h^2<ab.$$

2. The equation of the ellipse whose axes are parallel to the coordinate axes and whose centre is at the origin, is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with the following properties:

	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a<b$
Coordinates of the centre	$(0, 0)$	$(0, 0)$
Coordinates of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, -b)$ and $(0, -b)$
Coordinates of foci	$(ae, 0)$ and $(-ae, 0)$	$(0, be)$ and $(0, -be)$
Length of the major axis	$2a$	$2b$
Length of the minor axis	$2b$	$2a$
Equation of the major axis	$y=0$	$x=0$
Equation of the minor axis	$x=0$	$y=0$
Equations of the directrices	$x=\frac{a}{e}$ and $x=-\frac{a}{e}$	$y=\frac{b}{e}$ and $y=-\frac{b}{e}$
Eccentricity	$e=\sqrt{1-\frac{b^2}{a^2}}$	$e=\sqrt{1-\frac{a^2}{b^2}}$
Length of the latus-rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Focal distances of a point (x, y)	$a \pm ex$	$b \pm ey$

CHAPTER 27

HYPERBOLA

27.1 INTRODUCTION

We have discussed in earlier chapters that a hyperbola is the particular case of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ when $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 > ab$. The analytical definition of a hyperbola is as follows:

HYPERBOLA *A hyperbola is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed line (called directrix) is always constant which is always greater than unity.*

The constant ratio is generally denoted by e and is known as the *eccentricity* of the hyperbola.

If S is the focus, $Z\,Z'$ is the directrix and P is any point on the hyperbola, then by definition

$$\frac{SP}{PM} = e \Rightarrow SP = e\,.\,PM$$

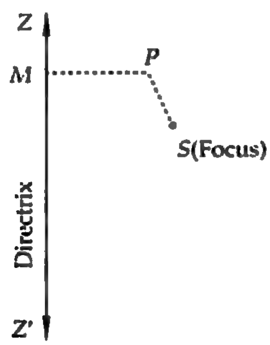


Fig. 27.1

ILLUSTRATION *Find the equation of the hyperbola whose focus is (1, 2), directrix the line $x + y + 1 = 0$ and eccentricity 3/2.*

SOLUTION Let $S(1, 2)$ be the focus and let $P\,(x, y)$ be a point on the hyperbola. Draw perpendicular PM from P on the directrix $x + y + 1 = 0$. Then,

$$SP = e\,PM \qquad \text{[By definition]}$$

$$\Rightarrow \sqrt{(x-1)^2 + (y-2)^2} = \frac{3}{2}\left|\frac{x+y+1}{\sqrt{1^2+1^2}}\right|$$

$$\Rightarrow (x-1)^2 + (y-2)^2 = \frac{9}{4}\left\{\frac{(x+y+1)^2}{2}\right\}$$

$$\Rightarrow 8\left\{(x-1)^2 + (y-2)^2\right\} = 9\,(x+y+1)^2$$

$$\Rightarrow 8x^2 + 8y^2 - 16x - 32y + 40 = 9x^2 + 9y^2 + 9 + 18xy + 18x + 18y$$

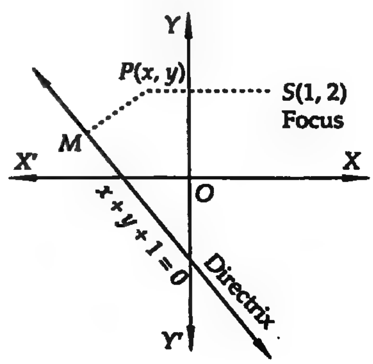


Fig. 27.2

$\Rightarrow \quad x^2 + y^2 + 18xy + 34x + 50y - 31 = 0$, which is the required equation of the hyperbola.

27.2 EQUATION OF THE HYPERBOLA IN STANDARD FORM

Let S be the focus, ZK be the directrix and e be the eccentricity of the hyperbola whose equation is required. Draw SK perpendicular from S on the directrix ZK and divide SK internally and externally at A and A' (on SK produced) respectively in the ratio $e : 1$. Then,

$$SA = e\,AK \quad ...(i)$$

and, $$SA' = e\,A'K \quad ...(ii)$$

Since A and A' are such points that their distances from the focus bear constant ratio e (> 1) to their respective distances from the directrix. Therefore, these points lie on the hyperbola.

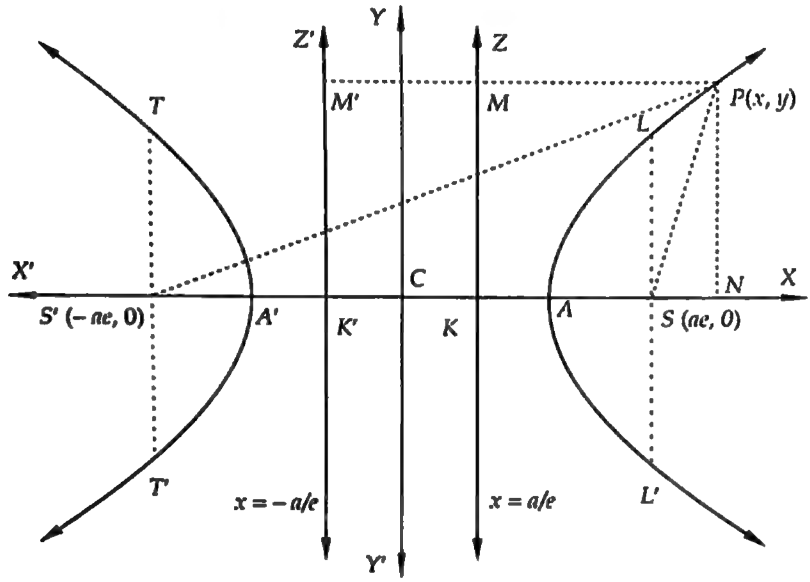


Fig. 27.3

Let $AA' = 2a$ and C be the middle point of AA'. Then, $CA = CA' = a$.

Adding (i) and (ii), we get

$$SA + SA' = e(AK + A'K)$$
$$\Rightarrow \quad CS - CA + CS + CA' = e(CA - CK + CA' + CK)$$
$$\Rightarrow \quad 2CS = 2ae$$
$$\Rightarrow \quad CS = ae$$

Subtracting (i) from (ii), we get

$$SA' - SA = e(A'K - AK)$$
$$\Rightarrow \quad (CS' + SA') - (CS + CA) = e(CA' + CK - CA + CK)$$
$$\Rightarrow \quad AA' = 2e(CK)$$
$$\Rightarrow \quad 2a = 2e(CK)$$
$$\Rightarrow \quad CK = \frac{a}{e}$$

Let C be the origin, CSX the axis of x and a straight line CY through C perpendicular to CX as the axis of Y. Let $P(x, y)$ be any point on the hyperbola and PM, PN be the perpendiculars from P on KZ and KX. By definition of hyperbola

$$SP = e\,PM$$
$$\Rightarrow \quad SP^2 = e^2\,PM^2$$
$$\Rightarrow \quad SP^2 = e^2\,KN^2$$

$$\Rightarrow \quad SP^2 = e^2 (CN - CK)^2$$

$$\Rightarrow \quad (x - ae)^2 + y^2 = e^2 \left(x - \frac{a}{e} \right)^2$$

$$\Rightarrow \quad x^2 (e^2 - 1) - y^2 = a^2 (e^2 - 1)$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{a^2 (e^2 - 1)} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2 (e^2 - 1)$$

This is the equation of the hyperbola in the standard form.

27.2.1 TRACING OF HYPERBOLA

The equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

$$\therefore \quad y = \pm \frac{b}{a} \sqrt{x^2 - a^2} \qquad \text{...(ii)} \qquad \text{and, } x = \pm \frac{a}{b} \sqrt{y^2 + b^2} \qquad \text{...(iii)}$$

In order to trace the graph of the hyperbola (i), we observe the following points:

(a) *Symmetry:* For every value of x there are equal and opposite values of y [see (ii)]. Similarly, for every value of y there are equal and opposite values of x [See (iii)]. So, the curve is symmetric about both the axes.

(b) *Origin:* The curve does not pass through the origin.

(c) *Intersection with the axes :* The curve meets x-axis at $y = 0$. Putting $y = 0$ in (iii), we get $x = \pm a$. So ,the curve meets x-axis at A $(a, 0)$ and A' $(-a, 0)$.
Putting $x = 0$ in (ii), we get imaginary values of y. So ,the curve does not meet y-axis.

(d) *Region:* From (ii), we find that for $-a < x < a$, the values of y are imaginary. So, the curve does not exist between the lines $x = -a$ and $x = a$.
From (ii), we find that $y = 0$ at $x = \pm a$ and if x increases and is greater than a, the values of y also increase. Similarly, if decreases and is less than $-a$, y also increases.
With the help of the above facts and by joining some convenient points on the hyperbola the general shape of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is as shown in Fig. 27.3.

27.2.2 SECOND FOCUS AND SECOND DIRECTRIX OF THE HYPERBOLA

Similar to ellipse it can be shown that the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, b^2 = a^2 (e^2 - 1)$ has second focus S' $(-ae, 0)$ and second directrix $Z'K'$ having equation $x = -\frac{a}{e}$.

27.2.3 VARIOUS ELEMENTS OF HYPERBOLA

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have following points:

VERTICES *In Fig. 27.3, the points A and A', where the curve meets the line joining the foci S and S', are called the vertices of the hyperbola. The coordinates of A and A' are (a, 0) and (– a, 0) respectively.*

TRANSVERSE AND CONJUGATE AXES *In Fig. 27.3, the straight line joining the vertices A and A' is called the transverse axis of the hyperbola. Its length AA' is generally taken to be 2a.*

The straight line through the centre which is perpendicular to the transverse axis does not meet the hyperbola in real points. But if B, B' be the points on this line such that $CB = CB' = b$, the line BB' is called the conjugate axis such that $BB' = 2b$.

FOCI *In Fig. 27.3, the points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the hyperbola.*

DIRECTRICES *In Fig. 27.3, ZK and $Z'K'$ are two directrices of the hyperbola and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively.*

CENTRE *In Fig. 27.3, the middle point C of AA' bisects every chord of the hyperbola passing through it and is called the centre of the hyperbola.*

27.2.4 ECCENTRICITY

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

$$b^2 = a^2(e^2 - 1)$$

$$\Rightarrow \quad e^2 = \frac{a^2 + b^2}{a^2} = 1 + \frac{b^2}{a^2}$$

$$\Rightarrow \quad e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{(2b)^2}{(2a)^2}} \Rightarrow e = \sqrt{1 + \frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}}$$

27.2.5 LENGTH OF THE LATUS-RECTUM

In Fig. 27.3, LSL' is the latus-rectum and LS is called the semi latus-rectum. $TS'T'$ is also a latus-rectum.

The coordinates of L are (ae, SL). As L lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the coordinates of L will satisfy the equation of the hyperbola.

$$\therefore \quad \frac{(ae)^2}{a^2} - \frac{(SL)^2}{b^2} = 1$$

$$\Rightarrow \quad (SL)^2 = b^2(e^2 - 1)$$

$$\Rightarrow \quad (SL)^2 = b^2\left(\frac{b^2}{a^2}\right) \qquad \left[\because b^2 = a^2(e^2 - 1) \Rightarrow e^2 - 1 = \frac{b^2}{a^2}\right]$$

$$\Rightarrow \quad SL = \frac{b^2}{a}$$

$$\therefore \quad SL = SL' = \frac{b^2}{a}$$

Hence, length of the latus-rectum $= 2(SL) = \frac{2b^2}{a} = 2a(e^2 - 1)$.

27.2.6 FOCAL DISTANCES OF A POINT

The distances of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from its foci are known as the focal distances of that point.

THEOREM *The difference of the focal distances of any point on a hyperbola is constant and equal to the length of the transverse axis of the hyperbola.*

PROOF Let $P(x, y)$ be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (See Fig. 27.3). Then, by definition, we have

$$SP = ePM \text{ and } S'P = e\,PM'.$$

Now, $SP = e\,PM \Rightarrow SP = e\,(NK) = e\,(CN - CK) = e\left(x - \frac{a}{e}\right) = ex - a.$

and, $S'P = e\,PM' \Rightarrow S'P = e\,(NK') = e\,(CN + CK') = e\left(x + \frac{a}{e}\right) = e\,x + a$

$\therefore$ $S'P - SP = (ex + a) - (ex - a) = 2\,a =$ Transverse axis.

Hence, the difference of the focal distances of a point on the hyperbola is constant and is equal to the length of the transverse axis of the hyperbola. Q.E.D

On account of this property, a second definition of the hyperbola may be given as follows:

A hyperbola is the locus of a point which moves in such a way that the difference of its distances from two fixed points (foci) is always constant.

27.2.7 CONJUGATE HYPERBOLA

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

The conjugate hyperbola of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Its shape is as shown in Fig. 27.4.

The eccentricity of the conjugate hyperbola is given by $a^2 = b^2(e^2 - 1)$ and the length of the latus-rectum is $\frac{2\,a^2}{b}$.

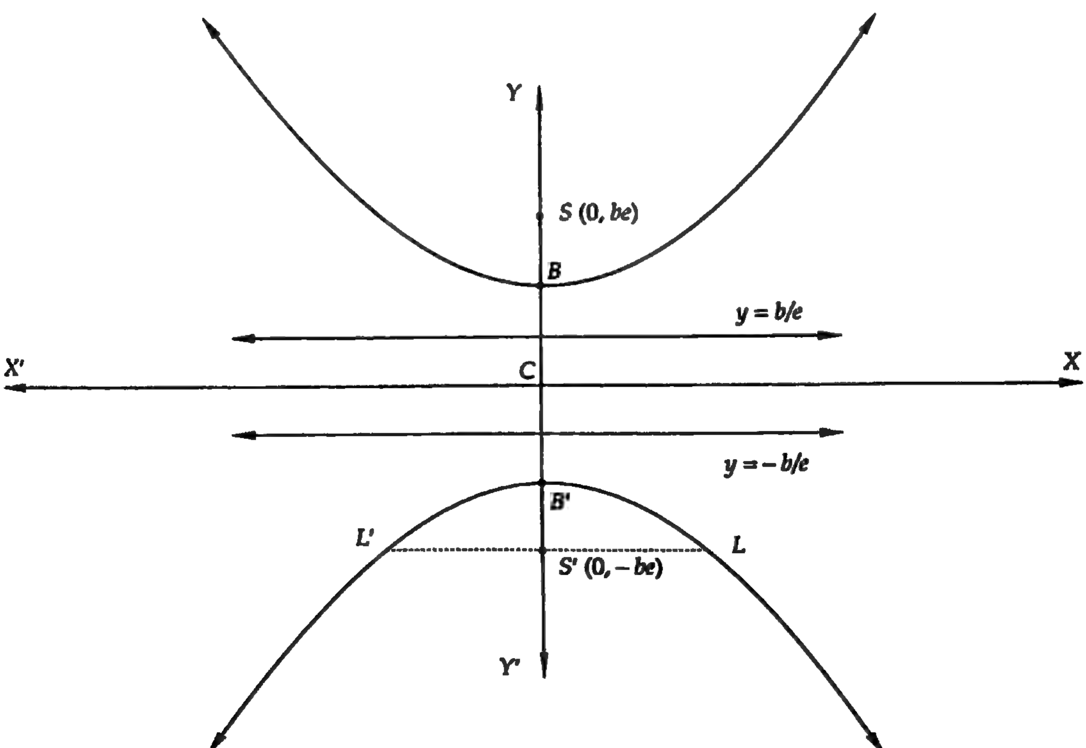


Fig. 27.4

Various results related to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and its conjugate $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given in the following table for ready reference.

	Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	*Conjugate hyperbola* $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
Coordinates of the centre	$(0, 0)$	$(0, 0)$
Coordiantes of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, b)$ and $(0, -b)$
Coordinates of foci	$(\pm ae, 0)$	$(0, \pm be)$
Length of the transverse axis	$2a$	$2b$
Length of the conjugate axis	$2b$	$2a$
Equations of the directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Eccentricity	$e=\sqrt{\frac{a^2+b^2}{a^2}}$ or, $b^2=a^2(e^2-1)$	$e=\sqrt{\frac{b^2+a^2}{b^2}}$ or, $a^2=b^2(e^2-1)$
Length of the latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equation of the transverse axis	$y=0$	$x=0$
Equation of the conjugate axis	$x=0$	$y=0$
Focal distances	$ex \pm a$	$ey \pm b$
Difference of the focal distances of a point	$2a$	$2b$

If the centre of the hyperbola is at the point (h, k) and the directions of the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE EQUATION OF A HYPERBOLA WHEN ITS FOCUS, DIRECTRIX AND ECCENTRICITY ARE GIVEN**

EXAMPLE 1 *Find the equation of the hyperbola whose directrix is* $2x+y=1$, *focus* $(1, 2)$ *and eccentricity* $\sqrt{3}$.

SOLUTION Let S (1, 2) be the fous and P (x, y) be a point on the hyperbola. Draw PM perpendicular from P on the directrix. Then,

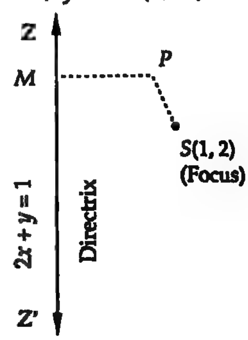


Fig. 27.5

$SP = e\,PM$ [By definition]

$$\Rightarrow \quad \sqrt{(x-1)^2+(y-2)^2} = \sqrt{3}\left|\frac{2x+y-1}{\sqrt{2^2+1^2}}\right|$$

$$\Rightarrow \quad (x-1)^2+(y-2)^2 = \frac{3(2x+y-1)^2}{5}$$

$$\Rightarrow \quad 5\{(x-1)^2+(y-2)^2\} = 3(2x+y-1)^2$$

$\Rightarrow \quad 5x^2 + 5y^2 - 10x - 20y + 25 = 3(4x^2 + y^2 + 1 + 4xy - 4x - 2y)$

$\Rightarrow \quad 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$, which is the required equation of the hyperbola.

Type II **ON FINDING THE CENTRE, LENGTHS OF TRANSVERSE AND CONJUGATE AXES, ECCENTRICITY, FOCI, VERTICES, LATUS-RECTUM, DIRECTRICES etc. OF A GIVEN HYPERBOLA**

EXAMPLE 2 *For the following hyperbolas find the lengths of transverse and conjugate axes, eccentricity and coordinates of foci and vertices; length of the latus-rectum, equations of the directrices:*

(i) $16x^2 - 9y^2 = 144$ (ii) $3x^2 - 6y^2 = -18$

SOLUTION (i) The equation $16x^2 - 9y^2 = 144$ can be written as $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a^2 = 9$ and $b^2 = 16$.

Length of the transverse axis: The length of the transverse axis $= 2a = 6$

Length of the conjugate axis: The length of the conjugate axis $= 2b = 8$

Eccentricity: The eccentricity e is given by $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$

Foci: The coordinates of the foci are $(\pm ae, 0)$ *i.e.* $(\pm 5, 0)$

Vertices: The coordinates of the vertices are $(\pm a, 0)$ *i.e.* $(\pm 3, 0)$.

Latus-rectum: The length of the latus-rectum $= \frac{2b^2}{a} = \frac{32}{3}$

Equations of the directrices: The equations of the directrices are $x = \pm \frac{a}{e}$ *i.e.* $x = \pm \frac{9}{5}$.

(ii) The equation $3x^2 - 6y^2 = -18$ can be written as $\frac{x^2}{6} - \frac{y^2}{3} = -1$.

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, where $a^2 = 6$ and $b^2 = 3$.

Length of the transverse axis: The length of the transverse axis $= 2b = 2\sqrt{3}$.

Length of the conjugate axis: The length of the conjugate axis $= 2a = 2\sqrt{6}$.

Eccentricity: The eccentricity e is given by $e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{6}{3}} = \sqrt{3}$

Foci: The coordinates of the foci are $(0, \pm be)$ i.e. $(0, \pm 3)$

Vertices: The coordinates of the vertices are $(0, \pm b)$ *i.e.* $(0, \pm \sqrt{3})$

Latusrectum: The length of the latusrectum $= \frac{2a^2}{b} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$.

Equations of the directrices: The equations of the directrices are $y = \pm b/e$ *i.e.* $y = \pm 1$

EXAMPLE 3 *Show that the equation $9x^2 - 16y^2 - 18x + 32y - 151 = 0$ represents a hyperbola. Find the coordinates of the centre, lengths of the axes, eccentricity, latus-rectum, coordinates of foci and vertices, equations of the directrices of the hyperbola.*

SOLUTION We have,

$$9x^2 - 16y^2 - 18x + 32y - 151 = 0$$

$$\Rightarrow \quad 9(x^2 - 2x) - 16(y^2 - 2y) = 151$$

$\Rightarrow \quad 9(x^2 - 2x + 1) - 16(y^2 - 2y + 1) = 151 + 9 - 16$

$\Rightarrow \quad 9(x-1)^2 - 16(y-1)^2 = 144$

$$\Rightarrow \quad \frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1 \qquad \text{...(i)}$$

Shifting the origin at (1, 1) without rotating the axes and denoting the new coordinates with respect to these axes by X and Y, we obtain

$$x = X + 1 \quad \text{and} \quad y = Y + 1 \qquad \text{...(ii)}$$

Using these relations, equations (i) reduces to

$$\frac{X^2}{16} - \frac{Y^2}{9} = 1 \qquad \text{...(iii)}$$

This is of the form $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$, where $a^2 = 16$ and $b^2 = 9$.

Centre: The coordinates of the centre with respect to the new axes are $(X = 0, Y = 0)$.

So, the coordinates of the centre with respect to the old axes are

$(1, 1)$ [Putting $X = 0, Y = 0$ in (ii)]

Transverse axis: Length of the transverse axis $= 2a = 8$

Conjugate axis: Length of the conjugate axis $= 2b = 6$

Eccentricity: The eccentricity e is given by

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$$

Latusrectum: Length of the latusrectum $= \frac{2b^2}{a} = \frac{18}{4} = \frac{9}{2}$.

Foci: The coordinates of foci with respect to the new axes are $(X = \pm ae, Y = 0)$ i.e. $(X = \pm 5,\ Y = 0)$. So, the coordinates of foci with respect to the old axes are

$(1 \pm 5, 1)$ *i.e.* $(6, 1)$ and $(-4, 1)$ [Putting $X = \pm 5,\ Y = 0$ in (ii)]

Vertices: The coordinates of the vertices with respect to the new axes are $(X = \pm a,\ Y = 0)$ i.e. $(X = \pm 4,\ Y = 0)$. So, the coordinates of the vertices with respect to the old axes are

$(\pm 4 + 1, 1)$ i.e., $(5, 1)$ and $(-3, 1)$ [Putting $X = \pm 4,\ Y = 0$ in (ii)]

Directrices: The equations of the directrices with respect to the new axes are $X = \pm \frac{a}{e}$ i.e. $X = \pm \frac{16}{5}$.

So, the equations of the directrices with respect to the old axes are

$x = \pm \frac{16}{5} + 1$ $\left[\text{Putting } X = \pm \frac{16}{5} \text{ in (ii)}\right]$

or, $\quad x = \frac{21}{5}$ and $x = -\frac{11}{5}$

EXAMPLE 4 *Show that the equation $x^2 - 2y^2 - 2x + 8y - 1 = 0$ represents a hyperbola. Find the coordinates of the centre, lengths of the axes, eccentricity, latusrectum, coordinates of foci and vertices and equations of directrices of the hyperbola.*

SOLUTION We have,

$$x^2 - 2y^2 - 2x + 8y - 1 = 0 \Rightarrow (x^2 - 2x) - 2(y^2 - 4y) = 1$$

$\Rightarrow \quad (x^2 - 2x + 1) - 2(y^2 - 4y + 4) = -6$

$\Rightarrow \quad (x-1)^2 - 2(y-2)^2 = -6$

$$\Rightarrow \quad \frac{(x-1)^2}{(\sqrt{6})^2} - \frac{(y-2)^2}{(\sqrt{3})} = -1 \qquad \text{...(i)}$$

Shifting the origin at (1, 2) without rotating the coordinate axes and denoting the new coordinates with respect to these axes by X and Y, we obtain

$$x = X + 1 \text{ and } y = Y + 2 \qquad \text{...(ii)}$$

Using these relations, equation (i) reduces to

$$\frac{X^2}{(\sqrt{6})^2} - \frac{Y^2}{(\sqrt{3})^2} = -1 \qquad \text{...(iii)}$$

This equation is of the form $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = -1$, where $a^2 = (\sqrt{6})^2$ and $b^2 = (\sqrt{3})^2$.

Centre: The coordinates of the centre with respect to the new axes are ($X = 0$, $Y = 0$). So, the coordinates of the centre with respect to the old axes are

$(1, 2)$ [Putting $X = 0, Y = 0$ in (ii)]

Lengths of the axes: Since the transverse axis of the hyperbola is along new Y-axis.

$\therefore$ Transverse axis $= 2b = 2\sqrt{3}$ and, Conjugate axis $= 2a = 2\sqrt{6}$.

Eccentricity: The eccentricity e is a given by

$$e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{6}{3}} = \sqrt{3}$$

Latusrectum: Length of the latusrectum $= \frac{2a^2}{b} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$

Foci: The coordinates of foci with respect to the new axes are ($X = 0$, $Y = \pm be$) i.e. ($X = 0$, $Y = \pm 3$). So, the coordinates of foci with respect to the old axes are

$(1, 2 \pm 3)$ *i.e.* $(1, 5)$ and $(1, -1)$ [Putting $X = 0, Y = \pm 3$ in (ii)]

Vertices: The coordinates of the vertices with respect to the new axes are $X = 0, Y = \pm b$) i.e. ($X = 0, Y = \pm\sqrt{3}$)

So, the coordinates of the vertices with respect to the old axes are

$(1, 2 \pm \sqrt{3})$ i.e. $(1, 2 + \sqrt{3})$ and $(1, 2 - \sqrt{3})$ [Putting $X = 0, Y = \pm\sqrt{3}$ in (ii)]

Directrices: The equations of the directrices with respect to the new axes are $Y = \pm b/e$ i.e. $Y = \pm 1$. So, the equations of the directrices with respect to the old axes are

$y = 2 \pm 1$ i.e. $y = 1$ and $y = 3$ [Putting $Y = \pm 2$ in (ii)]

Type III **ON FINDING THE EQUATION OF A HYPERBOLA WHEN SOME OF ITS PARTS ARE GIVEN**

EXAMPLE 5 *Find the equation of the hyperbola, referred to its principal axes as axes of coordinates, in the following cases:*

(i) *Vertices at* $(\pm 5, 0)$, *Foci at* $(\pm 7, 0)$ (ii) *Vertices at* $(0, \pm 7)$, $e = \frac{4}{3}$

SOLUTION (i) Since the vertices lie on x-axis, so let the equation of the required hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

The coordinates of its vertices and foci are $(\pm a, 0)$ and $(\pm ae, 0)$ respectively. But, the coordinates of vertices and foci are given as $(\pm 5, 0)$ and $(\pm 7, 0)$ respectively.

$$\therefore \quad a = 5 \text{ and } ae = 7 \Rightarrow e = \frac{7}{5}$$

Now, $b^2 = a^2(e^2 - 1) \Rightarrow b^2 = 25\left(\frac{49}{25} - 1\right) = 24.$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{25}-\frac{y^2}{24}=1$ as the equation of the required hyperbola.

(ii) Since the vertices of the required hyperbola lie on y-axis. So, let its equation be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = -1 \quad \text{...(i)}$$

The coordinates of vertices of this hyperbola are $(0, \pm b)$ and the coordinates of vertices are given as $(\pm 7, 0)$. So, $b = 7$.

Now, $a^2 = b^2(e^2-1) \Rightarrow a^2 = 49\left(\frac{16}{9}-1\right) \Rightarrow a^2 = 49\times\frac{7}{9} = \frac{343}{9}$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{9x^2}{343}-\frac{y^2}{49} = -1$ as the equation of the desired hyperbola.

EXAMPLE 6 *Referred to the principal axes as the axes of coordinates find the equation of the hyperbola whose foci are at* $(0, \pm\sqrt{10})$ *and which passes through the point* $(2, 3)$.

SOLUTION Since the vertices are on y-axis, so let the equation of the required hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = -1 \quad \text{...(i)}$$

It passes through $(2, 3)$.

$$\therefore \quad \frac{4}{a^2}-\frac{9}{b^2} = -1$$

$$\Rightarrow \quad \frac{4}{b^2(e^2-1)}-\frac{9}{b^2} = -1 \quad [\because a^2 = b^2(e^2-1)]$$

$$\Rightarrow \quad \frac{4}{b^2e^2-b^2}-\frac{9}{b^2} = -1 \quad \text{...(ii)}$$

The coordinates of foci are given to be $(0, \pm\sqrt{10})$.

$$\therefore \quad be = \sqrt{10} \Rightarrow b^2e^2 = 10 \quad \text{...(iii)}$$

From (ii) and (iii), we get

$$\frac{4}{10-b^2}-\frac{9}{b^2} = -1$$

$$\Rightarrow \quad 4b^2-9(10-b^2) = -b^2(10-b^2)$$

$$\Rightarrow \quad 13b^2-90 = -10b^2+b^4$$

$$\Rightarrow \quad b^4-23b^2+90 = 0 \Rightarrow (b^2-18)(b^2-5) = 0 \Rightarrow b^2 = 18 \text{ or, } b^2 = 5.$$

Now, $a^2 = b^2(e^2-1) \Rightarrow a^2 = (be)^2-b^2 \Rightarrow a^2 = 10-b^2$ $\quad [\because be = \sqrt{10}]$

If $b^2 = 18$, then $a^2 = 10-b^2 \Rightarrow a^2 = 10-18 = -8$, which is not possible.

$\therefore \quad b^2 = 5$ and hence $a^2 = 10-b^2 \Rightarrow a^2 = 10-5 = 5$.

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{5}-\frac{y^2}{5} = -1$ i.e. $x^2-y^2 = -5$ as the equation of the required hyperbola.

EXAMPLE 7 *Find the equation of the hyperbola, the length of whose latusrectum is 8 and eccentricity is* $3/\sqrt{5}$.

SOLUTION Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

The length of its latusrectum is $\frac{2b^2}{a}$. It is given that the length of its latusrectum is 8.

$\therefore \quad \frac{2b^2}{a} = 8$

$\Rightarrow \quad b^2 = 4a \Rightarrow a^2(e^2 - 1) = 4a \qquad \left[\because b^2 = a^2(e^2 - 1)\right]$

$\Rightarrow \quad a(e^2 - 1) = 4 \Rightarrow a\left(\frac{9}{5} - 1\right) = 4 \Rightarrow a = 5$

Putting $a = 5$ in $b^2 = 4a$, we get $b^2 = 20$.

Substituting the values of a and b in (i), we obtain $\frac{x^2}{25} - \frac{y^2}{20} = 1$ as the required equation of the hyperbola

EXAMPLE 8 *The foci of a hyperbola coincide with the foci of the ellipse* $\frac{x^2}{25} + \frac{y^2}{9} = 1$. *Find the equation of the hyperbola, if its eccentricity is 2.*

SOLUTION The equation of the ellipse is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 = 25$ and $b^2 = 9$.

Let e be the eccentricity of the ellipse. Then,

$$e = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow e = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

So, the coordinates of foci are $(\pm ae, 0)$ i.e. $(\pm 4, 0)$.

It is given that the foci of the hyperbola coincide with the foci of the ellipse. So, the coordinates of foci of the hyperbola are $(\pm 4, 0)$.

Let e' be the eccentricity of the required hyperbola and its equation be

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1 \qquad \text{...(i)}$$

The coordinates of its foci are $(\pm a'e', 0)$.

$\therefore \quad a'e' = 4 \Rightarrow 2a' = 4 \Rightarrow a' = 2 \qquad [\because e = 2]$

Also, $\quad b'^2 = a'^2(e'^2 - 1) \Rightarrow b'^2 = 4(4 - 1) = 12.$

Substituting the values of a' and b' in (i), we obtain $\frac{x^2}{4} - \frac{y^2}{12} = 1$ as the equation of the required hyperbola.

EXAMPLE 9 *Find the equation of the hyperbola whose conjugate axis is 5 and the distance between the foci is 13.*

SOLUTION Let $2a$ and $2b$ be the transverse and conjugate axes and e be the eccentricity. Let the centre be the origin and the transverse and the conjugate axes the coordinate axes. Then, the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

We have, $2b = 5$ and $2ae = 13$.

Now, $\quad b^2 = a^2(e^2 - 1)$

$\Rightarrow \quad b^2 = a^2e^2 - a^2$

$$\Rightarrow \quad \frac{25}{4} = \frac{169}{4} - a^2 \Rightarrow a^2 = \frac{144}{4} \Rightarrow a = 6.$$

Substituting the values of a and b in (i), the equation of the hyperbola is

$$\frac{x^2}{36} - \frac{y^2}{25/4} = 1 \Rightarrow 25x^2 - 144y^2 = 900.$$

EXAMPLE 10 *Find the equation of the hyperbola whose foci are (8, 3) and (0, 3) and eccentricity* $= \frac{4}{3}$.

SOLUTION The centre of the hyperbola is the mid-point of the line joining the two foci. So, the coordinates of the centre are $\left(\frac{8+0}{2}, \frac{3+3}{2}\right)$ i.e. $(4, 3)$.

Let $2a$ and $2b$ be the length of transverse and conjugate axes and let e be the eccentricity. Then, the equation of the hyperbola is

$$\frac{(x-4)^2}{a^2} - \frac{(y-3)^2}{b^2} = 1 \qquad \text{...(i)}$$

The coordinates of two foci are (8, 3) and (0, 3).

$\therefore$ Distance between two foci $= \sqrt{(8-0)^2 + (3-3)^2} = 8.$

But, the distance between the two foci is equal to $2ae$.

$$\therefore \quad 2ae = 8 \Rightarrow ae = 4 \Rightarrow \frac{4a}{3} = 4 \Rightarrow a = 3 \qquad \left[\because e = \frac{4}{3}\right]$$

Now, $b^2 = a^2(e^2 - 1) \Rightarrow b^2 = 9\left(-1 + \frac{16}{9}\right) = 7$

Thus, the equation of the hyperbola is

$$\frac{(x-4)^2}{9} - \frac{(y-3)^2}{7} = 1 \qquad \text{[Putting the values of } a \text{ and } b \text{ in (i)]}$$

or, $7x^2 - 9y^2 - 56x + 54y - 32 = 0.$

LEVEL-2

***Type IV* MISCELLANEOUS PROBLEMS ON HYPERBOLA**

EXAMPLE 11 *If e and e' be the eccentricities of a hyperbola and its conjugate, prove that* $\frac{1}{e^2} + \frac{1}{e'^2} = 1.$

SOLUTION Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(i)

Then, the equation of the hyperbola conjugate to (i) is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ ...(ii)

Now, $e = \text{Eccentricity of (i)} = \sqrt{1 + \left(\frac{\text{Conjugate axis}}{\text{Transverse axis}}\right)^2}$

$$\Rightarrow \quad e = \sqrt{1 + \left(\frac{2b}{2a}\right)^2}$$

$$\Rightarrow \quad e^2 = 1 + \frac{b^2}{a^2} \Rightarrow e^2 = \frac{a^2 + b^2}{a^2} \qquad \text{...(iii)}$$

and, $e' = \text{Eccentricity of (ii)} = \sqrt{1 + \left(\frac{\text{Conjugate axis}}{\text{Transverse axis}}\right)^2}$

$$\Rightarrow \quad e' = \sqrt{1+\left(\frac{2a}{2b}\right)^2}$$

$$\Rightarrow \quad e'^2 = 1+\frac{a^2}{b^2} \Rightarrow e'^2 = \frac{a^2+b^2}{b^2} \quad \text{...(iv)}$$

From (iii) and (iv), we have

$$\frac{1}{e^2}+\frac{1}{e'^2} = \frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} \Rightarrow \frac{1}{e^2}+\frac{1}{e'^2} = 1$$

EXAMPLE 12 *Find the locus of the point of intersection of the lines* $\sqrt{3}x-y-4\sqrt{3}\lambda=0$ *and* $\sqrt{3}\lambda x+\lambda y-4\sqrt{3}=0$ *for different values of* λ.

SOLUTION Let (h, k) be the point of intersection of the given lines. Then,

$$\sqrt{3}h-k-4\sqrt{3}\lambda=0 \text{ and } \sqrt{3}\lambda h+\lambda k-4\sqrt{3}=0$$

$$\Rightarrow \quad \sqrt{3}h-k=4\sqrt{3}\lambda \text{ and } \lambda(\sqrt{3}h+k)=4\sqrt{3}$$

$$\Rightarrow \quad (\sqrt{3}h-k)\,\lambda(\sqrt{3}h+k)=(4\sqrt{3}\lambda)(4\sqrt{3})$$

$$\Rightarrow \quad 3h^2-k^2=48$$

Hence, the locus of (h, k) is $3x^2-y^2=48$.

EXERCISE 27.1

LEVEL-1

1. The equation of the directrix of a hyperbola is $x-y+3=0$. Its focus is $(-1, 1)$ and eccentricity 3. Find the equation of the hyperbola.
2. Find the equation of the hyperbola whose
   (i) focus is $(0, 3)$, directrix is $x+y-1=0$ and eccentricity $=2$
   (ii) focus is $(1, 1)$, directrix is $3x+4y+8=0$ and eccentricity $=2$
   (iii) focus is $(1, 1)$ directrix is $2x+y=1$ and eccentricity $=\sqrt{3}$
   (iv) focus is $(2, -1)$, directrix is $2x+3y=1$ and eccentricity $=2$
   (v) focus is $(a, 0)$, directrix is $2x-y+a=0$ and eccentricity $=\frac{4}{3}$
   (vi) focus is $(2, 2)$, directrix is $x+y=9$ and eccentricity $=2$.
3. Find the eccentricity, coordinates of the foci, equations of directrices and length of the latus-rectum of the hyperbola
   (i) $9x^2-16y^2=144$ **[NCERT EXEMPLAR]** (ii) $16x^2-9y^2=-144$
   (iii) $4x^2-3y^2=36$ (iv) $3x^2-y^2=4$
   (v) $2x^2-3y^2=5$.
4. Find the axes, eccentricity, latus-rectum and the coordinates of the foci of the hyperbola $25x^2-36y^2=225$.
5. Find the centre, eccentricity, foci and directrices of the hyperbola
   (i) $16x^2-9y^2+32x+36y-164=0$ (ii) $x^2-y^2+4x=0$
   (iii) $x^2-3y^2-2x=8$.
6. Find the equation of the hyperbola, referred to its principal axes as axes of coordinates, in the following cases:
   (i) the distance between the foci = 16 and eccentricity $=\sqrt{2}$
   (ii) conjugate axis is 5 and the distance between foci = 13
   (iii) conjugate axis is 7 and passes through the point $(3, -2)$.

7. Find the equation of the hyperbola whose
   (i) foci are (6, 4) and (– 4, 4) and eccentricity is 2.
   (ii) vertices are (– 8, – 1) and (16, – 1) and focus is (17, – 1)
   (iii) foci are (4, 2) and (8, 2) and eccentricity is 2.
   (iv) vertices are at (0 ± 7) and foci at $\left(0, \pm \frac{28}{3}\right)$.
   (v) vertices are at $(\pm 6, 0)$ and one of the directrices is $x = 4$. [NCERT EXEMPLAR]
   (vi) foci at $(\pm 2, 0)$ and eccentricity is 3/2. [NCERT EXEMPLAR]
8. Find the eccentricity of the hyperbola, the length of whose conjugate axis is $\frac{3}{4}$ of the length of transverse axis.
9. Find the equation of the hyperboala whose
   (i) focus is at (5, 2), vertex at (4, 2)and centre at (3, 2)
   (ii) focus is at (4, 2), centre at (6, 2) and $e = 2$.
10. If P is any point on the hyperbola whose axis are equal, prove that $SP \,.\, S'P = CP^2$.
11. In each of the following find the equations of the hyperbola satisfying the given conditions:
   (i) vertices $(\pm 2, 0)$, foci $(\pm 3, 0)$ [NCERT]
   (ii) vertices $(0, \pm 5)$, foci $(0, \pm 8)$ [NCERT]
   (iii) vertices $(0, \pm 3)$, foci $(0, \pm 5)$ [NCERT]
   (iv) foci $(\pm 5, 0)$, transverse axis = 8 [NCERT]
   (v) foci $(0, \pm 13)$, conjugate axis = 24 [NCERT]
   (vi) foci $(\pm 3\sqrt{5}, 0)$, the latus-rectum = 8 [NCERT]
   (vii) foci $(\pm 4, 0)$, the latus-rectum = 12 [NCERT]
   (viii) vertices $(0, \pm 6)$, $e = \frac{5}{3}$ [NCERT EXEMPLAR]
   (ix) foci $(0, \pm \sqrt{10})$, passing through (2, 3) [NCERT]
   (x) foci $(0, \pm 12)$, latus-rectum = 36 [NCERT]
12. If the distance between the foci of a hyperbola is 16 and its ecentricity is $\sqrt{2}$, then obtain its equation. [NCERT EXEMPLAR]
13. Show that the set of all points such that the difference of their distances from (4, 0) and (– 4, 0) is always equal to 2 represents a hyperbola. [NCERT EXEMPLAR]

ANSWERS

1. $7(x^2 + y^2) - 18xy + 50x - 50y + 77 = 0$.
2. (i) $x^2 + y^2 + 4xy - 4x + 2y - 7 = 0$
   (ii) $11x^2 + 96xy + 39y^2 + 242x + 306y + 206 = 0$
   (iii) $7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$
   (iv) $3x^2 + 23y^2 + 48xy + 36x - 50y - 61 = 0$
   (v) $19x^2 - 64xy - 29y^2 + 154ax - 32ay - 29a^2 = 0$
   (vi) $x^2 + 4xy + y^2 - 32x - 32y + 154 = 0$.
3.

	Eccentricity	*Foci*	*Directrices*	*L.R.*
(i)	$\frac{5}{4}$	$(\pm 5, 0)$	$5x \mp 16 = 0$	$\frac{9}{2}$
(ii)	$\frac{5}{4}$	$(0, \pm 5)$	$5y \mp 16 = 0$	$\frac{9}{2}$

(iii)	$\frac{\sqrt{13}}{3}$	$(\pm\sqrt{13}, 0)$	$\sqrt{13}\,x \mp 3\sqrt{3} = 0$	$\frac{8}{\sqrt{3}}$
(iv)	2	$\left(\pm\frac{4}{\sqrt{3}}, 0\right)$	$\sqrt{3}\,x \mp 1 = 0$	$4\sqrt{3}$
(v)	$\sqrt{\frac{5}{3}}$	$\left(\pm\frac{5}{\sqrt{6}}, 0\right)$	$\sqrt{2}\,x \mp \sqrt{3} = 0$	$\frac{10}{3}\sqrt{\frac{2}{5}}$

4. Transverse axis = 6, conjugate axis = 5, $e = \frac{\sqrt{61}}{6}$, L.R. $= \frac{25}{6}$, foci $\left(\pm\frac{\sqrt{61}}{2}, 0\right)$.

5.

	Centre	*Eccentricity*	*Foci*	*Directrices*
(i)	$(-1, 2)$	$\frac{5}{3}$	$(4, 2), (-6, 2)$	$5x = 4, 5x + 14 = 0$
(ii)	$(-2, 0)$	$\sqrt{2}$	$(-2 \pm 2\sqrt{2}, 0)$	$x + 2 = \pm\sqrt{2}$
(iii)	$(1, 0)$	$\frac{2\sqrt{3}}{3}$	$(1 \pm 2\sqrt{3}, 0)$	$x = 1 \pm \frac{9}{2\sqrt{3}}$

6. (i) $x^2 - y^2 = 32$ (ii) $25x^2 - 144y^2 = 900$ (iii) $65x^2 - 36y^2 = 441$.

7. (i) $12x^2 - 4y^2 - 24x + 32y - 127 = 0$ (ii) $25x^2 - 144y^2 - 200x - 288y - 3344 = 0$

(iii) $3x^2 - y^2 - 36x + 4y + 101 = 0$ (iv) $\frac{9x^2}{343} - \frac{y^2}{49} = -1$

(v) $\frac{x^2}{36} - \frac{y^2}{45} = 1$ (vi) $\frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$

8. 5/4 9. (i) $3(x-3)^2 - (y-2)^2 = 3$ (ii) $3(x-6)^2 - (y-2)^2 = 3$

11. (i) $\frac{x^2}{4} - \frac{y^2}{5} = 1$ (ii) $\frac{x^2}{39} - \frac{y^2}{25} = -1$ (iii) $\frac{x^2}{16} - \frac{y^2}{9} = -1$ (iv) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

(v) $\frac{x^2}{144} - \frac{y^2}{25} = -1$ (vi) $\frac{x^2}{25} - \frac{y^2}{20} = 1$ (vii) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ (viii) $\frac{x^2}{49} - \frac{9y^2}{343} = 1$

(ix) $\frac{x^2}{5} - \frac{y^2}{5} = -1$ (x) $3y^2 - x^2 = 108$ 12. $x^2 - y^2 = 32$.

HINTS TO NCERT & SELECTED PROBLEMS

11. (i) Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

It is given that its vertices are at $(\pm 2, 0)$ and foci are at $(\pm 3, 0)$.

$\therefore$ $a = 2$ and $ae = 3$

Now,

$$b^2 = a^2(e^2 - 1) \Rightarrow b^2 = (ae)^2 - a^2 = 9 - 4 = 5.$$

Hence, the equation of the hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{5} = 1 \quad \text{[Substituting } a = 2, b^2 = 5 \text{ in (i)]}$$

(ii) Let the equation of the hyperbola be

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Its vertices are at $(0, \pm 5)$ and foci are at $(0, \pm 8)$.

$\therefore \quad b = 5$ and $be = 8.$

Now,

$$a^2 = b^2(e^2 - 1) \Rightarrow a^2 = (be)^2 - b^2 = 64 - 25 = 39$$

Substituting the values of a and b in (i), we get

$-\frac{x^2}{39} + \frac{y^2}{25} = 1$ as the equation of the hyperbola.

(iii) Proceed as in (ii)

(iv) Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad ...(i)$$

Its foci are at $(\pm 5, 0)$ and transverse axis is 8.

$\therefore \quad ae = 5$ and $2a = 8$

$\Rightarrow \quad a^2e^2 = 25$ and $a = 4$

$\Rightarrow \quad a^2 + b^2 = 25$ and $a = 4 \quad \left[\because b^2 = a^2(e^2 - 1) \Rightarrow a^2 e^2 = a^2 + b^2\right]$

$\Rightarrow \quad a = 4, b = 3$

Substituting the values of a and b in (i), we obtain that the equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

(v) The foci of the given hyperbola are on y-axis. So, let its equation be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad ...(i)$$

Its foci are at $(0, \pm 13)$ and conjugate axis is 24.

$\therefore \quad be = 13$ and $2a = 24$

Now, $a^2 = b^2(e^2 - 1) \Rightarrow a^2 = (be)^2 - b^2 \Rightarrow 144 = 169 - b^2 \Rightarrow b^2 = 25$

Substituting $a^2 = 144$ and $b^2 = 25$ in (i), we obtain $\frac{x^2}{144} - \frac{y^2}{25} = -1$ as the equation of the hyperbola.

(vi) Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad ...(i)$$

It is given that its foci are at $(\pm 3\sqrt{5}, 0)$ and latus-rectum = 8.

$\therefore \quad ae = 3\sqrt{5}$ and $\frac{2b^2}{a} = 8$

Now, $\frac{2b^2}{a} = 8$

$\Rightarrow \quad b^2 = 4a$

$\Rightarrow \quad a^2(e^2 - 1) = 4a$

$\Rightarrow \quad 45 - a^2 = 4a \Rightarrow a^2 + 4a - 45 = 0 \Rightarrow (a + 9)(a - 5) = 0 \Rightarrow a = 5$

Again, $\frac{2b^2}{a} = 8 \Rightarrow b^2 = 4a \Rightarrow b^2 = 20 \quad [\because a = 5]$

Substituting the values of a and b in (i), we obtain $\frac{x^2}{25} - \frac{y^2}{20} = 1$ as the equation of the hyperbola.

(vii) Proceed as in (vi)

(viii) Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = -1 \qquad ...(i)$$

Its vertices are at $(0, \pm 6)$ and $e = \frac{5}{3}$.

$\therefore \quad b = 6$ and $e = \frac{5}{3}$

Now, $a^2 = b^2(e^2-1) \Rightarrow a^2 = 36\left(\frac{25}{9}-1\right) = 64$

Substituting the values of a and b in (i), we obtain $\frac{x^2}{64}-\frac{9y^2}{36} = -1$ as the equation of hyperbola.

(ix) Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = -1 \qquad ...(i)$$

Its foci are at $(0, \pm\sqrt{10})$ and passes through (2, 3).

$\therefore \quad be = \sqrt{10}$ and $\frac{4}{a^2}-\frac{9}{b^2} = -1$

$\Rightarrow \quad b^2e^2 = 10$ and $\frac{4}{b^2(e^2-1)}-\frac{9}{b^2} = -1 \qquad \left[\because a^2 = b^2(e^2-1)\right]$

$\Rightarrow \quad \frac{4}{10-b^2}-\frac{9}{b^2} = -1$

$\Rightarrow \quad 4b^2-9(10-b^2) = -b^2(10-b^2)$

$\Rightarrow \quad b^4-23b^2+90 = 0$

$\Rightarrow \quad (b^2-5)(b^2-18) = 0 \Rightarrow b^2 = 5, 18$

Now, $a^2 = b^2(e^2-1)$ and $be = \sqrt{10} \Rightarrow a^2 = 10-b^2$

For $b^2 = 5$, $a^2 = 5$ and for $b^2 = 18$, $a^2 = -8$, which is absurd as $a^2 > 0$.

$\therefore \quad a^2 = 5$ and $b^2 = 5$.

So, the equation of the hyperbola is $\frac{x^2}{5}-\frac{y^2}{5} = -1$. $\qquad \left[\text{Substituting } a^2 = 5, b^2 = 5 \text{ in (i)}\right]$

(x) Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = -1 \qquad ...(i)$$

Its foci are at $(0, \pm 2)$ and latusrectum = 36

$\therefore \quad be = 12$ and $\frac{2a^2}{b} = 36 \Rightarrow be = 12$ and $a^2 = 18b$

Now, $a^2 = 18b$

$\Rightarrow \quad b^2(e^2-1) = 18b \qquad \left[\because a^2 = b^2(e^2-1)\right]$

$\Rightarrow \quad 144-b^2 = 18b \qquad [\because be = 12]$

$\Rightarrow \quad b^2+18b-144 = 0$

$\Rightarrow \quad (b+24)(b-6) = 0$

$\Rightarrow \quad b = 6$

$\therefore \quad a^2 = 18b \Rightarrow a^2 = 18 \times 6 = 108$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{108} - \frac{y^2}{36} = -1$ as the equation of the hyperbola.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the eccentricity of the hyperbola $9x^2 - 16y^2 = 144$.
2. Write the eccentricity of the hyperbola whose latus-rectum is half of its transverse axis.
3. Write the coordinates of the foci of the hyperbola $9x^2 - 16y^2 = 144$.
4. Write the equation of the hyperbola of eccentricity $\sqrt{2}$, if it is known that the distance between its foci is 16.
5. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, write the value of b^2.
6. Write the length of the latus-rectum of the hyperbola $16x^2 - 9y^2 = 144$.
7. If the latus-rectum through one focus of a hyperbola subtends a right angle at the farther vertex, then write the eccentricity of the hyperbola.
8. Write the distance between the directrices of the hyperbola
   $x = 8 \sec\theta, y = 8\tan\theta$.
9. Write the equation of the hyperbola whose vertices are $(\pm 3, 0)$ and foci at $(\pm 5, 0)$.
10. If e_1 and e_2 are respectively the eccentricities of the ellipse $\frac{x^2}{18} + \frac{y^2}{4} = 1$ and the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, then write the value of $2e_1^2 + e_2^2$.

ANSWERS

1. $\frac{5}{4}$ 2. $\frac{1}{\sqrt{2}}$ 3. $(\pm 5, 0)$ 4. $x^2 - y^2 = 32$ 5. 7 6. $\frac{4}{3}$
7. 2 8. $8\sqrt{2}$ 9. $16x^2 - 9y^2 = 144$ 10. 3

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. Equation of the hyperbola whose vertices are $(\pm 3, 0)$ and foci at $(\pm 5, 0)$, is
   (a) $16x^2 - 9y^2 = 144$ (b) $9x^2 - 16y^2 = 144$
   (c) $25x^2 - 9y^2 = 225$ (d) $9x^2 - 25y^2 = 81$
2. If e_1 and e_2 are respectively the eccentricities of the ellipse $\frac{x^2}{18} + \frac{y^2}{4} = 1$ and the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, then the relation between e_1 and e_2 is
   (a) $3e_1^2 + e_2^2 = 2$ (b) $e_1^2 + 2e_2^2 = 3$
   (c) $2e_1^2 + e_2^2 = 3$ (d) $e_1^2 + 3e_2^2 = 2$

3. The distance between the directrices of the hyperbola $x = 8 \sec \theta, y = 8 \tan \theta$, is
(a) $8\sqrt{2}$ (b) $16\sqrt{2}$ (c) $4\sqrt{2}$ (d) $6\sqrt{2}$

4. The equation of the conic with focus at $(1, -1)$ directrix along $x - y + 1 = 0$ and eccentricity $\sqrt{2}$ is
(a) $xy = 1$ (b) $2xy + 4x - 4y - 1 = 0$
(c) $x^2 - y^2 = 1$ (d) $2xy - 4x + 4y + 1 = 0$

5. The eccentricity of the conic $9x^2 - 16y^2 = 144$ is
(a) $\frac{5}{4}$ (b) $\frac{4}{3}$ (c) $\frac{4}{5}$ (d) $\sqrt{7}$

6. A point moves in a plane so that its distances PA and PB from two fixed points A and B in the plane satisfy the relation $PA - PB = k$ $(k \neq 0)$, then the locus of P is
(a) a hyperbola (b) a branch of the hyperbola
(c) a parabola (d) an ellipse

7. The eccentricity of the hyperbola whose latus-rectum is half of its transverse axis, is
(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{\frac{2}{3}}$ (c) $\sqrt{\frac{3}{2}}$ (d) none of these

8. The eccentricity of the hyperbola $x^2 - 4y^2 = 1$ is
(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{5}}{2}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{5}}$

9. The difference of the focal distances of any point on the hyperbola is equal to
(a) length of the conjugate axis (b) eccentricity
(c) length of the transverse axis (d) Latus-rectum

10. The foci of the hyperbola $9x^2 - 16y^2 = 144$ are
(a) $(\pm 4, 0)$ (b) $(0, \pm 4)$ (c) $(\pm 5, 0)$ (d) $(0, \pm 5)$

11. The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$, then equation of the hyperbola is
(a) $x^2 + y^2 = 32$ (b) $x^2 - y^2 = 16$ (c) $x^2 + y^2 = 16$ (d) $x^2 - y^2 = 32$

12. If e_1 is the eccentricity of the conic $9x^2 + 4y^2 = 36$ and e_2 is the eccentricity of the conic $9x^2 - 4y^2 = 36$, then
(a) $e_1^2 - e_2^2 = 2$ (b) $2 < e_2^2 - e_1^2 < 3$
(c) $e_2^2 - e_1^2 = 2$ (d) $e_2^2 - e_1^2 > 3$

13. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2 \alpha + y^2 = 25$, then $\alpha =$
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

14. The equation of the hyperbola whose foci are $(6, 4)$ and $(-4, 4)$ and eccentricity 2, is
(a) $\frac{(x-1)^2}{25/4} - \frac{(y-4)^2}{75/4} = 1$ (b) $\frac{(x+1)^2}{25/4} - \frac{(y+4)^2}{75/4} = 1$
(c) $\frac{(x-1)^2}{75/4} - \frac{(y-4)^2}{25/4} = 1$ (d) none of these

15. The length of the straight line $x - 3y = 1$ intercepted by the hyperbola $x^2 - 4y^2 = 1$ is
(a) $\frac{6}{\sqrt{5}}$ (b) $3\sqrt{\frac{2}{5}}$ (c) $6\sqrt{\frac{2}{5}}$ (d) none of these

16. The latus-rectum of the hyperbola $16x^2 - 9y^2 = 144$ is

(a) 16/3 (b) 32/3 (c) 8/3 (d) 4/3

17. The foci of the hyperbola $2x^2 - 3y^2 = 5$ are

(a) $(\pm 5/\sqrt{6}, 0)$ (b) $(\pm 5/6, 0)$ (c) $(\pm \sqrt{5}/6, 0)$ (d) none of these

18. The eccentricity the hyperbola $x = \frac{a}{2}\left(t + \frac{1}{t}\right), y = \frac{a}{2}\left(t - \frac{1}{t}\right)$ is

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $2\sqrt{3}$ (d) $3\sqrt{2}$

19. The equation of the hyperbola whose centre is (6, 2) one focus is (4, 2) and of eccentricity 2 is

(a) $3(x-6)^2 - (y-2)^2 = 3$ (b) $(x-6)^2 - 3(y-2)^2 = 1$

(c) $(x-6)^2 - 2(y-2)^2 = 1$ (d) $2(x-6)^2 - (y-2)^2 = 1$

20. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}\lambda = 0$ and $\sqrt{3}\lambda x + \lambda y - 4\sqrt{3} = 0$ is a hyperbola of eccentricity

(a) 1 (b) 2 (c) 3 (d) 4

ANSWERS

1. (a)	2. (c)	3. (a)	4. (d)	5. (a)	6. (a)	7. (c)	8. (b)
9. (c)	10. (c)	11. (d)	12. (b)	13. (b)	14. (a)	15. (c)	16. (d)
17. (a)	18. (a)	19. (a)	20. (b)				

SUMMARY

1. A hyperbola is the locus of a point in a plane which moves in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed line (called directrix) is always constant which is always greater than unity.
   The fixed point is called the focus, the fixed line is called the directrix and the constant ratio, generally denoted by e, is known as the eccentricity of the hyperbola.
   The general equation of the hyperbola is of the form
   $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, where $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 > ab$.
2. The equation of the hyperbola having its centre at the origin and axes along the coordinate axes is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with the following property:

	Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Conjugate hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Coordinates of the centre	$(0, 0)$	$(0, 0)$
Coordiantes of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, b)$ and $(0, -b)$
Coordinates of foci	$(\pm ae, 0)$	$(0, \pm be)$
Length of the transverse axis	$2a$	$2b$
Length of the conjugate axis	$2b$	$2a$
Equations of the directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
Eccentricity	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$	$e = \sqrt{\frac{b^2 + a^2}{b^2}}$
	or, $b^2 = a^2 (e^2 - 1)$	or, $a^2 = b^2 (e^2 - 1)$
Length of the latus-rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

3. A hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.
4. If the centre of the hyperbola is at the point (h, k) and the directions of the axes are parallel to the coordinate axes, then its equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

CHAPTER 28

INTRODUCTION TO THREE DIMENSIONAL COORDINATE GEOMETRY

28.1 INTRODUCTION

Uptill now, we have learnt about two-dimensional coordinate system, which is also denoted by R^2. Because we live in a three-dimensional coordinate system. We call this three-dimensional space and denote it by R^3. We introduce a coordinate system in three-dimensional space by choosing three mutually perpendicular axes as a frame of reference. The orientation of the reference system will be *right-handed* in the sense that if you stand at the origin with your right arm along the positive x-axis and your left arm along the positive y-axis, as shown in Fig. 28.1, your head will then point in the direction of positive z-axis.

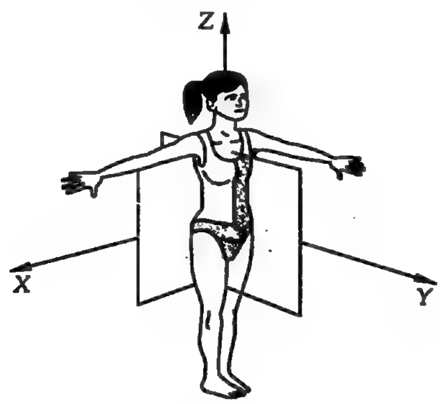


Fig. 28.1

In order to understand a three dimensional co-ordinate system, let us think of a room as shown in Fig. 28.2 and take x-axis and y-axis as lying in the plane of the floor and z-axis as a line

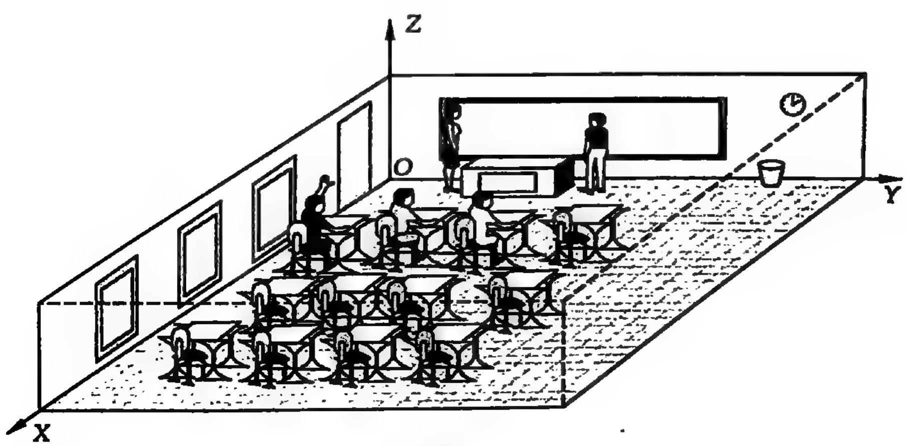


Fig. 28.2

perpendicular to the floor. We observe that the floor has two boundaries as x-axis and y-axis, so we say that it is situated in xy-plane. Similarly, front wall is in the yz-plane and left wall is in xz-plane. The xy, yz and zx-planes are called coordinate planes as shown in Fig. 28.3.

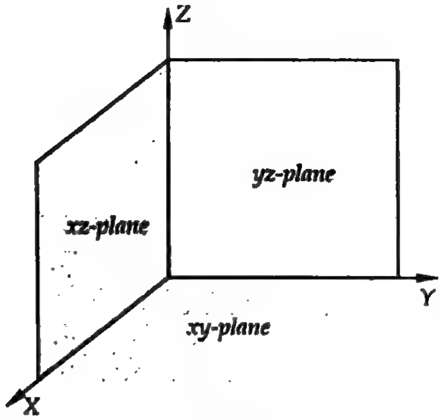


Fig. 28.3

28.2 COORDINATES OF A POINT IN SPACE

As we have studied in two dimensional geometry that two mutually perpendicular lines divide the plane containing them into four parts which are known as quadrants and the lines are known as the coordinate axes. Analogous to it three mutually perpendicular lines in space define three mutually perpendicular planes which in turn divide the space into eight parts known as *octants* and the lines are known as the coordinate axes.

Let $X'OX$, $Y'OY$ and $Z'OZ$ be three mutually perpendicular lines intersecting at O such that two of them viz. $Y'OY$ and $Z'OZ$ lie in the plane of the paper and the third $X'OX$ is perpendicular to the plane of the paper and is projecting out from the plane of the paper (see Fig. 28.4). Let O be the origin and the lines $X'OX$, $Y'OY$ and $Z'OZ$ be x-axis, y-axis and z-axis respectively. These three lines are also called the *rectangular axes of coordinates*. The planes containing the lines $X'OX$, $Y'OY$ and $Z'OZ$ in pairs, determine three mutually perpendicular planes XOY, YOZ and ZOX or simply XY, YZ and ZX which are called *rectangular coordinate planes*.

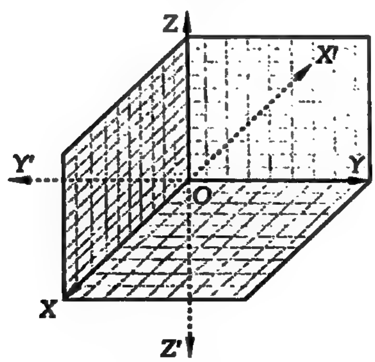


Fig. 28.4

Let P be a point in space (Fig. 28.5). Through P draw three planes parallel to the coordinate planes to meet the axes in A, B and C respectively. Let $OA = x$, $OB = y$ and $OC = z$. These three real numbers taken in this order determined by the point P are called the coordinates of the point P, written as (x, y, z), x, y, z are positive or negative according as they are measured along positive or negative directions of the coordinate axes.

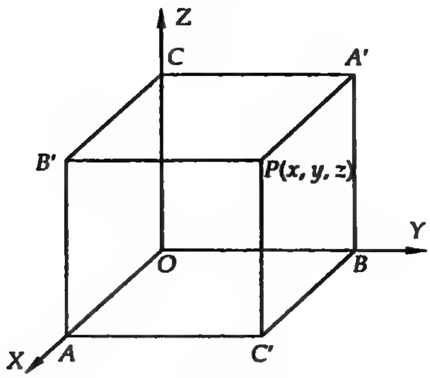


Fig. 28.5

Conversely, given an ordered triad (x, y, z) of real numbers we can always find the point whose coordinates are (x, y, z) in the following manner:

(i) Measure OA, OB, OC along x-axis, y-axis and z-axis respectively.

(ii) Through the points A, B, C draw planes parallel to the coordinate planes YOZ, ZOX and XOY respectively. The point of intersection of these planes is the required point P.

To give another explanation about the coordinates of a point P we draw three planes through P parallel to the coordinate planes. These three planes determine a rectangular parallelopiped which has three pairs of rectangular faces, viz. $PB'AC'$, $OCA'B$; $PA'BC'$, $OAB'C$; $PA'CB'$, $OAC'B$ as shown in Fig. 28.5. Then, we have

$$x = OA = CB' = PA' = \text{Perpendicular distance from } P \text{ on the } YOZ \text{ plane}$$
$$y = OB = A'C = PB' = \text{Perpendicular distance from } P \text{ on the } ZOX \text{ plane}$$
$$z = OC = A'B = PC' = \text{Perpendicular distance from } P \text{ on the } XOY \text{ plane.}$$

Thus, *the coordinates of the point P are the perpendicular distances from P on the three mutually rectangular coordinate planes YOZ, ZOX and XOY respectively.*

Further, since the line PA lies in the plane $PB'AC'$ which is perpendicular to the line OA, we have PA perpendicular to OA. Similarly, PB perpendicular to OB and PC perpendicular to OC.

Thus, *the coordinates of a point are the distances from the origin of the feet of the perpendiculars from the point on the respective coordinate axes.*

Alternatively, to find the coordinates of a point P in space, we first draw perpendicular PM on the xy-plane with M as the foot of this perpendicular as shown in Fig. 28.6. Now, from the point M, we draw perpendicular ML on x-axis with L as the foot of this perpendicular. If $OL = a$, $LM = b$ and $PM = c$, then we say that a, b and c are x, y, and z coordinates, respectively, of the point P in space. In such a case, we say that the point P has coordinates (a, b, c).

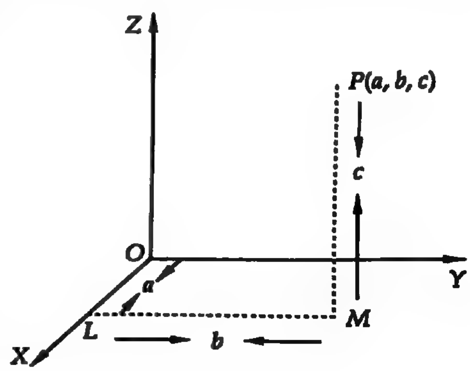


Fig. 28.6

Conversely, if we are given the co-ordinates (a, b, c) of a point P and we have to locate the point, then first fix the point L on x-axis such that $OL = a$. Now, find a point M on the perpendicular to x-axis at point L such that $LM = b$. We can say that M has coordinates (a, b) in xy- plane. Having reached the point M, we draw the perpendicular on xy-plane at point M and locate a point P on this perpendicular such that $PM = a$. The point P so obtained has the coordinates (a, b, c).

Thus, there is one-to-one correspondence between the points in space and the ordered triplets (x, y, z) of real numbers.

28.3 SIGNS OF COORDINATES OF A POINT

To determine the signs of the coordinates of a point in three dimension we follow the sign convention analogous to the sign convention in two dimensional geometry that all distances measured along or parallel to OX, OY, OZ will be positive and distances moved along or parallel to OX', OY', OZ' will be negative.

As discussed in previous section that three mutually perpendicular lines $X'OX, Y'OY$ and $Z'OZ$ determine three mutually perpendicular coordinate planes which in turn divide the space into eight compartments known as octants. The octant having OX, OY and OZ as its edges is denoted by $OXYZ$. Similarly, the other octants are denoted by $OX'YZ$, $OXY'Z$, $OX'Y'Z$, $OXYZ'$, $OXYZ'$, $OXY'Z'$, $OX'Y'Z'$. The signs of the coordinates of a point depend upon the octant in which it lies. Let P be a point and let A, B, C be the feet of the perpendiculars drawn from P on $X'OX, Y'OY$ and $Z'OZ$ respectively. If P lies in octant $OXYZ$, then clearly A, B, C lie on OX, OY and OZ respectively. Therefore, by our sign convention OA, OB and OC are positive. Thus, all the three coordinates of P are positive. If P lies in octant $OX'YZ$, then A, B and C lie on OX', OY and OZ respectively. Therefore, x-coordinate of P is negative and y and z-coordinates are positive.

The following table shows the signs of coordinates of points in various octants:

Octant / coordinate	OXYZ	OX'YZ	OXY'Z	OX'Y'Z	OXYZ'	OX'YZ'	OXY'Z'	OX'Y'Z'
x	+	−	+	−	+	−	+	−
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

REMARK 1 *If a point P lies in x y-plane, then by the definition of coordinates of a point, z-coordinate of P is zero. Therefore, the coordinates of a point on xy-plane are of the form $(x, y, 0)$ and we may take the equation of xy-plane as $z = 0$. Similarly, the coordinates of any point in yz and zx-planes are of the forms $(0, y, z)$ and $(x, 0, z)$ respectively and their equations may be taken as $x = 0$ and $y = 0$ respectively.*

REMARK 2 *If a point lies on the x-axis, then its y and z-coordinates are both zero. Therefore, the coordinates of a point on x-axis are of the form $(x, 0, 0)$ and we may take the equation of x-axis as $y = 0$, $z = 0$. Similarly, the coordinates of a point on y and z-axes are of the form $(0, y, 0)$ and $(0, 0, z)$ respectively and their equations may be taken as $x = 0, z = 0$ and $x = 0, y = 0$ respectively.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *In Fig. 28.7, if the coordinates of point P are (a, b, c), then*

(i) write the coordinates of points A, B, C, D, E and F.

(ii) write the coordinates of the feet of the perpendiculars from the point P to the coordinate axes.

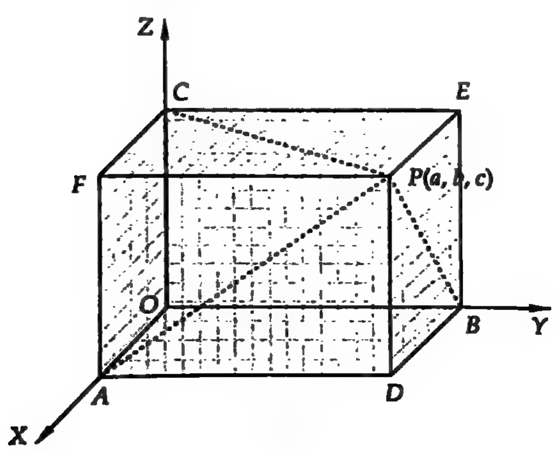


Fig. 28.7

(iii) *write the coordinates of the feet of the perpendicular from the point P on the coordinate planes XY, YZ and ZX.*

(iv) *find the perpendicular distances of point P from XY, YZ and ZX-planes.*

(v) *find the perpendicular distances of the point P from the coordinate axes.*

(vi) *find the coordinates of the reflection of P in XY, YZ and ZX-planes.*

SOLUTION (i) Since the coordinates of P are (a, b, c). Therefore, $OA = a$, $OB = b$ and $OC = c$.

Now, A lies on OX such that $OA = a$. So, the coordinates of A are $(a, 0, 0)$.

Similarly, coordinates of B and C are $(0, b, 0)$ and $(0, 0, c)$ respectively.

Since D lies in XY-plane such that $OA = a$ and $AD = OB = b$. So, the coordinates of D are $(a, b, 0)$.

Point E lies in YZ-plane such that $OB = b$ and $BE = OC = c$. So, the coordinates of E are $(0, b, c)$.

Similarly, coordinates of F are $(a, 0, c)$ as it lies in XZ-plane.

(ii) PA, PB and PC are perpendiculars from P on OX, OY and OZ respectively. So, A, B and C are the feet of perpendiculars from P on OX, OY and OZ respectively. Their coordinates are $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ as discussed in (i).

(iii) Clearly, PD, PE and PF are the perpendiculars from P on XY, YZ and ZX-planes respectively. So, D, E and F are the feet of the perpendiculars from P on XY, YZ and ZX-planes. The coordinates of D, E and F are $D(a, b, 0)$, $E(0, b, c)$ and $F(a, 0, c)$ respectvely as discussed in (i).

(iv) PD, PE and PF are the perpendicular distances of P from XY, YZ and ZX-planes respectively.

$$\therefore \quad PD = OC = c,\ PE = OA = a \text{ and } PF = OB = b.$$

Hence, the perpendicular distances of $P(a, b, c)$ from XY, YZ and ZX planes are c, a and b respectively.

(v) PA, PB and PC are the perpendicular distances of point P from OX, OY and OZ respectively.

In right-angled triangle ADP, we have

$$AP^2 = AD^2 + DP^2$$

$$\Rightarrow \quad PA = \sqrt{AD^2 + DP^2} = \sqrt{b^2 + c^2} \qquad [\because AD = OB = b \text{ and } PD = OC = c]$$

In right-angled triangle PDB right angled at D, we have

$$PB^2 = BD^2 + PD^2$$

$$\Rightarrow \quad PB = \sqrt{BD^2 + PD^2} = \sqrt{a^2 + c^2} \qquad [\because BD = OA = a \text{ and } PD = OC = c]$$

In right-angled triangle PCF right angled at F, we have

$$PC^2 = PF^2 + CF^2$$

$$\Rightarrow \quad PC = \sqrt{PF^2 + CF^2}$$

$$\Rightarrow \quad PC = \sqrt{b^2 + a^2} \qquad [\because PF = AD = OB = b \text{ and } CF = OA = a]$$

$$\Rightarrow \quad PC = \sqrt{a^2 + b^2}$$

(vi) The reflection or image of $P(a, b, c)$ in xy-plane will be as much below the xy-plane as point P is above it, that is, if P' is the reflection of P in xy-plane, then $P'D = PD = c$ and $P'D$ is parallel to OZ'. So, the coordinates of P' are $(a, b, -c)$.

The image of $P(a, b, c)$ in yz-plane will be as much on the back side of yz-plane as the point P is on its front side. Thus, if P'' is the image of P in yz-plane, then P'' lies on PE such that $PE = EP''$. But, $PE = OA = a$. So, the coordinates of P'' are $(-a, b, c)$.

The image of $P(a, b, c)$ in zx-plane will be as much as on the left side of xz-plane as the point P is on its right side. Thus, if P''' is the image of P in zx-plane, then P''' lies on PF produced such that $PF = FP'''$. But, $PF = OB = b$. So, the coordinates of P''' are $(a, -b, c)$.

EXAMPLE 2 *Planes are drawn parallel to the coordinate planes through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Find the length of the edges of the parallelopiped so formed.*

SOLUTION Clearly, PA, PB and PC are the lengths of the edges of the parallelopiped shown in Fig. 28.8.

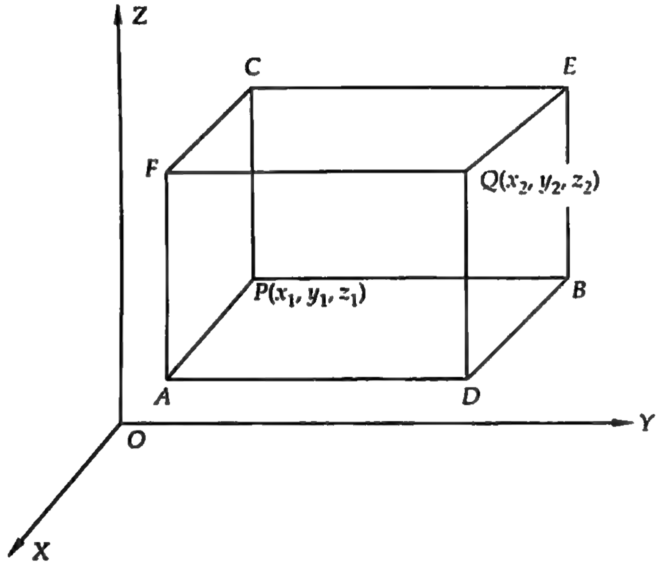


Fig. 28.8

Clearly, $PBEC$, $QDAF$ are planes parallel to yz-plane such that their distances from yz-plane are x_1 and x_2 respectively. So,

$$PA = (\text{Distance between the planes } PBEC \text{ and } QDAF) = x_2 - x_1.$$

PB is the distance between the planes $PAFC$ and $BDQE$ which are parallel to zx-plane and are at distances y_1 and y_2, respectively, from zx-plane.

$$\therefore \quad PB = y_2 - y_1$$

Similarly, PC is the distance between the parallel planes $PBDA$ and $CEQF$ which are at distances z_1 and z_2, respectively, from xy-plane.

$$\therefore \quad PC = z_2 - z_1$$

EXERCISE 28.1

1. Name the octants in which the following points lie:
   (i) $(5, 2, 3)$ (ii) $(-5, 4, 3)$ (iii) $(4, -3, 5)$ (iv) $(7, 4, -3)$
   (v) $(-5, -4, 7)$ (vi) $(-5, -3, -2)$ (vii) $(2, -5, -7)$ (viii) $(-7, 2-5)$ [NCERT]
2. Find the image of:
   (i) $(-2, 3, 4)$ in the yz-plane. (ii) $(-5, 4, -3)$ in the xz-plane.
   (iii) $(5, 2, -7)$ in the xy-plane. (iv) $(-5, 0, 3)$ in the xz-plane.
   (v) $(-4, 0, 0)$ in the xy-plane.
3. A cube of side 5 has one vertex at the point $(1, 0, -1)$, and the three edges from this vertex are, respectively, parallel to the negative x and y axes and positive z-axis. Find the coordinates of the other vertices of the cube.
4. Planes are drawn parallel to the coordinate planes through the points $(3, 0, -1)$ and $(-2, 5, 4)$. Find the lengths of the edges of the parallelopiped so formed.

5. Planes are drawn through the points $(5, 0, 2)$ and $(3, -2, 5)$ parallel to the coordinate planes. Find the lengths of the edges of the rectangular parallelopiped so formed.
6. Find the distances of the point $P(-4, 3, 5)$ from the coordinate axes.
7. The coordinates of a point are $(3, -2, 5)$. Write down the coordinates of seven points such that the absolute values of their coordinates are the same as those of the coordinates of the given point.

ANSWERS

1. (i) $XOYZ$ (ii) $X'OYZ$ (iii) $XOY'Z$ (iv) $XOYZ'$
 (v) $X'OY'Z$ (vi) $X'OY'Z'$ (vii) $XOY'Z'$ (viii) $X'OYZ'$
2. (i) $(2, 3, 4)$ (ii) $(-5, -4, -3)$ (iii) $(5, 2, 7)$ (iv) $(-5, 0, 3)$
 (v) $(-4, 0, 0)$
3. (i) $(1, 0, 4), (1, -5, -1), (1, -5, 4), (-4, 0, -1), (-4, -5, 4), (-4, -5, -1), (4, 0, 4)$
4. 5,5, 5 5. 2, 2, 3 6. x-axis : $\sqrt{34}$; y-axis : $\sqrt{41}$; z-axis: 5
7. $(-3, -2, -5), (-3, -2, 5), (3, -2, -5), (-3, 2-5), (3, 2, 5), (3, 2, -5), (-3, 2, 5)$

28.4 DISTANCE FORMULA

THEOREM *The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by*

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

PROOF Let O be the origin and let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points in space. Let L and M be the feet of the perpendiculars from P and Q on the XOY plane. Then in the XOY plane the coordinates of L and M are (x_1, y_1) and (x_2, y_2) respectively. Therefore, by using distance formula in two dimensional geometry, we get

$$LM = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{...(i)}$$

Draw PN perpendicular to QM. Then

$$PN = LM \quad \text{and} \quad NQ = z_2 - z_1$$

Clearly, ΔPNQ is a right triangle right angled at N.

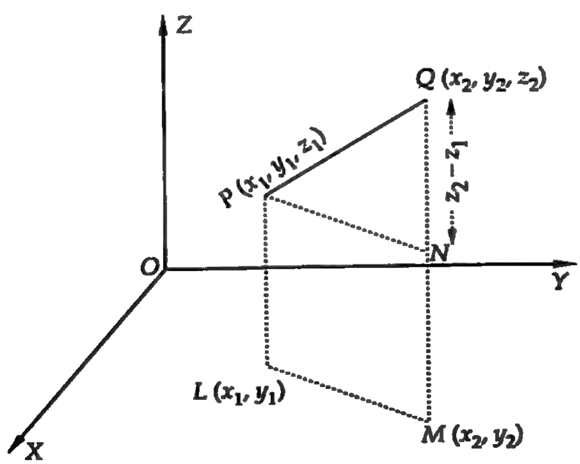


Fig. 28.9

So, by Pythagoras theorem, we obtain

$$PQ^2 = PN^2 + NQ^2$$

$$\Rightarrow \quad PQ^2 = LM^2 + NQ^2 \quad [\because PN = LM]$$

$$\Rightarrow \quad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad \text{[Using (i)]}$$

Thus, the distance between points $P(x_1, y_1, z_1)$ and, $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_1 - y_1)^2 + (z_2 - z_1)^2}$$

Q.E.D.

REMARK *If O is the origin and $P(x, y, z)$ is a point in space, then*

$$OP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the distance between the points $P(-2, 4, 1)$ and $Q(1, 2, -5)$.*

SOLUTION Here, $x_1 = -2, y_1 = 4, z_1 = 1, x_2 = 1, y_2 = 2$ and $z_2 = -5$

$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(1-(-2))^2 + (2-4)^2 + (-5-1)^2}$

$\Rightarrow \quad PQ = \sqrt{9 + 4 + 36} = 7$ units

EXAMPLE 2 *Prove by using distance formula that the points $P(1, 2, 3)$, $Q(-1, -1, -1)$ and $R(3, 5, 7)$ are collinear.*

SOLUTION Using distance formula, we obtain

$$PQ = \sqrt{(-1-1)^2 + (-1-2)^2 + (-1-3)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$$

$$QR = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2} = \sqrt{16 + 36 + 64} = \sqrt{116} = 2\sqrt{29}$$

and, $\quad PR = \sqrt{(3-1)^2 + (5-2)^2 + (7-3)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$

Clearly, $QR = PQ + PR$. Therefore, points Q, P, R are collinear and P lies between Q and R.

EXAMPLE 3 *Determine the point in XY-plane which is equidistant from three points $A(2, 0, 3)$, $B(0, 3, 2)$ and $C(0, 0, 1)$.*

SOLUTION We know that z-coordinate of every point on xy-plane is zero. So, let $P(x, y, 0)$ be a point on xy-plane such that $PA = PB = PC$.

Now, $\quad PA = PB$

$\Rightarrow \quad PA^2 = PB^2$

$\Rightarrow \quad (x-2)^2 + (y-0)^2 + (0-3)^2 = (x-0)^2 + (y-3)^2 + (0-2)^2$

$\Rightarrow \quad 4x - 6y = 0 \Rightarrow 2x - 3y = 0 \quad$ ...(i)

$\quad PB = PC$

$\Rightarrow \quad PB^2 = PC^2$

$\Rightarrow \quad (x-0)^2 + (y-3)^2 + (0-2)^2 = (x-0)^2 + (y-0)^2 + (0-1)^2$

$\Rightarrow \quad -6y + 12 = 0 \Rightarrow y = 2 \quad$ ...(ii)

Putting $y = 2$ in (i), we obtain $x = 3$. Hence, the required point has the coordinates $(3, 2, 0)$.

EXAMPLE 4 *Find the coordinates of a point on Y-axis which is at a distance of $5\sqrt{2}$ from the point $P(3, -2, 5)$.* [NCERT]

SOLUTION Let the point be $A(0, y, 0)$. Then,

$\quad AP = 5\sqrt{2}$

$\Rightarrow \quad \sqrt{(0-3)^2 + (y+2)^2 + (0-5)^2} = 5\sqrt{2}$

$\Rightarrow \quad 9 + (y+2)^2 + 25 = 50 \Rightarrow (y+2)^2 = 16 \Rightarrow y + 2 = \pm 4 \Rightarrow y = 2, -6$

Hence, the coordinates of the required point are $(0, 2, 0)$ and $(0, -6, 0)$.

EXAMPLE 5 *Show that the points $A(0, 1, 2)$, $B(2, -1, 3)$ and $C(1, -3, 1)$ are vertices of an isosceles right-angled triangle.*

SOLUTION Using distance formula, we obtain

$$AB = \sqrt{(2-0)^2 + (-1-1)^2 + (3-2)^2} = \sqrt{4+4+1} = 3$$

$$BC = \sqrt{(1-2)^2 + (-3+1)^2 + (1-3)^2} = \sqrt{1+4+4} = 3$$

and, $$CA = \sqrt{(1-0)^2 + (-3-1)^2 + (1-2)^2} = \sqrt{1+16+1} = 3\sqrt{2}$$

Clearly, $AB = BC$ and $AB^2 + BC^2 = AC^2$. Hence, triangle ABC is an isosceles right-angled triangle which is right angled at B.

EXAMPLE 6 *Find the locus of the point which is equidistant from the points $A(0, 2, 3)$ and $(2, -2, 1)$.*

SOLUTION Let $P(x, y, z)$ be any point which is equidistant from $A(0, 2, 3)$ and $B(2, -2, 1)$. Then,

$$PA = PB$$

$$\Rightarrow \quad PA^2 = PB^2$$

$$\Rightarrow \quad \sqrt{(x-0)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-2)^2 + (y+2)^2 + (z-1)^2}$$

$$\Rightarrow \quad 4x - 8y - 4z + 4 = 0 \Rightarrow x - 2y - z + 1 = 0$$

Hence, the required locus is $x - 2y - z + 1 = 0$.

EXAMPLE 7 *Find the coordinates of a point equidistant from the four points $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.*

SOLUTION Let $P(x, y, z)$ be the required point. Then ,$OP = PA = PB = PC$.

Now, $OP = PA$

$$\Rightarrow \quad OP^2 = PA^2$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = (x-a)^2 + (y-0)^2 + (z-0)^2$$

$$\Rightarrow \quad 0 = -2ax + a^2 \Rightarrow x = a/2$$

Similarly, $OP = PB \Rightarrow y = b/2$ and $OP = PC \Rightarrow z = c/2$.

Hence, the coordinates of the required point are $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

EXERCISE 28.2

LEVEL-1

1. Find the distance between the following pairs of points:
   (i) $P(1, -1, 0)$ and $Q(2, 1, 2)$ (ii) $A(3, 2, -1)$ and $B(-1, -1, -1)$.
2. Find the distance between the points P and Q having coordinates $(-2, 3, 1)$ and $(2, 1, 2)$.
3. Using distance formula prove that the following points are collinear:
   (i) $A(4, -3, -1)$, $B(5, -7, 6)$ and $C(3, 1, -8)$
   (ii) $P(0, 7, -7)$, $Q(1, 4, -5)$ and $R(-1, 10, -9)$.
   (iii) $A(3, -5, 1)$, $B(-1, 0, 8)$ and $C(7, -10, -6)$
4. Determine the points in (i) xy-plane (ii) yz-plane and (iii) zx-plane which are equidistant from the points $A(1, -1, 0)$, $B(2, 1, 2)$ and $C(3, 2, -1)$.
5. Determine the point on z-axis which is equidistant from the points $(1, 5, 7)$ and $(5, 1, -4)$.
6. Find the point on y-axis which is equidistant from the points $(3, 1, 2)$ and $(5, 5, 2)$.
7. Find the points on z-axis which are at a distance $\sqrt{21}$ from the point $(1, 2, 3)$.
8. Prove that the triangle formed by joining the three points whose coordinates are $(1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$ is an equilateral triangle.

9. Show that the points (0, 7, 10), (– 1, 6, 6) and (– 4, 9, 6) are the vertices of an isosceles right-angled triangle.
10. Show that the points A (3, 3, 3), B (0, 6, 3), C (1, 7, 7) and D (4, 4, 7) are the vertices of a square.
11. Prove that the point A (1, 3, 0), B (–5, 5, 2), C (–9, –1, 2) and D (–3, –3, 0) taken in order are the vertices of a parallelogram. Also, show that $ABCD$ is not a rectangle.
12. Show that the points A (1, 3, 4), B (–1, 6, 10), C (–7, 4, 7) and D (–5, 1, 1) are the vertices of a rhombus.
13. Prove that the tetrahedron with vertices at the points O (0, 0, 0), A (0, 1, 1), B (1, 0, 1) and C (1, 1, 0) is a regular one.
14. Show that the points (3, 2, 2), (–1, 4, 2), (0, 5, 6), (2, 1, 2) lie on a sphere whose centre is (1, 3, 4). Find also its radius.
15. Find the coordinates of the point which is equidistant from the four points O (0, 0, 0), A (2, 0, 0), B (0, 3, 0) and C (0, 0, 8).
16. If A (–2, 2, 3) and B (13, –3, 13) are two points. Find the locus of a point P which moves in such a way that $3PA = 2PB$.
17. Find the locus of P if $PA^2 + PB^2 = 2k^2$, where A and B are the points (3, 4, 5) and (–1, 3, –7). [NCERT]
18. Show that the points (a, b, c), (b, c, a) and (c, a, b) are the vertices of an equilateral triangle.
19. Are the points A (3, 6, 9), B (10, 20, 30) and C (25, –41, 5), the vertices of a right-angled triangle? [NCERT]
20. Verify the following:
    (i) (0, 7, –10), (1, 6, –6) and (4, 9, –6) are vertices of an isosceles triangle. [NCERT]
    (ii) (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) are vertices of a right-angled triangle [NCERT]
    (iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are vertices of a parallelogram. [NCERT]
    (iv) (5, –1, 1), (7, –4, 7), (1, –6, 10) and (–1, –3, 4) are the vertices of a rhombus.
21. Find the locus of the points which are equidistant from the points (1, 2, 3) and (3, 2, –1). [NCERT]
22. Find the locus of the point, the sum of whose distances from the points A (4, 0, 0) and B (–4, 0, 0) is equal to 10. [NCERT]
23. Show that the points A (1, 2, 3), B (–1, –2, –1), C (2, 3, 2) and D (4, 7, 6) are the vertices of a parallelogram $ABCD$, but not a rectangle. [NCERT]
24. Find the equation of the set of the points P such that its distances from the points A (3, 4, –5) and B (–2, 1, 4) are equal. [NCERT]

ANSWERS

1. (i) 3 (ii) 5
2. $\sqrt{21}$
4. (i) $\left(\frac{3}{2}, 1, 0\right)$ (ii) $\left(0, \frac{31}{16}, -\frac{3}{16}\right)$ (iii) $\left(\frac{31}{10}, 0, \frac{1}{5}\right)$
5. $\left(0, 0, \frac{3}{2}\right)$
6. (0, 5, 0)
7. (0, 0, 7), (0, 0, –1)
15. $\left(1, \frac{3}{2}, 4\right)$
16. $5(x^2 + y^2 + z^2) + 140x - 60y + 50z - 1235 = 0$
17. $2(x^2 + y^2 + z^2) - 4x - 14y + 4z + 109 - 2k^2 = 0$
19. No
21. $x - 2z = 0$
22. $9x^2 + 25y^2 + 25z^2 - 225 = 0$
23. $10x + 6y - 18z - 29 = 0$

HINTS TO NCERT & SELECTED PROBLEMS

10. Show that $AB = BC = CD = DA$, $AC^2 = AB^2 + BC^2$ and, $BD^2 = AB^2 + AD^2$.

11. Show that $AB = CD$, $BC = DA$ and $AC \neq BD$.

12. Show that $AB = BC = CD = DA$.

13. Show that $OA = OB = OC = AB = BC = CA$

17. Let the coordinates of P be (α, β, γ). Then,

$$PA^2 + PB^2 = 2k^2$$

$$\Rightarrow \quad (\alpha-3)^2 + (\beta-4)^2 + (\gamma-5)^2 + (\alpha+1)^2 + (\beta-3)^2 + (\gamma+7)^2 = 2k^2$$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 - 2\alpha - 7\beta + 2\gamma + \frac{109}{2} = k^2$$

Hence, the locus of (α, β, γ) is

$$x^2 + y^2 + z^2 - 2x - 7y + 2z + \frac{109}{2} = k^2$$

or, $\quad 2(x^2 + y^2 + z^2) - 4x - 14y + 4z + 109 - 2k^2 = 0.$

19. $$AB = \sqrt{(10-3)^2 + (20-6)^2 + (30-9)^2} = \sqrt{686}$$

$$BC = \sqrt{(25-10)^2 + (-41-20)^2 + (5-30)^2} = \sqrt{4571}$$

and, $\quad CA = \sqrt{(25-3)^2 + (-41-6)^2 + (5-9)^2} = \sqrt{2709}$

Clearly, $BC^2 \neq AB^2 + CA^2$, $AB^2 \neq BC^2 + CA^2$ and $AC^2 \neq AB^2 + BC^2$.

Hence, ΔABC is not a right triangle.

20. (i) Let $A(0, 7, -10)$, $B(1, 6, -6)$ and $C(4, 9, -6)$ be the given points. Then,

$AB = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$, $BC = \sqrt{9+9+0} = \sqrt{18} = 3\sqrt{2}$

and, $AC = \sqrt{16+4+16} = 6$

We observe that $AB + BC > AC$, $AB + AC > BC$ and $BC + CA > AB$. So, points, A, B, C form a triangle.

Also, $AB = BC$. So, triangle ABC is isosceles.

(ii) Let $P(0, 7, 10)$, $Q(-1, 6, 6)$ and $R(-4, 9, 6)$ be the given points. Then,

$PQ = 3\sqrt{2}$, $QR = 3\sqrt{2}$ and $PR = 6$.

Clearly, $PQ + QR = 6\sqrt{2} > PR$, $QR + PR > PQ$ and $PQ + PR > QR$.

So, given points form a triangle.

Also, $PR^2 = PQ^2 + QR^2$.

So, ΔPQR is a right triangle right angled at Q.

(iii) Let $A(-1, 2, 1)$ $B(1, -2, 5)$, $C(4, -7, 8)$ and $D(2, -3, 4)$ be the given points. Then,

$AB = 6$, $BC = \sqrt{43}$, $CD = 6$ and $AD = \sqrt{43}$

$\therefore \quad AB = CD = 6$ and $BC = AD = \sqrt{43}$

Hence, $ABCD$ is a parallelogram.

21. Let $P(\alpha, \beta, \gamma)$ be a point equidistant from the points $A(1, 2, 3)$ and $B(3, 2, -1)$. Then,

$$PA = PB$$

$$\Rightarrow \quad PA^2 = PB^2$$

$$\Rightarrow \quad (\alpha-1)^2 + (\beta-2)^2 + (\gamma-3)^2 = (\alpha-3)^2 + (\beta-2)^2 + (\gamma+1)^2$$

$$\Rightarrow \quad 4\alpha - 8\gamma = 0$$

$$\Rightarrow \quad \alpha - 2\gamma = 0$$

Hence, the locus of $P(\alpha, \beta, \gamma)$ is $x - 2z = 0$.

22. Let $P(\alpha, \beta, \gamma)$ be the point such that the sum of its distances from the points $A(4, 0, 0)$ and $B(-4, 0, 0)$ is equal to 10.

i.e. $PA + PB = 10$

$\Rightarrow \sqrt{(\alpha-4)^2+\beta^2+\gamma^2} + \sqrt{(\alpha+4)^2+\beta^2+\gamma^2} = 10$

$\Rightarrow \sqrt{(\alpha+4)^2+\beta^2+\gamma^2} = 10 - \sqrt{(\alpha-4)^2+\beta^2+\gamma^2}$

$\Rightarrow (\alpha+4)^2+\beta^2+\gamma^2 = 100 - 20\sqrt{(\alpha-4)^2+\beta^2+\gamma^2} + (\alpha-4)^2+\beta^2+\gamma^2$

$\Rightarrow 16\,\alpha = 100 - 20\sqrt{(\alpha-4)^2+\beta^2+\gamma^2}$

$\Rightarrow (4\,\alpha - 25) = 5\sqrt{(\alpha-4)^2+\beta^2+\gamma^2}$

$\Rightarrow (4\,\alpha-25)^2 = 25\left\{(\alpha-4)^2+\beta^2+\gamma^2\right\}$

$\Rightarrow 9\,\alpha^2 + 25\beta^2 + 25\gamma^2 - 225 = 0$

Hence, the locus of (α, β, γ) is $9x^2 + 25y^2 + 25z^2 - 225 = 0$.

23. Given points are $A(1, 2, 3)$, $B(-1, -2, -1)$, $C(2, 3, 2)$ and $D(4, 7, 6)$.

$\therefore \quad AB = 6, BC = \sqrt{43}, CD = 6$ and $AD = \sqrt{43}$

$\Rightarrow \quad AB = CE = 6$ and $BC = AD = \sqrt{43}$

So, $ABCD$ is a parallelogram.

Also,

$AB^2 + BC^2 = 36 + 43 = 79$ and $AC = \sqrt{3}$

$\therefore \quad AB^2 + BC^2 \neq AC^2$ i.e. $\angle B$ is not a right angle.

Hence, $ABCD$ is not a rectangle.

24. Let $P(\alpha, \beta, \gamma)$ be one of the points equidistant from $A(3, 4, -5)$ and $B(-2, 1, 4)$. Then,

$PA = PB$

$\Rightarrow PA^2 = PB^2$

$\Rightarrow (\alpha-3)^2 + (\beta-4)^2 + (\gamma+5)^2 = (\alpha+2)^2 + (\beta-1)^2 + (\gamma-4)^2$

$\Rightarrow 10\,\alpha + 6\,\beta - 18\,\gamma - 29 = 0$

Hence, the locus of P is $10x + 6y - 18z - 29 = 0$.

28.5 SECTION FORMULAE

THEOREM (FOR INTERNAL DIVISION) *Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in space and let R be a point on the line segment joining P and Q such that it divides the join of P and Q internally in the ratio $m_1 : m_2$. Then, the coordinates of R are:*

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right)$$

PROOF Let the coordinates of R be (x, y, z). Let PL, QM and RN be perpendiculars from P, Q and R respectively on XOY plane. Clearly, PL, QM and RN lie in a plane which contains the line PQ and is perpendicular to XOY plane. Therefore, points L, M, N are in a straight line which is the intersection of this plane with XOY-plane. Through R draw a line parallel to LM and meeting LP produced in L' and MQ in M'. Clearly, triangles RPL' and $RM'Q$ are equiangular and hence similar;

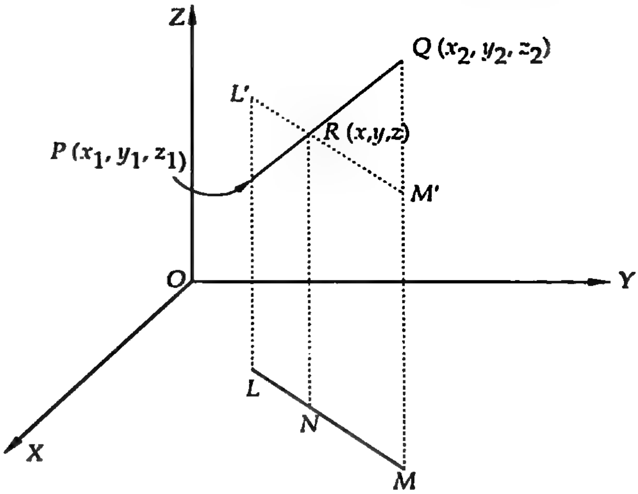


Fig. 28.10

$$\therefore \quad \frac{PL'}{M'Q} = \frac{PR}{PQ}$$

$$\Rightarrow \quad \frac{LL' - LP}{MQ - MM'} = \frac{m_1}{m_2}$$

$$\Rightarrow \quad \frac{NR - LP}{MQ - NR} = \frac{m_1}{m_2}$$

$$\Rightarrow \quad \frac{z - z_1}{z_2 - z} = \frac{m_1}{m_2}$$

$$\Rightarrow \quad z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$$

Similarly, by drawing planes containing PQ and perpendicular to YOZ and ZOX planes respectively, we get

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \quad y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

Hence, the coordinates of R are $\left(\dfrac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \dfrac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \dfrac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right)$

REMARK 1 *If R is the mid-point of the segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then $m_1 = m_2 = 1$ and so the coordinates of R are given by*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

REMARK 2. *If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points, and let R be a point on PQ produced dividing it externally in the ratio $m_1 : m_2$ $(m_1 \neq m_2)$. Then, the coordinates of R are*

$$\left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 + m_2}\right).$$

REMARK 3 *The xy, yz and zx planes divide the segment joining points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $-z_1 : z_2$, $-x_1 : x_2$ and $-y_1 : y_2$ respectively.*

REMARK 4 *The line segment joining points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is divided by the plane $ax + by + cz + d = 0$ in the ratio $-\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the coordinates of the point which divides the join of $P(2,-1,4)$ and $Q(4,3,2)$ in the ratio 2 : 3 (i) internally (ii) externally.*

SOLUTION Let $R(x, y, z)$ be the required point.

(i) If R divides PQ internally in the ratio $2:3$, then

$$x = \frac{2\times 4 + 3\times 2}{2+3}, y = \frac{2\times 3 + 3\times -1}{2+3}, z = \frac{2\times 2 + 3\times 4}{2+3}$$

$$\Rightarrow \quad x = \frac{14}{5}, y = \frac{3}{5}, z = \frac{16}{5}$$

So, the coordinates of point R are $\left(\frac{14}{5}, \frac{3}{5}, \frac{16}{5}\right)$

(ii) If R divides PQ externally in the ratio $2:3$, then

$$x = \frac{2\times 4 - 3\times 2}{2-3}, \ y = \frac{2\times 3 - 3\times -1}{2-3}, \ z = \frac{2\times 2 - 3\times 4}{2-3}$$

$$\Rightarrow \quad x = -2, y = -9, z = 8.$$

So, the coordinates of R are $(-2,-9,8)$.

EXAMPLE 2 *Find the ratio in which the line joining the points $(1,2,3)$ and $(-3,4,-5)$ is divided by the xy-plane. Also, find the coordinates of the point of division.*

SOLUTION Suppose the line joining the points $P(1, 2, 3)$ and $Q(-3, 4 -5)$ is divided be the xy-plane at a point R in the ratio $\lambda:1$. Then, the coordinates of R are

$$\left(\frac{-3\lambda+1}{\lambda+1}, \frac{4\lambda+2}{\lambda+1}, \frac{-5\lambda+3}{\lambda+1}\right) \quad ...(i)$$

Since R lies on xy-plane i.e. $z = 0$. Therefore,

$$\frac{-5\lambda+3}{\lambda+1} = 0 \Rightarrow \lambda = \frac{3}{5}$$

So, the required ratio is $\frac{3}{5}:1$ or, $3:5$. Putting $\lambda = \frac{3}{5}$ in (i), we obtain the coordinates of R as $(-1/2, 11/4, 0)$.

ALITER We know that the xy-plane divides the line segment joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $-z_1 : z_2$. Hence, the required ratio is $-3:-5$ i.e. $3:5$ internally and the coordinates of the point of division are

$$\left(\frac{3\times -3+5\times 1}{3+5}, \frac{3\times 4+5\times 2}{3+5}, \frac{3\times -5+5\times 3}{3+5}\right) = \left(-\frac{1}{2}, \frac{11}{4}, 0\right)$$

EXAMPLE 3 *Find the ratio in which the join the $A(2,1,5)$ and $B(3,4,3)$ is divided by the plane $2x+2y-2z=1$. Also, find the coordinates of the point of division.*

SOLUTION Suppose the plane $2x+2y-2z=1$ divides the line joining the points $A(2,1,5)$ and $B(3,4,3)$ at a point C in the ratio $\lambda:1$. Then, the coordinates of C are

$$\left(\frac{3\lambda+2}{\lambda+1}, \frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+5}{\lambda+1}\right) \quad ...(i)$$

Since point C lies on the plane $2x+2y-2z=1$. Therefore, coordinates of C must satisfy the equation of the plane

i.e. $$2\left(\frac{3\lambda+2}{\lambda+1}\right)+2\left(\frac{4\lambda+1}{\lambda+1}\right)-2\left(\frac{3\lambda+5}{\lambda+1}\right)=1 \Rightarrow 8\lambda-4=\lambda+1 \Rightarrow \lambda=\frac{5}{7}$$

So, the required ratio is $\frac{5}{7}:1$ or $5:7$. Putting $\lambda=\frac{5}{7}$ in (i), the coordinates of the point C of division are $\left(\frac{29}{12},\frac{9}{4},\frac{25}{6}\right)$.

ALITER We know that the line segment joining points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is divided by the plane $ax+by+cz+d=0$ in the ratio $-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}$. Therefore, the line segment joining points $A(2,1,5)$ and $B(3,4,3)$ is divided by the plane $2x+2y-2z-1=0$ in the ratio

$$-\frac{2\times 2+2\times 1-2\times 5-1}{2\times 3+2\times 4-2\times 3-1}=\frac{5}{7} \text{ i.e. } 5:7.$$

The coordinates of the point of division are

$$\left(\frac{5\times 3+7\times 2}{5+7},\frac{5\times 4+7\times 1}{5+7},\frac{5\times 3+7\times 5}{5+7}\right)=\left(\frac{29}{12},\frac{9}{4},\frac{25}{6}\right)$$

EXAMPLE 4 *Using section formula, prove that the three points $A(-2,3,5)$, $B(1,2,3)$ and $C(7,0,-1)$ are collinear.*

SOLUTION Suppose the given points are collinear and C divides AB in the ratio $\lambda:1$. Then, coordinates of C are

$$\left(\frac{\lambda-2}{\lambda+1},\frac{2\lambda+3}{\lambda+1},\frac{3\lambda+5}{\lambda+1}\right)$$

But, coordinates of C are given as $(7, 0, -1)$. Therefore,

$$\frac{\lambda-2}{\lambda+1}=7,\ \frac{2\lambda+3}{\lambda+1}=0 \text{ and, } \frac{3\lambda+5}{\lambda+1}=-1$$

From each of these equations, we obtain $\lambda=-\frac{3}{2}$.

Since each of these equations give the same value of λ. Therefore, the given points are collinear and C divides AB externally in the ratio 3 : 2.

EXAMPLE 5 *The mid-points of the sides of a triangle are $(1, 5, -1)$, $(0, 4, -2)$ and $(2, 3, 4)$. Find its vertices.*

SOLUTION Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of the given triangle, and let $D(1,5,-1)$, $E(0,4,-2)$ and $F(2,3,4)$ be the mid-points of the sides BC, CA and AB respectively.

D is the mid-point of BC

$$\therefore \quad \frac{x_2+x_3}{2}=1,\ \frac{y_2+y_3}{2}=5,\ \frac{z_2+z_3}{2}=-1$$

$$\Rightarrow \quad x_2+x_3=2,\ y_2+y_3=10,\ z_2+z_3=-2 \qquad \text{...(i)}$$

E is the mid-point of CA

$$\therefore \quad \frac{x_1+x_3}{2}=0,\ \frac{y_1+y_3}{2}=4,\ \frac{z_1+z_3}{2}=-2$$

$$\Rightarrow \quad x_1+x_3=0,\ y_1+y_3=8,\ z_1+z_3=-4 \qquad \text{...(ii)}$$

F is the mid-point of AB

$$\therefore \quad \frac{x_1+x_2}{2}=2,\ \frac{y_1+y_2}{2}=3,\ \frac{z_1+z_2}{2}=4$$

$$\Rightarrow \quad x_1+x_2=4,\ y_1+y_2=6,\ z_1+z_2=8 \qquad \text{...(iii)}$$

Adding first three equations in (i), (ii) and (iii), we obtain

$$2(x_1 + x_2 + x_3) = 2 + 0 + 4 \Rightarrow x_1 + x_2 + x_3 = 3.$$

Solving first equations in (i), (ii) and (iii) with $x_1 + x_2 + x_3 = 3$, we obtain:

$$x_1 = 1, x_2 = 3, x_3 = -1.$$

Adding second equations in (i), (ii) and (iii), we obtain

$$2(y_1 + y_2 + y_3) = 10 + 8 + 6 \Rightarrow y_1 + y_2 + y_3 = 12$$

Solving second equations in (i), (ii) and (iii) with $y_1 + y_2 + y_3 = 12$, we obtain

$$y_1 = 2,\ y_2 = 4,\ y_3 = 6.$$

Adding last equations in (i), (ii) and (iii), we obtain

$$2(z_1 + z_2 + z_3) = -2 - 4 + 8 \Rightarrow z_1 + z_2 + z_3 = 1.$$

Solving last equations in (i), (ii) and (iii) with $z_1 + z_2 + z_3 = 1$, we obtain

$$z_1 = 3,\ z_2 = 5,\ z_3 = -7.$$

Thus, the vertices of the triangle are $A(1, 2, 3)$, $B(3, 4, 5)$ and $C(-1, 6, -7)$.

EXAMPLE 6 *Given that $P(3, 2, -4)$, $Q(5, 4, -6)$ and $R(9, 8, -10)$ are collinear. Find the ratio in which Q divides PR.* [NCERT]

SOLUTION Suppose Q divides PR in the ratio $\lambda : 1$. Then, coordinates of Q are

$$\left(\frac{9\lambda + 3}{\lambda + 1}, \frac{8\lambda + 2}{\lambda + 1}, \frac{-10\lambda - 4}{\lambda + 1}\right)$$

But, coordinates of Q are $(5, 4, -6)$. Therefore,

$$\frac{9\lambda + 3}{\lambda + 1} = 5,\ \frac{8\lambda + 2}{\lambda + 1} = 4,\ \frac{-10\lambda - 4}{\lambda + 1} = 6.$$

These three equations give $\lambda = \frac{1}{2}$. So, Q divides PR in the ratio $\frac{1}{2} : 1$ or $1 : 2$.

EXAMPLE 7 *Find the coordinates of the points which trisect the line segment AB, given that $A(2, 1, -3)$ and $B(5, -8, 3)$.*

SOLUTION Let P and Q be the points which trisect AB. Then, $AP = PQ = QB$. Therefore, P divides AB in the ratio $1 : 2$ and Q divides it in the ratio $2 : 1$.

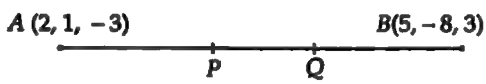


Fig. 28.11

As P divides AB in the ratio $1 : 2$, so coordinates of P are

$$\left(\frac{1 \times 5 + 2 \times 2}{1 + 2}, \frac{1 \times -8 + 2 \times 1}{1 + 2}, \frac{1 \times 3 + 2 \times -3}{1 + 2}\right) = (3, -2, -1)$$

Since Q divides AB in the ratio $2 : 1$, so coordinates of Q are

$$\left(\frac{2 \times 5 + 1 \times 2}{2 + 1}, \frac{2 \times -8 + 1 \times 1}{2 + 1}, \frac{2 \times 3 + 1 \times -3}{1 + 2}\right) = (4, -5, 1)$$

EXAMPLE 8 *Three vertices of a parallelogram $ABCD$ are $A(3, -1, 2)$, $B(1, 2, -4)$ and $C(-1, 1, 2)$. Find the coordinates of the fourth vertex.* [NCERT]

SOLUTION Let the coordinates of the fourth vertex D be (x, y, z). Since diagonals of a parallelogram bisect each other. Therefore, mid-point of AC and BD coincide.

$$\therefore \quad \left(\frac{3 - 1}{2}, \frac{-1 + 1}{2}, \frac{2 + 2}{2}\right) = \left(\frac{1 + x}{2}, \frac{2 + y}{2}, \frac{-4 + z}{2}\right)$$

$\Rightarrow \quad (1, 0, 2) = \left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z-4}{2}\right)$

$\Rightarrow \quad \frac{x+1}{2} = 1, \frac{y+2}{2} = 0, \frac{z-4}{2} = 2$

$\Rightarrow \quad x = 1, y = -2,$ and $z = 8$

Hence, the coordinates of the fourth vertex are $(1, -2, 8)$.

EXAMPLE 9 *Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0, 4, 0) and C (6, 0, 0).* [NCERT]

SOLUTION Let D, E and F be the mid-points of sides BC, CA and AB respectively.

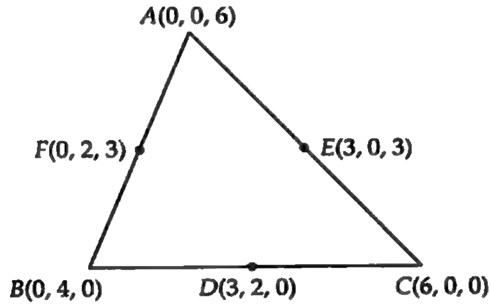


Fig. 28.12

The coordiantes of D, E and F are

$$\left(\frac{0+6}{2}, \frac{4+0}{2}, \frac{0+0}{2}\right) = (3, 2, 0),\ E\left(\frac{6+0}{2}, \frac{0+0}{2}, \frac{0+6}{2}\right) = (0, 2, 3)$$

and $F\left(\frac{0+0}{2}, \frac{0+4}{2}, \frac{6+0}{2}\right) = (0, 2, 3)$ respectively.

$$\therefore \quad AD = \sqrt{(0-3)^2 + (0-2)^2 + (6-0)^2} = \sqrt{9+4+36} = 7$$

$$BE = \sqrt{(0-3)^2 + (4-0)^2 + (0-3)^2} = \sqrt{9+16+9} = \sqrt{34}$$

and, $CF = \sqrt{(6-0)^2 + (0-2)^2 + (0-3)^2} = \sqrt{36+4+9} = 7$

EXAMPLE 10 *Let A (3, 2, 0), B (5, 3, 2), C (−9, 6, −3) be three points forming a triangle. The bisector AD of ∠BAC meets side BC in D. Find the coordinates of D.* [NCERT]

SOLUTION The bisector AD of $\angle BAC$ divides BC in the ratio $AB : AC$.

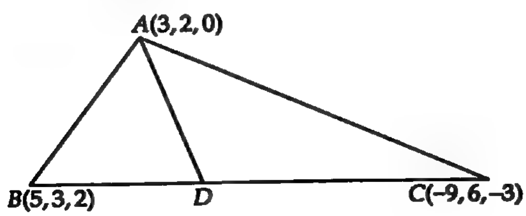


Fig. 28.13

Now, $AB = \sqrt{(3-5)^2 + (2-3)^2 + (0-2)^2} = 3$ and, $AC = \sqrt{(3+9)^2 + (2-6)^2 + (0+3)^2} = 13$

Thus, D divides BC in the ratio $AB : AC$ i.e. $3 : 13$. Hence, the coordinates of D are

$$\left(\frac{3 \times -9 + 13 \times 5}{3+13}, \frac{3 \times 6 + 3 \times 13}{3+13}, \frac{3 \times -3 + 13 \times 2}{3+13}\right) = \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$$

EXAMPLE 11 *If the origin is the centroid of the triangle with vertices $P(2a, 2, 6)$, $Q(-4, 3b, -10)$ and $R(8, 14, 2c)$, find the values of a, b and c.* [NCERT]

SOLUTION The coordinates of the centroid of ΔPQR are

$$\left(\frac{2a-4+8}{3}, \frac{2+3b+14}{3}, \frac{6-10+2c}{3}\right)=\left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right)$$

It is given that the origin is the centroid of ΔPQR.

$$\therefore \quad \frac{2a+4}{3}=0, \frac{16+3b}{3}=0, \frac{2c-4}{3}=0$$

$$\Rightarrow \quad 2a+4=0, 16+3b=0, 2c-4=0$$

$$\Rightarrow \quad a=-2, b=-\frac{16}{3} \text{ and } c=2$$

EXAMPLE 12 *A point R with x-coordinate 4 lies on the line segment joining the points $P(2, -3, 4)$ and $Q(8, 0, 10)$. Find the coordinates of the point R.* [NCERT]

SOLUTION Suppose R divides PQ in the ratio $\lambda : 1$. Then, the coordinates of R are

$$\left(\frac{8\lambda+2}{\lambda+1}, \frac{-3}{\lambda+1}, \frac{10\lambda+4}{\lambda+1}\right)$$

Fig. 28.14

Since x-coordinate of R is 4.

$$\therefore \quad \frac{8\lambda+2}{\lambda+1}=4 \Rightarrow 8\lambda+2=4\lambda+4 \Rightarrow 4\lambda=2 \Rightarrow \lambda=\frac{1}{2}$$

Hence, the coordinates of R are $(4, -2, 6)$.

LEVEL-2

EXAMPLE 13 *Show that the coordinates of the centroid of the triangle with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$* [NCERT]

SOLUTION *Let D be the mid-point of AC. Then, coordinates of D are*

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right).$$

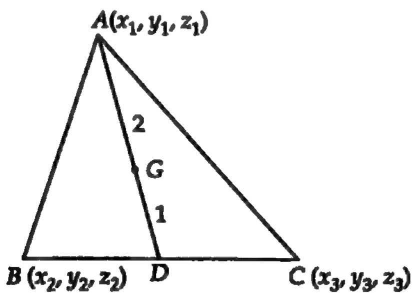


Fig. 28.15

Let G be the centroid of ΔABC. Then, G divides AD in the ratio $2:1$.
So, coordinates of D are

$$\left(\frac{1\cdot x_1+2\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{1\cdot y_1+2\left(\frac{y_2+y_3}{2}\right)}{1+2}, \frac{1\cdot z_1+2\left(\frac{z_2+z_3}{2}\right)}{1+2}\right)$$

i.e., $$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

EXAMPLE 14 *Let P and Q be any two points. Find the coordinates of the point R which divides PQ externally in the ratio $2:1$ and verify that Q is the mid point of PR.*

SOLUTION Let the coordinates of points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then, the coordinates of the point R which divides PQ externally in the ratio $2:1$ are

$$\left(\frac{2x_2 - x_1}{2-1}, \frac{2y_2 - y_1}{2-1}, \frac{2z_2 - z_1}{2-1}\right) = (2x_2 - x_1, 2y_2 - y_1, 2z_2 - z_1)$$

The coordinates of the mid-point of PR are

$$\left(\frac{x_1 + 2x_2 - x_1}{2}, \frac{y + 2y_2 - y_1}{2}, \frac{z_1 + 2z_2 - z_1}{2}\right) = (x_2, y_2, z_2)$$

Clearly, these are the coordinates of point Q. Hence, Q is the mid-point of PR.

EXAMPLE 15 *Prove that the lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent.*

SOLUTION Let $ABCD$ be a tetrahedron such that the coordinates of its vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$. The coordinates of the centroids of faces ABC, DAB, DBC and DCA are respectively

$$G_1\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

$$G_2\left(\frac{x_1 + x_2 + x_4}{3}, \frac{y_1 + y_2 + y_4}{3}, \frac{z_1 + z_2 + z_4}{3}\right)$$

$$G_3\left(\frac{x_2 + x_3 + x_4}{3}, \frac{y_2 + y_3 + y_4}{3}, \frac{z_2 + z_3 + z_4}{3}\right)$$

and, $$G_4\left(\frac{x_4 + x_3 + x_1}{3}, \frac{y_4 + y_3 + y_1}{3}, \frac{z_4 + z_3 + z_1}{3}\right)$$

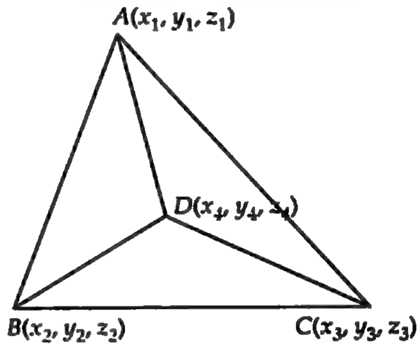


Fig. 28.16

Now, coordinates of point G dividing DG_1 in the ratio $3:1$ are

$$\left(\frac{1.x_4 + 3\left(\frac{x_1 + x_2 + x_3}{3}\right)}{1+3}, \frac{1.y_4 + 3\left(\frac{y_1 + y_2 + y_3}{3}\right)}{1+3}, \frac{1.z_4 + 3\left(\frac{z_1 + z_2 + z_3}{3}\right)}{1+3}\right)$$

$$= \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

Similarly, the points dividing CG_2, AG_3 and BG_4 in the ratio $3:1$ have the same coordinates.

Thus, the point $G\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$ is common to DG_1, CG_2, AG_3 and BG_4. Hence, they are concurrent.

EXERCISE 28.3

LEVEL-1

1. The vertices of the triangle are $A(5, 4, 6)$, $B(1, -1, 3)$ and $C(4, 3, 2)$. The internal bisector of angle A meets BC at D. Find the coordinates of D and the length AD.
2. A point C with z-coordinate 8 lies on the line segment joining the points $A(2, -3, 4)$ and $B(8, 0, 10)$. Find its coordinates.

3. Show that the three points $A(2, 3, 4)$, $B(-1, 2-3)$ and $C(-4, 1, -10)$ are collinear and find the ratio in which C divides AB.

4. Find the ratio in which the line joining $(2, 4, 5)$ and $(3, 5, 4)$ is divided by the yz-plane.

5. Find the ratio in which the line segment joining the points $(2, -1, 3)$ and $(-1, 2, 1)$ is divided by the plane $x + y + z = 5$.

6. If the points $A(3, 2, -4)$, $B(9, 8, -10)$ and $C(5, 4, -6)$ are collinear, find the ratio in which C divides AB.

7. The mid-points of the sides of a triangle ABC are given by $(-2, 3, 5)$, $(4, -1, 7)$ and $(6, 5, 3)$. Find the coordinates of A, B and C.

8. $A(1, 2, 3)$, $B(0, 4, 1)$, $C(-1, -1, -3)$ are the vertices of a triangle ABC. Find the point in which the bisector of the angle $\angle BAC$ meets BC.

9. Find the ratio in which the sphere $x^2 + y^2 + z^2 = 504$ divides the line joining the points $(12, -4, 8)$ and $(27, -9, 18)$.

10. Show that the plane $ax + by + cz + d = 0$ divides the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $-\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$.

11. Find the centroid of a triangle, mid-points of whose sides are $(1, 2, -3)$, $(3, 0, 1)$ and $(-1, 1, -4)$.

12. The centroid of a triangle ABC is at the point $(1, 1, 1)$. If the coordinates of A and B are $(3, -5, 7)$ and $(-1, 7, -6)$ respectively, find the coordinates of the point C. [NCERT]

13. Find the coordinates of the points which tisect the line segment joining the points $P(4, 2, -6)$ and $Q(10, -16, 6)$. [NCERT]

14. Using section formula, show that the points $A(2, -3, 4)$, $B(-1, 2, 1)$ and $C(0, 1/3, 2)$ are collinear. [NCERT]

15. Given that $P(3, 2, -4)$, $Q(5, 4, -6)$ and $R(9, 8, -10)$ are collinear. Find the ratio in which Q divides PR. [NCERT]

16. Find the ratio in which the line segment joining the points $(4, 8, 10)$ and $(6, 10, -8)$ is divided by the yz-plane. [NCERT]

ANSWERS

1. $\left(\frac{23}{8}, \frac{3}{2}, \frac{19}{8}\right), \frac{\sqrt{1530}}{8}$
2. $(6, -1, 8)$
3. 2 : 1 externally
4. 2 : 3 externally
5. 1 : 3 externally
6. 1 : 2
7. $A(12, 1, 5)$, $B(0, 9, 1)$, $C(-4, -3, 9)$
8. $\left(-\frac{3}{10}, \frac{5}{2}, -\frac{1}{5}\right)$
9. 2 : 3, −2 : 3
11. $(1, 1, -2)$
12. $(1, 1, 2)$
13. $(6, -4, -2)$, $(8, -10, 2)$
15. 1 :2
16. 2 : 3 externally

HINTS TO NCERT & SELECTED PROBLEMS

1. Use the fact that D divides BC in the ratio $AB : AC$.

2. Suppose C divides AB in the ratio $\lambda : 1$. Then, the coordinates of C are

$$\left(\frac{8\lambda + 2}{\lambda + 1}, \frac{-3}{\lambda + 1}, \frac{10\lambda + 4}{\lambda + 1}\right).$$

It is given that the z-coordinate of C is 8.

$$\therefore \quad \frac{10\lambda+4}{\lambda+1}=8 \Rightarrow \lambda=2.$$

Hence, the coordinates of C are (6, −1, 8).

12. Let the coordinates of C be (α, β, γ). Then centroid of triangle ABC has the coordinates $\left(\frac{\alpha+2}{3}, \frac{\beta+2}{3}, \frac{\gamma+1}{3}\right)$. But, coordinates of the centroid are given as (1, 1, 1).

$$\therefore \quad \frac{\alpha+2}{3}=1, \frac{\beta+2}{3}=1, \frac{\gamma+1}{3}=1 \Rightarrow \alpha=1, \beta=1, \gamma=2$$

Hence, the coordinates of C are (1, 1, 2).

13. Let A and B be the points of trisection of PQ. Then, A divides PQ internally in the ratio 1 : 2. So, coordinates of A are

$$\left(\frac{1\times 10+2\times 4}{1+2}, \frac{1\times -16+2\times 2}{1+2}, \frac{1\times 6+2\times -6}{1+2}\right)=(6,-4,-2).$$

Clearly, B is the mid-point of AQ. So, its coordinates are

$$\left(\frac{6+10}{2}, \frac{-4-16}{2}, \frac{-2+6}{2}\right)=(8,-10,2).$$

14. Suppose $C\left(0, \frac{1}{3}, 2\right)$ divides the segment joining A (2, − 3, 4) and B (−1, 2, 1) in the ratio $\lambda:1$. Then, the coordinates of C are $\left(\frac{-\lambda+2}{\lambda+1}, \frac{2\lambda-3}{\lambda+1}, \frac{\lambda+4}{\lambda+1}\right)$.

But, the coordinates of C are $\left(0, \frac{1}{3}, 2\right)$.

$$\therefore \quad \frac{-\lambda+2}{\lambda+1}=0, \frac{2\lambda-3}{\lambda+1}=\frac{1}{3} \text{ and } \frac{\lambda+4}{\lambda+1}=2$$

All these equations give the same value of λ. Hence, A, B, C are collinear points.

15. Suppose Q divides PR in the ratio $\lambda:1$. Then, the coordinates of Q are

$$\left(\frac{9\lambda+3}{\lambda+1}, \frac{8\lambda+2}{\lambda+1}, \frac{-10\lambda-4}{\lambda+1}\right)$$

But, the coordinates of Q are (5, 4, − 6).

$$\therefore \quad \frac{9\lambda+3}{\lambda+1}=5, \frac{8\lambda+2}{\lambda+1}=4, \frac{-10\lambda-4}{\lambda+1}=-6$$

All these equations give $\lambda=\frac{1}{2}$. Hence, Q divides PR in the ratio $\frac{1}{2}:1$ i.e. 1 : 2.

16. Let the required ratio be $\lambda:1$. Then, the coordinates of point of division are $\left(\frac{6\lambda+4}{\lambda+1}, \frac{10\lambda+8}{\lambda+1}, \frac{-8\lambda+10}{\lambda+1}\right)$. This point lies of yz-plane. So, its x-coordinate must be zero.

$$\text{i.e.} \quad \frac{6\lambda+4}{\lambda+1}=0 \Rightarrow \lambda=-\frac{2}{3}$$

Hence, yz-plane divides the segment joining given points externally in the ratio 2 : 3.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the distance of the point $P(2, 3, 5)$ from the xy-plane.
2. Write the distance of the point $P(3, 4, 5)$ from z-axis.
3. If the distance between the points $P(a, 2, 1)$ and $Q(1, -1, 1)$ is 5 units, find the value of a.
4. The coordinates of the mid-points of sides AB, BC and CA of ΔABC are $D(1, 2, -3)$, $E(3, 0, 1)$ and $F(-1, 1, -4)$ respectively. Write the coordinates of its centroid.
5. Write the coordinates of the foot of the perpendicular from the point $(1, 2, 3)$ on y-axis.
6. Write the length of the perpendicular drawn from the point $P(3, 5, 12)$ on x-axis.
7. Write the coordinates of third vertex of a triangle having centroid at the origin and two vertices at $(3, -5, 7)$ and $(3, 0, 1)$.
8. What is the locus of a point (x, y, z) for which $y = 0, z = 0$?
9. Find the ratio in which the line segment joining the points $(2, 4, 5)$ and $(3, -5, 4)$ is divided by the yz-plane.
10. Find the point on y-axis which is at a distance of $\sqrt{10}$ units from the point $(1, 2, 3)$.
11. Find the point on x-axis which is equidistant from the points $A(3, 2, 2)$ and $B(5, 5, 4)$.
12. Find the coordinates of a point equidistant from the origin and points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.
13. Write the coordinates of the point P which is five-sixth of the way from $A(-2, 0, 6)$ to $B(10, -6, -12)$.
14. If a parallelopiped is formed by the planes drawn through the points $(2, 3, 5)$ and $(5, 9, 7)$ parallel to the coordinate planes, then write the lengths of edges of the parallelopiped and length of the diagonal.
15. Determine the point on yz-plane which is equidistant from points $A(2, 0, 3)$, $B(0, 3, 2)$ and $C(0, 0, 1)$.
16. If the origin is the centroid of a triangle ABC having vertices $A(a, 1, 3)$, $B(-2, b-5)$ and $C(4, 7, c)$, find the values of a, b, c.

ANSWERS

1. 5 2. 5 3. $5, -3$ 4. $(1, 1, -2)$ 5. $(0, 2, 0)$
6. 13 7. $(-6, 5, -8)$ 8. x-axis 9. $4 : 5$ internally
10. $(0, 2, 0)$ 11. $(49/4, 0, 0)$ 12. $(a/2, b/2, c/2)$ 13. $(8, -5, -9)$
14. $3, 6, 2, 7$ 15. $(0, 1, 3)$ 16. $a = -2, b = -8, c = 2$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternatives in each of the following:

1. The ratio in which the line joining $(2, 4, 5)$ and $(3, 5, -9)$ is divided by the yz-plane is
   (a) $2:3$ (b) $3:2$ (c) $-2:3$ (d) $4:-3$
2. The ratio in which the line joining the points (a, b, c) and $(-a, -c, -b)$ is divided by the xy-plane is
   (a) $a:b$ (b) $b:c$ (c) $c:a$ (d) $c:b$
3. If $P(0, 1, 2)$, $Q(4, -2, 1)$ and $O(0, 0, 0)$ are three points, then $\angle POQ =$
   (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$
4. If the extremities of the diagonal of a square are $(1, -2, 3)$ and $(2, -3, 5)$, then the length of the side is

(a) $\sqrt{6}$ (b) $\sqrt{3}$ (c) $\sqrt{5}$ (d) $\sqrt{7}$

5. The points $(5, -4, 2)$, $(4, -3, 1)$, $(7, 6, 4)$ and $(8, -7, 5)$ are the vertices of
(a) a rectangle (b) a square (c) a parallelogram (d) none of these

6. In a three dimensional space the equation $x^2 - 5x + 6 = 0$ represents
(a) points (b) planes (c) curves (d) pair of straight lines

7. Let $(3, 4, -1)$ and $(-1, 2, 3)$ be the end points of a diameter of a sphere. Then, the radius of the sphere is equal to
(a) 2 (b) 3 (c) 6 (d) 7

8. XOZ-plane divides the join of $(2, 3, 1)$ and $(6, 7, 1)$ in the ratio
(a) $3:7$ (b) $2:7$ (c) $-3:7$ (d) $-2:7$

9. What is the locus of a point for which $y = 0, z = 0$?
(a) x-axis (b) y-axis (c) z-axis (d) yz-plane

10. The coordinates of the foot of the perpendicular drawn from the point $P(3, 4, 5)$ on the yz-plane are
(a) $(3, 4, 0)$ (b) $(0, 4, 5)$ (c) $(3, 0, 5)$ (d) $(3, 0, 0)$

11. The coordinates of the foot of the perpendicular from a point $P(6, 7, 8)$ on x-axis are
(a) $(6, 0, 0)$ (b) $(0, 7, 0)$ (c) $(0, 0, 8)$ (d) $(0, 7, 8)$

12. The perpendicular distance of the point $P(6, 7, 8)$ from xy-plane is
(a) 8 (b) 7 (c) 6 (d) 10

13. The length of the perpendicular drawn from the point $P(3, 4, 5)$ on y-axis is
(a) 10 (b) $\sqrt{34}$ (c) $\sqrt{113}$ (d) $5\sqrt{2}$

14. The perpendicular distance of the point $P(3, 3, 4)$ from the x-axis is
(a) $3\sqrt{2}$ (b) 5 (c) 3 (d) 4

15. The length of the perpendicular drawn from the point $P(a, b, c)$ from z-axis is
(a) $\sqrt{a^2 + b^2}$ (b) $\sqrt{b^2 + c^2}$ (c) $\sqrt{a^2 + c^2}$ (d) $\sqrt{a^2 + b^2 + c^2}$

ANSWERS

1. (c)	2. (d)	3. (d)	4. (b)	5. (a)	6. (b)	7. (c)	8. (c)
9. (a)	10. (b)	11. (a)	12. (a)	13. (b)	14. (b)	15. (a)	

SUMMARY

1. In three dimensions, the coordinate axes of a rectangular Cartesian coordinate system are three mutually perpendicular lines. The axes are called the x, y and z axes.

2. The three planes determined by the pair of axes are the coordinate planes. These planes are called xy, yz and zx planes and they divide the space into eight regions known as octants.

3. The coordinates of a point P in the space are the perpendicular distances from P on three mutaually perpendicular coordinates planes YZ, ZX and XY respectively. The coordinates of a point P are written in the form of triplet like (x, y, z).

4. The coordinates of a point are also the distances from the origin of the feet of the perpendiculars from the point on the respective coordinate axes.

5. The coordinates of any point on:
(i) x-axis are of the form $(x, 0, 0)$
(ii) y-axis are of the form $(0, y, 0)$
(iii) z-axis are of the form $(0, 0, z)$
(iv) xy-plane are of the form $(x, y, 0)$
(v) yz-plane are of the form $(0, y, z)$
(vi) zx – plane are of the form $(x, 0, z)$

6. The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

7. The distance of a point $P(x, y, z)$ from the origin $O(0, 0, 0)$ is given by 3 $OP = \sqrt{x^2 + y^2 + z^2}$.

8. The coordinates of the point R which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally and externally in the ratio $m : n$ are given by

$$\left(\frac{m x_2 + n x_1}{m+n}, \frac{m y_2 + n y_1}{m+n}, \frac{m z_2 + n z_1}{m+n}\right) \text{and,} \left(\frac{m x_2 - n x_1}{m-n}, \frac{m y_2 - n y_1}{m-n}, \frac{m z_2 - n z_1}{m-n}\right)$$

respectively.

9. The coordinates of the mid-point of the line segment joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

10. The coordinates of the centroid of the triangle whose vertices are $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and $(x_3 + y_3 + z_3)$ are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.

CHAPTER 29

LIMITS

29.1 INFORMAL APPROACH TO LIMIT

Consider the function $f(x) = \dfrac{x^2 - 4}{x - 2}$.

Clearly, this function is defined for all x except at $x = 2$ as it assumes the form $\dfrac{0}{0}$ (known as an indeterminate form) at $x = 2$. However, if $x \neq 2$, then

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2$$

The following table exhibits the values of $f(x)$ at points which are close to 2 on its two sides *viz.* left and right on the real line.

x	1.4	1.5	1.6	1.7	1.8	1.9	1.99	2	2.01	2.1	2.2	2.3	2.4	2.5	2.6
$f(x)$	3.4	3.5	3.6	3.7	3.8	3.9	3.99	$\frac{0}{0}$	4.01	4.1	4.2	4.3	4.4	4.5	4.6

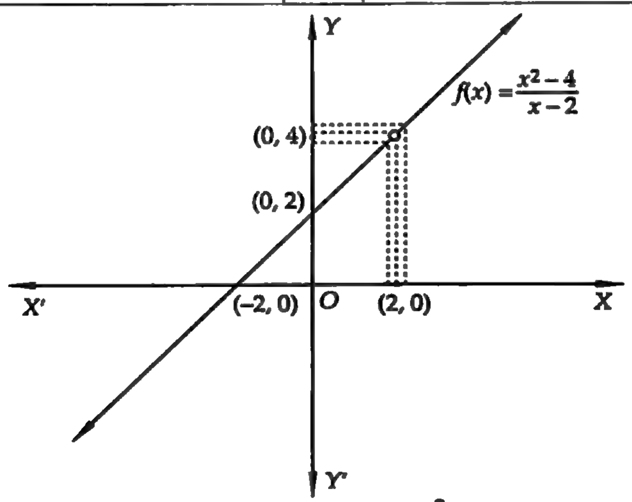


Fig. 29.1 Graph of $f(x) = \dfrac{x^2 - 4}{x - 2}$

The graph of this function is shown in Fig. 29.1.

It is evident from the above table and the graph of $f(x)$ that as x increases and comes closer to 2 from left hand side of 2, the values of $f(x)$ increase and come closer to 4. This is interpreted as: *When x approaches to 2 from its left hand side, the function f(x) tends to the limit 4.*

It we use the notation '$x \to 2^-$' to denote 'x tends to 2 from left hand side', the above statement can be restated as:

as $x \to 2^-$, $f(x) \to 4$

or, $\lim_{x \to 2^-} f(x) = 4$

or, Left hand limit of $f(x)$ *at* $x = 2$ is 4.

Thus, $\lim_{x \to 2^-} f(x) = 4$ means that as x tends to 2 from left hand side, the values of $f(x)$ are tending to 4.

From the above table as well as the graph of $f(x)$, shown in Fig. 29.1, we observe that as x decreases and comes closer to 2 from right hand side, the values of $f(x)$ decrease and come closer to 4. This is interpreted as:

When x approaches to 2 from its right hand side, the function f(x) tends to the limit 4.

Using the notation '$x \to 2^+$' to denote 'x tends to 2 from right hand side', the above statement can be re-stated as:

as $x \to 2^+$, $f(x) \to 4$

or, $\lim_{x \to 2^+} f(x) = 4$

or, Right hand limit of $f(x)$ at $x = 2$ is 4.

Thus, $\lim_{x \to 2^+} f(x) = 4$ means that as x tends to 2 from right hand side, the values of $f(x)$ are tending to 4.

It follows from the above discussion that for the function $f(x)$ given by $f(x) = \dfrac{x^2 - 4}{x - 2}$:

(i) $\lim_{x \to 2^-} f(x) = 4$ (ii) $\lim_{x \to 2^+} f(x) = 4$

(iii) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ (iv) $f(2)$ does not exist i.e. $f(x)$ is not defined at $x = 2$.

Now, consider the function $f(x) = \dfrac{|x - 3|}{x - 3}$.

This function is defined for all x except $x = 3$, as it assumes the form $\dfrac{0}{0}$ (an indeterminate form) at $x = 3$. The graph of $f(x)$ is shown in Fig. 29.2.

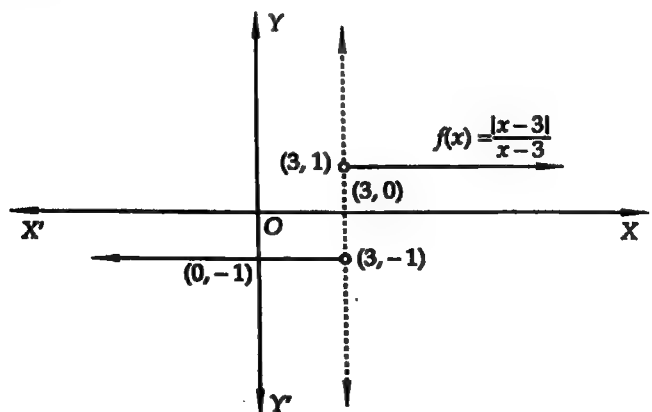


Fig. 29.2 Graph of $f(x) = \dfrac{|x - 3|}{x - 3}$

The following table shows the values of $f(x)$ at points which are close to 3 and are on its two sides.

x	2.3	2.4	2.5	2.6	2.7	2.8	2.9	2.99	3	3.01	3.1	3.2	3.3	3.4	3.5	3.6
$f(x)$	-1	-1	-1	-1	-1	-1	-1	-1	$\frac{0}{0}$	1	1	1	1	1	1	1

It is evident from the table and the graph of $f(x)$ that as $x \to 3$ from its left hand side the values of $f(x)$ are everywhere -1.

i.e. $\lim_{x \to 3^-} f(x) = -1$ or, Left hand limit (LHL) of $f(x)$ at $x = 3$ is -1.

We also observe that at every point on the right hand side of 3, the function assumes válue 1.

$\therefore \quad \lim_{x \to 3^+} f(x) = 1$

Let us now consider the function $f(x) = \frac{1}{x-4}$, $x \neq 4$. Here also the function is undefined at $x = 4$ as $f(4)$ assumes the form $\frac{1}{0}$. In this case it is evident from the graph shown in Fig. 29.3 that as x approaches to 4 from the left hand side, $f(x)$ decreases to $-\infty$.

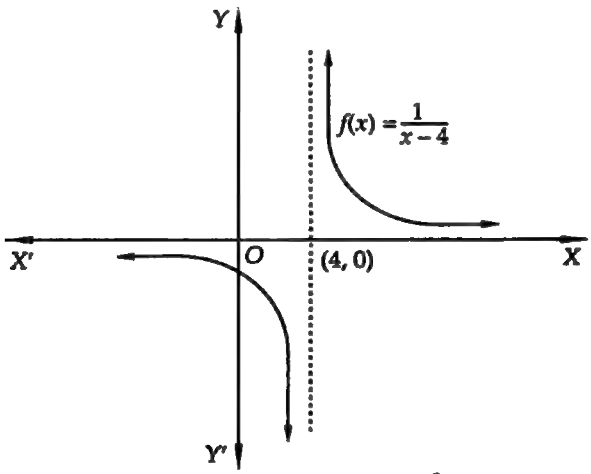


Fig. 29.3 Graph of $f(x) = \frac{1}{x-4}$

i.e. $\lim_{x \to 4^-} f(x) = -\infty$

Also, we observe that $f(x)$ increases to $+\infty$ as x approaches to 4 from the right

i.e. $\lim_{x \to 4^+} f(x) = +\infty$

So, we say that $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$ both do not exist.

It follows from the above discussion that as we can approach to a given number 'a' (say) on the real line either from its left hand side by increasing numbers which are less than 'a' or from right hand side by decreasing numbers which are greater than 'a'. So, there are two types of limits viz. (i) left hand limit and, (ii) right hand limit. We also observe that for some functions at a given point 'a' (say) left hand and right hand limits are equal whereas for some functions these two limits are not equal and even sometimes either left hand limit or right hand limit or both do not exist.

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ i.e. (LHL at $x = a$) = (RHL at $x = a$), then we say that $\lim_{x \to a} f(x)$ exists.

Otherwise, $\lim_{x \to a} f(x)$ does not exist.

29.2 EVALUATION OF LEFT HAND AND RIGHT HAND LIMITS

In the previous sections, we have learnt that a real number l_1 is the left hand limit of function $f(x)$ at $x = a$ if the values of $f(x)$ can be made as close as desired to the number l_1 at points closed to a and on the left of a. In such a case, we write $\lim\limits_{x \to a^-} f(x) = l_1$. Also, a real number l_2 is the right hand limit of $f(x)$ at $x = a$ i.e. $\lim\limits_{x \to a^+} f(x) = l_2$, if the values of $f(x)$ can be made as close as desired to the number l_2 at points close to 'a' on the right of 'a'.

In this section, we shall discuss methods of evaluation of left hand and right hand limits of a function at a given point.

As discussed earlier that statement $x \to a^-$ means that x is tending to a from the left hand side i.e. x is a number less than a but very very close to a. Therefore, $x \to a^-$ is equivalent to $x = a - h$ where $h > 0$ such that $h \to 0$.

Similarly, $x \to a^+$ is equivalent to $x = a + h$ where $h \to 0$. Thus, we have the following algorithms for finding left hand and right hand limits at $x = a$.

ALGORITHM

<u>STEP I</u> *Write* $\lim\limits_{x \to a^-} f(x)$

<u>STEP II</u> *Put* $x = a - h$ *and replace* $x \to a^-$ by $h \to 0$ *to obtain* $\lim\limits_{x \to a^-} = f(x) = \lim\limits_{h \to 0} f(a-h)$.

<u>STEP III</u> *Simplify* $\lim\limits_{h \to 0} f(a-h)$ *by using the formula for the given function.*

<u>STEP IV</u> *The value obtain in step III is the LHL of* $f(x)$ *at* $x = a$.

ILLUSTRATION 1 *Evaluate the left hand limit of the function*

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases} \quad \text{at } x = 4.$$

SOLUTION We have,

(LHL of $f(x)$ at $x = 4$)

$$= \lim_{x \to 4^-} f(x) \qquad \text{(Step I)}$$

$$= \lim_{h \to 0} f(4-h) \qquad \text{(Step II)}$$

$$= \lim_{h \to 0} \frac{|4-h-4|}{4-h-4} = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1. \qquad \text{(Step III)}$$

To evaluate RHL of $f(x)$ at $x = a$ i.e. $\lim\limits_{x \to a^+} f(x)$ we use the following algorithm.

ALGORITHM

<u>STEP I</u> *Write* $\lim\limits_{x \to a^+} f(x)$

<u>STEP II</u> *Put* $x = a + h$ *and replace* $x \to a^+$ by $h \to 0$ *to obtain* $\lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a+h)$.

<u>STEP III</u> *Simplify* $\lim\limits_{h \to 0} f(a+h)$ *by using the formula for the given function.*

<u>STEP IV</u> *The value obtained in step III is the RHL of* $f(x)$ *at* $x = a$.

ILLUSTRATION 2 *Evaluate the right hand limit of the function*

$$f(x)=\begin{cases}\dfrac{|x-4|}{x-4} & , x\neq 4\\ 0 & , x=4\end{cases} \quad \text{at } x=4.$$

SOLUTION We have,

(RHL of $f(x)$ at $x=4$)

$= \lim\limits_{x\to 4^+} f(x)$ (Step I)

$= \lim\limits_{h\to 0} f(4+h)$ (Step II)

$= \lim\limits_{h\to 0} \dfrac{|4+h-4|}{4+h-4} = \lim\limits_{h\to 0} \dfrac{|h|}{h} = \lim\limits_{h\to 0} \dfrac{h}{h} = \lim\limits_{h\to 0} 1 = 1$ (Step III)

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate the left hand and right hand limits of the function defined by*

$$f(x) = \begin{cases}1+x^2, & \text{if } 0\le x\le 1\\ 2-x, & \text{if } x>1\end{cases} \quad \text{at } x=1.$$

Also, show that $\lim\limits_{x\to 1} f(x)$ *does not exist.*

SOLUTION (LHL of $f(x)$ at $x=1$)

$$= \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 1+(1-h)^2 = \lim_{h\to 0} 2-2h+h^2 = 2.$$

and, (RHL of $f(x)$ at $x=1$)

$$= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 2-(1+h) = \lim_{h\to 0} 1-h = 1.$$

Clearly, $\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$.

Hence, $\lim\limits_{x\to 1} f(x)$ does not exist.

EXAMPLE 2 *If* $f(x) = \begin{cases}\dfrac{x-|x|}{x}, & x\neq 0\\ 2, & x=0\end{cases}$ *show that* $\lim\limits_{x\to 0} f(x)$ *does not exist.*

SOLUTION We have,

(LHL of $f(x)$ at $x=0$)

$$= \lim_{x\to 0^-} f(x)$$

$$= \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{-h-|-h|}{(-h)} = \lim_{h\to 0} \frac{-h-h}{-h} = \lim_{h\to 0} \frac{-2h}{-h} = \lim_{h\to 0} 2 = 2.$$

(RHL of $f(x)$ at $x=0$)

$$= \lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0} f(0+h) = \lim_{h\to 0} \frac{h-|h|}{h} = \lim_{h\to 0} \frac{h-h}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0.$$

Clearly, $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$.

Hence, $\lim\limits_{x\to 0} f(x)$ does not exist.

EXAMPLE 3 *If* $f(x) = \begin{cases} 5x-4, & 0 < x \le 1 \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$, *show that* $\lim\limits_{x \to 1} f(x)$ *exists.*

SOLUTION We have,

(LHL of $f(x)$ at $x = 1$)

$$= \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} 5(1-h) - 4 = \lim_{h \to 0} 1 - 5h = 1.$$

(RHL of $f(x)$ at $x = 1$)

$$= \lim_{x \to 1^+} f(x)$$

$$= \lim_{h \to 0} f(1+h) = \lim_{h \to 0} 4(1+h)^3 - 3(1+h) = 4(1)^3 - 3(1) = 1$$

Clearly, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$.

Hence, $\lim\limits_{x \to 1} f(x)$ exists and is equal to 1.

EXAMPLE 4 *Discuss the existence of each of the following limits:*

(i) $\lim\limits_{x \to 0} \dfrac{1}{x}$ (ii) $\lim\limits_{x \to 0} \dfrac{1}{|x|}$

SOLUTION (i) The graph of $f(x) = \dfrac{1}{x}$ is as shown in Fig. 29.4. We observe that as x approaches to 0 from the LHS i.e. x is negative and very close to zero, then the values of $1/x$ are negative and very large in magnitude.

$$\therefore \quad \lim_{x \to 0^-} \frac{1}{x} \to -\infty.$$

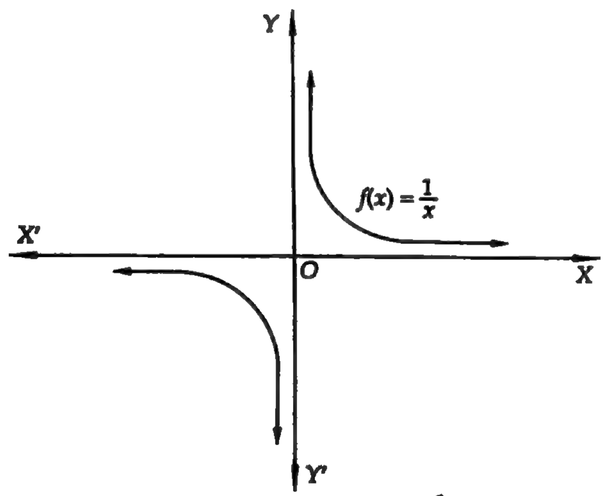


Fig. 29.4 Graph of $f(x) = \dfrac{1}{x}$

Similarly, when x approaches to 0 from the right i.e. x is positive and very close to 0, then the values of $\dfrac{1}{x}$ are very large and positive.

$$\therefore \quad \lim_{x \to 0^+} \frac{1}{x} \to \infty.$$

Thus we have, $\lim\limits_{x \to 0^-} \dfrac{1}{x} \ne \lim\limits_{x \to 0^+} \dfrac{1}{x}$. Hence, $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist.

(ii) The graph of $f(x) = \dfrac{1}{|x|}$ is shown in Fig. 29.5. We observe that as x approaches to 0 from LHS i.e. x is negative and close to 0, then $|x|$ is close to zero and is positive. Consequently, $\dfrac{1}{|x|}$ is large and positive.

$$\therefore \qquad \lim_{x \to 0^-} \frac{1}{|x|} \to \infty$$

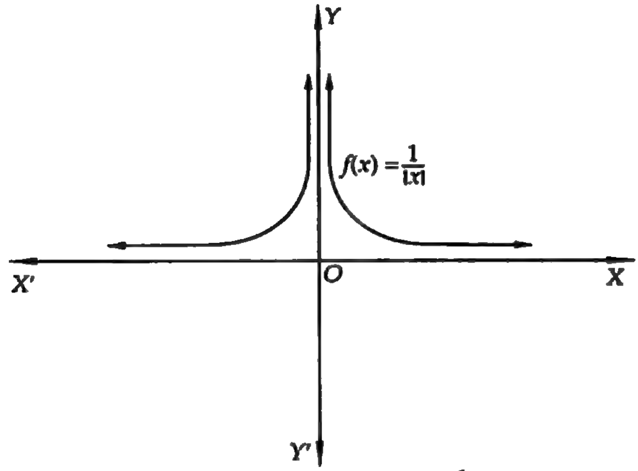


Fig. 29.5 Graph of $f(x) = \dfrac{1}{|x|}$

Also, if x approaches to 0 from RHS i.e. x is positive and close to 0, then $|x|$ is close to zero and is positive.

Consequently, $\dfrac{1}{|x|}$ is large and positive.

$$\therefore \qquad \lim_{x \to 0^+} \frac{1}{|x|} \to \infty$$

Thus, we have

$$\lim_{x \to 0^-} \frac{1}{|x|} = \lim_{x \to 0^+} \frac{1}{|x|}$$

Hence, $\lim\limits_{x \to 0} \dfrac{1}{|x|}$ exists and it tends to infinity.

EXAMPLE 5 *Let* $f(x) = \begin{cases} \cos x, & \text{if } x > 0 \\ x + k, & \text{if } x < 0 \end{cases}$. *Find the value of constant k, given that* $\lim\limits_{x \to 0} f(x)$ *exists.*

SOLUTION It is given that

$$\lim_{x \to 0} f(x) \text{ exists}$$

$$\Rightarrow \qquad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$\Rightarrow \qquad \lim_{x \to 0} x + k = \lim_{x \to 0} \cos x \qquad \text{[Using definition of } f(x)\text{]}$$

$$\Rightarrow \qquad 0 + k = \cos 0 \Rightarrow k = 1$$

EXAMPLE 6 *Let $f(x)$ be a function defined by* $f(x) = \begin{cases} 4x-5, & \text{if } x \le 2 \\ x-\lambda, & \text{if } x > 2 \end{cases}$

Find λ, if $\lim\limits_{x \to 2} f(x)$ exists.

SOLUTION We have,

$$f(x) = \begin{cases} 4x-5, & \text{if } x \le 2 \\ x-\lambda, & \text{if } x > 2 \end{cases}$$

$\therefore$ $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{h \to 0} f(2-h) = \lim\limits_{h \to 0} 4(2-h)-5 = \lim\limits_{h \to 0} 3-4h = 3$

and, $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{h \to 0} f(2+h) = \lim\limits_{h \to 0} 2+h-\lambda = 2-\lambda$

If $\lim\limits_{x \to 2} f(x)$ exists, then

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \Rightarrow 3 = 2-\lambda \Rightarrow \lambda = -1.$$

EXAMPLE 7 If $f(x) = \begin{cases} mx^2+n, & x < 0 \\ nx+m, & 0 \le x \le 1 \\ nx^3+m, & x > 1 \end{cases}$ [NCERT]

For what values of integers m and n does the limits $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$ exist.

SOLUTION It is given that

$\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$ both exist

$\Leftrightarrow$ $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$ and, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$

$\Leftrightarrow$ $\lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(0+h)$ and, $\lim\limits_{h \to 0} f(1-h) = \lim\limits_{h \to 0} f(1+h)$

$\Leftrightarrow$ $\lim\limits_{h \to 0} m(-h)^2+n = \lim\limits_{h \to 0} n(h)+m$ and, $\lim\limits_{h \to 0} n(1-h)+m = \lim\limits_{h \to 0} n(1+h)^3+m$

$\Leftrightarrow$ $n = m$, and $n+m = n+m$

$\Leftrightarrow$ $m = n$

Hence, $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$ both sides for $n = m$.

EXAMPLE 8 *If* $f(x) = \begin{cases} |x|+1, & x < 0 \\ 0, & x = 0 \\ |x|-1, & x > 0 \end{cases}$. *For what value (s) of 'a' does $\lim\limits_{x \to a} f(x)$ exist?*

SOLUTION We have, [NCERT]

$$f(x) = \begin{cases} |x|+1, & x < 0 \\ 0, & x = 0 \\ |x|-1, & x > 0 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -x+1, & x < 0 \\ 0, & x = 0 \\ x-1, & x > 0 \end{cases} \qquad \left[\because |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}\right]$$

Clearly, $\lim\limits_{x \to a} f(x)$ exists for all $a \ne 0$. So, let us see whether $\lim\limits_{x \to 0} f(x)$ exist or not.

We observe that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} f(0-h) = \lim_{h \to 0} -(-h)+1 = 1$$

and, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} h-1 = -1$

$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

So, $\lim_{x \to 0} f(x)$ does not exist. Hence, $\lim_{x \to a} f(x)$ exists for all $a \neq 0$.

EXAMPLE 9 *Suppose* $f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax, & x>1 \end{cases}$

and, if $\lim_{x \to 1} f(x) = f(1)$. *What are possible values of a and b?* [NCERT]

SOLUTION We have,

$$\lim_{x \to 1} f(x) = f(1)$$

$$\Leftrightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\Leftrightarrow \quad \lim_{x \to 1^-} f(x) = f(1) \text{ and, } \lim_{x \to 1^+} f(x) = f(1)$$

$$\Leftrightarrow \quad \lim_{h \to 0} f(1-h) = 4 \text{ and, } \lim_{h \to 0} f(1+h) = 4$$

$$\Leftrightarrow \quad \lim_{h \to 0} \left\{a+b(1-h)\right\} = 4 \text{ and, } \lim_{h \to 0} \left\{b-a(1+h)\right\} = 4$$

$$\Leftrightarrow \quad a+b = 4 \text{ and, } b-a = 4$$

$$\Leftrightarrow \quad a = 0,\ b = 4$$

LEVEL-2

EXAMPLE 10 *Find the left hand and right hand limits of the greatest integer function $f(x) = [x]$ = greatest integer less than or equal to x, at $x = k$, where k is an integer. Also, show that $\lim_{x \to k} f(x)$ does not exist.*

SOLUTION We have,

$$(\text{LHL at } x = k) = \lim_{x \to k^-} f(x) = \lim_{h \to 0} f(k-h) = \lim_{h \to 0} [k-h]$$

$$= \lim_{h \to 0} k-1 = k-1 \qquad [\because\ k-1 < k-h < k\ \therefore [k-h] = k-1]$$

$$(\text{RHL at } x = k) = \lim_{x \to k^+} f(x) = \lim_{h \to 0} f(k+h) = \lim_{h \to 0} [k+h]$$

$$= \lim_{h \to 0} k = k \qquad [\because\ k < k+h < k+1\ \therefore [k+h] = k]$$

Clearly, $\lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x)$.

Hence, $\lim_{x \to k} f(x)$ does not exist.

EXAMPLE 11 *Prove that* $\lim_{x \to a^+} [x] = [a]$ *for all* $a \in R$, *where* [.] *denotes the greatest integer function.*

SOLUTION Since $a \in R$. Therefore, there exists an integer k such that $k \leq a < k+1$.

Now, $\lim_{x \to a^+} [x] = \lim_{h \to 0} [a+h] = k \qquad [\because\ k \leq a < k+1\ \therefore k \leq a+h < k+1 \Rightarrow [a+h] = k]$

$= [a] \qquad [\because\ k \leq a < k+1 \Rightarrow [a] = k]$

EXAMPLE 12 *Show that* $\lim\limits_{x\to 0} \dfrac{e^{1/x}-1}{e^{1/x}+1}$ *does not exist.*

SOLUTION Let $f(x) = \dfrac{e^{1/x}-1}{e^{1/x}+1}$. Then,

(LHL of $f(x)$ at $x=0$)

$$= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1}$$

$$= \lim_{h\to 0}\left(\frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1}\right) = \frac{0-1}{0+1} = -1 \qquad \left[\because h\to 0 \Rightarrow \frac{1}{h}\to\infty \Rightarrow e^{1/h}\to\infty \Rightarrow \frac{1}{e^{1/h}}\to 0\right]$$

and, (RHL of $f(x)$ at $x=0$)

$$= \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{e^{1/h}-1}{e^{1/h}+1}$$

$$= \lim_{h\to 0}\left(\frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{1/h}}}\right) = \frac{1-0}{1+0} = 1$$

Clearly, $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$. Hence, $\lim\limits_{x\to 0} f(x)$ does not exist.

EXAMPLE 13 *If f is an odd function and if $\lim\limits_{x\to 0} f(x)$ exists. Prove that this limit must be zero.*

SOLUTION It is given that

$$\lim_{x\to 0} f(x) \text{ exists}$$

$$\Rightarrow \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$\Rightarrow \quad \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h)$$

$$\Rightarrow \quad \lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h)$$

$$\Rightarrow \quad -\lim_{h\to 0} f(h) = \lim_{h\to 0} f(h) \qquad [\because f(x) \text{ is odd } \therefore f(-h) = -f(h)]$$

$$\Rightarrow \quad 2\lim_{h\to 0} f(h) = 0 \Rightarrow \lim_{h\to 0} f(h) = 0 \Rightarrow \lim_{x\to 0} f(x) = 0$$

EXAMPLE 14 *If f is an even function, then prove that* $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$.

SOLUTION Clearly,

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} f(h) \qquad [\because f \text{ is even } \therefore f(-h) = f(h)]$$

$$= \lim_{h\to 0} f(0+h) = \lim_{x\to 0^+} f(x).$$

EXERCISE 29.1

LEVEL-1

1. Show that $\lim\limits_{x \to 0} \dfrac{x}{|x|}$ does not exist. [NCERT]

2. Find k so that $\lim\limits_{x \to 2} f(x)$ may exist, where $f(x) = \begin{cases} 2x+3, & x \le 2 \\ x+k, & x > 2 \end{cases}$.

3. Show that $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist.

4. Let $f(x)$ be a function defined by $f(x) = \begin{cases} \dfrac{3x}{|x|+2x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.

   Show that $\lim\limits_{x \to 0} f(x)$ does not exist.

5. Let $f(x) = \begin{cases} x+1, & \text{if } x > 0 \\ x-1, & \text{if } x < 0 \end{cases}$. Prove that $\lim\limits_{x \to 0} f(x)$ does not exist.

6. Let $f(x) = \begin{cases} x+5, & \text{if } x > 0 \\ x-4, & \text{if } x < 0 \end{cases}$. Prove that $\lim\limits_{x \to 0} f(x)$ does not exist.

7. Find $\lim\limits_{x \to 3} f(x)$, where $f(x) = \begin{cases} 4, & \text{if } x > 3 \\ x+1, & \text{if } x < 3 \end{cases}$

8. If $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$. Find $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$. [NCERT]

9. Find $\lim\limits_{x \to 1} f(x)$, if $f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x^2-1, & x > 1 \end{cases}$. [NCERT]

10. Evaluate $\lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ [NCERT]

11. Let $a_1, a_2, \ldots, a_n$ be fixed real numbers such that $f(x) = (x-a_1)(x-a_2)\ldots(x-a_n)$

    What is $\lim\limits_{x \to a_1} f(x)$? For $a \ne a_1, a_2, \ldots, a_n$ compute $\lim\limits_{x \to a} f(x)$. [NCERT]

12. Find $\lim\limits_{x \to 1^+} \dfrac{1}{x-1}$.

13. Evaluate the following one sided limits:

    (i) $\lim\limits_{x \to 2^+} \dfrac{x-3}{x^2-4}$ (ii) $\lim\limits_{x \to 2^-} \dfrac{x-3}{x^2-4}$ (iii) $\lim\limits_{x \to 0^+} \dfrac{1}{3x}$

    (iv) $\lim\limits_{x \to -8^+} \dfrac{2x}{x+8}$ (v) $\lim\limits_{x \to 0^+} \dfrac{2}{x^{1/5}}$ (vi) $\lim\limits_{x \to \frac{\pi}{2}^-} \tan x$

    (vii) $\lim\limits_{x \to -\pi/2^+} \sec x$ (viii) $\lim\limits_{x \to 0^-} \dfrac{x^2-3x+2}{x^3-2x^2}$ (ix) $\lim\limits_{x \to -2^+} \dfrac{x^2-1}{2x+4}$

    (x) $\lim\limits_{x \to 0^-} (2-\cot x)$ (xi) $\lim\limits_{x \to 0^-} 1+\operatorname{cosec} x$

LEVEL-2

14. Show that $\lim_{x \to 0} e^{-1/x}$ does not exist.

15. Find:

(i) $\lim_{x \to 2} [x]$ (ii) $\lim_{x \to \frac{5}{2}} [x]$ (iii) $\lim_{x \to 1} [x]$

16. Prove that $\lim_{x \to a^+} [x] = [a]$ for all $a \in R$. Also, prove that $\lim_{x \to 1^-} [x] = 0$.

17. Show that $\lim_{x \to 2^-} \frac{x}{[x]} \neq \lim_{x \to 2^+} \frac{x}{[x]}$.

18. Find $\lim_{x \to 3^+} \frac{x}{[x]}$. Is it equal to $\lim_{x \to 3^-} \frac{x}{[x]}$.

19. Find $\lim_{x \to -5/2} [x]$.

20. Evaluate $\lim_{x \to 2} f(x)$ (if it exists), where $f(x) = \begin{cases} x - [x] & , \quad x < 2 \\ 4 & , \quad x = 2 \\ 3x - 5 & , \quad x > 2 \end{cases}$.

21. Show that $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

22. Let $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x} & , \text{ where } x \neq \dfrac{\pi}{2} \\ 3 & , \text{ where } x \neq \dfrac{\pi}{2} \end{cases}$ and if $\lim_{x \to \pi/2} f(x) = f\left(\frac{\pi}{2}\right)$, find the value of k.

ANSWERS

2. $k = 5$ 7. 4 8. 3, Does not exist 9. Does not exist 10. Does not exist
11. 0, $(a - a_1)(a - a_2) \dots (a - a_n)$ 12. ∞ 13. (i) $-\infty$ (ii) ∞ (iii) ∞ (iv) $-\infty$
(v) ∞ (vi) ∞ (vii) $-\infty$ (viii) $-\infty$ (ix) ∞ (x) ∞ (xi) $-\infty$
15. (i) Does not exist (ii) 2 (iii) Does not exist
18. 1, No 19. -3 20. 1 22. $k = 6$

HINTS TO NCERT & SELECTED PROBLEMS

1. $\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} \frac{x}{-x} = -1$ and, $\lim_{x \to 0^+} = \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{x} = 1$

$$\therefore \quad \lim_{x \to 0^-} \frac{x}{|x|} \neq \lim_{x \to 0^+} \frac{x}{|x|}$$

Hence, $\lim_{x \to 0} \frac{x}{|x|}$ does not exist.

3. We have,

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \frac{1}{-h} = -\infty \text{ and, } \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{1}{h} = \infty$$

Clearly, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$. Hence, $\lim_{x \to 0} f(x)$ does not exist.

8. We have,

$$f(x)=\begin{cases}2x+3, & x\le 0\\ 3(x+1) & x\ge 0\end{cases}$$

$\therefore$ $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}(2x+3)=2\times 0+3=3$

and, $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}3(x+1)=3(0+1)=3$

So, $\lim\limits_{x\to 0} f(x)$ exists and is equal to 3.

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-} 2x+3=2\times 1+3=5$

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+} 3(x+1)=3(1+1)=6.$

$\therefore$ $\lim\limits_{x\to 1^-} f(x)\ne \lim\limits_{x\to 1^+} f(x)$

Hence, $\lim\limits_{x\to 1} f(x)$ does not exist.

9. We have,

$$f(x)=\begin{cases}x^2-1, & x\le 1\\ -x^2-1, & x\ge 1\end{cases}$$

$\therefore$ $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-} x^2-1=1^2-1=0$ and, $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+} -x^2-1=-1-1=-2$

$\therefore$ $\lim\limits_{x\to 1^-} f(x)\ne \lim\limits_{x\to 1^+} f(x).$

Hence, $\lim\limits_{x\to 1} f(x)$ does not exist.

10. We have,

$$f(x)=\begin{cases}\dfrac{|x|}{x}, & x\ne 0\\ 0, & x=0\end{cases}$$

$\therefore$ $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\dfrac{|x|}{x}=\lim\limits_{x\to 0^-}\dfrac{-x}{x}=-1$ and, $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\dfrac{|x|}{x}=\lim\limits_{x\to 0}\dfrac{x}{x}=1.$

$\therefore$ $\lim\limits_{x\to 0^-} f(x)\ne \lim\limits_{x\to 0^+} f(x).$

Hence, $\lim\limits_{x\to 0} f(x)$ does not exist.

11. We have,

$$f(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$$

$\therefore \quad \lim_{x \to a_1^-} f(x) = \lim_{x \to 0} f(a_1 - h) = \lim_{x \to 0} -h(a_1 - h - a_2)(a_1 - h - a_3)\ldots(a_1 - h - a_n)$

$= 0(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n) = 0$

$\lim_{x \to a_1^+} f(x) = \lim_{x \to 0} (a_1 + h - a_1)(a_1 + h - a_2)(a_1 + h - a_3)\ldots(a_1 + h - a_n)$

$= \lim_{x \to 0} h(a_1 + h - a_2)(a_1 + h - a_3)\ldots(a_1 + h - a_n)$

$= 0(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n) = 0$

$\therefore \quad \lim_{x \to a_1^-} f(x) = \lim_{x \to a_1^+} f(x) = 0.$

Hence, $\lim_{x \to a_1} f(x) = 0.$

For any $a \neq a_1, a_2, \ldots a_n$.

$\lim_{x \to a} f(x) = \lim_{x \to a} (x - a_1)(x - a_2)(x - a_3)\ldots(x - a_n) = (a - a_1)(a - a_2)(a - a_3)\ldots(a - a_n)$

12. $\lim_{x \to 1^+} \frac{1}{x - 1} = \lim_{h \to 0} \frac{1}{1 + h - 1} = \lim_{h \to 0} \frac{1}{h} = \infty.$

14. We have,

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} e^{1/h} = \infty$

and, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} e^{-1/h} = \lim_{h \to 0} \frac{1}{e^{1/h}} = 0$

21. We have,

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h - 0} \sin\left(\frac{1}{-h}\right) = -\lim_{h \to 0} \sin\frac{1}{h}$

$\Rightarrow \quad \lim_{x \to 0^-} f(x) = -$(An oscillating number which oscillates between -1 and 1).

So, $\lim_{x \to 0^-} f(x)$ does not exist. Similarly, $\lim_{x \to 0^+} f(x)$ does not exist.

29.3 DIFFERENCE BETWEEN THE VALUE OF A FUNCTION AT A POINT AND THE LIMIT AT THAT POINT

Let $f(x)$ be a function and let a be a point. Then, we have the following possibilities:

(i) $\lim_{x \to a} f(x)$ *exists but* $f(a)$ *(the value of* $f(x)$ *at* $x = a$*) does not exist*:

Consider the function $f(x)$ defined by $f(x) = \frac{x^2 - 9}{x - 3}$.

Clearly, this function is not defined at $x = 3$ i.e. $f(3)$ does not exist, because it attains the form $\frac{0}{0}$. But, it can be easily seen that $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 6$. So, $\lim_{x \to 3} f(x)$ exists.

Thus, the $\lim_{x \to 3} f(x)$ exists but $f(3)$ does not exist.

(ii) *The value $f(a)$ exists but $\lim\limits_{x \to a} f(x)$ does not exist:*

In example 4 on page 29.6, we have seen that $\lim\limits_{x \to k} f(x)$ does not exist but $f(k) = k$ exists.

(iii) $\lim\limits_{x \to a} f(x)$ *and $f(a)$ both exist but are unequal:*

Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$$

It can be easily seen that $\lim\limits_{x \to 2^-} f(x) = 4 = \lim\limits_{x \to 2^+} f(x)$.

So, $\lim\limits_{x \to 2} f(x)$ exists and is equal to 4. Also, the value $f(2)$ exists and is equal to 3.

Thus, $\lim\limits_{x \to 2} f(x)$ and $f(2)$ both exist but are unequal.

(iv) $\lim\limits_{x \to a} f(x)$ *and $f(a)$ both exist and are equal*

Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

For this function, it can be easily seen that $\lim\limits_{x \to 2} f(x)$ and $f(2)$ both exist and are equal to 4.

29.4 THE ALGEBRA OF LIMITS

Let f and g be two real functions with common domain D. In the chapter on functions, we have defined four new function $f \pm g, fg, f/g$ on domain D by setting

$$(f \pm g)(x) = f(x) \pm g(x),$$
$$(fg)(x) = f(x)\, g(x)$$
$$(f/g)(x) = f(x)/g(x), \text{ if } g(x) \neq 0 \text{ for any } x \in D.$$

Following are some results concerning the limits of these functions.

Let $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$.

If l and m exist, then

(i) $\lim\limits_{x \to a} (f \pm g)(x) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = l \pm m$

(ii) $\lim\limits_{x \to a} (fg)(x) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = lm$

(iii) $\lim\limits_{x \to a} \left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{l}{m}$, provided $m \neq 0$.

(iv) $\lim\limits_{x \to a} k\, f(x) = k \lim\limits_{x \to a} f(x)$, where k is constant

(v) $\lim\limits_{x \to a} |f(x)| = |\lim\limits_{x \to a} f(x)| = |l|$

(vi) $\lim_{x \to a} \left\{f(x)\right\}^{g(x)} = l^m$

(vii) If $f(x) \le g(x)$ for every x in the deleted neighbourhood *of a*, then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$

(viii) If $f(x) \le g(x) \le h(x)$ for every x in the deleted neighbourhood of a and

$\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = l$.

This result is often stated as *Sandwich Theorem*.

(ix) If $\lim_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \to a} \frac{1}{f(x)} = 0$.

29.5 INDETERMINATE FORMS AND EVALUATION OF LIMITS

Uptill now we have been discussing left hand and right hand limits and the existence of limits. In what follows, we will be assuming that the limit of a function at a given point exists. In the previous section, we have stated that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ provided that } \lim_{x \to a} g(x) \ne 0.$$

An interesting situation now arises. If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$, which is undefined or meaningless. But, this does not imply that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is meaningless or it does not exist. In fact, in many cases this limit exists and has a finite value. The determination of limit in such a case is traditionally referred to as the evaluation of the indeterminate form $\frac{0}{0}$, though literally speaking nothing is indeterminate involved here. Sometimes $\frac{0}{0}$ is referred to as undetermined form or illusory form. In addition to $\frac{0}{0}$ there are six other indeterminate forms, namely, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0 and 1^∞. Among all these seven indeterminate forms $\frac{0}{0}$ is the fundamental one because all the remaining six forms can easily be reduced to this form. In this chapter, we shall study how to evaluate a limit which belongs to one of following in determinate forms:

$$\frac{0}{0},\ 0 \times \infty \text{ and } \infty - \infty.$$

To facilitate the job of evaluation of limits we categorize problems on limits in the following categories:

(i) Algebraic Limits. (ii) Non-algebraic Limits.

If a problem on limits does not involve trigonometric, inverse trigonometric, exponential and logarithmic function, then it is a problem on algebraic limits, otherwise, it is a problem on non-algebraic limits.

For example,

(i) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$ (ii) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$ (iii) $\lim_{x \to 2} \frac{x^5 - 32}{x - 2}$

(iv) $\lim_{x \to \infty} \frac{x^2 + x + 1}{2x^2 + 5}$ etc. are problems on algebraic limits.

Following are some examples of non-algebraic limits:

(i) $\lim_{x \to 0} \frac{e^{\sin x} - 1}{x}$ (ii) $\lim_{x \to 0} \frac{3 \sin^{-1} 2x}{\sin x}$ (iii) $\lim_{x \to \pi/4} \frac{\sin x - \cos x}{4x - \pi}$

(iv) $\lim_{x \to 0} \frac{2^x - 3^x}{x}$

29.6 EVALUATION OF ALGEBRAIC LIMITS

In order to evaluate algebraic limits we have the following methods.

(i) Direct substitution method.
(ii) Factorisation method.
(iii) Rationalisation method.
(iv) By using some standard limits.
(v) Method of evaluation of algebraic limits at infinity.

We shall now discuss these methods with suitable illustrations in the following sub- sections.

29.6.1 DIRECT SUBSTITUTION METHOD

Consider the following limits:

(i) $\lim_{x \to a} f(x)$ (ii) $\lim_{x \to a} \frac{\Phi(x)}{\psi(x)}$

If $f(a)$ and $\frac{\varphi(a)}{\psi(a)}$ exist and are fixed real numbers, then we say that

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} \frac{\varphi(x)}{\psi(a)} = \frac{\phi(a)}{\psi(a)}$$

In other words, if the direct substitution of the point, to which the variable tends to, we obtain a fixed real number, then the number obtained is the limit of the function. In fact, if the point to which the variable tends to is a point in the domain of the function, then the value of the function at that point is its limit.

Following examples will illustrate the above method.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate :* $\lim_{x \to 1} 3x^2 + 4x + 5$.

SOLUTION $\lim_{x \to 1} 3x^2 + 4x + 5 = 3(1)^2 + 4(1) + 5 = 12$.

EXAMPLE 2 *Evaluate:* $\lim_{x \to 2} \frac{x^2 - 4}{x + 3}$

SOLUTION Using direct substitution method, we obtain

$$\lim_{x \to 2} \frac{x^2 - 4}{x + 3} = \frac{4 - 4}{2 + 3} = \frac{0}{5} = 0.$$

EXAMPLE 3 *Evaluate* $\lim_{x \to 0} \frac{\sqrt{1+x}+\sqrt{1-x}}{1+x}$

SOLUTION Using direct substitution method, we obtain

$$\lim_{x \to 0} \frac{\sqrt{1+x}+\sqrt{1-x}}{1+x} = \frac{\sqrt{1+0}+\sqrt{1-0}}{1+0} = \frac{1+1}{1} = 2.$$

EXERCISE 29.2

LEVEL-1

Evaluate the following limits:

1. $\lim_{x \to 1} \frac{x^2+1}{x+1}$
2. $\lim_{x \to 0} \frac{2x^2+3x+4}{x^2+3x+2}$
3. $\lim_{x \to 3} \frac{\sqrt{2x+3}}{x+3}$
4. $\lim_{x \to 1} \frac{\sqrt{x+8}}{\sqrt{x}}$
5. $\lim_{x \to a} \frac{\sqrt{x}+\sqrt{a}}{x+a}$
6. $\lim_{x \to 1} \frac{1+(x-1)^2}{1+x^2}$
7. $\lim_{x \to 0} \frac{x^{2/3}-9}{x-27}$
8. $\lim_{x \to 0} 9$
9. $\lim_{x \to 2} (3-x)$
10. $\lim_{x \to -1} (4x^2+2)$
11. $\lim_{x \to 1} \frac{x^3-3x+1}{x-1}$
12. $\lim_{x \to 0} \frac{3x+1}{x+3}$
13. $\lim_{x \to 3} \frac{x^2-9}{x+2}$
14. $\lim_{x \to 0} \frac{ax+b}{cx+d}, d \neq 0$

ANSWERS

1. 1 2. 2 3. $\frac{1}{2}$ 4. 3 5. $\frac{1}{\sqrt{a}}$ 6. $\frac{1}{2}$ 7. $\frac{1}{3}$ 8. 9

9. 1 10. 6 11. $\frac{-3}{2}$ 12. $\frac{1}{3}$ 13. 0 14 $\frac{b}{d}$

29.6.2 FACTORIZATION METHOD

Consider the limit: $\lim_{x \to a} \frac{f(x)}{g(x)}$.

If by substituting $x = a$, $\frac{f(x)}{g(x)}$, reduces to the form $\frac{0}{0}$, then $(x-a)$ is a factor of $f(x)$ and $g(x)$ both. So, we first factorize $f(x)$ and $g(x)$ and then cancel out the common factor to evaluate the limit.

Following algorithm may be used to evaluate the limit by factorization method.

ALGORITHM

STEP I Obtain the problem, say, $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = 0$ *and* $\lim_{x \to a} g(x) = 0$.

STEP II Factorize $f(x)$ and $g(x)$.

STEP III Cancel out the common factor(s) of $f(x)$ and $g(x)$.

STEP IV Use direct substitution method to obtain the limit.

Some useful results to remember:

(i) $a^2 - b^2 = (a-b)(a+b)$ (ii) $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

(iii) $a^3 + b^3 = (a+b)(a^2-ab+b^2)$ (iv) $a^4 - b^4 = (a^2-b^2)(a^2+b^2) = (a+b)(a-b)(a^2+b^2)$

(v) If $f(\alpha) = 0$, then $x - \alpha$ is a factor of $f(x)$.

Following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate :* $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 4}$.

SOLUTION When $x = 2$ the expression $\dfrac{x^2 - 5x + 6}{x^2 - 4}$ assumes the indeterminate form $\dfrac{0}{0}$.

Therefore, $(x-2)$ is a common factor in numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{(x-2)(x-3)}{(x+2)(x-2)} = \lim_{x \to 2} \frac{x-3}{x+2} = \frac{2-3}{2+2} = -\frac{1}{4}.$$

EXAMPLE 2 *Evaluate:* $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

SOLUTION When $x = 1$ the expression $\dfrac{x^3 - 1}{x - 1}$ assumes the indeterminate form $\dfrac{0}{0}$. Therefore, $(x-1)$ is a common factor in numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x \to 1} x^2 + x + 1 = 1^2 + 1 + 1 = 3.$$

EXAMPLE 3 *Evaluate:* $\lim\limits_{x \to 2} \dfrac{x^3 - 3x^2 + 4}{x^4 - 8x^2 + 16}$.

SOLUTION When $x = 2$, the expression $\dfrac{x^3 - 3x^2 + 4}{x^4 - 8x^2 + 16}$ assumes the indeterminate form $\dfrac{0}{0}$.

Therefore, $(x-2)$ is a factor common to numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 2} \frac{x^3 - 3x^2 + 4}{x^4 - 8x^2 + 16}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2 - x - 2)}{(x^2 - 4)^2}$$

$$= \lim_{x \to 2} \frac{(x-2)(x-2)(x+1)}{(x-2)^2 (x+2)^2} = \lim_{x \to 2} \frac{x+1}{(x+2)^2} = \frac{2+1}{(2+2)^2} = \frac{3}{16}$$

EXAMPLE 4 *Evaluate:* $\lim_{x \to 2} \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$.

SOLUTION When $x = 2$, the expression $\frac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$ assumes the form $\frac{0}{0}$. Therefore, $(x - 2)$ is a factor common to numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to 2} \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$$

$$= \lim_{x \to 2} \frac{(x-1)(x-2)(x-3)}{(x-2)(x-4)} = \lim_{x \to 2} \frac{(x-1)(x-3)}{(x-4)} = \frac{(2-1)(2-3)}{(2-4)} = \frac{1}{2}.$$

EXAMPLE 5 *Evaluate:* $\lim_{x \to 1/2} \frac{8x^3 - 1}{16x^4 - 1}$.

SOLUTION When $x = 1/2$, the expression $\frac{8x^3 - 1}{16x^4 - 1}$ assumes the form $\frac{0}{0}$. Therefore, $\left(x - \frac{1}{2}\right)$ or, $2x - 1$ is a factor common to numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 1/2} \frac{8x^3 - 1}{16x^4 - 1} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1/2} \frac{(2x)^3 - 1^3}{(4x^2)^2 - 1^2}$$

$$= \lim_{x \to 1/2} \frac{(2x-1)(4x^2 + 2x + 1)}{(4x^2 + 1)(4x^2 - 1)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1/2} \frac{(2x-1)(4x^2 + 2x + 1)}{(4x^2 + 1)(2x-1)(2x+1)} = \lim_{x \to 1/2} \frac{4x^2 + 2x + 1}{(4x^2 + 1)(2x+1)} = \frac{3}{4}.$$

EXAMPLE 6 *Evaluate:* $\lim_{x \to 1} \left(\frac{2}{1 - x^2} + \frac{1}{x - 1}\right)$.

SOLUTION We have,

$$\lim_{x \to 1} \left(\frac{2}{1 - x^2} + \frac{1}{x - 1}\right) = \lim_{x \to 1} \left(\frac{2}{1 - x^2} - \frac{1}{1 - x}\right)$$

When $x = 1$, the expression $\frac{2}{1 - x^2} - \frac{1}{1 - x}$ assumes the form $\infty - \infty$. So, we need some simplification to express it in the form $\frac{0}{0}$. Taking LCM, we get

$$\lim_{x \to 1} \left(\frac{2}{1 - x^2} - \frac{1}{1 - x}\right) \qquad (\infty - \infty \text{ form})$$

$$= \lim_{x \to 1} \frac{2 - (1 + x)}{1 - x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{1 - x}{1 - x^2} = \lim_{x \to 1} \frac{1}{1 + x} = \frac{1}{2}.$$

EXAMPLE 7 *Evaluate:* $\lim_{x \to 1} \left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1} \right)$.

SOLUTION When $x = 1$, the expression $\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1}$ assumes the indeterminate form $\infty - \infty$. So, we need simplification to reduce the expression in the indeterminate form $\frac{0}{0}$.

$$\lim_{x \to 1} \left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1} \right) \qquad (\infty - \infty \text{ form})$$

$$= \lim_{x \to 1} \left\{ \frac{1}{(x+2)(x-1)} - \frac{x}{(x-1)(x^2+x+1)} \right\} \qquad (\infty - \infty \text{ form})$$

$$= \lim_{x \to 1} \left\{ \frac{(x^2+x+1) - x(x+2)}{(x+2)(x-1)(x^2+x+1)} \right\} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{-(x-1)}{(x+2)(x-1)(x^2+x+1)} = \lim_{x \to 1} \frac{-1}{(x+2)(x^2+x+1)} = -\frac{1}{9}.$$

EXAMPLE 8 *Evaluate:* $\lim_{x \to \sqrt{2}} \frac{x^4 - 4}{x^2 + 3x\sqrt{2} - 8}$. **[NCERT EXEMPLAR]**

SOLUTION When $x = \sqrt{2}$, the expression $\frac{x^4 - 4}{x^2 + 3x\sqrt{2} - 8}$ assumes the indeterminate form $\frac{0}{0}$. So, $(x - \sqrt{2})$ is a factor of numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to \sqrt{2}} \frac{x^4 - 4}{x^2 + 3x\sqrt{2} - 8} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \sqrt{2}} \frac{(x^2 - 2)(x^2 + 2)}{(x + 4\sqrt{2})(x - \sqrt{2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)}{(x + 4\sqrt{2})(x - \sqrt{2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \sqrt{2}} \frac{(x + \sqrt{2})(x^2 + 2)}{(x + 4\sqrt{2})} = \frac{(2\sqrt{2})(2 + 2)}{5\sqrt{2}} = \frac{8}{5}.$$

EXAMPLE 9 *Evaluate:* $\lim_{x \to 4} \frac{(x^2 - x - 12)^{18}}{(x^3 - 8x^2 + 16x)^9}$.

SOLUTION When $x = 4$, the expression $\frac{(x^2 - x - 12)^{18}}{(x^3 - 8x^2 + 16x)^9}$ assumes the form $\frac{0}{0}$. So, $(x - 4)$ is a common factor in numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to 4} \frac{(x^2 - x - 12)^{18}}{(x^3 - 8x^2 + 16x)^9}$$

$$= \lim_{x \to 4} \frac{[(x-4)(x+3)]^{18}}{[x(x^2 - 8x + 16)]^9} = \lim_{x \to 4} \frac{[(x-4)(x+3)]^{18}}{x^9 (x-4)^{18}}$$

$$= \lim_{x \to 4} \frac{(x-4)^{18}(x+3)^{18}}{x^9 (x-4)^{18}} = \lim_{x \to 4} \frac{(x+3)^{18}}{x^9} = \frac{7^{18}}{4^9}$$

LEVEL-2

EXAMPLE 10 *Evaluate:* $\lim_{x \to 3} \frac{x^3 - 7x^2 + 15x - 9}{x^4 - 5x^3 + 27x - 27}$.

SOLUTION When $x = 3$, the expression $\frac{x^3 - 7x^2 + 15x - 9}{x^4 - 5x^3 + 27x - 27}$ assumes the form $\frac{0}{0}$. So, $(x-3)$ is a factor of numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to 3} \frac{x^3 - 7x^2 + 15x - 9}{x^4 - 5x^3 + 27x - 27} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{(x-3)(x^2 - 4x + 3)}{(x-3)(x^3 - 2x^2 - 6x + 9)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{x^2 - 4x + 3}{x^3 - 2x^2 - 6x + 9} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{(x-3)(x-1)}{(x-3)(x^2 + x - 3)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{x-1}{x^2 + x - 3} = \frac{3-1}{9+3-3} = \frac{2}{9}.$$

EXAMPLE 11 *Evaluate:*

$$\lim_{x \to \sqrt{2}} \frac{x^9 - 3x^8 + x^6 - 9x^4 - 4x^2 - 16x + 84}{x^5 - 3x^4 - 4x + 12}$$

SOLUTION When $x = \sqrt{2}$, the expression $\frac{x^9 - 3x^8 + x^6 - 9x^4 - 4x^2 - 16x + 84}{x^5 - 3x^4 - 4x + 12}$ assume the form $\frac{0}{0}$. Therefore, $(x - \sqrt{2})$ is a factor of numerator and denominator. But, irrational roots occur in pairs. So, $(x + \sqrt{2})$ will also be a factor of both numerator and denominator, consequently, $(x^2 - 2)$ will be a common factor of numerator and denominator, Dividing numerator and denominator by $(x^2 - 2)$, we get

$$\lim_{x \to \sqrt{2}} \frac{x^9 - 3x^8 + x^6 - 9x^4 - 4x^2 - 16x + 84}{x^5 - 3x^4 - 4x + 12}$$

$$= \lim_{x \to \sqrt{2}} \frac{(x^2 - 2)(x^7 - 3x^6 + 2x^5 - 5x^4 + 4x^3 - 19x^2 + 8x - 42)}{(x^2 - 2)(x^3 - 3x^2 + 2x - 6)}$$

$$= \lim_{x \to \sqrt{2}} \frac{x^7 - 3x^6 + 2x^5 - 5x^4 + 4x^3 - 19x^2 + 8x - 42}{x^3 - 3x^2 + 2x - 6}$$

$$= \frac{8\sqrt{2} - 24 + 8\sqrt{2} - 20 + 8\sqrt{2} - 38 + 8\sqrt{2} - 42}{2\sqrt{2} - 6 + 2\sqrt{2} - 6} = \frac{8\sqrt{2} - 31}{\sqrt{2} - 3}.$$

EXERCISE 29.3

LEVEL-1

Evaluate the following limits:

1. $\lim_{x\to 5} \dfrac{2x^2+9x-5}{x+5}$

2. $\lim_{x\to 3} \dfrac{x^2-4x+3}{x^2-2x-3}$

3. $\lim_{x\to 3} \dfrac{x^4-81}{x^2-9}$

4. $\lim_{x\to 2} \dfrac{x^3-8}{x^2-4}$

5. $\lim_{x\to -1/2} \dfrac{8x^3+1}{2x+1}$

6. $\lim_{x\to 4} \dfrac{x^2-7x+12}{x^2-3x-4}$

7. $\lim_{x\to 2} \dfrac{x^4-16}{x-2}$

8. $\lim_{x\to 5} \dfrac{x^2-9x+20}{x^2-6x+5}$

9. $\lim_{x\to -1} \dfrac{x^3+1}{x+1}$

10. $\lim_{x\to 5} \dfrac{x^3-125}{x^2-7x+10}$

11. $\lim_{x\to\sqrt{2}} \dfrac{x^2-2}{x^2+\sqrt{2}x-4}$

12. $\lim_{x\to\sqrt{3}} \dfrac{x^2-3}{x^2+3\sqrt{3}\,x-12}$

13. $\lim_{x\to\sqrt{3}} \dfrac{x^4-9}{x^2+4\sqrt{3}x-15}$

14. $\lim_{x\to 2} \left(\dfrac{x}{x-2}-\dfrac{4}{x^2-2x}\right)$

15. $\lim_{x\to 1} \left(\dfrac{1}{x^2+x-2}-\dfrac{x}{x^3-1}\right)$

16. $\lim_{x\to 3} \left(\dfrac{1}{x-3}-\dfrac{2}{x^2-4x+3}\right)$

17. $\lim_{x\to 2} \left(\dfrac{1}{x-2}-\dfrac{2}{x^2-2x}\right)$

18. $\lim_{x\to 1/4} \dfrac{4x-1}{2\sqrt{x}-1}$

19. $\lim_{x\to 4} \dfrac{x^2-16}{\sqrt{x}-2}$

20. $\lim_{x\to 0} \dfrac{(a+x)^2-a^2}{x}$

21. $\lim_{x\to 2} \left(\dfrac{1}{x-2}-\dfrac{4}{x^3-2x^2}\right)$

22. $\lim_{x\to 3} \left(\dfrac{1}{x-3}-\dfrac{3}{x^2-3x}\right)$

23. $\lim_{x\to 1} \left(\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right)$

24. $\lim_{x\to 3} (x^2-9)\left(\dfrac{1}{x+3}+\dfrac{1}{x-3}\right)$

LEVEL-2

25. $\lim_{x\to 1} \dfrac{x^4-3x^3+2}{x^3-5x^2+3x+1}$

26. $\lim_{x\to 2} \dfrac{x^3+3x^2-9x-2}{x^3-x-6}$

27. $\lim_{x\to 1} \dfrac{1-x^{-1/3}}{1-x^{-2/3}}$

28. $\lim_{x\to 3} \dfrac{x^2-x-6}{x^3-3x^2+x-3}$

29. $\lim_{x\to -2} \dfrac{x^3+x^2+4x+12}{x^3-3x+2}$

30. $\lim_{x\to 1} \dfrac{x^3+3x^2-6x+2}{x^3+3x^2-3x-1}$

31. $\lim_{x \to 2} \left\{ \frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x} \right\}$ [NCERT EXEMPLAR]

32. $\lim_{x \to 1} \frac{\sqrt{x^2-1} + \sqrt{x-1}}{\sqrt{x^2-1}}, x > 1$

33. $\lim_{x \to 1} \left\{ \frac{x-2}{x^2-x} - \frac{1}{x^3 - 3x^2 + 2x} \right\}$ [NCERT]

34. $\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$ [NCERT EXEMPLAR]

ANSWERS

1. -11 2. $\frac{1}{2}$ 3. 18 4. 3 5. 3 6. $\frac{1}{5}$ 7. 32 8. $\frac{1}{4}$
9. 3 10. 25 11. $\frac{2}{3}$ 12. $\frac{2}{5}$ 13. 2 14. 2 15. $-\frac{1}{9}$ 16. $\frac{1}{2}$
17. $\frac{1}{2}$ 18. 2 19. 32 20. $2a$ 21. 1 22. $\frac{1}{3}$ 23. $\frac{1}{2}$ 24. 6
25. $\frac{5}{4}$ 26. $\frac{15}{11}$ 27. $\frac{1}{2}$ 28. $\frac{1}{2}$ 29. $\frac{4}{3}$ 30. $\frac{1}{2}$ 31. $-\frac{1}{2}$ 32. $\frac{\sqrt{2}+1}{\sqrt{2}}$
33. 2 34. 1

HINTS TO NCERT & SELECTED PROBLEM

33. $\lim_{x \to 1} \left(\frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x} \right)$

$$= \lim_{x \to 1} \left\{ \frac{x-2}{x(x-1)} - \frac{1}{x(x-1)(x-2)} \right\}$$

$$= \lim_{x \to 1} \frac{(x-2)^2 - 1^2}{x(x-1)(x-2)} = \lim_{x \to 1} \frac{(x-3)(x-1)}{x(x-1)(x-2)} = \lim_{x \to 1} \frac{x-3}{x(x-2)} = \frac{-2}{-1} = 2$$

34. $\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = \lim_{x \to 1} \frac{(x-1)(x^6 + x^5 - x^4 - x^3 - x^2 - x - 1)}{(x-1)(x^2 - 2x - 2)}$

$$= \lim_{x \to 1} \frac{(x^6 + x^5 - x^4 - x^3 - x^2 - x - 1)}{x^2 - 2x - 2} = \frac{-3}{-3} = 1$$

29.6.3 RATIONALISATION METHOD

This is particularly used when either the numerator or denominator or both involve expression consisting of square roots and substituting the value of x the rational expression takes the form $\frac{0}{0}, \frac{\infty}{\infty}$ etc.

Following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate:* $\lim_{x \to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$.

SOLUTION When $x = 0$, the expression $\dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ takes the form $\dfrac{0}{0}$.

Rationalising the numerator, we get

$$\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x \to 0} \frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x(\sqrt{2+x}+\sqrt{2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

EXAMPLE 2 *Evaluate:* $\lim_{x \to 0} \dfrac{x}{\sqrt{a+x}-\sqrt{a-x}}$.

SOLUTION When $x = 0$, the expression $\dfrac{x}{\sqrt{a+x}-\sqrt{a-x}}$ takes the form $\dfrac{0}{0}$.

Rationalising the denominator, we get

$$\lim_{x \to 0} \frac{x}{\sqrt{a+x}-\sqrt{a-x}} = \lim_{x \to 0} \frac{x}{(\sqrt{a+x}-\sqrt{a-x})} \times \frac{(\sqrt{a+x}+\sqrt{a-x})}{(\sqrt{a+x}+\sqrt{a-x})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{x(\sqrt{a+x}+\sqrt{a-x})}{(a+x-a+x)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{(\sqrt{a}+x+\sqrt{a-x})}{2} = \frac{2\sqrt{a}}{2} = \sqrt{a}$$

EXAMPLE 3 *Evaluate:* $\lim_{x \to 0} \dfrac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{x^2}$.

SOLUTION When $x = 0$, the expression $\dfrac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{x^2}$ takes the form $\dfrac{0}{0}$.

Rationalising the numerator, we get

$$\lim_{x \to 0} \frac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{x^2} = \lim_{x \to 0} \frac{(\sqrt{a^2+x^2}-\sqrt{a^2-x^2})}{x^2} \frac{(\sqrt{a^2+x^2}+\sqrt{a^2-x^2})}{(\sqrt{a^2+x^2}+\sqrt{a^2-x^2})} \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{a^2+x^2-a^2+x^2}{x^2(\sqrt{a^2+x^2}+\sqrt{a^2-x^2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2}{(\sqrt{a^2+x^2}+\sqrt{a^2-x^2})} = \frac{2}{\sqrt{a^2}+\sqrt{a^2}} = \frac{1}{a}$$

EXAMPLE 4 *Evaluate:* $\lim_{x \to 4} \dfrac{x^2-16}{\sqrt{x^2+9}-5}$.

SOLUTION When $x = 4$, the expression $\dfrac{x^2-16}{\sqrt{x^2+9}-5}$ assumes the form $\dfrac{0}{0}$.

Rationalising the denominator, we get

$$\lim_{x \to 4} \frac{x^2-16}{\sqrt{x^2+9}-5} = \lim_{x \to 4} \frac{(x^2-16)}{(\sqrt{x^2+9}-5)} \frac{(\sqrt{x^2+9}+5)}{(\sqrt{x^2+9}+5)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{(x^2-16)(\sqrt{x^2+9}+5)}{(x^2+9-25)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \left(\sqrt{x^2+9}+5\right) = \left(\sqrt{16+9}+5\right) = 5+5 = 10$$

EXAMPLE 5 *Evaluate:* $\lim_{x \to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$. **[NCERT EXEMPLAR]**

SOLUTION When $x = a$, the expression $\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ assumes the form $\frac{0}{0}$.

Rationalising the numerator, we get

$$\lim_{x \to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} = \lim_{x \to a} \frac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{a+2x}+\sqrt{3x})}{(\sqrt{3a+x}-2\sqrt{x})(\sqrt{3a+x}+2\sqrt{x})} \frac{(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{a+2x}+\sqrt{3x})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{(a+2x-3x)}{(3a+x-4x)} \frac{(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \lim_{x \to a} \frac{\sqrt{3a+x}+2\sqrt{x}}{3\left\{\sqrt{a+2x}+\sqrt{3x}\right\}} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \frac{\sqrt{3a+a}+2\sqrt{a}}{3(\sqrt{a+2a}+\sqrt{3a})} = \frac{1}{3} \times \frac{4\sqrt{a}}{2\sqrt{3a}} = \frac{2}{3\sqrt{3}}$$

EXAMPLE 6 *Evaluate:* $\lim_{x \to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$.

SOLUTION When $x = 4$, the expression $\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$ assumes the form $\frac{0}{0}$.

Rationalising the numerator and denominator, we get

$$\lim_{x \to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}} = \lim_{x \to 4} \frac{(3-\sqrt{5+x})(3+\sqrt{5+x})}{(1-\sqrt{5-x})(1+\sqrt{5-x})} \frac{(1+\sqrt{5-x})}{(3+\sqrt{5+x})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{(9-5-x)}{(1-5+x)} \left(\frac{1+\sqrt{5-x}}{3+\sqrt{5+x}}\right) \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{-(x-4)(1+\sqrt{5-x})}{(x-4)(3+\sqrt{5+x})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{-(1+\sqrt{5-x})}{(3+\sqrt{5+x})} = \frac{-(1+1)}{(3+3)} = -\frac{1}{3}$$

EXAMPLE 7 *Evaluate:* $\lim_{x \to 2} \frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$. **[NCERT EXEMPLAR]**

SOLUTION When $x = 2$, the expression $\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$ assumes the form $\frac{0}{0}$.

Rationalising the denominator, we get

$$\lim_{x \to 2} \frac{x^2 - 4}{\sqrt{3x-2} - \sqrt{x+2}} = \lim_{x \to 2} \frac{(x-2)(x+2)\left(\sqrt{3x-2} + \sqrt{x+2}\right)}{\left(\sqrt{3x-2} - \sqrt{x+2}\right)\left(\sqrt{3x-2} + \sqrt{x+2}\right)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)\left(\sqrt{3x-2} + \sqrt{x+2}\right)}{(3x-2-x-2)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)\left(\sqrt{3x-2} + \sqrt{x+2}\right)}{2(x-2)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{(x+2)\left(\sqrt{3x-2} + \sqrt{x+2}\right)}{2} = \frac{(2+2)(2+2)}{2} = 8.$$

EXAMPLE 8 *Evaluate:* $\lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$.

SOLUTION When $x = 1$, the expression $\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$ takes the form $\frac{0}{0}$.

Rationalising $(\sqrt{x}-1)$ in the numerator, we get

$$\lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3} = \lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}+1)(2x+3)(x-1)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{(2x-3)(x-1)}{(\sqrt{x}+1)(2x+3)(x-1)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{2x-3}{(\sqrt{x}+1)(2x+3)} = -\frac{1}{10}.$$

LEVEL-2

EXAMPLE 9 *Evaluate:* $\lim_{x \to \sqrt{10}} \frac{\sqrt{7-2x} - (\sqrt{5}-\sqrt{2})}{x^2-10}$

SOLUTION We have,

$$\lim_{x \to \sqrt{10}} \frac{\sqrt{7-2x} - (\sqrt{5}-\sqrt{2})}{x^2-10}$$

$$= \lim_{x \to \sqrt{10}} \frac{\sqrt{7-2x} - \sqrt{(\sqrt{5}-\sqrt{2})^2}}{x^2-10} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \sqrt{10}} \frac{\sqrt{7-2x} - \sqrt{7-2\sqrt{10}}}{x^2-10} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \sqrt{10}} \frac{\sqrt{7-2x} - \sqrt{7-2\sqrt{10}}}{x^2-10} \times \frac{\sqrt{7-2x} + \sqrt{7-2\sqrt{10}}}{\sqrt{7-2x} + \sqrt{7-2\sqrt{10}}}$$

$$= \lim_{x \to \sqrt{10}} \frac{(7-2x) - (7-2\sqrt{10})}{(x-\sqrt{10})(x+\sqrt{10})\left\{\sqrt{7-2x} + \sqrt{7-2\sqrt{10}}\right\}}$$

$$= \lim_{x \to \sqrt{10}} \frac{-2x+2\sqrt{10}}{(x-\sqrt{10})(x+\sqrt{10})\left\{\sqrt{7-2x} + \sqrt{7-2\sqrt{10}}\right\}}$$

$$= \lim_{x \to \sqrt{10}} \frac{-2(x-\sqrt{10})}{(x-\sqrt{10})(x+\sqrt{10})\left\{\sqrt{7-2x}+\sqrt{7-2\sqrt{10}}\right\}}$$

$$= \lim_{x \to \sqrt{10}} \frac{-2}{(x+\sqrt{10})\left\{\sqrt{7-2x}+\sqrt{7-2\sqrt{10}}\right\}}$$

$$= \lim_{x \to \sqrt{10}} \frac{-2}{2\sqrt{10}\left\{\sqrt{7-2\sqrt{10}}+\sqrt{7-2\sqrt{10}}\right\}}$$

$$= \frac{-1}{\sqrt{10}\times 2\times\sqrt{7-2\sqrt{10}}} = \frac{-1}{2\sqrt{10}(\sqrt{5}-\sqrt{2})} \qquad \left[\because (\sqrt{5}-\sqrt{2})^2 = 7-2\sqrt{10}\right]$$

$$= \frac{-1}{2\sqrt{10}} \times \frac{(\sqrt{5}+\sqrt{2})}{3} = -\frac{(\sqrt{5}+\sqrt{2})}{6\sqrt{10}}$$

EXERCISE 29.4

LEVEL-1

Evaluate the following limits:

1. $\lim_{x \to 0} \frac{\sqrt{1+x+x^2}-1}{x}$
2. $\lim_{x \to 0} \frac{2x}{\sqrt{a+x}-\sqrt{a-x}}$
3. $\lim_{x \to 0} \frac{\sqrt{a^2+x^2}-a}{x^2}$
4. $\lim_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$
5. $\lim_{x \to 2} \frac{\sqrt{3-x}-1}{2-x}$
6. $\lim_{x \to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$
7. $\lim_{x \to 0} \frac{x}{\sqrt{1+x}-\sqrt{1-x}}$
8. $\lim_{x \to 1} \frac{\sqrt{5x-4}-\sqrt{x}}{x-1}$
9. $\lim_{x \to 1} \frac{x-1}{\sqrt{x^2+3}-2}$
10. $\lim_{x \to 3} \frac{\sqrt{x+3}-\sqrt{6}}{x^2-9}$
11. $\lim_{x \to 1} \frac{\sqrt{5x-4}-\sqrt{x}}{x^2-1}$
12. $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$ [NCERT]
13. $\lim_{x \to 2} \frac{\sqrt{x^2+1}-\sqrt{5}}{x-2}$
14. $\lim_{x \to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}}$
15. $\lim_{x \to 7} \frac{4-\sqrt{9+x}}{1-\sqrt{8-x}}$
16. $\lim_{x \to 0} \frac{\sqrt{a+x}-\sqrt{a}}{x\sqrt{a^2+ax}}$
17. $\lim_{x \to 5} \frac{x-5}{\sqrt{6x-5}-\sqrt{4x+5}}$
18. $\lim_{x \to 1} \frac{\sqrt{5x-4}-\sqrt{x}}{x^3-1}$
19. $\lim_{x \to 2} \frac{\sqrt{1+4x}-\sqrt{5+2x}}{x-2}$
20. $\lim_{x \to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}$

21. $\lim_{x \to 0} \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x}$

22. $\lim_{x \to 0} \frac{\sqrt{1+x+x^2}-\sqrt{x+1}}{2x^2}$

23. $\lim_{x \to 4} \frac{2-\sqrt{x}}{4-x}$

24. $\lim_{x \to a} \frac{x-a}{\sqrt{x}-\sqrt{a}}$

25. $\lim_{x \to 0} \frac{\sqrt{1+3x}-\sqrt{1-3x}}{x}$

26. $\lim_{x \to 0} \frac{\sqrt{2-x}-\sqrt{2+x}}{x}$

27. $\lim_{x \to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}$

28. $\lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6}$

29. $\lim_{x \to 0} \frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}$

30. $\lim_{x \to 1} \frac{x^2-\sqrt{x}}{\sqrt{x}-1}$ [NCERT EXEMPLAR]

31. $\lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}, x \neq 0$ [NCERT EXEMPLAR]

LEVEL-2

32. $\lim_{x \to \sqrt{10}} \frac{\sqrt{7+2x}-(\sqrt{5}+\sqrt{2})}{x^2-10}$

33. $\lim_{x \to \sqrt{6}} \frac{\sqrt{5+2x}-(\sqrt{3}+\sqrt{2})}{x^2-6}$

34. $\lim_{x \to \sqrt{2}} \frac{\sqrt{3+2x}-(\sqrt{2}+1)}{x^2-2}$

ANSWERS

1. $\frac{1}{2}$ 2. $2\sqrt{a}$ 3. $\frac{1}{2a}$ 4. $\frac{1}{2}$ 5. $\frac{1}{2}$ 6. 1 7. 1 8. 2

9. 2 10. $\frac{1}{12\sqrt{6}}$ 11. 1 12. $\frac{1}{2}$ 13. $\frac{2}{\sqrt{5}}$ 14. $2\sqrt{2}$ 15. $-\frac{1}{4}$ 16. $\frac{1}{2a\sqrt{a}}$

17. 5 18. $\frac{2}{3}$ 19. $\frac{1}{3}$ 20. $\frac{1}{4}$ 21. 0 22. $\frac{1}{4}$ 23. $\frac{1}{4}$ 24. $2\sqrt{a}$

25. 3 26. $-\frac{1}{\sqrt{2}}$ 27. $\frac{1}{4}$ 28. $-\frac{1}{18}$ 29. 1 30. 3 31. $\frac{1}{2\sqrt{x}}$ 32. $\frac{(\sqrt{5}-\sqrt{2})}{6\sqrt{10}}$

33. $\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{6}}$ 34. $\frac{\sqrt{2}-1}{2\sqrt{2}}$

HINTS TO NCERT & SELECTED PROBLEM

12. $\lim_{x \to 0} = \frac{\sqrt{1+x}-1}{x} = \lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \lim_{x \to 0} \frac{1+x-1}{x(\sqrt{1+x}+1} = \lim_{x \to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$

33. $\lim_{x \to \sqrt{6}} = \frac{\sqrt{5+2x}-(\sqrt{3}+\sqrt{2})}{x^2-6} = \lim_{x \to \sqrt{6}} \frac{\sqrt{5+2x}-\sqrt{(\sqrt{3}+\sqrt{2})^2}}{x^2-6} = \lim_{x \to \sqrt{6}} \frac{\sqrt{5+2x}-\sqrt{5+2\sqrt{6}}}{x^2-6}$

$$= \lim_{x \to \sqrt{6}} \frac{(\sqrt{5+2x})-(\sqrt{5+2\sqrt{6}})}{(x-\sqrt{6})(x+\sqrt{6})(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}})} = \lim_{x \to \sqrt{6}} \frac{2(x-\sqrt{6})}{(x-\sqrt{6})(x+\sqrt{6})(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}})}$$

$$= \lim_{x \to \sqrt{6}} \frac{2}{(x+\sqrt{6})(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}})} = \frac{2}{(2\sqrt{6})(2\sqrt{5+2\sqrt{6}})} = \frac{1}{2\sqrt{6}(\sqrt{3}+\sqrt{2})}$$

29.6.4 EVALUATION OF ALGEBRAIC LIMITS BY USING SOME STANDARD LIMITS

Following theorem will be used to evaluate some algebraic limits.

THEOREM *If $n \in Q$, then* $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$.

PROOF We have,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a^+} \frac{x^n - a^n}{x - a} \qquad \left[\because \lim_{x \to a} f(x) \text{ exists } \therefore \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x)\right]$$

$$= \lim_{h \to 0} \frac{(a+h)^n - a^n}{a + h - a}$$

$$= \lim_{h \to 0} \frac{a^n \left\{\left(1 + \frac{h}{a}\right)^n - 1\right\}}{h}$$

$$= a^n \lim_{h \to 0} \frac{\left\{1 + n\frac{h}{a} + \frac{n(n-1)}{2!}\frac{h^2}{a^2} + \dots - 1\right\}}{h} \qquad \left[\because (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots\right]$$

$$= a^n \lim_{h \to 0} \left\{\frac{n}{a} + \frac{n(n-1)}{2!}\frac{h}{a^2} + \dots\right\} = a^n \times \frac{n}{a} = n\,a^{n-1}$$

Following examples will illustrate the use of the above result in evaluating algebraic limits.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate:* $\lim\limits_{x \to 2} \dfrac{x^{10} - 1024}{x - 2}$

SOLUTION When $x = 2$, the expression $\dfrac{x^{10} - 1024}{x - 2}$ assumes the form $\dfrac{0}{0}$.

Now, $\lim\limits_{x \to 2} \dfrac{x^{10} - 1024}{x - 2} = \lim\limits_{x \to 2} \dfrac{x^{10} - 2^{10}}{x - 2} = 10(2^{10-1}) = 5120$

EXAMPLE 2 *Evaluate:* $\lim\limits_{x \to 2} \dfrac{x^{10} - 1024}{x^5 - 32}$.

SOLUTION When $x = 2$, the expression $\dfrac{x^{10} - 1024}{x^5 - 32}$ assume s the indeterminate form $\dfrac{0}{0}$.

Now, $\lim\limits_{x \to 2} \dfrac{x^{10} - 1024}{x^5 - 32}$ $\qquad \left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x \to 2} \frac{x^{10} - 2^{10}}{x^5 - 2^5} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to 2} \frac{\frac{x^{10}-2^{10}}{x-2}}{\frac{x^5-2^5}{x-2}} = \lim_{x\to 2} \frac{x^{10}-2^{10}}{x-2} \div \lim_{x\to 2} \frac{x^5-2^5}{x-2} = 10\cdot 2^{10-1} \div 5\cdot 2^{5-1} = 64.$$

EXAMPLE 3 *Evaluate:* $\lim\limits_{x\to 9} \dfrac{x^{3/2}-27}{x-9}$.

SOLUTION When $x = 9$, the expression $\dfrac{x^{3/2}-27}{x-9}$ assumes the form $\dfrac{0}{0}$

Now, $\lim\limits_{x\to 9} \dfrac{x^{3/2}-27}{x-9} = \lim\limits_{x\to 9} \dfrac{x^{3/2}-9^{3/2}}{x-9} = \dfrac{3}{2}(9)^{3/2-1} = \dfrac{3}{2}(3) = \dfrac{9}{2}$

EXAMPLE 4 *Evaluate:* $\lim\limits_{x\to a} \dfrac{x\sqrt{x}-a\sqrt{a}}{x-a}$.

SOLUTION We have,

$$\lim_{x\to a} \frac{x\sqrt{x}-a\sqrt{a}}{x-a} = \lim_{x\to a} \frac{x^{3/2}-a^{3/2}}{x-a} = \frac{3}{2}a^{3/2-1} = \frac{3}{2}\sqrt{a}$$

EXAMPLE 5 *Evaluate:* $\lim\limits_{x\to a} \dfrac{x^m-a^m}{x^n-a^n}$.

SOLUTION We have,

$$\lim_{x\to a} \frac{x^m-a^m}{x^n-a^n} = \lim_{x\to a} \left\{\frac{x^m-a^m}{x-a}\cdot\frac{x-a}{x^n-a^n}\right\} = \lim_{x\to a} \left\{\frac{x^m-a^m}{x-a} \div \frac{x^n-a^n}{x-a}\right\}$$

$$= \lim_{x\to a} \frac{x^m-a^m}{x-a} \div \lim_{x\to a} \frac{x^n-a^n}{x-a} = ma^{m-1} \div na^{n-1} = \frac{m}{n}a^{m-n}.$$

EXAMPLE 6 *Evaluate:* $\lim\limits_{x\to 2} \dfrac{x-2}{\sqrt[3]{x}-\sqrt[3]{2}}$.

SOLUTION We have,

$$\lim_{x\to 2} \frac{x-2}{x^{1/3}-2^{1/3}} = \frac{1}{\lim\limits_{x\to 2} \dfrac{x^{1/3}-2^{1/3}}{x-2}} = \frac{1}{\frac{1}{3}\left(2^{1/3-1}\right)} = \frac{1}{\frac{1}{3}\times(2^{-2/3})} = 3\,(2^{2/3})$$

EXAMPLE 7 *Evaluate:* $\lim\limits_{x\to 0} \dfrac{(1-x)^n-1}{x}$.

SOLUTION We have,

$$\lim_{x\to 0} \frac{(1-x)^n-1}{x} = -\lim_{x\to 0} \frac{(1-x)^n-1}{(1-x)-1}$$

$$= -\lim_{y\to 1} \frac{y^n-1^n}{y-1}, \text{ where } y = 1-x. \qquad [\because\ x\to 0 \Rightarrow y\to 1]$$

$$= -n(1)^{n-1} = -n.$$

EXAMPLE 8 *Evaluate:* $\lim_{x \to a} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{x-a}$.

SOLUTION We have,

$$\lim_{x \to a} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{x-a}$$

$$= \lim_{x \to a} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{(x+2) - (a+2)}$$

$$= \lim_{y \to b} \frac{y^{5/3} - b^{5/3}}{y-b}, \text{ where } x+2 = y \text{ and } a+2 = b.$$

$$= \frac{5}{3} b^{5/3-1} = \frac{5}{3} b^{2/3} = \frac{5}{3}(a+2)^{2/3}$$

EXAMPLE 9 *Find the value of k, if* $\lim_{x \to 1} \frac{x^4 - 1}{x-1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$. [NCERT EXEMPLAR]

SOLUTION We have,

$$\lim_{x \to 1} \frac{x^4 - 1}{x-1} = \lim_{x \to 1} \frac{x^4 - 1^4}{x-1} = 4(1)^{4-1} = 4$$

and, $$\lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2} = \lim_{x \to k} \frac{x^3 - k^3}{x-k} \times \frac{x-k}{x^2 - k^2} = \lim_{x \to k} \frac{x^3 - k^3}{x-k} \div \frac{x^2 - k^2}{x-k}$$

$$= \lim_{x \to k} \frac{x^3 - k^3}{x-k} \div \lim_{x \to k} \frac{x^2 - k^2}{x-k} = 3k^{3-1} \div 2k^{2-1} = \frac{3}{2}k$$

$\therefore$ $$\lim_{x \to 1} \frac{x^4 - 1}{x-1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$$

$\Rightarrow$ $$4 = \frac{3k}{2} \Rightarrow k = \frac{8}{3}$$

EXAMPLE 10 *If* $\lim_{x \to 2} \frac{x^n - 2^n}{x-2} = 80$ *and* $n \in N$, *find n.* [NCERT EXEMPLAR]

SOLUTION We have,

$$\lim_{x \to 2} \frac{x^n - 2^n}{x-2} = 80 \Rightarrow n \cdot 2^{n-1} = 80 \Rightarrow n \cdot 2^{n-1} = 5 \cdot 2^{5-1} \Rightarrow n = 5.$$

LEVEL-2

EXAMPLE 11 *If* $\lim_{x \to -a} \frac{x^9 + a^9}{x+a} = 9$, *find the real values of a.*

SOLUTION We have,

$$\lim_{x \to -a} \frac{x^9 + a^9}{x+a} = 9$$

$\Rightarrow$ $$\lim_{x \to -a} \frac{x^9 - (-a)^9}{x - (-a)} = 9 \Rightarrow 9(-a)^{9-1} = 9 \Rightarrow 9a^8 = 9 \Rightarrow a^8 = 1 \Rightarrow a = \pm 1$$

EXAMPLE 12 *Evaluate:* $\lim\limits_{x \to 1} \dfrac{(x + x^2 + x^3 + \ldots + x^n) - n}{x - 1}$.

SOLUTION We have,

$$\lim_{x \to 1} \frac{(x + x^2 + x^3 + \ldots + x^n) - n}{x - 1} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 1} \frac{(x - 1) + (x^2 - 1) + (x^3 - 1) + \ldots + (x^n - 1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{x - 1}{x - 1} + \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1} + \lim_{x \to 1} \frac{x^3 - 1}{x - 1} + \ldots + \lim_{x \to 1} \frac{x^n - 1^n}{x - 1}$$

$$= 1 + 2(1)^{2-1} + 3(1)^{3-1} + \ldots + n(1)^{n-1} = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

EXERCISE 29.5

LEVEL-1

Evaluate the following limits:

1. $\lim\limits_{x \to a} \dfrac{(x + 2)^{5/2} - (a + 2)^{5/2}}{x - a}$

2. $\lim\limits_{x \to a} \dfrac{(x + 2)^{3/2} - (a + 2)^{3/2}}{x - a}$

3. $\lim\limits_{x \to 0} \dfrac{(1 + x)^6 - 1}{(1 + x)^2 - 1}$

4. $\lim\limits_{x \to a} \dfrac{x^{2/7} - a^{2/7}}{x - a}$

5. $\lim\limits_{x \to a} \dfrac{x^{5/7} - a^{5/7}}{x^{2/7} - a^{2/7}}$

6. $\lim\limits_{x \to -1/2} \dfrac{8x^3 + 1}{2x + 1}$

7. $\lim\limits_{x \to 27} \dfrac{(x^{1/3} + 3)(x^{1/3} - 3)}{x - 27}$

8. $\lim\limits_{x \to 4} \dfrac{x^3 - 64}{x^2 - 16}$

9. $\lim\limits_{x \to 1} \dfrac{x^{15} - 1}{x^{10} - 1}$

10. $\lim\limits_{x \to -1} \dfrac{x^3 + 1}{x + 1}$

11. $\lim\limits_{x \to a} \dfrac{x^{2/3} - a^{2/3}}{x^{3/4} - a^{3/4}}$

12. If $\lim\limits_{x \to 3} \dfrac{x^n - 3^n}{x - 3} = 108$, find the value of n.

13. If $\lim\limits_{x \to a} \dfrac{x^9 - a^9}{x - a} = 9$, find all possible values of a.

14. If $\lim\limits_{x \to a} \dfrac{x^5 - a^5}{x - a} = 405$, find all possible values of a.

15. If $\lim\limits_{x \to a} \dfrac{x^9 - a^9}{x - a} = \lim\limits_{x \to 5} (4 + x)$, find all possible values of a.

16. If $\lim\limits_{x \to a} \dfrac{x^3 - a^3}{x - a} = \lim\limits_{x \to 1} \dfrac{x^4 - 1}{x - 1}$, find all possible values of a.

ANSWERS

1. $\frac{5}{2}(a + 2)^{3/2}$ 2. $\frac{3}{2}(a + 2)^{1/2}$ 3. 3 4. $\frac{2}{7}a^{-5/7}$ 5. $\frac{5}{2}a^{3/7}$

6. 3 7. $\frac{2}{9}$ 8. 6 9. $\frac{3}{2}$ 10. 3 11. $\frac{8}{9}a^{-1/12}$ 12. 4

13. 1, −1 14. $a = 3, -3$ 15. 1, −1 16. $\pm\frac{2}{\sqrt{3}}$

HINTS TO SELECTED PROBLEM

9. $\lim_{x \to 1} \frac{x^{15}-1}{x^{10}-1} = \lim_{x \to 1} \frac{\frac{x^{15}-1^{15}}{x-1}}{\frac{x^{10}-1^{10}}{x-1}} = \frac{15(1)^{15-1}}{10(1)^{10-1}} = \frac{3}{2}$

29.6.5 METHOD OF EVALUATION OF ALGEBRAIC LIMITS AT INFINITY

Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$. Graphs of these functions are shown in Figures 29.6 and 29.7.

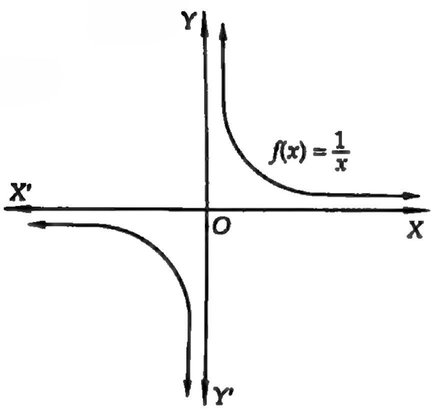


Fig. 29.6 Graph of $f(x) = \frac{1}{x}$

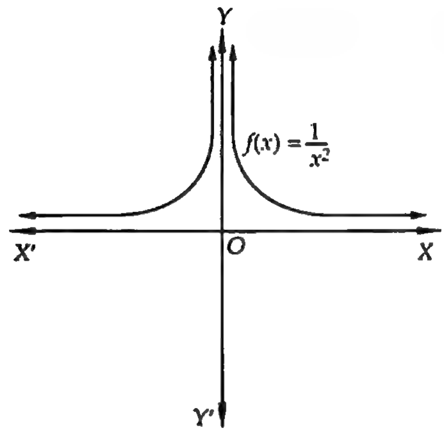


Fig. 29.7 Graph of $f(x) = \frac{1}{x^2}$

We observe from the graphs that as x increases, the values of $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ decrease rapidly and when x is indefinitely large $\frac{1}{x}$ and $\frac{1}{x^2}$ are indefinitely small i.e. very close to zero. In such cases, we write

$$\lim_{x \to +\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to +\infty} \frac{1}{x^2} = 0.$$

We also observe from the graphs of these two functions that as x decreases and is very small negative real number, then also the values of $\frac{1}{x}$ and $\frac{1}{x^2}$ approach to zero. So, we write

$$\lim_{x \to -\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^2} = 0.$$

It follows from the above discussion that:

(i) $\lim_{x \to \infty} c = c$ (ii) $\lim_{x \to -\infty} c = c$

(iii) $\lim_{x \to \infty} \frac{c}{x^n} = 0, n > 0$ (iv) $\lim_{x \to -\infty} \frac{c}{x^n} = 0, n \in N$

From the graphs of real functions, we obtain the following useful results:

(i) $\lim_{x \to +\infty} x \to +\infty$ (ii) $\lim_{x \to -\infty} x \to -\infty$

(iii) $\lim_{x \to +\infty} x^2 \to +\infty$ (iv) $\lim_{x \to -\infty} x^2 \to +\infty$ and so on.

(v) $\lim_{x \to \infty} e^x \to \infty$ or, $\lim_{x \to -\infty} e^{-x} \to \infty$

(vi) $\lim_{x \to \infty} e^{-x} \to 0$ or, $\lim_{x \to -\infty} e^x \to 0$

(vii) $\lim_{x \to \infty} a^x \to 0$, if $|a| < 1$ (viii) $\lim_{x \to \infty} a^x \to \infty$, if $a > 1$

(ix) $\lim_{x \to 0^+} \log_a x \to -\infty$ and $\lim_{x \to \infty} \log_a x \to \infty$, where $a > 1$

(x) $\lim_{x \to 0^+} \log_a x \to \infty$ and $\lim_{x \to \infty} \log_a x \to -\infty$, if $0 < a < 1$.

We use these results to evaluate limits at infinity. Following algorithm may be used to evaluate algebraic limits at infinity.

ALGORITHM

STEP I *Write down the given expression in the form of a rational function. i.e.* $\frac{f(x)}{g(x)}$*, if it is not so.*

STEP II *If k is the highest power of x in numerator and denominator both, then divide each term in numerator and denominator by* x^k.

STEP III *Use the results* $\lim_{x \to \infty} \frac{c}{x^n} = 0$ and $\lim_{x \to \infty} c = c$, where $n > 0$.

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate:* $\lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f}$.

SOLUTION Here the expression assumes the form $\frac{\infty}{\infty}$. We notice that the highest power of x in both the numerator and denominator is 2. So we divide each term in both the numerator and denominator by x^2.

$$\therefore \quad \lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f} = \lim_{x \to \infty} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x} + \frac{f}{x^2}} = \frac{a + 0 + 0}{d + 0 + 0} = \frac{a}{d}.$$

EXAMPLE 2 *Evaluate:* $\lim_{x \to \infty} \frac{5x - 6}{\sqrt{4x^2 + 9}}$.

SOLUTION We have,

$$\lim_{x \to \infty} \frac{5x - 6}{\sqrt{4x^2 + 9}} = \lim_{x \to \infty} \frac{5 - 6/x}{\sqrt{4 + 9/x^2}} \quad \text{[Dividing each term in } N^r \text{ and } D^r \text{ by } x]$$

$$= \frac{5 - 0}{\sqrt{4 + 0}} = \frac{5}{2}$$

EXAMPLE 3 *Evaluate:* $\lim_{x \to \infty} \frac{\sqrt{3x^2 - 1} + \sqrt{2x^2 - 1}}{4x + 3}$

SOLUTION Dividing each term in the numerator and denominator by x, we get

$$\lim_{x \to \infty} \frac{\sqrt{3x^2-1}+\sqrt{2x^2-1}}{4x+3} = \lim_{x \to \infty} \frac{\sqrt{3-1/x^2}+\sqrt{2-1/x^2}}{4+3/x} = \frac{\sqrt{3}+\sqrt{2}}{4}$$

EXAMPLE 4 *Evaluate:* $\lim\limits_{x \to \infty} \sqrt{x}\,(\sqrt{x+c}-\sqrt{x})$.

SOLUTION The given expression is of the form $\infty - \infty$. So we first write it in the rational form $\frac{f(x)}{g(x)}$. So that it reduces to either $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form.

$$\therefore \quad \lim_{x \to \infty} \sqrt{x}\left\{\sqrt{x+c}-\sqrt{x}\right\} = \lim_{x \to \infty} \frac{\sqrt{x}\left\{\sqrt{x+c}-\sqrt{x}\right\}\left\{\sqrt{x+c}+\sqrt{x}\right\}}{\left\{\sqrt{x+c}+\sqrt{x}\right\}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x}\,(x+c-x)}{\sqrt{x+c}+\sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{c\sqrt{x}}{\sqrt{x+c}+\sqrt{x}} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{c}{\sqrt{1+\frac{c}{x}}+1} \qquad [\text{Diving } N^r \text{ and } D^r \text{ by } \sqrt{x}]$$

$$= \frac{c}{\sqrt{1+0}+1} = \frac{c}{2}$$

EXAMPLE 5 *Evaluate:* $\lim\limits_{x \to \infty} \left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)$

SOLUTION Here the expression assumes the form $\infty - \infty$ as $x \to \infty$. So, we first reduce it to the rational form $\frac{f(x)}{g(x)}$.

$$\lim_{x \to \infty} \sqrt{x^2+x+1}-\sqrt{x^2+1} = \lim_{x \to \infty} \frac{\left\{\sqrt{x^2+x+1}-\sqrt{x^2+1}\right\}}{\left\{\sqrt{x^2+x+1}+\sqrt{x^2+1}\right\}}\left\{\sqrt{x^2+x+1}+\sqrt{x^2+1}\right\}$$

$$= \lim_{x \to \infty} \frac{x^2+x+1-x^2-1}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$= \lim_{x \to \infty} \frac{x}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}} \qquad [\text{Dividing } N^r \text{ and } D^r \text{ by } x]$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

EXAMPLE 6 *Evaluate:* $\lim\limits_{n \to \infty} \frac{1+2+3+\ldots+n}{n^2}$.

SOLUTION We have,

$$\lim_{n \to \infty} \frac{1+2+3+\ldots+n}{n^2} = \lim_{n \to \infty} \frac{1}{n^2} \times \frac{n(n+1)}{2} \qquad \left[\because 1+2+\ldots+n = \frac{n(n+1)}{2}\right]$$

$$= \lim_{n \to \infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

EXAMPLE 7 *Evaluate:* $\lim_{n \to \infty} \frac{n!}{(n+1)!-n!}$

SOLUTION We have,

$$\lim_{n \to \infty} \frac{n!}{(n+1)!-n!} = \lim_{n \to \infty} \frac{n!}{(n+1)\,n!-n!} = \lim_{n \to \infty} \frac{1}{n+1-1} = \lim_{n \to \infty} \frac{1}{n} = 0$$

EXAMPLE 8 *Let* $f(x) = \frac{ax+b}{x+1}$, $\lim_{x \to 0} f(x) = 2$ *and* $\lim_{x \to \infty} f(x) = 1$ *prove that* $f(-2) = 0$.

SOLUTION We have,

$$\lim_{x \to 0} f(x) = 2 \Rightarrow \lim_{x \to 0} \frac{ax+b}{x+1} = 2 \Rightarrow \frac{b}{1} = 2 \Rightarrow b = 2$$

It is also given that

$$\lim_{x \to \infty} f(x) = 1 \Rightarrow \lim_{x \to \infty} \frac{ax+b}{x+1} = 1 \Rightarrow \lim_{x \to \infty} \frac{a+\frac{b}{x}}{1+\frac{1}{x}} = 1 \Rightarrow \frac{a+0}{1+0} = 1 \Rightarrow a = 1.$$

Substituting the values of a and b in $f(x) = \frac{ax+b}{x+1}$, we obtain

$$f(x) = \frac{x+2}{x+1} \Rightarrow f(-2) = \frac{-2+2}{-2+1} = 0.$$

LEVEL-2

EXAMPLE 9 *Evaluate:* $\lim_{x \to -\infty} \left(\sqrt{x^2-x+1}+x\right)$.

SOLUTION We have,

$$\lim_{x \to -\infty} \left(\sqrt{x^2-x+1}+x\right) = \lim_{y \to \infty} \left(\sqrt{y^2+y+1}-y\right), \text{ where } y = -x$$

$$= \lim_{y \to \infty} \frac{\left\{\sqrt{y^2+y+1}-y\right\}\left\{\sqrt{y^2+y+1}+y\right\}}{\left\{\sqrt{y^2+y+1}+y\right\}}$$

$$= \lim_{y \to \infty} \frac{y^2+y+1-y^2}{\sqrt{y^2+y+1}+y}$$

$$= \lim_{y \to \infty} \frac{y+1}{\sqrt{y^2+y+1}+y} = \lim_{y \to \infty} \frac{1+\frac{1}{y}}{\sqrt{1+\frac{1}{y}+\frac{1}{y^2}}+1} = \frac{1}{2}$$

<u>ALITER</u> We have,

$$\lim_{x \to -\infty} \left(\sqrt{x^2-x+1}+x\right) = \lim_{x \to -\infty} \frac{\left\{\sqrt{x^2-x+1}+x\right\}\left\{\sqrt{x^2-x+1}-x\right\}}{\left\{\sqrt{x^2-x+1}-x\right\}}$$

$$= \lim_{x \to -\infty} \frac{x^2 - x + 1 - x^2}{\left\{\sqrt{x^2 - x + 1} - x\right\}}$$

$$= \lim_{x \to -\infty} \frac{-x + 1}{\sqrt{x^2 - x + 1} - x}$$

$$= \lim_{x \to -\infty} \frac{-\frac{x}{|x|} + \frac{1}{|x|}}{\frac{\sqrt{x^2 - x + 1}}{|x|} - \frac{x}{|x|}} \quad \left[\text{Dividing } N^r \text{ and } D^r \text{ by } |x|\right]$$

$$= \lim_{x \to -\infty} \frac{\frac{-x}{-x} - \frac{1}{x}}{\sqrt{\frac{x^2}{x^2} - \frac{x}{x^2} + \frac{1}{x^2}} + \frac{x}{x}} \quad \left[\because \sqrt{x^2} = |x| = -x \text{ for } x < 0\right]$$

$$= \lim_{x \to -\infty} \frac{1 - \frac{1}{x}}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + 1} = \frac{1}{2}$$

EXAMPLE 10 *Evaluate:* $\lim_{x \to \infty} \frac{(x+1)^{10} + (x+2)^{10} + \dots + (x+100)^{10}}{x^{10} + 10^{10}}$

SOLUTION We have,

$$\lim_{x \to \infty} \frac{(x+1)^{10} + (x+2)^{10} + \dots + (x+100)^{10}}{x^{10} + 10^{10}}$$

$$= \lim_{x \to \infty} \frac{\left(1 + \frac{1}{x}\right)^{10} + \left(1 + \frac{2}{x}\right)^{10} + \dots + \left(1 + \frac{100}{x}\right)^{10}}{1 + \left(\frac{10}{x}\right)^{10}}$$

$$= \frac{1 + 1 + \dots + 1 \text{ (100 - times)}}{1 + 0} = \frac{100}{1} = 100$$

EXERCISE 29.6

LEVEL-1

Evaluate the following limits:

1. $\lim_{x \to \infty} \frac{(3x-1)(4x-2)}{(x+8)(x-1)}$
2. $\lim_{x \to \infty} \frac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 - 5x + 7}$
3. $\lim_{x \to \infty} \frac{5x^3 - 6}{\sqrt{9 + 4x^6}}$
4. $\lim_{x \to \infty} \sqrt{x^2 + cx} - x$
5. $\lim_{x \to \infty} \sqrt{x+1} - \sqrt{x}$
6. $\lim_{x \to \infty} \sqrt{x^2 + 7x} - x$
7. $\lim_{x \to \infty} \frac{x}{\sqrt{4x^2 + 1} - 1}$
8. $\lim_{n \to \infty} \frac{n^2}{1 + 2 + 3 + \dots + n}$

9. $\lim\limits_{x \to \infty} \dfrac{3x^{-1} + 4x^{-2}}{5x^{-1} + 6x^{-2}}$

10. $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2}}{\sqrt{x^2 + c^2} - \sqrt{x^2 + d^2}}$

11. $\lim\limits_{n \to \infty} \dfrac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$

12. $\lim\limits_{x \to \infty} x\left\{\sqrt{x^2+1} - \sqrt{x^2-1}\right\}$

13. $\lim\limits_{x \to \infty} \left\{\sqrt{x+1} - \sqrt{x}\right\}\sqrt{x+2}$

14. $\lim\limits_{n \to \infty} \dfrac{1^2 + 2^2 + \ldots + n^2}{n^3}$

15. $\lim\limits_{n \to \infty} \left(\dfrac{1}{n^2} + \dfrac{2}{n^2} + \dfrac{3}{n^2} + \ldots + \dfrac{n-1}{n^2}\right)$

16. $\lim\limits_{n \to \infty} \dfrac{1^3 + 2^3 + \ldots + n^3}{n^4}$

17. $\lim\limits_{n \to \infty} \dfrac{1^3 + 2^3 + \ldots + n^3}{(n-1)^4}$

18. $\lim\limits_{x \to \infty} \sqrt{x}\left\{\sqrt{x+1} - \sqrt{x}\right\}$

19. $\lim\limits_{n \to \infty} \left(\dfrac{1}{3} + \dfrac{1}{3^2} + \dfrac{1}{3^3} + \ldots + \dfrac{1}{3^n}\right)$

20. $\lim\limits_{x \to \infty} \dfrac{x^4 + 7x^3 + 46x + a}{x^4 + 6}$, where a is a non-zero real number.

21. $f(x) = \dfrac{ax^2 + b}{x^2 + 1}$, $\lim\limits_{x \to 0} f(x) = 1$ and $\lim\limits_{x \to \infty} f(x) = 1$, then prove that $f(-2) = f(2) = 1$.

22. Show that $\lim\limits_{x \to \infty} (\sqrt{x^2 + x + 1} - x) \neq \lim\limits_{x \to \infty} (\sqrt{x^2 + 1} - x)$

LEVEL-2

23. $\lim\limits_{x \to -\infty} \left(\sqrt{4x^2 - 7x} + 2x\right)$

24. $\lim\limits_{x \to -\infty} \left(\sqrt{x^2 - 8x} + x\right)$

25. Evaluate : $\lim\limits_{n \to \infty} \dfrac{1^4 + 2^4 + 3^4 + \ldots + n^4}{n^5} - \lim\limits_{n \to \infty} \dfrac{1^3 + 2^3 + \ldots + n^3}{n^5}$

26. Evaluate : $\lim\limits_{n \to \infty} \dfrac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1)}{n^3}$

ANSWERS

1. 12 2. $\frac{3}{2}$ 3. $\frac{5}{2}$ 4. $\frac{c}{2}$ 5. 0 6. $\frac{7}{2}$ 7. $\frac{1}{2}$ 8. 2 9. $\frac{3}{5}$

10. $\frac{a^2 - b^2}{c^2 - d^2}$ 11. 1 12. 1 13. $\frac{1}{2}$ 14. $\frac{1}{3}$ 15. $\frac{1}{2}$ 16. $\frac{1}{4}$ 17. $\frac{1}{4}$

18. $\frac{1}{2}$ 19. $\frac{1}{2}$ 20. 1 23. $\frac{7}{4}$ 24. 4 25. $\frac{1}{5}$ 26. $\frac{1}{3}$

29.7 EVALUATION OF TRIGONOMETRIC LIMITS

In this section, we will be studying various methods of evaluating trigonometric limits. In order to evaluate trigonometric limits we will be using the following results which are stated and proved in the following theorem.

THEOREM *If angle θ is measured in radians, then*

(i) $\lim_{\theta \to 0} \sin \theta = 0$ and $\lim_{\theta \to 0} \cos \theta = 1$

(ii) $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, *where* θ *is measured in radians*

(iii) $\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$

(iv) $\lim_{\theta \to a} \frac{\sin (\theta - a)}{\theta - a} = 1$

(v) $\lim_{\theta \to a} \frac{\tan (\theta - a)}{\theta - a} = 1$

PROOF (i) Let ABC be a right angled triangle such that $\angle C = \frac{\pi}{2}$ and $\angle ABC = \theta$. Then,

$$\sin \theta = \frac{CA}{BA} \text{ and, } \cos \theta = \frac{BC}{BA}.$$

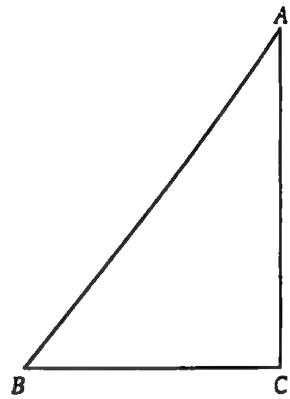


Fig. 29.8

Now, if we keep BC fixed and go on decreasing angle θ, then we find that A goes on coming nearer and nearer to C.

$\therefore \quad A \to C$ as $\theta \to 0$

This means that $CA \to 0$ and $BA \to BC$ as $\theta \to 0$.

$\Rightarrow \quad \frac{CA}{BA} \to 0$ and $\frac{BC}{BA} \to 1$ as $\theta \to 0$

$\Rightarrow \quad \sin \theta \to 0$ and $\cos \theta \to 1$ as $\theta \to 0$

(ii) Consider a circle of radius r. Let O be the centre of the circle such that $\angle AOB = \theta$ where θ is measured in radians and it is very small. Suppose the tangent at A meets OB produced at P.

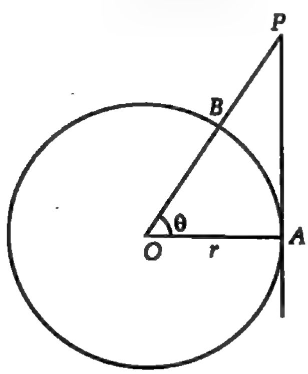


Fig. 29.9

From the figure 29.9, we have

Area of $\Delta OAB <$ Area of sector $OAB <$ Area of ΔOAP

$\Rightarrow \quad \frac{1}{2} OA \cdot OB \sin \theta < \frac{1}{2}(OA)^2 \theta < \frac{1}{2} OA \cdot AP$

$\Rightarrow \quad \frac{1}{2} r^2 \sin \theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan \theta$ [In ΔOAP, $AP = OA \tan \theta$]

$\Rightarrow \quad \sin \theta < \theta < \tan \theta$

$\Rightarrow \quad 1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$ [$\because$ θ is small $\sin \theta > 0$]

$\Rightarrow \quad 1 > \dfrac{\sin \theta}{\theta} > \cos \theta$

$\Rightarrow \quad 1 \geq \lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta} \geq \lim\limits_{\theta \to 0} \cos \theta$ or, $\lim\limits_{\theta \to 0} \cos \theta \leq \lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta} \leq 1$

$\Rightarrow \quad 1 \leq \lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta} \leq 1$

$\Rightarrow \quad \lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$

(iii) We have,

$$\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \times \frac{1}{\cos \theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \times \lim_{\theta \to 0} \frac{1}{\cos \theta} = 1 \times 1 = 1$$

(iv) We have,

$$\lim_{\theta \to a} \frac{\sin (\theta - a)}{\theta - a} = \lim_{h \to 0} \frac{\sin (a + h - a)}{(a + h - a)} \qquad \left[\text{Using: } \lim_{\theta \to a} \frac{\sin (\theta - a)}{\theta - a} = \lim_{\theta \to a^+} \frac{\sin (\theta - a)}{\theta - a}\right]$$

$$= \lim_{h \to 0} \frac{\sin h}{h} = 1$$

(v) We have,

$$\lim_{\theta \to a} \frac{\tan (\theta - a)}{\theta - a} = \lim_{h \to 0} \frac{\tan (a + h - a)}{a + h - a} \qquad \left[\text{Using: } \lim_{\theta \to a} \frac{\tan (\theta - a)}{\theta - a} = \lim_{\theta \to a^+} \frac{\tan (\theta - a)}{\theta - a}\right]$$

$$= \lim_{h \to 0} \frac{\tan h}{h} = 1$$

29.7.1 EVALUATION OF TRIGONOMETRIC LIMITS WHEN THE VARIABLE TENDS TO ZERO

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate the following limits:*

(i) $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$ (ii) $\lim\limits_{x \to 0} \dfrac{\sin 5x}{2x}$ (iii) $\lim\limits_{x \to 0} \dfrac{\sin ax}{\sin bx}$ [NCERT]

(iv) $\lim\limits_{x \to 0} \dfrac{\sin^2 ax}{\sin^2 bx}$ (v) $\lim\limits_{x \to 0} \dfrac{\sin^2 3x}{x^2}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \left(3 \times \frac{\sin 3x}{3x}\right) = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = 3(1) = 3 \qquad \left[\because \lim_{x \to 0} \frac{\sin 3x}{3x} = 1\right]$$

(ii) We have,

$$\lim_{x \to 0} \frac{\sin 5x}{2x} = \lim_{x \to 0} \left(\frac{5}{2} \times \frac{\sin 5x}{5x}\right) = \frac{5}{2} \lim_{x \to 0} \frac{\sin 5x}{5x} = \frac{5}{2}(1) = \frac{5}{2} \qquad \left[\because \lim_{x \to 0} \frac{\sin 5x}{5x} = 1\right]$$

(iii) We have,

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)ax}{\left(\frac{\sin bx}{bx}\right)bx}=\frac{a}{b}\,\frac{\lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim_{x\to 0}\left(\frac{\sin bx}{bx}\right)}=\frac{a}{b}\,\frac{(1)}{(1)}=\frac{a}{b}$$

(iv) We have,

$$\lim_{x\to 0}\frac{\sin^2 ax}{\sin^2 bx}=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)(ax)\left(\frac{\sin ax}{ax}\right)ax}{\left(\frac{\sin bx}{bx}\right)(bx)\left(\frac{\sin bx}{bx}\right)bx}$$

$$=\frac{a^2}{b^2}\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)\left(\frac{\sin ax}{ax}\right)}{\left(\frac{\sin bx}{bx}\right)\left(\frac{\sin bx}{bx}\right)}=\frac{a^2}{b^2}\times\frac{1}{1}=\frac{a^2}{b^2}$$

(v) We have,

$$\lim_{x\to 0}\frac{\sin^2 3x}{x^2}=\lim_{x\to 0}\frac{\sin 3x}{x}\times\frac{\sin 3x}{x}$$

$$=\lim_{x\to 0}3\left(\frac{\sin 3x}{3x}\right)\times 3\left(\frac{\sin 3x}{3x}\right)$$

$$=3\lim_{x\to 0}\frac{\sin 3x}{3x}\times 3\lim_{x\to 0}\frac{\sin 3x}{3x}=(3)(3)=9$$

EXAMPLE 2 *Evaluate the following limits:*

(i) $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$ [NCERT EXEMPLAR] (ii) $\lim_{x\to 0}\frac{1-\cos 2x}{x}$

(iii) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (iv) $\lim_{x\to 0}\frac{1-\cos 2mx}{1-\cos 2nx}$ [NCERT EXEMPLAR]

(v) $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}$ [NCERT EXEMPLAR]

SOLUTION (i) We have,

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}$$

$$=2\lim_{x\to 0}\left(\frac{\sin x}{x}\times\frac{\sin x}{x}\right)=2\lim_{x\to 0}\frac{\sin x}{x}\times\lim_{x\to 0}\frac{\sin x}{x}=2(1)(1)=2.$$

(ii) We have,

$$\lim_{x\to 0}\frac{1-\cos 2x}{x}=\lim_{x\to 0}\frac{2\sin^2 x}{x}$$

$$=2\lim_{x\to 0}\left(\frac{\sin x}{x}\times\sin x\right)=2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\sin x\right)=2(1)(0)=0$$

(iii) We have,

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{2\sin^2 x/2}{x^2}$$

$$= 2 \lim_{x \to 0} \left(\frac{\sin x/2}{x} \times \frac{\sin x/2}{x} \right)$$

$$= 2 \lim_{x \to 0} \left(\frac{\sin x/2}{x/2} \times \frac{1}{2} \times \frac{1}{2} \frac{\sin x/2}{x/2} \right)$$

$$= \frac{2}{4} \left(\lim_{x \to 0} \frac{\sin x/2}{x/2} \right) \times \left(\lim_{x \to 0} \frac{\sin x/2}{x/2} \right) = \frac{2}{4} (1)(1) = \frac{1}{2}$$

(iv) We have,

$$\lim_{x \to 0} \frac{1-\cos 2mx}{1-\cos 2nx}$$

$$= \lim_{x \to 0} \frac{2\sin^2 mx}{2\sin^2 nx} = \lim_{x \to 0} \frac{\sin^2 mx}{\sin^2 nx}$$

$$= \lim_{x \to 0} \left\{ \frac{\frac{\sin mx}{mx} \times mx}{\frac{\sin nx}{nx} \times nx} \times \frac{\frac{\sin mx}{mx} \times mx}{\frac{\sin nx}{nx} \times nx} \right\} = \frac{m^2}{n^2} \left\{ \lim_{x \to 0} \frac{\frac{\sin mx}{mx}}{\frac{\sin nx}{nx}} \right\} \times \left\{ \lim_{x \to 0} \frac{\frac{\sin mx}{mx}}{\frac{\sin nx}{nx}} \right\}$$

$$= \frac{m^2}{n^2} \left\{ \frac{\lim_{x \to 0} \frac{\sin mx}{mx}}{\lim_{x \to 0} \frac{\sin nx}{nx}} \right\} \times \left\{ \frac{\lim_{x \to 0} \frac{\sin mx}{mx}}{\lim_{x \to 0} \frac{\sin nx}{nx}} \right\} = \frac{m^2}{n^2} \left(\frac{1}{1}\right)\left(\frac{1}{1}\right) = \frac{m^2}{n^2}$$

(v) We have,

$$\lim_{x \to 0} \frac{1-\cos mx}{1-\cos nx}$$

$$= \lim_{x \to 0} \frac{2\sin^2\left(\frac{mx}{2}\right)}{2\sin^2\left(\frac{nx}{2}\right)} = \lim_{x \to 0} \left(\frac{\sin \frac{mx}{2}}{\sin \frac{nx}{2}} \right)^2$$

$$= \lim_{x \to 0} \left\{ \frac{\left(\frac{\sin \frac{mx}{2}}{\frac{mx}{2}} \right) \frac{mx}{2}}{\left(\frac{\sin \frac{nx}{2}}{\frac{nx}{2}} \right) \cdot \frac{nx}{2}} \right\}^2 = \left\{ \frac{m}{n} \frac{\lim_{x \to 0} \left(\frac{\sin \frac{mx}{2}}{\frac{mx}{2}} \right)}{\lim_{x \to 0} \left(\frac{\sin \frac{nx}{2}}{\frac{nx}{2}} \right)} \right\}^2 = \left(\frac{m \times 1}{n \times 1} \right)^2 = \frac{m^2}{n^2}$$

EXAMPLE 3 *Evaluate the following limits:*

(i) $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$ (ii) $\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$ **[NCERT EXEMPLAR]**

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$$

$$= \lim_{x \to 0} \left(\frac{\sin x - \sin x \cos x}{x^3 \cos x} \right)$$

$$= \lim_{x \to 0} \left\{ \frac{\sin x (1 - \cos x)}{x^3 \cos x} \right\} = \lim_{x \to 0} \left\{ \frac{\sin x}{x} \times \frac{1 - \cos x}{x^2} \times \frac{1}{\cos x} \right\}$$

$$= \left\{ \lim_{x \to 0} \frac{\sin x}{x} \right\} \times \left\{ \lim_{x \to 0} \frac{2 \sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2 \times 4} \right\} \times \lim_{x \to 0} \frac{1}{\cos x}$$

$$= \left\{ \lim_{x \to 0} \frac{\sin x}{x} \right\} \times \frac{1}{2} \times \left\{ \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \right\} \times \lim_{x \to 0} \frac{1}{\cos x} = 1 \times \frac{1}{2} (1)^2 \times \frac{1}{1} = \frac{1}{2}$$

(ii) We have,

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$$

$$= \lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} = \lim_{x \to 0} \frac{\sin x (1 - \cos x)}{\cos x \sin^3 x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\cos x \sin^2 x} = \lim_{x \to 0} \frac{1 - \cos x}{\cos x (1 - \cos^2 x)}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\cos x (1 + \cos x)(1 - \cos x)} = \lim_{x \to 0} \frac{1}{\cos x (1 + \cos x)} = \frac{1}{2}$$

EXAMPLE 4 *Evaluate:* $\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^3}$

SOLUTION We have,

$$\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\frac{\sin 2x}{\cos 2x} - \sin 2x}{x^3} = \lim_{x \to 0} \frac{\sin 2x (1 - \cos 2x)}{x^3 \cos 2x} = \lim_{x \to 0} \frac{\sin 2x \times 2 \sin^2 x}{x^3 \cos 2x}$$

$$= 2 \lim_{x \to 0} \frac{\tan 2x \sin^2 x}{x^3} = 4 \lim_{x \to 0} \left(\frac{\tan 2x}{2x} \right) \left(\frac{\sin x}{x} \right)^2 = 4 (1) (1)^2 = 4.$$

EXAMPLE 5 *Evaluate the following limits:*

(i) $\lim_{x \to 0} \frac{\cos Ax - \cos Bx}{x^2}$ (ii) $\lim_{x \to 0} \frac{\sin 2x + \sin 3x}{2x + \sin 3x}$ [NCERT EXEMPLAR]

(iii) $\lim_{x \to 0} \frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{\cos Ax - \cos Bx}{x^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2 \sin\left(\frac{A+B}{2}\right)x \ \sin\left(\frac{B-A}{2}\right)x}{x^2} \qquad \left[\because \cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}\right]$$

$$= 2 \lim_{x \to 0} \left\{ \frac{\sin\left(\frac{A+B}{2}\right)x}{\left(\frac{A+B}{2}\right)x} \times \left(\frac{A+B}{2}\right) \frac{\sin\left(\frac{B-A}{2}\right)x}{\left(\frac{B-A}{2}\right)x} \times \left(\frac{B-A}{2}\right) \right\} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= 2\left(\frac{B+A}{2}\right)\left(\frac{B-A}{2}\right) \left\{ \lim_{x \to 0} \frac{\sin\left(\frac{A+B}{2}\right)x}{\left(\frac{A+B}{2}\right)x} \right\} \times \left\{ \lim_{x \to 0} \frac{\sin\left(\frac{B-A}{2}\right)x}{\left(\frac{B-A}{2}\right)x} \right\}$$

$$= \left(\frac{B^2 - A^2}{2}\right)(1)(1) = \frac{B^2 - A^2}{2}$$

(ii) We have,

$$\lim_{x \to 0} \frac{\sin 2x + \sin 3x}{2x + \sin 3x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\frac{\sin 2x}{x} + \frac{\sin 3x}{x}}{2 + \frac{\sin 3x}{x}} \qquad [\text{Dividing } N^r \text{ and } D^r \text{ by } x]$$

$$= \lim_{x \to 0} \frac{2\left(\frac{\sin 2x}{2x}\right) + 3\left(\frac{\sin 3x}{3x}\right)}{2 + 3\left(\frac{\sin 3x}{3x}\right)} = \frac{2 \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right) + 3 \lim_{x \to 0} \left(\frac{\sin 3x}{3x}\right)}{2 + 3 \lim_{x \to 0} \left(\frac{\sin 3x}{3x}\right)}$$

$$= \frac{2 \times 1 + 3 \times 1}{2 + 3 \times 1} = \frac{5}{5} = 1$$

(iii) We have,

$$\lim_{x \to 0} \frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x} = \lim_{x \to 0} \frac{2 \sin 4x \cos 2x}{2 \sin x \cos 4x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \left\{ \frac{\frac{\sin 4x}{4x} \times 4x \times \cos 2x}{\frac{\sin x}{x} \times x \times \cos 4x} \right\} = 4 \, \frac{\lim_{x \to 0} \frac{\sin 4x}{4x} \times \lim_{x \to 0} \cos 2x}{\lim_{x \to 0} \frac{\sin x}{x} \times \lim_{x \to 0} \cos 4x} = 4\left(\frac{1 \times 1}{1 \times 1}\right) = 4$$

EXAMPLE 6 *Evaluate the following limits:*

(i) $\lim_{y \to 0} \frac{(x+y)\sec(x+y) - x \sec x}{y}$ **[NCERT EXEMPLAR]** (ii) $\lim_{x \to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x}$

SOLUTION (i) We have,

$$\lim_{y \to 0} \frac{(x+y)\sec(x+y) - x \sec x}{y} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{y \to 0} \frac{x(\sec(x+y) - \sec x) + y \sec(x+y)}{y} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{y \to 0} x \left\{ \frac{\sec (x+y) - \sec x}{y} \right\} + \lim_{y \to 0} \frac{y \sec (x+y)}{y}$$

$$= \lim_{y \to 0} x \left\{ \frac{\cos x - \cos (x+y)}{y \cos x \cos (x+y)} \right\} + \lim_{y \to 0} \sec (x+y)$$

$$= \lim_{y \to 0} \left\{ \frac{\cos x - \cos (x+y)}{y} \times \frac{x}{\cos x \cos (x+y)} \right\} + \lim_{y \to 0} \sec (x+y)$$

$$= \lim_{y \to 0} \left\{ \frac{2 \sin \left(x + \frac{y}{2} \right) \sin \left(\frac{y}{2} \right)}{2 \left(\frac{y}{2} \right)} \times \frac{x}{\cos x \cos (x+y)} \right\} + \lim_{y \to 0} \sec (x+y)$$

$$= \lim_{y \to 0} \sin \left(x + \frac{y}{2} \right) \times \lim_{y \to 0} \frac{\sin \left(\frac{y}{2} \right)}{\left(\frac{y}{2} \right)} \times \lim_{y \to 0} \frac{x}{\cos x \cos (x+y)} + \lim_{y \to 0} \sec (x+y)$$

$$= \sin x \times 1 \times \frac{x}{\cos^2 x} + \sec x = x \tan x \sec x + \sec x$$

(ii) We have,

$$\lim_{x \to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x} = \lim_{x \to 0} \left(\frac{\frac{\cos 2x - \cos 4x}{\cos 2x \cos 4x}}{\frac{\cos x - \cos 3x}{\cos x \cos 3x}} \right) \qquad \left(\text{form } \frac{0}{0} \right)$$

$$= \lim_{x \to 0} \left(\frac{\cos 2x - \cos 4x}{\cos x - \cos 3x} \times \frac{\cos x \cos 3x}{\cos 2x \cos 4x} \right) = \lim_{x \to 0} \left(\frac{2 \sin 3x \sin x}{2 \sin 2x \sin x} \times \frac{\cos x \cos 3x}{\cos 2x \cos 4x} \right)$$

$$= \lim_{x \to 0} \left(\frac{\frac{\sin 3x}{3x} \times 3x}{\frac{\sin 2x}{2x} \times 2x} \times \frac{\cos x \cos 3x}{\cos 2x \cos 4x} \right) = \frac{3}{2} \frac{\lim_{x \to 0} \frac{\sin 3x}{3x}}{\lim_{x \to 0} \frac{\sin 2x}{2x}} \times \lim_{x \to 0} \frac{\cos x \cos 3x}{\cos 2x \cos 4x} = \frac{3}{2} \left(\frac{1}{1} \times \frac{1}{1} \right) = \frac{3}{2}$$

EXAMPLE 7 *Evaluate:* $\lim_{x \to 0} \frac{\cot 2x - \operatorname{cosec} 2x}{x}$

SOLUTION We have,

$$\lim_{x \to 0} \frac{\cot 2x - \operatorname{cosec} 2x}{x}$$

$$= \lim_{x \to 0} \frac{\frac{\cos 2x}{\sin 2x} - \frac{1}{\sin 2x}}{x} = \lim_{x \to 0} \frac{\cos 2x - 1}{x \sin 2x}$$

$$= \lim_{x \to 0} \frac{-(1 - \cos 2x)}{x \sin 2x} = - \lim_{x \to 0} \frac{2 \sin^2 x}{x (2 \sin x \cos x)} = - \lim_{x \to 0} \frac{\tan x}{x} = -1$$

EXAMPLE 8 *Evaluate:* $\lim_{x \to 0} \frac{\sin x - 2 \sin 3x + \sin 5x}{x}$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\lim_{x \to 0} \frac{\sin x - 2 \sin 3x + \sin 5x}{x} = \lim_{x \to 0} \frac{(\sin x - \sin 3x) + (\sin 5x - \sin 3x)}{x}$$

$$= \lim_{x \to 0} \frac{-2 \sin x \cos 2x + 2 \sin x \cos 4x}{x} = \lim_{x \to 0} \frac{2 \sin x (\cos 4x - \cos 2x)}{x}$$

$$= 2 \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \times \lim_{x \to 0} (\cos 4x - \cos 2x) = 2 \times 1 \times 0 = 0$$

ALITER $\lim_{x \to 0} \frac{\sin x - 2 \sin 3x + \sin 5x}{x}$

$$= \lim_{x \to 0} \left\{\left(\frac{\sin x}{x}\right) - 2\left(\frac{\sin 3x}{x}\right) + \left(\frac{\sin 5x}{x}\right)\right\}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} - 2 \lim_{x \to 0} \frac{\sin 3x}{x} + \lim_{x \to 0} \frac{\sin 5x}{x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} - 2 \times 3 \left(\lim_{x \to 0} \frac{\sin 3x}{3x}\right) + 5 \left(\lim_{x \to 0} \frac{\sin 5x}{5x}\right) = 1 - 2 \times 3 \times 1 + 5 \times 1 = 0$$

LEVEL-2

EXAMPLE 9 *Evaluate:* $\lim_{x \to 0} \frac{\tan x + 4 \tan 2x - 3 \tan 3x}{x^2 \tan x}$

SOLUTION We have,

$$\lim_{x \to 0} \frac{\tan x + 4 \tan 2x - 3 \tan 3x}{x^2 \tan x}$$

$$= \lim_{x \to 0} \frac{\tan x + 4\left(\dfrac{2 \tan x}{1 - \tan^2 x}\right) - 3\left(\dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}\right)}{x^2 \tan x}$$

$$= \lim_{x \to 0} \frac{1 + \dfrac{8}{1 - \tan^2 x} - 3\left(\dfrac{3 - \tan^2 x}{1 - 3 \tan^2 x}\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{(1 - \tan^2 x)(1 - 3 \tan^2 x) + 8(1 - 3 \tan^2 x) - 3(3 - \tan^2 x)(1 - \tan^2 x)}{x^2 (1 - \tan^2 x)(1 - 3 \tan^2 x)}$$

$$= \lim_{x \to 0} \frac{1 - 4 \tan^2 x + 3 \tan^4 x + 8 - 24 \tan^2 x - 9 + 12 \tan^2 x - 3 \tan^4 x}{x^2 (1 - \tan^2 x)(1 - 3 \tan^2 x)}$$

$$= \lim_{x \to 0} \frac{-16 \tan^2 x}{x^2 (1 - \tan^2 x)(1 - 3 \tan^2 x)}$$

$$= -16 \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^2 \times \frac{1}{1 - \tan^2 x} \times \frac{1}{1 - 3 \tan^2 x} = -16$$

EXAMPLE 10 *Evaluate:* $\lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2}$

SOLUTION We have,

$$\lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2} = \lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2} \times \frac{1 + \cos x \sqrt{\cos 2x}}{1 + \cos x \sqrt{\cos 2x}}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x \cos 2x}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x (2\cos^2 x - 1)}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{1 - 2\cos^4 x + \cos^2 x}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{(1 - \cos^2 x)\ (1 + 2\cos^2 x)}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2} \times \lim_{x \to 0} \frac{1 + 2\cos^2 x}{\left\{1 + \cos x \sqrt{\cos 2x}\right\}} = 1 \times \left(\frac{1+2}{1+1}\right) = \frac{3}{2}$$

EXAMPLE 11 *Evaluate:* $\lim\limits_{x \to 0} \dfrac{8}{x^8}\left(1 - \cos\dfrac{x^2}{2} - \cos\dfrac{x^2}{4} + \cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right)$

SOLUTION We have,

$$\lim_{x \to 0} \frac{8}{x^8}\left(1 - \cos\frac{x^2}{2} - \cos\frac{x^2}{4} + \cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$$

$$= \lim_{x \to 0} \frac{8}{x^8}\left\{\left(1 - \cos\frac{x^2}{2}\right) - \cos\frac{x^2}{4}\left(1 - \cos\frac{x^2}{2}\right)\right\}$$

$$= \lim_{x \to 0} \frac{8}{x^8}\left(1 - \cos\frac{x^2}{2}\right)\left(1 - \cos\frac{x^2}{4}\right)$$

$$= \lim_{x \to 0} 8\left\{\frac{1 - \cos\frac{x^2}{2}}{x^4}\right\}\left\{\frac{1 - \cos\frac{x^2}{4}}{x^4}\right\}$$

$$= \lim_{x \to 0} 8 \times \frac{2\sin^2\frac{x^2}{4}}{x^4} \times \frac{2\sin^2\frac{x^2}{8}}{x^4}$$

$$= \lim_{x \to 0} 32 \times \left\{\frac{\sin\frac{x^2}{4}}{x^2}\right\}^2 \times \left\{\frac{\sin\frac{x^2}{8}}{x^2}\right\}^2$$

$$= 32 \lim_{x \to 0} \left\{\frac{\sin\frac{x^2}{4}}{4\left(\frac{x^2}{4}\right)}\right\}^2 \times \left\{\frac{\sin\frac{x^2}{8}}{8\left(\frac{x^2}{8}\right)}\right\}^2 = 32 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{8}\right)^2 = 32 \times \frac{1}{16} \times \frac{1}{64} = \frac{1}{32}$$

EXAMPLE 12 *Evaluate:* $\lim\limits_{x \to 0} \dfrac{1 - \cos x \cos 2x \cos 3x}{\sin^2 2x}$

SOLUTION Clearly,

$$\cos x \cos 2x \cos 3x = \frac{1}{2}\{2\cos x \cos 2x \cos 3x\} = \frac{1}{2}\{(2\cos x \cos 2x)\cos 3x\}$$

$$= \frac{1}{2}\left\{(\cos 3x + \cos x)\cos 3x\right\} = \frac{1}{2}\left\{\cos^2 3x + \cos 3x \cos x\right\}$$

$$= \frac{1}{4}\left\{2\cos^2 3x + 2\cos 3x \cos x\right\} = \frac{1}{4}\left\{1 + \cos 6x + \cos 4x + \cos 2x\right\}$$

$$\therefore \quad \lim_{x \to 0} \frac{1 - \cos x \cos 2x \cos 3x}{\sin^2 2x}$$

$$= \lim_{x \to 0} \frac{1 - \frac{1}{4}(1 + \cos 6x + \cos 4x + \cos 2x)}{\sin^2 2x}$$

$$= \lim_{x \to 0} \frac{4 - 1 - \cos 6x - \cos 4x - \cos 2x}{4\sin^2 2x}$$

$$= \lim_{x \to 0} \frac{(1 - \cos 6x) + (1 - \cos 4x) + (1 - \cos 2x)}{4\sin^2 2x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 3x + 2\sin^2 2x + 2\sin^2 x}{4\sin^2 2x}$$

$$= \lim_{x \to 0} \frac{\frac{\sin^2 3x}{x^2} + \frac{\sin^2 2x}{x^2} + \frac{\sin^2 x}{x^2}}{2\left(\frac{\sin^2 2x}{x^2}\right)}$$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin 3x}{x}\right)^2 + \left(\frac{\sin 2x}{x}\right)^2 + \left(\frac{\sin x}{x}\right)^2}{2\left(\frac{\sin 2x}{x}\right)^2}$$

$$= \lim_{x \to 0} \frac{9 \times \left(\frac{\sin 3x}{3x}\right)^2 + 4 \times \left(\frac{\sin 2x}{2x}\right)^2 + \left(\frac{\sin x}{x}\right)^2}{2 \times 4\left(\frac{\sin 2x}{2x}\right)^2}$$

$$= \frac{9 \times 1 + 4 \times 1 + 1}{8} = \frac{14}{8} = \frac{7}{4}$$

EXERCISE 29.7

LEVEL-1

Evaluate the following limits:

1. $\lim_{x \to 0} \frac{\sin 3x}{5x}$
2. $\lim_{x \to 0} \frac{\sin x^0}{x}$
3. $\lim_{x \to 0} \frac{x^2}{\sin x^2}$
4. $\lim_{x \to 0} \frac{\sin x \cos x}{3x}$

5. $\lim_{x \to 0} \frac{3 \sin x - 4 \sin^3 x}{x}$

6. $\lim_{x \to 0} \frac{\tan 8x}{\sin 2x}$

7. $\lim_{x \to 0} \frac{\tan mx}{\tan nx}$

8. $\lim_{x \to 0} \frac{\sin 5x}{\tan 3x}$

9. $\lim_{x \to 0} \frac{\sin x^\circ}{x^\circ}$

10. $\lim_{x \to 0} \frac{7x \cos x - 3 \sin x}{4x + \tan x}$

11. $\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx}$

12. $\lim_{x \to 0} \frac{\tan^2 3x}{x^2}$

13. $\lim_{x \to 0} \frac{1 - \cos mx}{x^2}$

14. $\lim_{x \to 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x}$

15. $\lim_{x \to 0} \frac{\cos 3x - \cos 7x}{x^2}$

16. $\lim_{\theta \to 0} \frac{\sin 3\theta}{\tan 2\theta}$

17. $\lim_{x \to 0} \frac{\sin x^2 (1 - \cos x^2)}{x^6}$

18. $\lim_{x \to 0} \frac{\sin^2 4x^2}{x^4}$

19. $\lim_{x \to 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$

20. $\lim_{x \to 0} \frac{2x - \sin x}{\tan x + x}$

21. $\lim_{x \to 0} \frac{5x \cos x + 3 \sin x}{3x^2 + \tan x}$

22. $\lim_{x \to 0} \frac{\sin 3x - \sin x}{\sin x}$

23. $\lim_{x \to 0} \frac{\sin 5x - \sin 3x}{\sin x}$

24. $\lim_{x \to 0} \frac{\cos 3x - \cos 5x}{x^2}$

25. $\lim_{x \to 0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$

26. $\lim_{x \to 0} \frac{\sin (2 + x) - \sin (2 - x)}{x}$

27. $\lim_{h \to 0} \frac{(a + h)^2 \sin (a + h) - a^2 \sin a}{h}$

28. $\lim_{x \to 0} \frac{\tan x - \sin x}{\sin 3x - 3 \sin x}$

29. $\lim_{x \to 0} \frac{\sec 5x - \sec 3x}{\sec 3x - \sec x}$

30. $\lim_{x \to 0} \frac{1 - \cos 2x}{\cos 2x - \cos 8x}$

31. $\lim_{x \to 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x}$

32. $\lim_{x \to 0} \frac{\sin (a + x) + \sin (a - x) - 2 \sin a}{x \sin x}$

33. $\lim_{x \to 0} \frac{x^2 - \tan 2x}{\tan x}$

34. $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$ [NCERT EXEMPLAR]

35. $\lim_{x \to 0} \frac{x \tan x}{1 - \cos x}$

36. $\lim_{x \to 0} \frac{x^2 + 1 - \cos x}{x \sin x}$

37. $\lim_{x \to 0} \frac{\sin 2x (\cos 3x - \cos x)}{x^3}$

38. $\lim_{x \to 0} \frac{2 \sin x^\circ - \sin 2x^\circ}{x^3}$

39. $\lim_{x \to 0} \frac{x^3 \cot x}{1 - \cos x}$

40. $\lim_{x \to 0} \frac{x \tan x}{1 - \cos 2x}$

41. $\lim_{x \to 0} \frac{\sin (3 + x) - \sin (3 - x)}{x}$

42. $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$

43. $\lim_{x \to 0} \frac{3 \sin^2 x - 2 \sin x^2}{3x^2}$

44. $\lim_{x \to 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x}$

45. $\lim_{x \to 0} \frac{1 - \cos 4x}{x^2}$

46. $\lim_{x \to 0} \frac{x \cos x + \sin x}{x^2 + \tan x}$

47. $\lim_{x \to 0} \frac{1 - \cos 2x}{3 \tan^2 x}$

48. $\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$

49. $\lim_{x \to 0} \frac{ax + x \cos x}{b \sin x}$ [NCERT]

50. $\lim_{\theta \to 0} \frac{\sin 4\theta}{\tan 3\theta}$

51. $\lim_{x \to 0} \frac{2 \sin x - \sin 2x}{x^3}$

52. $\lim_{x \to 0} \frac{1 - \cos 5x}{1 - \cos 6x}$

53. $\lim_{x \to 0} \frac{\operatorname{cosec} x - \cot x}{x}$

54. $\lim_{x \to 0} \frac{\sin 3x + 7x}{4x + \sin 2x}$

55. $\lim_{x \to 0} \frac{5x + 4 \sin 3x}{4 \sin 2x + 7x}$

56. $\lim_{x \to 0} \frac{3 \sin x - \sin 3x}{x^3}$

57. $\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^3}$

58. $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$ [NCERT]

LEVEL-2

59. $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$

60. $\lim_{x \to 0} \frac{\{\sin(\alpha + \beta) x + \sin(\alpha - \beta) x + \sin 2\alpha x\}}{\cos^2 \beta x - \cos^2 \alpha x}$

61. $\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$

62. $\lim_{h \to 0} \frac{(a + h)^2 \sin(a + h) - a^2 \sin a}{h}$

63. If $\lim_{x \to 0} kx \operatorname{cosec} x = \lim_{x \to 0} x \operatorname{cosec} kx$, find k.

ANSWERS

1. $\frac{3}{5}$ 2. $\frac{\pi}{180}$ 3. 1 4. $\frac{1}{3}$ 5. 3 6. 4 7. $\frac{m}{n}$ 8. $\frac{5}{3}$

9. 1 10. $\frac{4}{5}$ 11. $\frac{a^2 - b^2}{c^2 - d^2}$ 12. 9 13. $\frac{m^2}{2}$ 14. $\frac{8}{9}$ 15. 20

16. $\frac{3}{2}$ 17. $\frac{1}{2}$ 18. 16 19. 3 20. $\frac{1}{2}$ 21. 8 22. 2 23. 2

24. 8 25. $\frac{1}{3}$ 26. $2 \cos 2$ 27. $2a \sin a + a^2 \cos a$ 28. $\frac{-1}{8}$

29. 2 30. $\frac{1}{15}$ 31. 3 32. $-\sin a$ 33. -2 34. $\frac{1}{4\sqrt{2}}$ 35. 2

36. $\frac{3}{2}$ 37. -8 38. $\left(\frac{\pi}{180}\right)^3$ 39. 2 40. $\frac{1}{2}$ 41. $2 \cos 3$ 42. 4 43. $\frac{1}{3}$

44. 1 45. 8 46. 2 47. $\frac{2}{3}$ 48. $\frac{4}{9}$ 49. $\frac{a+1}{b}$ 50. $\frac{4}{3}$ 51. 1

52. $\frac{25}{36}$ 53. $\frac{1}{2}$ 54. $\frac{5}{3}$ 55. $\frac{17}{15}$ 56. 4 57. 4 58. 1 59. 0

60. $\frac{2\alpha}{\alpha^2 - \beta^2}$ 61. $\frac{a^2 - b^2}{c^2}$ 62. $a^2 \cos a + 2a \sin a$ 63. $k = \pm 1$

HINTS TO NCERT & SELECTED PROBLEMS

42. $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1} = \lim_{x \to 0} \frac{(2\cos^2 x - 1) - 1}{\cos x - 1} = \lim_{x \to 0} \frac{2(\cos^2 x - 1)}{\cos x - 1}$

$$= \lim_{x \to 0} \frac{2(\cos x - 1)(\cos x + 1)}{\cos x - 1} = \lim_{x \to 0} 2(\cos x + 1) = 4$$

49. $\lim_{x \to 0} \frac{ax + x\cos x}{b \sin x} = \lim_{x \to 0} \frac{a + \cos x}{b\left(\frac{\sin x}{x}\right)} = \frac{a+1}{b \times 1} = \frac{a+1}{b}$

58. $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx} = \lim_{x \to 0} \frac{\frac{\sin ax}{x} + b}{a + \frac{\sin bx}{x}}$ [Dividing N^r and D^r by x]

$$= \lim_{x \to 0} \frac{a\left(\frac{\sin ax}{ax}\right) + b}{a + b\left(\frac{\sin bx}{bx}\right)} = \frac{a \times 1 + b}{a + b \times 1} = \frac{a+b}{a+b} = 1$$

29.7.2 EVALUATION OF TRIGONOMETRIC LIMITS WHEN VARIABLE TENDS TO A NON-ZERO QUANTITY

So far we have been discussing trigonometric limits when $x \to 0$. Now we will discuss evaluation of trigonometric limits when x tends to a non-zero real number. As we have already assumed that $\lim_{x \to a} f(x)$ always exists.

$$\therefore \quad \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) \qquad \left[\because \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \Leftrightarrow \lim_{x \to a} f(x)\right]$$

$$= \lim_{h \to 0} f(a+h)$$

Thus, we have the following algorithm to evaluate the said type of limits.

ALGORITHM

<u>STEP I</u> *Obtain the problem. Suppose $x \to a$, where a is a non-zero real number.*

<u>STEP II</u> *Replace $x \to a$ by $h \to 0$ and x by $(a+h)$.*

<u>STEP III</u> *Solve the problem by using formulae discussed in the previous section.*

Following examples will illustrate the above procedure.

<u>REMARK</u> *In order to evaluate $\lim_{x \to a} f(x)$, where a is a non-zero real number, we may use the following results:*

(i) $\lim_{x \to a} \frac{\sin (x-a)}{x-a} = 1$ (ii) $\lim_{x \to a} \frac{\tan (x-a)}{x-a} = 1$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate:*

(i) $\lim_{x \to \pi} \frac{\sin x}{\pi - x}$ (ii) $\lim_{x \to \pi} \frac{\sin 3x - 3\sin x}{(\pi - x)^3}$

(iii) $\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\pi - 2x}$ (iv) $\lim_{x \to \frac{\pi}{2}} \frac{\cos 3x + 3\cos x}{\left(\frac{\pi}{2} - x\right)^3}$

SOLUTION (i) $\lim\limits_{x\to\pi} \dfrac{\sin x}{(\pi-x)} = \lim\limits_{h\to 0} \dfrac{\sin(\pi+h)}{(\pi-(\pi+h))} = \lim\limits_{h\to 0} \dfrac{-\sin h}{-h} = \lim\limits_{h\to 0} \dfrac{\sin h}{h} = 1.$

<u>ALITER</u> $\lim\limits_{x\to\pi} \dfrac{\sin x}{\pi-x} = \lim\limits_{x\to\pi} \dfrac{\sin(\pi-x)}{(\pi-x)} = 1.$

(ii) $$\lim_{x\to\pi} \frac{\sin 3x-3\sin x}{(\pi-x)^3} = \lim_{x\to\pi} \frac{3\sin x-4\sin^3 x-3\sin x}{(\pi-x)^3}$$

$$= -4 \lim_{x\to\pi} \frac{\sin^3 x}{(\pi-x)^3}$$

$$= -4 \lim_{h\to 0} \frac{\sin^3(\pi+h)}{\{\pi-(\pi+h)\}^3}$$

$$= -4 \lim_{h\to 0} \frac{(-\sin h)^3}{(-h)^3} = -4 \lim_{h\to 0}\left(\frac{\sin h}{h}\right)^3 = -4\times(1)^3 = -4$$

<u>ALITER</u> $$\lim_{x\to\pi} \frac{\sin 3x-3\sin x}{(\pi-x)^3} = \lim_{x\to\pi} \frac{3\sin x-4\sin^3 x-3\sin x}{(\pi-x)^3}$$

$$= -4 \lim_{x\to\pi} \frac{\sin^3(\pi-x)}{(\pi-x)^3} = -4 \lim_{x\to\pi}\left\{\frac{\sin(\pi-x)}{\pi-x}\right\}^3 = -4\times(1)^3 = -4$$

(iii) $$\lim_{x\to\pi/2} \frac{\cos x}{\pi-2x} = \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)} = \lim_{h\to 0} \frac{-\sin h}{-2h} = \frac{1}{2}\lim_{h\to 0} \frac{\sin h}{h} = \frac{1}{2}\times 1 = \frac{1}{2}$$

<u>ALITER</u> $$\lim_{x\to\pi/2} \frac{\cos x}{\pi-2x} = \lim_{x\to\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{2\left(\frac{\pi}{2}-x\right)} = \frac{1}{2}\lim_{x\to\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)} = \frac{1}{2}\times 1 = \frac{1}{2}$$

(iv) $$\lim_{x\to\pi/2} \frac{\cos 3x+3\cos x}{\left(\frac{\pi}{2}-x\right)^3} = \lim_{x\to\pi/2} \frac{4\cos^3 x-3\cos x+3\cos x}{\left(\frac{\pi}{2}-x\right)^3}$$

$$= 4\lim_{x\to\frac{\pi}{2}} \frac{\cos^3 x}{\left(\frac{\pi}{2}-x\right)^3} = 4\lim_{h\to 0} \frac{\cos^3\left(\frac{\pi}{2}+h\right)}{\left\{\frac{\pi}{2}-\left(\frac{\pi}{2}+h\right)\right\}}$$

$$= 4\lim_{h\to 0} \frac{(-\sin h)^3}{(-h)^3} = 4\lim_{h\to 0}\left(\frac{\sin h}{h}\right)^3 = 4\times(1)^3 = 4$$

<u>ALITER</u> $$\lim_{x\to\pi/2} \frac{\cos 3x+3\cos x}{\left(\frac{\pi}{2}-x\right)^3} = \lim_{x\to\pi/2} \frac{4\cos^3 x-3\cos x+3\cos x}{\left(\frac{\pi}{2}-x\right)^3} = 4\lim_{x\to\pi/2} \frac{\cos^3 x}{\left(\frac{\pi}{2}-x\right)^3}.$$

$$= 4\lim_{x\to\pi/2} \frac{\sin^3\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)^3} = 4\lim_{x\to\pi/2}\left\{\frac{\sin\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)}\right\}^3 = 4\times(1)^3 = 4$$

EXAMPLE 2 *Evaluate:*

(i) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x}{\frac{\pi}{2} - x}$ (ii) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan 2x}{x - \frac{\pi}{2}}$

(iii) $\lim\limits_{x \to \frac{\pi}{2}} \left(\dfrac{\pi}{2} - x\right) \tan x$ [NCERT]

SOLUTION (i) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x}{\left(\frac{\pi}{2} - x\right)} = \lim\limits_{h \to 0} \dfrac{\cot\left(\frac{\pi}{2} + h\right)}{\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)} = \lim\limits_{h \to 0} \dfrac{-\tan h}{-h} = \lim\limits_{h \to 0} \dfrac{\tan h}{h} = 1$

ALITER $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x}{\left(\frac{\pi}{2} - x\right)} = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)} = 1$

(ii) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan 2x}{x - \frac{\pi}{2}} = \lim\limits_{h \to 0} \dfrac{\tan 2\left(\frac{\pi}{2} + h\right)}{\left(\frac{\pi}{2} + h\right) - \frac{\pi}{2}} = \lim\limits_{h \to 0} \dfrac{\tan(\pi + 2h)}{h} = -\lim\limits_{h \to 0} \dfrac{\tan 2h}{h}$

$$= -2 \lim_{h \to 0} \frac{\tan 2h}{2h} = -2 \times 1 = -2$$

ALITER $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan 2x}{x - \frac{\pi}{2}} = \lim\limits_{x \to \frac{\pi}{2}} -\dfrac{\tan(\pi - 2x)}{x - \frac{\pi}{2}}$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\tan 2\left(\frac{\pi}{2} - x\right)}{-\left(\frac{\pi}{2} - x\right)} = -2 \lim_{x \to \frac{\pi}{2}} \frac{\tan 2\left(\frac{\pi}{2} - x\right)}{2\left(\frac{\pi}{2} - x\right)} = -2 \times 1 = -2$$

(iii) $\lim\limits_{x \to \frac{\pi}{2}} \left(\dfrac{\pi}{2} - x\right) \tan x$ $(0 \times \infty \text{ form})$

$$= \lim_{h \to 0} \left\{\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)\right\} \tan\left(\frac{\pi}{2} + h\right) = \lim_{h \to 0} -h \times -\cot h = \lim_{h \to 0} \frac{h}{\tan h} = 1$$

ALITER $\lim\limits_{x \to \frac{\pi}{2}} \left(\dfrac{\pi}{2} - x\right) \tan x$ $(0 \times \infty \text{ form})$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)}{\cot x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{\tan\left(\frac{\pi}{2} - x\right)} = 1.$$

EXAMPLE 3 *Evaluate the following limits:*

(i) $\lim_{n \to \infty} 2^n \sin \frac{a}{2^n}$ (ii) $\lim_{x \to \infty} x \tan \frac{1}{x}$

(iii) $\lim_{x \to \infty} 2^{x-1} \tan\left(\frac{a}{2^x}\right)$

SOLUTION (i) $\lim_{n\to\infty} 2^n \sin \frac{a}{2n}$ $(\infty \times 0 \text{ form})$

$$= \lim_{n\to\infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\left(\frac{1}{2^n}\right)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{n\to\infty} \left\{\frac{\sin\left(\frac{a}{2^n}\right)}{\left(\frac{a}{2^n}\right)}\right\} \times a = 1 \times a = a$$

(ii) $\lim_{x\to\infty} x \tan \frac{1}{x}$ $(\infty \times 0 \text{ form})$

$$= \lim_{x\to\infty} \left\{\frac{\tan\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\right\} = 1$$

(iii) $\lim_{x\to\infty} 2^{x-1} \tan\left(\frac{a}{2^x}\right) = \frac{a}{2} \lim_{x\to\infty} \frac{\tan\left(\frac{a}{2x}\right)}{\left(\frac{a}{2x}\right)} = \frac{a}{2} \times 1 = \frac{a}{2}$

EXAMPLE 4 *Evaluate the following elements:*

(i) $\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$ [NCERT EXEMPLAR]

(ii) $\lim_{x \to \pi/6} \frac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}}$ [NCERT EXEMPLAR]

(iii) $\lim_{x \to a} \frac{\sin x - \sin a}{x - a}$

(iv) $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$ [NCERT EXEMPLAR]

SOLUTION (i) We have,

$$\lim_{\pi \to \pi/2} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{\pi}{4} + h\right) - \cos\left(\frac{\pi}{4} + h\right)}{\frac{\pi}{4} + h - \frac{\pi}{4}} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{\sin\frac{\pi}{4}\cos h + \cos\frac{\pi}{4}\sin h - \cos\frac{\pi}{4}\cos h + \sin\frac{\pi}{4}\sin h}{h} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{2\left(\frac{1}{\sqrt{2}} \sin h\right)}{h} = \frac{2}{\sqrt{2}} \lim_{h \to 0} \frac{\sin h}{h} = \sqrt{2}\,(1) = \sqrt{2}$$

ALITER $$\lim_{x \to \pi/4} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} = \lim_{x \to \pi/4} \frac{\sqrt{2}\left\{\frac{1}{\sqrt{2}} \sin x - \frac{1}{\sqrt{2}} \cos x\right\}}{x - \frac{\pi}{4}}$$

$$= \sqrt{2} \lim_{x \to \pi/4} \frac{\sin x \cos \frac{\pi}{4} - \cos x \sin \frac{\pi}{4}}{x - \frac{\pi}{4}}$$

$$= \sqrt{2} \lim_{x \to \pi/4} \frac{\sin\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \sqrt{2} \times 1 = \sqrt{2}$$

(ii) We have,

$$\lim_{x \to \frac{\pi}{6}} \frac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}} = 2 \lim_{x \to \pi/6} \frac{\left(\sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6}\right)}{x - \frac{\pi}{6}} = 2 \lim_{x \to \pi/6} \frac{\sin\left(x - \frac{\pi}{6}\right)}{\left(x - \frac{\pi}{6}\right)} = 2.$$

(iii) We have,

$$\lim_{x \to a} \frac{\sin x - \sin a}{x - a} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{\sin(a+h) - \sin a}{a + h - a}$$

$$= \lim_{h \to 0} \frac{\sin a \cos h + \cos a \sin h - \sin a}{h} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \cos a \frac{\sin h}{h} - \sin a \left(\frac{1 - \cos h}{h}\right)$$

$$= \lim_{h \to 0} \cos a \frac{\sin h}{h} - \sin a \lim_{h \to 0} \left(\frac{1 - \cos h}{h}\right)$$

$$= \cos a \left(\lim_{h \to 0} \frac{\sin h}{h}\right) - \sin a \left(\lim_{h \to 0} \frac{2 \sin^2 \frac{h}{2}}{h}\right)$$

$$= \cos a \lim_{h \to 0} \frac{\sin h}{h} - 2 \sin a \lim_{h \to 0} \left(\frac{\frac{\sin h/2}{h}}{h/2}\right)^2 \times \frac{h}{4} = \cos a \times 1 - 2 \sin a\,(1)^2 \times 0 = \cos a$$

ALITER $$\lim_{x \to a} \frac{\sin x - \sin a}{x - a} = \lim_{x \to a} \frac{2 \sin\left(\frac{x-a}{2}\right) \cos\left(\frac{x+a}{2}\right)}{2\left(\frac{x-a}{2}\right)}$$

$$= \lim_{x \to a} \left\{\frac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}}\right\} \times \cos\left(\frac{x+a}{2}\right) = \cos a$$

(iv) We have,

$$\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$$

$$= \lim_{x \to a} \frac{\sin x - \sin a}{x - a} (\sqrt{x} + \sqrt{a})$$

$$= \lim_{x \to a} \frac{2 \sin\left(\frac{x-a}{2}\right) \cos\left(\frac{x+a}{2}\right)}{x - a} (\sqrt{x} + \sqrt{a})$$

$$= \lim_{x \to a} \frac{2 \sin\left(\frac{x-a}{2}\right)}{2\left(\frac{x-a}{2}\right)} \times \cos\left(\frac{x+a}{2}\right) \times (\sqrt{x} + \sqrt{a}) = 1 \times \cos a \times (\sqrt{a} + \sqrt{a}) = 2\sqrt{a} \cos a$$

EXAMPLE 5 *Evaluate the following limits:*

(i) $\lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2}$ (ii) $\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$

SOLUTION (i) We have,

$$\lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{h \to 0} \frac{1 + \cos 2\left(\frac{\pi}{2} + h\right)}{\left\{\pi - 2\left(\frac{\pi}{2} + h\right)\right\}^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{1 + \cos(\pi + 2h)}{4h^2}$$

$$= \lim_{h \to 0} \frac{1 - \cos 2h}{4h^2} = \lim_{h \to 0} \frac{2 \sin^2 h}{4h^2} = \frac{2}{4} \lim_{h \to 0} \left(\frac{\sin h}{h}\right)\left(\frac{\sin h}{h}\right)$$

$$= \frac{1}{2}\left(\lim_{h \to 0} \frac{\sin h}{h}\right)\left(\lim_{h \to 0} \frac{\sin h}{h}\right) = \frac{1}{2}$$

<u>ALITER</u> $$\lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{x \to \pi/2} \frac{1 - \cos(\pi - 2x)}{(\pi - 2x)^2} = \lim_{x \to \pi/2} \frac{2 \sin^2\left(\frac{\pi}{2} - x\right)}{4\left(\frac{\pi}{2} - x\right)^2}$$

$$= \frac{1}{2} \lim_{x \to \pi/2} \left\{\frac{\sin\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)}\right\}^2 = \frac{1}{2} \times (1)^2 = \frac{1}{2}$$

(ii) We have,

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{h \to 0} \frac{1 - \sin\left(\frac{\pi}{2} + h\right)}{\left\{\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)\right\}^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{1 - \cos h}{h^2} = \lim_{h \to 0} 2 \frac{\sin^2 \frac{h}{2}}{h^2} = \frac{2}{4} \lim_{h \to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)\left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right) = \frac{1}{2}(1)(1) = \frac{1}{2}$$

ALITER $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2} = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{1-\cos\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)^2} = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{4\left(\frac{\pi}{4}-\frac{x}{2}\right)^2}$

$$= \frac{2}{4} \lim_{x \to \frac{\pi}{2}} \left\{\frac{\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right\}^2 = \frac{2}{4} \times (1)^2 = \frac{1}{2}$$

EXAMPLE 6 *Evaluate the following limits:*

(i) $\lim\limits_{x \to a} \dfrac{\cos x - \cos a}{\cot x - \cot a}$ (ii) $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$ **[NCERT EXEMPLAR]**

(iii) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x - \cos x}{(\pi - 2x)^3}$

SOLUTION (i) We have,

$$\lim_{x \to a} \frac{\cos x - \cos a}{\cot x - \cot a} = \lim_{x \to a} \frac{\cos x - \cos a}{\cos x \sin a - \sin x \cos a} \cdot \sin x \sin a \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{-2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x-a}{2}\right)} \sin x \sin a$$

$$= \lim_{x \to a} \frac{\sin\left(\frac{x+a}{2}\right)}{\cos\left(\frac{x-a}{2}\right)} \sin x \sin a = \frac{\sin a}{1} \times \sin a \sin a = \sin^3 a$$

(ii) We have,

$$\lim_{x \to \pi/2} (\sec x - \tan x) \qquad (\text{form } \infty - \infty)$$

$$= \lim_{x \to \pi/2} \left(\frac{1-\sin x}{\cos x}\right) \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \left\{\frac{1-\sin(\pi/2+h)}{\cos\left(\frac{\pi}{2}+h\right)}\right\}$$

$$= \lim_{h \to 0} \frac{1-\cos h}{-\sin h} = \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2}}{-2\sin h/2 \cos h/2} = -\lim_{h \to 0} \tan h/2 = -\tan 0 = 0$$

ALITER $\lim\limits_{x \to \pi/2} (\sec x - \tan x) = \lim\limits_{x \to \pi/2} \dfrac{1-\sin x}{\cos x} = \lim\limits_{x \to \pi/2} \dfrac{1-\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}$

$$= \lim_{x \to \pi/2} \frac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\pi}{2}\right)} = \lim_{x \to \pi/2} \tan\left(\frac{\pi}{4}-\frac{x}{2}\right) = 0$$

(iii) We have,

$$\lim_{x \to \pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3}$$

$$= \lim_{h \to 0} \frac{\cot\left(\frac{\pi}{2}+h\right) - \cos\left(\frac{\pi}{2}+h\right)}{\left\{\pi - 2\left(\frac{\pi}{2}+h\right)\right\}^3} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{-\tan h + \sin h}{-8h^3}$$

$$= \lim_{h \to 0} \frac{-\sin h\,(1-\cos h)}{\cos h\,(-8\,h^3)}$$

$$= \frac{1}{8} \lim_{h \to 0} \frac{\tan h}{h} \times \frac{1-\cos h}{h^2} = \frac{1}{8} \lim_{h \to 0} \frac{\tan h}{h} \times \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2}}{h^2}$$

$$= \frac{1}{8} \times \frac{2}{4} \lim_{h \to 0} \frac{\sin^2 \frac{h}{2}}{\left(\frac{h}{2}\right)^2} = \frac{1}{16}$$

LEVEL-2

EXAMPLE 7 *Evaluate:* $\lim_{x \to \frac{\pi}{6}} \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2}$

SOLUTION We have,

$$\lim_{x \to \frac{\pi}{6}} \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2} = \lim_{h \to 0} \frac{2 - \sqrt{3}\cos\left(\frac{\pi}{6}+h\right) - \sin\left(\frac{\pi}{6}+h\right)}{\left\{6\left(\frac{\pi}{6}+h\right) - \pi\right\}^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{2 - \sqrt{3}\left(\cos\frac{\pi}{6}\cos h - \sin\frac{\pi}{6}\sin h\right) - \left(\sin\frac{\pi}{6}\cos h + \cos\frac{\pi}{6}\sin h\right)}{36\,h^2}$$

$$= \lim_{h \to 0} \frac{2 - \frac{3}{2}\cos h + \frac{\sqrt{3}}{2}\sin h - \frac{1}{2}\cos h - \frac{\sqrt{3}}{2}\sin h}{36\,h^2} = \lim_{h \to 0} \frac{2\,(1-\cos h)}{36\,h^2}$$

$$= \frac{1}{18} \lim_{h \to 0} \frac{2\sin^2\left(\frac{h}{2}\right)}{h^2} = \frac{1}{9} \lim_{h \to 0} \left(\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right)^2 \times \frac{1}{4} = \frac{1}{9} \times (1)^2 \times \frac{1}{4} = \frac{1}{36}$$

<u>ALITER</u> $\lim_{x \to \pi/6} \frac{2 - \sqrt{3}\cos x - \sin x}{(6\,x - \pi)^2}$

$$= \lim_{x \to \pi/6} \frac{2 - 2\left\{\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right\}}{(6x-\pi)^2}$$

$$= \lim_{x \to \pi/6} \frac{2 - 2\left\{\cos x \cos\frac{\pi}{6} + \sin x \sin\frac{\pi}{6}\right\}}{(6x-\pi)^2}$$

$$= \lim_{x \to \pi/6} \frac{2 - 2\cos\left(x - \frac{\pi}{6}\right)}{(6x-\pi)^2}$$

$$= \lim_{x \to \pi/6} \frac{2 \times 2\sin^2\left(\frac{x}{2} - \frac{\pi}{12}\right)}{144\left(\frac{x}{2} - \frac{\pi}{12}\right)^2} = \frac{4}{144} \lim_{x \to \pi/6} \left\{\frac{\sin\left(\frac{x}{2} - \frac{\pi}{12}\right)}{\left(\frac{x}{2} - \frac{\pi}{12}\right)}\right\}^2 = \frac{1}{36} \times (1)^2 = \frac{1}{36}$$

EXAMPLE 8 *Prove that:* $\lim_{x \to \pi/4} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)} = -4$ [NCERT EXEMPLAR]

SOLUTION We have,

$$\lim_{x \to \pi/4} \frac{\tan^3 x - \tan x}{\cos(x + \pi/4)} = \lim_{x \to \pi/4} \frac{\tan x (\tan x - 1)(\tan x + 1)}{\cos(x + \pi/4)}$$

$$= \lim_{x \to \pi/4} \frac{\tan x (\sin x - \cos x)(\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\lim_{x \to \pi/4} \frac{\tan x (\cos x - \sin x)(\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\sqrt{2} \lim_{x \to \pi/4} \frac{\tan x\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)(\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\sqrt{2} \lim_{x \to \pi/4} \frac{\tan x \cos(x + \pi/4)\ (\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\sqrt{2} \times \lim_{x \to \pi/4} \frac{\tan x (\tan x + 1)}{\cos x} = -\sqrt{2} \times 2\sqrt{2} = -4.$$

EXAMPLE 9 *Evaluate:* $\lim_{n \to \infty} \left(\cos\frac{x}{2} \cos\frac{x}{2^2} \cos\frac{x}{2^3} \ldots \cos\frac{x}{2^n}\right)$

SOLUTION We have,

$$\lim_{n \to \infty} \left(\cos\frac{x}{2} \cos\frac{x}{2^2} \cos\frac{x}{2^3} \ldots \cos\frac{x}{2^n}\right)$$

$$= \lim_{n \to \infty} \frac{\sin\left(2^n \times \frac{x}{2^n}\right)}{2^n \sin\left(\frac{x}{2^n}\right)} \qquad \left[\because \cos A \cos 2A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}\right]$$

$$= \lim_{n \to \infty} \frac{\sin x}{x \left\{ \dfrac{\sin\left(\dfrac{x}{2^n}\right)}{\left(\dfrac{x}{2^n}\right)} \right\}} = \frac{\sin x}{x} \qquad \left[\because \lim_{n \to \infty} \frac{\sin\left(\dfrac{x}{2^n}\right)}{\left(\dfrac{x}{2^n}\right)} = 1 \right]$$

EXAMPLE 10 *Evaluate :* $\lim_{x \to \pi/4} \dfrac{4\sqrt{2} - (\cos x + \sin x)^5}{1 - \sin 2x}$

SOLUTION We have,

$$\lim_{x \to \pi/4} \frac{4\sqrt{2} - (\cos x + \sin x)^5}{1 - \sin 2x} = \lim_{x \to \pi/4} \frac{2^{5/2} - \left\{(\cos x + \sin x)^2\right\}^{5/2}}{2 - (1 + \sin 2x)}$$

$$= \lim_{x \to \pi/4} \frac{2^{5/2} - (1 + \sin 2x)^{5/2}}{2 - (1 + \sin 2x)}$$

$$= \lim_{x \to \pi/4} \frac{(1 + \sin 2x)^{5/2} - 2^{5/2}}{(1 + \sin 2x) - 2}$$

$$= \lim_{u \to 2} \frac{u^{5/2} - 2^{5/2}}{u - 2}, \text{ where } u = 1 + \sin 2x$$

$$= \frac{5}{2} \times (2)^{5/2 - 1} = \frac{5}{2} \times 2^{3/2} = 5\sqrt{2}$$

EXAMPLE 11 *If* α, β *are the roots of* $ax^2 + bx + c$, *then evaluate* $\lim_{x \to \beta} \dfrac{1 - \cos(ax^2 + bx + c)}{(x - \beta)^2}$.

SOLUTION It is given that α and β are the roots of the given equation $ax^2 + bx + c = 0$.

$\therefore \quad ax^2 + bx + c = a(x - \alpha)(x - \beta)$

Now,

$$\lim_{x \to \beta} \frac{1 - \cos(ax^2 + bx + c)}{(x - \beta)^2} = \lim_{x \to \beta} \frac{1 - \cos\{a(x - \alpha)(x - \beta)\}}{(x - \beta)^2}$$

$$= \lim_{x \to \beta} \frac{2\sin^2\left\{\dfrac{a(x - \alpha)(x - \beta)}{2}\right\}}{(x - \beta)^2}$$

$$= 2 \lim_{x \to \beta} \left[\frac{\sin\left\{\dfrac{a(x - \alpha)(x - \beta)}{2}\right\}}{\left\{\dfrac{a(x - \alpha)(x - \beta)}{2}\right\}} \right]^2 \times \frac{a^2(x - \alpha)^2}{4}$$

$$= 2 \times \frac{a^2(\beta - \alpha)^2}{4} = \frac{a^2(\beta - \alpha)^2}{2}$$

EXERCISE 29.8

LEVEL-1

Evaluate the following limits:

1. $\lim\limits_{x \to \pi/2} \left(\frac{\pi}{2} - x\right) \tan x$

2. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\sin 2x}{\cos x}$

3. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{1 - \sin x}$

4. $\lim\limits_{x \to \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2}\,(\pi/3 - x)}$

5. $\lim\limits_{x \to a} \frac{\cos x - \cos a}{x - a}$

6. $\lim\limits_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{x - \frac{\pi}{4}}$

7. $\lim\limits_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$

8. $\lim\limits_{x \to \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x}$

9. $\lim\limits_{x \to a} \frac{a \sin x - x \sin a}{ax^2 - xa^2}$

10. $\lim\limits_{x \to \pi/2} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$

11. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2}$

12. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{\left(\frac{\pi}{4} - x\right)^2}$

13. $\lim\limits_{x \to \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3}$

14. $\lim\limits_{x \to a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}}$

15. $\lim\limits_{x \to \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2}$

16. $\lim\limits_{x \to a} \frac{\cos \sqrt{x} - \cos \sqrt{a}}{x - a}$

17. $\lim\limits_{x \to a} \frac{\sin \sqrt{x} - \sin \sqrt{a}}{x - a}$

18. $\lim\limits_{x \to 1} \frac{1 - x^2}{\sin 2\pi x}$

19. $\lim\limits_{x \to \frac{\pi}{4}} \frac{f(x) - f\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}}$, where $f(x) = \sin 2x$

20. $\lim\limits_{x \to 1} \frac{1 + \cos \pi x}{(1 - x)^2}$

21. $\lim\limits_{x \to 1} \frac{1 - x^2}{\sin \pi x}$

22. $\lim\limits_{x \to \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x}$

23. $\lim\limits_{x \to \pi} \frac{1 + \cos x}{\tan^2 x}$

24. $\lim\limits_{n \to \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right)$

25. $\lim\limits_{n \to \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right)$

26. $\lim\limits_{n \to \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)}$

27. $\lim\limits_{x \to -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)}$

28. $\lim\limits_{x \to 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x - 2)}$

29. $\lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$

30. $\lim_{x \to \frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x}$

31. $\lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi - x)^2}$

32. $\lim_{x \to \pi/4} \frac{\sqrt{\cos x}-\sqrt{\sin x}}{x-\pi/4}$

33. $\lim_{x \to 1} \frac{1-\frac{1}{x}}{\sin \pi (x-1)}$

34. $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$ [NCERT EXEMPLAR]

LEVEL-2

35. $\lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}-\cos x - \sin x}{(4x-\pi)^2}$

36. $\lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)\sin x - 2\cos x}{\left(\frac{\pi}{2}-x\right)+\cot x}$

37. $\lim_{x \to \frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4}-x\right)(\cos x + \sin x)}$

38. $\lim_{x \to \pi} \frac{1-\sin\frac{x}{2}}{\cos\frac{x}{2}\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)}$ [NCERT EXEMPLAR]

ANSWERS

1. 1 2. 2 3. 2 4. 3 5. $-\sin a$ 6. -2 7. $\frac{1}{2}$ 8. $\frac{4}{3}$

9. $\frac{a\cos a - \sin a}{a^2}$ 10. $\frac{1}{4\sqrt{2}}$ 11. $\frac{1}{4}$ 12. $\frac{1}{\sqrt{2}}$ 13. $\frac{1}{16}$ 14. $-2\sqrt{a}\sin a$

15. $\frac{1}{8}$ 16. $-\frac{1}{2\sqrt{a}}\sin\sqrt{a}$ 17. $\frac{1}{2\sqrt{a}}\cos\sqrt{a}$ 18. $\frac{-1}{\pi}$ 19. 0 20. $\frac{\pi^2}{2}$

21. $\frac{2}{\pi}$ 22. $\frac{1}{4}$ 23. $\frac{1}{2}$ 24. $\frac{\pi}{4}$ 25. $\frac{a}{2}$ 26. $\frac{a}{b}$ 27. ∞ 28. 1

29. $\frac{2}{\pi}$ 30. 2 31. $\frac{1}{4}$ 32. $-\frac{1}{2^{1/4}}$ 33. $\frac{1}{\pi}$ 34. 4 35. $\frac{1}{16\sqrt{2}}$

36. $-\frac{1}{2}$ 37. 1 38. $\sqrt{2}$

29.7.3 EVALUATION OF TRIGONOMETRIC LIMITS BY FACTORISATION

Sometimes trigonometric limits can also be evaluated by factorisation method as illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate the following limits:*

(i) $\lim_{x \to \pi/4} \frac{\sec^2 x - 2}{\tan x - 1}$

(ii) $\lim_{x \to \pi} \frac{1+\sec^3 x}{\tan^2 x}$

(iii) $\lim_{x \to \pi/2} \frac{1-\sin^3 x}{\cos^2 x}$

(iv) $\lim_{x \to \pi} \frac{1+\cos^3 x}{\sin^2 x}$

(v) $\lim_{x \to \pi/2} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\sqrt{2}\cos^2 x}$

(vi) $\lim_{x \to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

SOLUTION (i) We have,

$$\lim_{x \to \pi/4} \frac{\sec^2 x - 2}{\tan x - 1} = \lim_{x \to \pi/4} \frac{1 + \tan^2 x - 2}{\tan x - 1} = \lim_{x \to \pi/4} (\tan x + 1) = 2.$$

(ii) We have,

$$\lim_{x \to \pi} \frac{1 + \sec^3 x}{\tan^2 x} = \lim_{x \to \pi} \frac{(1 + \sec^3 x)}{(\sec^2 x - 1)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \pi} \frac{(\sec x + 1)(\sec^2 x - \sec x + 1)}{(\sec x + 1)(\sec x - 1)}$$

$$= \lim_{x \to \pi} \frac{\sec^2 x - \sec x + 1}{\sec x - 1} = \frac{1 + 1 + 1}{-2} = -\frac{3}{2}$$

(iii) We have,

$$\lim_{x \to \pi/2} \frac{1 - \sin^3 x}{\cos^2 x}$$

$$= \lim_{x \to \pi/2} \frac{(1 - \sin x)(1 + \sin x + \sin^2 x)}{(1 - \sin x)(1 + \sin x)} = \lim_{x \to \pi/2} \frac{1 + \sin x + \sin^2 x}{1 + \sin x} = \frac{1 + 1 + 1}{1 + 1} = \frac{3}{2}$$

(iv) We have,

$$\lim_{x \to \pi} \frac{1 + \cos^3 x}{\sin^2 x}$$

$$= \lim_{x \to \pi} \frac{(1 + \cos x)(1 - \cos x + \cos^2 x)}{(1 - \cos x)(1 + \cos x)} = \lim_{x \to \pi} \frac{1 - \cos x + \cos^2 x}{1 - \cos x} = \frac{1 + 1 + 1}{1 + 1} = \frac{3}{2}$$

(v) We have,

$$\lim_{x \to \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\sqrt{2} \cos^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\sqrt{2} \cos^2 x} \times \frac{\sqrt{2} + \sqrt{1 + \sin x}}{\sqrt{2} + \sqrt{1 + \sin x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2 - (1 + \sin x)}{\sqrt{2}(1 - \sin^2 x)} \times \frac{1}{\sqrt{2} + \sqrt{1 + \sin x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x)}{\sqrt{2}(1 - \sin x)(1 + \sin x)} \times \frac{1}{\sqrt{2} + \sqrt{1 + \sin x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{\sqrt{2}(1 + \sin x)} \times \frac{1}{\sqrt{2} + \sqrt{1 + \sin x}}$$

$$= \frac{1}{2\sqrt{2}} \times \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{8}$$

(vi) We have,

$$\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{\sqrt{2} - \sqrt{2} \cos \frac{x}{2}}{(1 - \cos x)(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sqrt{2}\left(1 - \cos \frac{x}{2}\right)}{2 \sin^2 \frac{x}{2} (1 + \cos x)} = \frac{1}{\sqrt{2}} \lim_{x \to 0} \frac{1 - \cos \frac{x}{2}}{\left(1 - \cos \frac{x}{2}\right)\left(1 + \cos \frac{x}{2}\right)(1 + \cos x)}$$

$$= \frac{1}{\sqrt{2}} \lim_{x \to 0} \frac{1}{\left(1 + \cos \frac{x}{2}\right)(1 + \cos x)} = \frac{1}{\sqrt{2}} \frac{1}{(1+1)(1+1)} = \frac{1}{4\sqrt{2}}$$

EXAMPLE 2 *Evaluate:* $\lim_{x \to \pi/6} \frac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1}$

SOLUTION We have,

$$\lim_{x \to \pi/6} \frac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1} = \lim_{x \to \pi/6} \frac{(2 \sin x - 1)(\sin x + 1)}{(2 \sin x - 1)(\sin x - 1)} = \lim_{x \to \pi/6} \frac{\sin x + 1}{\sin x - 1} = \frac{\frac{1}{2} + 1}{\frac{1}{2} - 1} = -3$$

EXERCISE 29.9

LEVEL-1

Evaluate the following limits:

1. $\lim_{x \to \pi} \frac{1 + \cos x}{\tan^2 x}$
2. $\lim_{x \to \frac{\pi}{4}} \frac{\text{cosec}^2 x - 2}{\cot x - 1}$
3. $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\text{cosec}\, x - 2}$
4. $\lim_{x \to \frac{\pi}{4}} \frac{2 - \text{cosec}^2 x}{1 - \cot x}$
5. $\lim_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$
6. $\lim_{x \to \frac{3\pi}{2}} \frac{1 + \text{cosec}^3 x}{\cot^2 x}$

ANSWERS

1. $\frac{1}{2}$ 2. 2 3. 4 4. 2 5. $-\frac{3}{2}$ 6. $\frac{1}{4}$

29.8 EVALUATION OF EXPONENTIAL AND LOGARITHMIC LIMITS

Sometimes, following expansions are useful in evaluating limits. Students are advised to learn these expansions.

1. $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots$
2. $\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots$
3. $\log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \dots$
4. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$
5. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \dots$

6. $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \ldots$

7. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$

8. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$

9. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

THEOREM Prove that: (i) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$. (ii) $\lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1$.

PROOF (i) Using expansion of a^x, we obtain

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \frac{1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \ldots - 1}{x}$$

$$\Rightarrow \quad \lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \left\{\log_e a + \frac{x}{2!}(\log_e a)^2 + \ldots\right\} = \log_e a.$$

(ii) Using expansion of $\log(1+x)$, we obtain

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots}{x} = \lim_{x \to 0} 1 - \frac{x}{2} + \frac{x^2}{3} \ldots = 1.$$

COROLLARY Putting $a = e$ in (i) we get

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \log_e e = 1.$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate the following limits:*

(i) $\lim_{x \to 0} \frac{a^x - b^x}{x}$ (ii) $\lim_{x \to 0} \frac{a^x - b^x}{\sin x}$

(iii) $\lim_{x \to 0} \frac{2^x - 1}{\sqrt{1+x} - 1}$ (iv) $\lim_{x \to 0} \frac{3^x - 2^x}{\tan x}$

(v) $\lim_{x \to 0} \frac{a^{\sin x} - 1}{\sin x}$ (vi) $\lim_{x \to 1} \frac{a^{x-1} - 1}{\sin \pi x}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{a^x - b^x}{x} = \lim_{x \to 0} \frac{(a^x - 1) - (b^x - 1)}{x} = \lim_{x \to 0} \frac{a^x - 1}{x} - \lim_{x \to 0} \frac{b^x - 1}{x} = \log a - \log b = \log\left(\frac{a}{b}\right).$$

ALITER $$\lim_{x \to 0} \frac{a^x - b^x}{x} = \lim_{x \to 0} \frac{b^x\left\{\left(\frac{a}{b}\right)^x - 1\right\}}{x} = b^0 \times \log\left(\frac{a}{b}\right) = \log\left(\frac{a}{b}\right)$$

(ii) We have,

$$\lim_{x \to 0} \frac{a^x - b^x}{\sin x} = \lim_{x \to 0} \left\{\left(\frac{a^x - 1}{\sin x}\right) - \left(\frac{b^x - 1}{\sin x}\right)\right\}$$

$$= \lim_{x \to 0} \left(\frac{a^x - 1}{x} \times \frac{x}{\sin x}\right) - \lim_{x \to 0} \left(\frac{b^x - 1}{x} \times \frac{x}{\sin x}\right)$$

$$= \lim_{x \to 0} \frac{a^x - 1}{x} \times \lim_{x \to 0} \frac{x}{\sin x} - \lim_{x \to 0} \frac{b^x - 1}{x} \times \lim_{x \to 0} \frac{x}{\sin x}$$

$$= (\log a) \times 1 - (\log b) \times 1 = \log\left(\frac{a}{b}\right).$$

(iii) We have,

$$\lim_{x \to 0} \frac{2^x - 1}{\sqrt{1+x} - 1}$$

$$= \lim_{x - 0} \frac{2^x - 1}{\sqrt{1+x} - 1} \times \frac{(\sqrt{1+x} + 1)}{(\sqrt{1+x} + 1)} = \lim_{x - 0} \frac{2^x - 1}{x} \times \left\{\sqrt{1+x} + 1\right\}$$

$$= \lim_{x \to 0} \frac{2^x - 1}{x} \times \lim_{x \to 0} (\sqrt{1+x} + 1) = (\log 2)\, 2 = 2 \log 2.$$

(iv) We have,

$$\lim_{x \to 0} \frac{3^x - 2^x}{\tan x} = \lim_{x \to 0} \frac{(3^x - 1) - (2^x - 1)}{x} \times \frac{x}{\tan x}$$

$$= \left\{\lim_{x \to 0} \frac{3^x - 1}{x} - \lim_{x \to 0} \frac{2^x - 1}{x}\right\} \times \lim_{x \to 0} \frac{x}{\tan x} = (\log 3 - \log 2) \times 1 = \log\left(\frac{3}{2}\right)$$

(v) Let $y = \sin x$. Then, $y \to 0$ as $x \to 0$.

$$\therefore \quad \lim_{x \to 0} \frac{a^{\sin x} - 1}{\sin x} = \lim_{y \to 0} \frac{a^y - 1}{y} = \log a$$

(vi) We have,

$$\lim_{x \to 1} \frac{a^{x-1} - 1}{\sin \pi x} = \lim_{h \to 0} \frac{a^{1+h-1} - 1}{\sin \pi (1+h)} = \lim_{h \to 0} \frac{a^h - 1}{-\sin \pi h}$$

$$= \frac{-1}{\pi} \lim_{h \to 0} \left(\frac{a^h - 1}{h}\right)\left(\frac{\pi h}{\sin \pi h}\right) = -\frac{1}{\pi}\left(\lim_{h \to 0} \frac{a^h - 1}{h}\right)\left(\lim_{h \to 0} \frac{\pi h}{\sin \pi h}\right) = -\frac{1}{\pi} \log a$$

<u>ALITER</u> $\displaystyle \lim_{x \to 1} \frac{a^{x-1} - 1}{\sin \pi x} = \lim_{x \to 1} \frac{a^{x-1} - 1}{x - 1} \times \frac{x - 1}{\sin (\pi - \pi x)}$

$$= \lim_{x \to 1} \frac{a^{x-1} - 1}{x - 1} \times \frac{\pi (x - 1)}{-\pi \sin \pi (x - 1)} = \log_e a \times \frac{1}{-\pi} = -\frac{1}{\pi} \log_e a$$

EXAMPLE 2 *Evaluate the following limits:*

(i) $\displaystyle \lim_{x \to 0} \frac{10^x - 2^x - 5^x + 1}{x \tan x}$ (ii) $\displaystyle \lim_{x \to 0} \frac{3^x + 3^{-x} - 2}{x^2}$

(iii) $\displaystyle \lim_{x \to 0} \frac{2^{3x} - 3^x}{\sin 3x}$ (iv) $\displaystyle \lim_{x \to 0} \frac{3^{2x} - 2^{3x}}{x}$

(v) $\lim_{x \to 0} \frac{3^{2x}-1}{2^{3x}-1}$ (vi) $\lim_{x \to 0} \frac{\sin 3x}{3^x - 1}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{10^x - 2^x - 5^x + 1}{x \tan x}$$

$$= \lim_{x \to 0} \frac{5^x \cdot 2^x - 2^x - 5^x + 1}{x \tan x}$$

$$= \lim_{x \to 0} \frac{5^x (2^x - 1) - (2^x - 1)}{x \tan x} = \lim_{x \to 0} \frac{(5^x - 1)(2^x - 1)}{x \tan x} = \lim_{x \to 0} \frac{5^x - 1}{x} \times \frac{2^x - 1}{x} \times \frac{x}{\tan x}$$

$$= \lim_{x \to 0} \frac{5^x - 1}{x} \times \lim_{x \to 0} \frac{2^x - 1}{x} \times \lim_{x \to 0} \frac{x}{\tan x} = (\log 5)(\log 2)(1) = (\log 5)(\log 2)$$

(ii) We have,

$$\lim_{x \to 0} \frac{3^x + 3^{-x} - 2}{x^2}$$

$$= \lim_{x \to 0} \frac{3^{2x} - 2 \times 3^x + 1}{3^x \times x^2} = \lim_{x \to 0} \left(\frac{3^x - 1}{x}\right)^2 \times \frac{1}{3^x} = (\log 3)^2 \times \left(\frac{1}{3}\right)^0 = (\log 3)^2$$

(iii) We have,

$$\lim_{x \to 0} \frac{2^{3x} - 3^x}{\sin 3x}$$

$$= \lim_{x \to 0} \left\{\frac{(2^{3x} - 1)}{\sin 3x} - \frac{(3^x - 1)}{\sin 3x}\right\}$$

$$= \lim_{x \to 0} \left(\frac{2^{3x} - 1}{3x} \times \frac{3x}{\sin 3x}\right) - \left(\lim_{x \to 0} \frac{3^x - 1}{3x} \times \frac{3x}{\sin 3x}\right)$$

$$= (\log 2) \times 1 - \frac{1}{3}(\log 3) = \log 2 - \frac{1}{3} \log 3.$$

(iv) We have,

$$\lim_{x \to 0} \frac{3^{2x} - 2^{3x}}{x}$$

$$= \lim_{x \to 0} \left\{\left(\frac{3^{2x} - 1}{x}\right) - \left(\frac{2^{3x} - 1}{x}\right)\right\}$$

$$= \lim_{x \to 0} \left(\frac{3^{2x} - 1}{2x} \times 2\right) - \lim_{x \to 0} \left(\frac{2^{3x} - 1}{3x} \times 3\right) = 2 \log 3 - 3 \log 2 = \log 9 - \log 8 = \log\left(\frac{9}{8}\right)$$

(v) We have,

$$\lim_{x \to 0} \frac{3^{2x} - 1}{2^{3x} - 1}$$

$$= \lim_{x \to 0} \frac{\left(\frac{3^{2x} - 1}{2x}\right) \times 2x}{\left(\frac{2^{3x} - 1}{3x}\right) \times 3x} = \frac{2}{3} \times \frac{\lim_{x \to 0} \left(\frac{3^{2x} - 1}{2x}\right)}{\lim_{x \to 0} \left(\frac{2^{3x} - 1}{3x}\right)} = \frac{2}{3}\left(\frac{\log 3}{\log 2}\right) = \frac{\log 3^2}{\log 2^3} = \frac{\log 9}{\log 8}.$$

(vi) We have,

$$\lim_{x \to 0} \frac{\sin 3x}{3^x - 1} = \lim_{x \to 0} \frac{\sin 3x}{3x} \times \frac{3x}{3^x - 1} = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} \times \lim_{x \to 0} \frac{x}{3^x - 1} = \frac{3}{\log 3}$$

EXAMPLE 3 *Evaluate the following limits:*

(i) $\lim_{x \to 0} \frac{e^{-x} - 1}{x}$ (ii) $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$

(iii) $\lim_{x \to 1} \frac{x - 1}{\log_e x}$ (iv) $\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2}$

SOLUTION (i) Putting $-x = y$, we obtain

$$\lim_{x \to 0} \frac{e^{-x} - 1}{x} = \lim_{y \to 0} \frac{e^y - 1}{-y} = -\lim_{y \to 0} \frac{e^y - 1}{y} = -1$$

(ii) We have,

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{x} = \lim_{x \to 0} \left\{ \left(\frac{e^x - 1}{x} \right) - \left(\frac{e^{-x} - 1}{x} \right) \right\} = \left(\lim_{x \to 0} \frac{e^x - 1}{x} \right) + \left(\lim_{x \to 0} \frac{e^{-x} - 1}{-x} \right) = 1 + 1 = 2$$

(iii) We have,

$$\lim_{x \to 1} \frac{x - 1}{\log_e x} = \lim_{h \to 0} \frac{1 + h - 1}{\log_e (1 + h)} = \lim_{h \to 0} \frac{h}{\log_e (1 + h)} = \frac{1}{\lim_{h \to 0} \frac{\log (1 + h)}{h}} = \frac{1}{1} = 1$$

<u>ALITER</u> $$\lim_{x \to 1} \frac{x - 1}{\log_e x} = \lim_{x \to 1} \frac{x - 1}{\log_e \{1 + (x - 1)\}} = \lim_{x \to 1} \frac{1}{\frac{\log_e \{1 + (x - 1)\}}{x - 1}} = \frac{1}{1} = 1$$

(iv) We have,

$$\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2}$$

$$= \lim_{x \to 0} \frac{e^{2x} + 1 - 2e^x}{x^2 e^x} = \lim_{x \to 0} \left(\frac{e^x - 1}{x} \right)^2 \times e^x = \lim_{x \to 0} \left(\frac{e^x - 1}{x} \right)^2 \times \lim_{x \to 0} e^x = (1^2) \times e^0 = 1.$$

EXAMPLE 4 *Evaluate the following limits:*

(i) $\lim_{x \to 0} \frac{\log (5 + x) - \log (5 - x)}{x}$ (ii) $\lim_{x \to 5} \frac{\log x - \log 5}{x - 5}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{\log (5 + x) - \log (5 - x)}{x}$$

$$= \lim_{x \to 0} \frac{\log \left\{ 5 \left(1 + \frac{x}{5} \right) \right\} - \log \left\{ 5 \left(1 - \frac{x}{5} \right) \right\}}{x}$$

$$= \lim_{x \to 0} \frac{\left\{ \log 5 + \log \left(1 + \frac{x}{5} \right) \right\} - \left\{ \log 5 + \log \left(1 + \frac{x}{5} \right) \right\}}{x}$$

$$= \lim_{x \to 0} \frac{\log \left(1 + \frac{x}{5} \right) - \log \left(1 - \frac{x}{5} \right)}{x}$$

$$= \lim_{x \to 0} \frac{1}{5} \times \frac{\log\left(1+\frac{x}{5}\right)}{x/5} - \lim_{x \to 0} \frac{\log\left(1-\frac{x}{5}\right)}{-x/5} \times \frac{1}{(-5)} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

(ii) $$\lim_{x \to 5} \frac{\log x - \log 5}{x-5} = \lim_{h \to 0} \frac{\log(5+h) - \log 5}{h} = \lim_{h \to 0} \frac{\log\left(\frac{5+h}{5}\right)}{h} = \lim_{h \to 0} \frac{\log\left(1+\frac{h}{5}\right)}{\frac{h}{5}} \times \frac{1}{5} = \frac{1}{5}$$

<u>ALITER</u> $$\lim_{x \to 5} \frac{\log x - \log 5}{x-5} = \lim_{x \to 5} \frac{\log\left(\frac{x}{5}\right)}{x-5} = \lim_{x \to 5} \frac{\log\left(1+\frac{x}{5}-1\right)}{x-5}$$

$$= \lim_{x \to 5} \frac{\log\left(1+\frac{x-5}{5}\right)}{\left(\frac{x-5}{5}\right)} \times \frac{1}{5} = 1 \times \frac{1}{5} = \frac{1}{5}$$

LEVEL-2

EXAMPLE 5 *Prove that:* $\lim_{x \to 2} \frac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}} = -\frac{4}{3}$.

SOLUTION We have,

$$\lim_{x \to 2} \frac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}}$$

$$= \lim_{x \to 2} \frac{(3^x)^2 - 12(3^x) + 27}{3^3 - (3^{x/2})^3}$$

$$= \lim_{x \to 2} \frac{(3^x - 3)\ (3^x - 9)}{(3 - 3^{x/2})\ (9 + 3 \times 3^{x/2} + 3^x)}$$

$$= -\lim_{x \to 2} \frac{(3^x - 3)(3^{x/2} - 3)(3^{x/2} + 3)}{(3^{x/2} - 3)(3^x + 3 \times 3^{x/2} + 9)}$$

$$= -\lim_{x \to 2} \frac{(3^x - 3)\ (3^{x/2} + 3)}{(3^x + 3 \times 3^{x/2} + 9)} = -\frac{(9-3)(3+3)}{(9+9+9)} = -\frac{4}{3}$$

EXAMPLE 6 *Evaluate:* $\lim_{x \to e} \frac{\log x - 1}{x - e}$

SOLUTION We have,

$$\lim_{x \to e} \frac{\log x - 1}{x - e} = \lim_{h \to 0} \frac{\log(e+h) - 1}{e+h-e} = \lim_{h \to 0} \frac{\log(e+h) - \log e}{h} \qquad [\because \log e = 1]$$

$$= \lim_{h \to 0} \frac{\log\left(1+\frac{h}{e}\right)}{h} = \lim_{h \to 0} \frac{\log\left(1+\frac{h}{e}\right)}{\frac{h}{e}} \times \frac{1}{e}$$

$$= 1 \times \frac{1}{e} = \frac{1}{e} \qquad \left[\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right]$$

ALITER $\lim_{x \to e} \frac{\log x - 1}{x - e} = \lim_{x \to e} \frac{\log x - \log e}{x - e} = \lim_{x \to e} \frac{\log\left(\frac{x}{e}\right)}{x - e}$

$$= \lim_{x \to e} \frac{\log\left(1 + \frac{x}{e} - 1\right)}{x - e} = \lim_{x \to e} \frac{\log\left(1 + \frac{x - e}{e}\right)}{\left(\frac{x - e}{e}\right)e} = \frac{1}{e} \times 1 = \frac{1}{e}$$

EXERCISE 29.10

LEVEL-1

Evaluate the following limits:

1. $\lim_{x \to 0} \frac{5^x - 1}{\sqrt{4 + x} - 2}$
2. $\lim_{x \to 0} \frac{\log(1 + x)}{3^x - 1}$
3. $\lim_{x \to 0} \frac{a^x + a^{-x} - 2}{x^2}$
4. $\lim_{x \to 0} \frac{a^{mx} - 1}{b^{nx} - 1}$, $n \neq 0$
5. $\lim_{x \to 0} \frac{a^x + b^x - 2}{x}$
6. $\lim_{x \to 0} \frac{9^x - 2.6^x + 4^x}{x^2}$
7. $\lim_{x \to 0} \frac{8^x - 4^x - 2^x + 1}{x^2}$
8. $\lim_{x \to 0} \frac{a^{mx} - b^{nx}}{x}$
9. $\lim_{x \to 0} \frac{a^x + b^x + c^x - 3}{x}$
10. $\lim_{x \to 2} \frac{x - 2}{\log_a (x - 1)}$
11. $\lim_{x \to 0} \frac{5^x + 3^x + 2^x - 3}{x}$
12. $\lim_{x \to \infty} (a^{1/x} - 1)\, x$
13. $\lim_{x \to 0} \frac{a^{mx} - b^{nx}}{\sin kx}$
14. $\lim_{x \to 0} \frac{a^x + b^x - c^x - d^x}{x}$
15. $\lim_{x \to 0} \frac{e^x - 1 + \sin x}{x}$
16. $\lim_{x \to 0} \frac{\sin 2x}{e^x - 1}$
17. $\lim_{x \to 0} \frac{e^{\sin x} - 1}{x}$
18. $\lim_{x \to 0} \frac{e^{2x} - e^x}{\sin 2x}$
19. $\lim_{x \to a} \frac{\log x - \log a}{x - a}$
20. $\lim_{x \to 0} \frac{\log(a + x) - \log(a - x)}{x}$
21. $\lim_{x \to 0} \frac{\log(2 + x) + \log 0.5}{x}$
22. $\lim_{x \to 0} \frac{\log(a + x) - \log a}{x}$
23. $\lim_{x \to 0} \frac{\log(3 + x) - \log(3 - x)}{x}$
24. $\lim_{x \to 0} \frac{8^x - 2^x}{x}$
25. $\lim_{x \to 0} \frac{x(2^x - 1)}{1 - \cos x}$
26. $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{\log(1 + x)}$
27. $\lim_{x \to 0} \frac{\log|1 + x^3|}{\sin^3 x}$
28. $\lim_{x \to \pi/2} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$
29. $\lim_{x \to 0} \frac{e^x - 1}{\sqrt{1 - \cos x}}$
30. $\lim_{x \to 5} \frac{e^x - e^5}{x - 5}$

31. $\lim_{x \to 0} \frac{e^{x+2} - e^2}{x}$

32. $\lim_{x \to \pi/2} \frac{e^{\cos x} - 1}{\cos x}$

33. $\lim_{x \to 0} \frac{e^{3+x} - \sin x - e^3}{x}$

34. $\lim_{x \to 0} \frac{e^x - x - 1}{2}$

35. $\lim_{x \to 0} \frac{e^{3x} - e^{2x}}{x}$

36. $\lim_{x \to 0} \frac{e^{\tan x} - 1}{\tan x}$

37. $\lim_{x \to 0} \frac{e^{bx} - e^{ax}}{x}$ where $0 < a < b$

38. $\lim_{x \to 0} \frac{e^{\tan x} - 1}{x}$

39. $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

40. $\lim_{x \to 0} \frac{3^{2+x} - 9}{x}$

41. $\lim_{x \to 0} \frac{a^x - a^{-x}}{x}$

42. $\lim_{x \to 0} \frac{x(e^x - 1)}{1 - \cos x}$

43. $\lim_{x \to \pi/2} \frac{2^{-\cos x} - 1}{x\left(x - \frac{\pi}{2}\right)}$

ANSWERS

1. $4 \log 5$	2. $\frac{1}{\log_e 3}$	3. $(\log_e a)^2$	4. $\frac{m}{n} \frac{\log a}{\log b}$
5. $\log (ab)$	6. $\left(\log \frac{3}{2}\right)^2$	7. $(\log 4)(\log 2)$	8. $\log \left(\frac{a^m}{b^n}\right)$
9. $\log (abc)$	10. $\log a$	11. $\log 30$	12. $\log a$
13. $\frac{1}{k} \log \left(\frac{a^m}{b^n}\right)$	14. $\log \left(\frac{ab}{cd}\right)$	15. 2	16. 2
17. 1	18. $\frac{1}{2}$	19. $\frac{1}{a}$	20. $\frac{2}{a}$
21. $\frac{1}{2}$	22. $\frac{1}{a}$	23. $\frac{2}{3}$	24. $\log 4$
25. $\log 4$	26. $\frac{1}{2}$	27. 1	28. $\log a$
29. Does not exist	30. e^5	31. e^2	32. 1
33. $e^3 - 1$	34. 0	35. 1	36. 1
37. $(b - a)$	38. 1	39. 1	40. $9 . \log_e 3$
41. $2 \log_e 2$	42. 2	43. $\frac{2}{\pi} \log_e 2$	

HINTS TO SELECTED PROBLEMS

10. $\lim_{x \to 2} \frac{x-2}{\log_a (x-1)} = \lim_{x \to 2} \frac{(x-2)}{\log_e (x-1) \times \log_a e}$

$$= \lim_{x \to 2} \frac{1}{\frac{\log_e \{1 + (x-2)\}}{x-2}} \times \log_e a = \frac{1}{1} \times \log_e a = \log_e a$$

29. We have,

$$\lim_{x\to 0} \frac{e^x-1}{\sqrt{1-\cos x}} = \lim_{x\to 0} \frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|}$$

Now, $$\lim_{x\to 0^-} \frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|} = -\frac{1}{\sqrt{2}}\lim_{x\to 0}\frac{e^x-1}{\sin\frac{x}{2}} = -\frac{2}{\sqrt{2}}\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)\times\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right) = -\sqrt{2}$$

and, $$\lim_{x\to 0^+} \frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|} = \frac{2}{\sqrt{2}}\lim_{x\to 0^+}\left(\frac{e^x-1}{x}\right)\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right) = \sqrt{2}.$$

$\therefore$ $$\lim_{x\to 0^-} \frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|} \neq \lim_{x\to 0^+} \frac{e^x-1}{\sqrt{2}\left(\sin\frac{x}{2}\right)}$$

33. $$\lim_{x\to 0}\left\{\frac{e^{3+x}-e^3}{x}-\frac{\sin x}{x}\right\} = \lim_{x\to 0}\left\{e^3\left(\frac{e^x-1}{x}\right)-\frac{\sin x}{x}\right\} = e^3\times 1-1 = e^3-1$$

39. $$\lim_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x} = \lim_{x\to 0} e^{\sin x}\left(\frac{e^{x-\sin x}-1}{x-\sin x}\right) = e^0\times 1 = 1$$

42. $$\lim_{x\to 0} x\left(\frac{e^x-1}{1-\cos x}\right) = \lim_{x\to 0} x^2\left(\frac{\frac{e^x-1}{x}}{2\sin^2\frac{x}{2}}\right) = 2\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^2 = 2\times 1\times 1^2 = 2.$$

43. $$\lim_{x\to \pi/2}\frac{2^{-\cos x}-1}{x\left(x-\frac{\pi}{2}\right)} = \lim_{x\to\pi/2}\frac{2^{\sin(x-\pi/2)}-1}{x-\frac{\pi}{2}}\times\frac{1}{x}$$

$$= \lim_{x\to\pi/2}\frac{2^{\sin\left(x-\frac{\pi}{2}\right)}-1}{\sin\left(x-\frac{\pi}{2}\right)}\times\frac{\sin\left(x-\frac{\pi}{2}\right)}{x-\frac{\pi}{2}}\times\frac{1}{x} = \log_e 2\times 1\times\frac{2}{\pi} = \frac{2}{\pi}\log_e 2$$

29.9 EVALUATION OF LIMITS OF THE FORM 1^∞

To evaluate exponential limits of the form 1^∞, we use the following result which is stated and proved as a theorem.

THEOREM *If* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *such that* $\lim_{x\to a}\frac{f(x)}{g(x)}$ *exists, then*

$$\lim_{x\to a}\left\{1+f(x)\right\}^{\frac{1}{g(x)}} = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$$

<u>PROOF</u> Let $A = \lim_{x\to a}\left\{1+f(x)\right\}^{\frac{1}{g(x)}}$. Then,

$$\log_e A = \lim_{x \to a} \frac{\log\{1 + f(x)\}}{g(x)}$$

$$\Rightarrow \quad \log_e A = \lim_{x \to a} \frac{\log\{1 + f(x)\}}{f(x)} \times \frac{f(x)}{g(x)}$$

$$\Rightarrow \quad \log_e A = \lim_{x \to a} \frac{f(x)}{g(x)} \qquad \left[\because \lim_{x \to a} \frac{\log\{1 + f(x)\}}{f(x)} = 1\right]$$

$$\Rightarrow \quad A = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$$

Q.E.D

REMARK *The above result can also be restated in the following form:*

If $\lim_{x \to a} f(x) = 1$ *and* $\lim_{x \to a} g(x) = \infty$ *such that* $\lim_{x \to a} \{f(x) - 1\} g(x)$ *exists, then*

$$\lim_{x \to a} \{f(x)\}^{g(x)} = e^{\lim_{x \to a} \{f(x) - 1\} g(x)}$$

PARTICULAR CASES

(i) $\lim_{x \to 0} (1 + x)^{1/x} = e$ (ii) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(iii) $\lim_{x \to 0} (1 + \lambda x)^{1/x} = e^{\lambda}$ (iv) $\lim_{x \to \infty} \left(1 + \frac{\lambda}{x}\right)^x = e^{\lambda}$

Following examples will illustrate the applications of the above results.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate the following limits:*

(i) $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$ (ii) $\lim_{x \to 0} (1 + \sin x)^{2 \cot x}$

(iii) $\lim_{x \to 1} (\log_3 3x)^{\log_x 3}$ (iv) $\lim_{x \to 0} (\cos x)^{\cot x}$

SOLUTION (i) We have,

$$\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x = e^{\lim_{x \to \infty} \frac{2}{x} \times x} = e^2$$

(ii) We have,

$$\lim_{x \to 0} (1 + \sin x)^{2 \cot x} = e^{\lim_{x \to 0} \sin x \times 2 \cot x} = e^{\lim_{x \to 0} 2 \cos x} = e^2$$

(iii) We have,

$$\lim_{x \to 1} (\log_3 3x)^{\log_x 3} = \lim_{x \to 1} (\log_3 3 + \log_3 x)^{\log_x 3}$$

$$= \lim_{x \to 1} (1 + \log_3 x)^{\frac{1}{\log_3 x}} = e^{\lim_{x \to 1} \log_3 x \times \frac{1}{\log_3 x}} = e^1 = e$$

(iv) We have,

$$\lim_{x\to 0} (\cos x)^{\cot x}$$

$$= \lim_{x\to 0} \{1+\cos x - 1\}^{\cot x}$$

$$= \lim_{x\to 0} \{1-(1-\cos x)\}^{\cot x}$$

$$= \lim_{x\to 0} \left\{1-2\sin^2\left(\frac{x}{2}\right)\right\}^{\cot x}$$

$$= e^{\lim_{x\to 0} -2\sin^2(x/2)\times\cot x}$$

$$= e^{\lim_{x\to 0} \frac{-2\sin^2(x/2)\ \cos x}{2\sin(x/2)\cos x/2}} = e^{\lim_{x\to 0} -\tan(x/2)\times\cos x} = e^0 = 1$$

EXAMPLE 2 *Evaluate the following limits:*

(i) $\lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$ (ii) $\lim_{x\to a} \left(2-\frac{a}{x}\right)^{\tan \pi x/2a}$

SOLUTION (i) We have,

$$\lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$$

$$\lim_{x\to 0} \left\{1+\frac{a^x+b^x+c^x-3}{3}\right\}^{1/x} = \lim_{x\to 0} \left\{1+\frac{(a^x-1)+(b^x-1)+(c^x-1)}{3}\right\}^{1/x}$$

$$= e^{\lim_{x\to 0} \frac{a^x-1}{3x}+\frac{b^x-1}{3x}+\frac{c^x-1}{3x}} = e^{\frac{1}{3}\left\{\lim_{x\to 0}\frac{a^x-1}{x}+\lim_{x\to 0}\frac{b^x-1}{x}+\lim_{x\to 0}\frac{c^x-1}{x}\right\}}$$

$$= e^{\frac{1}{3}\{\log a+\log b+\log c\}} = e^{\log(abc)^{1/3}} = (abc)^{1/3}$$

(ii) We have,

$$\lim_{x\to a} \left(2-\frac{a}{x}\right)^{\tan \pi x/2a}$$

$$= \lim_{x\to a} \left\{1+\left(1-\frac{a}{x}\right)\right\}^{\tan \pi x/2a}$$

$$= e^{\lim_{x\to a}\left(1-\frac{a}{x}\right)\tan\frac{\pi x}{2a}} = e^{\lim_{x\to a}\left(\frac{x-a}{x}\right)\tan\frac{\pi x}{2a}} = e^{l}, \text{ where } l = \lim_{x\to a}\left(\frac{x-a}{x}\right)\tan\frac{\pi x}{2a}$$

Now, $l = \lim_{x\to a} \left(\frac{x-a}{x}\right)\tan\frac{\pi x}{2a} = \lim_{x\to a}\left(\frac{x-a}{x}\right)\cot\left(\frac{\pi}{2}-\frac{\pi x}{2a}\right)$

$$\Rightarrow \quad l = \lim_{x\to a}\left(\frac{x-a}{x}\right)\cot\frac{\pi}{2a}(a-x)$$

$$\Rightarrow \quad l = \lim_{x\to a}\left(\frac{x-a}{x}\right)\times\frac{1}{\tan\frac{\pi}{2a}(a-x)}$$

$$\Rightarrow \quad l = -\lim_{x \to a} \frac{(a-x)}{\tan \frac{\pi}{2a}(a-x)} \times \frac{1}{x} = -\lim_{x \to a} \frac{\frac{\pi}{2a}(a-x)}{\tan \frac{\pi}{2a}(a-x)} \times \frac{2a}{\pi x} = -\frac{2}{\pi}$$

Hence, $\lim_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan \frac{\pi x}{2a}} = e^{-2/\pi}$.

LEVEL-2

EXAMPLE 3 *Evaluate the following limits:*

(i) $\lim_{x \to \infty} \left(\frac{x+5}{x-1}\right)^x$ (ii) $\lim_{x \to \infty} \left(\frac{x^2+4x+3}{x^2+2x-5}\right)^x$

SOLUTION (i) We have,

$$\lim_{x \to \infty} \left(\frac{x+5}{x-1}\right)^x = \lim_{x \to \infty} \left(1 + \frac{6}{x-1}\right)^x = e^{\lim_{x \to \infty} \frac{6x}{x-1}} = e^6$$

(ii) We have

$$\lim_{x \to \infty} \left(\frac{x^2+4x-3}{x^2+2x-5}\right)^x = \lim_{x \to \infty} \left\{1 + \frac{2x+8}{x^2+2x-5}\right\}^x = e^{\lim_{x \to \infty} \frac{2x^2+8x}{x^2+2x-5}} = e^2$$

EXAMPLE 4 *Evaluate the following limits:*

(i) $\lim_{x \to 0} \left\{\tan\left(\frac{\pi}{4} + x\right)\right\}^{1/x}$ (ii) $\lim_{x \to 0} (\cos 2x)^{1/x^2}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \left\{\tan\left(\frac{\pi}{4} + x\right)\right\}^{1/x}$$

$$= \lim_{x \to 0} \left\{\frac{1+\tan x}{1-\tan x}\right\}^{1/x}$$

$$= \lim_{x \to 0} \left\{1 + \frac{2\tan x}{1-\tan x}\right\}^{1/x} = e^{\lim_{x \to 0} \frac{2\tan x}{1-\tan x} \times \frac{1}{x}} = e^{\lim_{x \to 0} \frac{2}{1-\tan x} \times \frac{\tan x}{x}} = e^2$$

(ii) We have,

$$\lim_{x \to 0} (\cos 2x)^{1/x^2} = \lim_{x \to 0} \left\{1 + (\cos 2x - 1)\right\}^{1/x^2} = e^{\lim_{x \to 0} \frac{\cos 2x - 1}{x^2}} = e^{-\lim_{x \to 0} \frac{2\sin^2 x}{x^2}} = e^{-2}$$

EXERCISE 29.11

LEVEL-1

Evaluate the following limits:

1. $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$
2. $\lim_{x \to 0^+} \left\{1 + \tan^2 \sqrt{x}\right\}^{1/2x}$
3. $\lim_{x \to 0} (\cos x)^{1/\sin x}$
4. $\lim_{x \to 0} (\cos x + \sin x)^{1/x}$

5. $\lim_{x \to 0} (\cos x + a \sin bx)^{1/x}$

6. $\lim_{x \to \infty} \left\{\dfrac{x^2 + 2x + 3}{2x^2 + x + 5}\right\}^{\frac{3x-2}{3x+2}}$

7. $\lim_{x \to 1} \left\{\dfrac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3}\right\}^{\frac{1-\cos(x-1)}{(x-1)^2}}$

8. $\lim_{x \to 0} \left\{\dfrac{e^x + e^{-x} - 2}{x^2}\right\}^{1/x^2}$

9. $\lim_{x \to a} \left\{\dfrac{\sin x}{\sin a}\right\}^{\frac{1}{x-a}}$

10. $\lim_{x \to \infty} \left\{\dfrac{3x^2 + 1}{4x^2 - 1}\right\}^{\frac{x^3}{1+x}}$

ANSWERS

1. e^x 2. $\sqrt{e}$ 3. 1 4. e 5. e^{ab} 6. $\frac{1}{2}$ 7. $\sqrt{\frac{5}{6}}$ 8. $e^{1/12}$ 9. $e^{\cot a}$ 10. 0

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the value of $\lim_{x \to 0} \dfrac{\sqrt{1 - \cos 2x}}{x}$.
2. Write the value of $\lim_{x \to 0^-} [x]$.
3. Write the value of $\lim_{x \to 0^+} [x]$.
4. Write the value of $\lim_{x \to 1^-} x - [x]$.
5. Write the value of $\lim_{x \to 0^-} \dfrac{\sin [x]}{[x]}$.
6. Write the value of $\lim_{x \to \pi} \dfrac{\sin x}{x - \pi}$.
7. Write the value of $\lim_{x \to \infty} \dfrac{\sin x}{x}$.
8. Write the value of $\lim_{x \to 2} \dfrac{|x-2|}{x-2}$.
9. Write the value of $\lim_{x \to 0} \dfrac{\sin x^\circ}{x}$.
10. Write the value of $\lim_{x \to 0^-} \dfrac{\sin x}{\sqrt{x}}$.
11. Write the value of $\lim_{x \to 0} \dfrac{\sin x}{\sqrt{1+x} - 1}$.
12. Write the value of $\lim_{x \to -\infty} (3x + \sqrt{9x^2 - x})$.
13. Write the value of $\lim_{n \to \infty} \dfrac{n! + (n+1)!}{(n+1)! + (n+2)!}$.
14. Write the value of $\lim_{x \to \pi/2} \dfrac{2x - \pi}{\cos x}$.
15. Write the value of $\lim_{n \to \infty} \dfrac{1 + 2 + 3 + \ldots + n}{n^2}$.

ANSWERS

1. Does not exit 2. -1 3. 1 4. 1 5. sin 1 6. -1 7. 0
8. Does not exist 9. $\frac{\pi}{180}$ 10. Does not exist 11. 2 12. $\frac{1}{6}$ 13. 0
14. -2 15. $\frac{1}{2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. $\lim_{n \to \infty} \dfrac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}$ is equal to

(a) 1 (b) 1/2 (c) 1/3 (d) 0

2. $\lim_{x \to 0} \frac{\sin 2x}{x}$ is equal to

(a) 0 (b) 1 (c) 1/2 (d) 2

3. If $f(x) = x \sin(1/x), x \neq 0$, then $\lim_{x \to 0} f(x) =$

(a) 1 (b) 0 (c) -1 (d) does not exist

4. $\lim_{x \to 0} \frac{1 - \cos 2x}{x}$ is

(a) 0 (b) 1 (c) 2 (d) 4

5. $\lim_{x \to 0} \frac{(1 - \cos 2x) \sin 5x}{x^2 \sin 3x} =$

(a) 10/3 (b) 3/10 (c) 6/5 (d) 5/6

6. $\lim_{x \to 0} \frac{x}{\tan x}$ is

(a) 0 (b) 1 (c) 4 (d) not defined

7. $\lim_{n \to \infty} \left\{ \frac{1}{1-n^2} + \frac{2}{1-n^2} + \dots + \frac{n}{1-n^2} \right\}$ is equal to

(a) 0 (b) $-1/2$ (c) 1/2 (d) none of these

8. $\lim_{x \to \infty} \frac{\sin x}{x}$ equals

(a) 0 (b) ∞ (c) 1 (d) does not exist

9. $\lim_{x \to 0} \frac{\sin x^0}{x}$ is equal to

(a) 1 (b) π (c) x (d) $\pi / 180$

10. $\lim_{x \to 3} \frac{x-3}{|x-3|}$, is equal to

(a) 1 (b) -1 (c) 0 (d) does not exist

11. $\lim_{x \to a} \frac{x^n - a^n}{x - a}$ is equal to

(a) na^n (b) na^{n-1} (c) na (d) 1

12. $\lim_{x \to \pi/4} \frac{\sqrt{2} \cos x - 1}{\cot x - 1}$ is equal to

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{2\sqrt{2}}$ (d) 1

13. $\lim_{x \to \infty} \frac{\sqrt{x^2 - 1}}{2x + 1}$ is equal to

(a) 1 (b) 0 (c) -1 (d) 1/2

14. $\lim_{h \to 0} 2 \left\{ \frac{\sqrt{3} \sin(\pi/6 + h) - \cos(\pi/6 + h)}{\sqrt{3}\, h (\sqrt{3} \cos h - \sin h)} \right\}$ is equal to

(a) 2/3 (b) 4/3 (c) $-2\sqrt{3}$ (d) $-4/3$

15. $\lim_{h \to 0} \left\{ \frac{1}{h \sqrt[3]{8+h}} - \frac{1}{2h} \right\} =$

(a) $-1/12$ (b) $-4/3$ (c) $-16/3$ (d) $-1/48$

16. $\lim_{n \to \infty} \left\{ \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n+1)(2n+3)} \right\}$ is equal to

(a) 0 (b) $1/2$ (c) $1/9$ (d) 2

17. $\lim_{x \to 1} \frac{\sin \pi x}{x-1}$ is equal to

(a) $-\pi$ (b) π (c) $-\frac{1}{\pi}$ (d) $\frac{1}{\pi}$

18. If $\lim_{x \to 1} \frac{x + x^2 + x^3 + \ldots + x^n - n}{x-1} = 5050$, then n equals

(a) 10 (b) 100 (c) 150 (d) none of these

19. The value of $\lim_{x \to \infty} \frac{\sqrt{1+x^4} + (1+x^2)}{x^2}$ is

(a) -1 (b) 1 (c) 2 (d) none of these

20. $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$ is equal to

(a) $\frac{1}{2}$ (b) 2 (c) 0 (d) 1

21. $\lim_{x \to \pi/3} \frac{\sin\left(\frac{\pi}{3} - x\right)}{2\cos x - 1}$ is equal to

(a) $\sqrt{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

22. $\lim_{x \to 3} \frac{\sum_{r=1}^{n} x^r - \sum_{r=1}^{n} 3^r}{x-3}$ is equal to

(a) $\frac{(2n-1) \times 3^n}{4}$ (b) $\frac{(2n-1) \times 3^n + 1}{4}$ (c) $(2n-1)\, 3^n + 1$ (d) $\frac{(2n-1) \times 3^n - 1}{4}$

23. $\lim_{n \to \infty} \frac{1 - 2 + 3 - 4 + 5 - 6 + \ldots + (2n-1) - 2n}{\sqrt{n^2+1} + \sqrt{n^2-1}}$ is equal to

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1 (d) -1

24. If $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $\lim_{x \to 0} f(x)$ equals

(a) 1 (b) 0 (c) -1 (d) none of these.

25. $\lim_{n \to \infty} \dfrac{n!}{(n+1)! + n!}$ is equal to

(a) $\dfrac{1}{2}$ (b) 0 (c) 2 (d) 1

26. $\lim_{x \to \pi/4} \dfrac{4\sqrt{2} - (\cos x + \sin x)^5}{1 - \sin 2x}$ is equal to

(a) $5\sqrt{2}$ (b) $3\sqrt{2}$ (c) $\sqrt{2}$ (d) none of these

27. $\lim_{x \to 2} \dfrac{\sqrt{1 + \sqrt{2 + x}} - \sqrt{3}}{x - 2}$ is equal to

(a) $\dfrac{1}{8\sqrt{3}}$ (b) $\dfrac{1}{\sqrt{3}}$ (c) $8\sqrt{3}$ (d) $\sqrt{3}$

28. $\lim_{x \to \infty} a^x \sin\left(\dfrac{b}{a^x}\right)$, $a, b > 1$ is equal to

(a) b (b) a (c) $a \log_e b$ (d) $b \log_e a$

29. $\lim_{x \to 0} \dfrac{8}{x^8}\left\{1 - \cos\dfrac{x^2}{2} - \cos\dfrac{x^2}{4} + \cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right\}$ is equal to

(a) $1/16$ (b) $-1/16$ (c) $1/32$ (d) $-1/32$

30. If α is a repeated root of $ax^2 + bx + c = 0$, then $\lim_{x \to \alpha} \dfrac{\tan(ax^2 + bx + c)}{(x - \alpha)^2}$ is

(a) a (b) b (c) c (d) 0

31. The value of $\lim_{x \to 0} \dfrac{\sqrt{a^2 - ax + x^2} - \sqrt{a^2 + ax + x^2}}{\sqrt{a + x} - \sqrt{a - x}}$ is

(a) a (b) $\sqrt{a}$ (c) $-a$ (d) $-\sqrt{a}$

32. The value of $\lim_{x \to 0} \dfrac{1 - \cos x + 2\sin x - \sin^3 x - x^2 + 3x^4}{\tan^3 x - 6\sin^2 x + x - 5x^3}$ is

(a) 1 (b) 2 (c) -1 (d) -2

33. $\lim_{\theta \to \pi/2} \dfrac{1 - \sin\theta}{(\pi/2 - \theta)\cos\theta}$ is equal to

(a) 1 (b) -1 (c) $1/2$ (d) $-1/2$

34. The value of $\lim_{x \to \pi/2} (\sec x - \tan x)$ is

(a) 2 (b) -1 (c) 1 (d) 0

35. The value of $\lim_{x \to \infty} \dfrac{n!}{(n+1)! - n!}$ is

(a) 0 (b) 1 (c) -1 (d) none of these

36. The value of $\lim_{n \to \infty} \dfrac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$ is

(a) 0 (b) -1 (c) 1 (d) none of these

37. The value of $\lim_{x \to \infty} \dfrac{(x+1)^{10} + (x+2)^{10} + \ldots + (x+100)^{10}}{x^{10} + 10^{10}}$ is

(a) 10 (b) 100 (c) 10^{10} (d) none of these

38. The value of $\lim\limits_{n \to \infty} \left\{ \dfrac{1+2+3+\ldots+n}{n+2} - \dfrac{n}{2} \right\}$ is

(a) 1/2 (b) 1 (c) −1 (d) −1/2

39. $\lim\limits_{x \to 1} [x-1]$, where [·] is the greatest integer function, is equal to

(a) 1 (b) 2 (c) 0 (d) does not exist

40. $\lim\limits_{x \to \infty} \dfrac{|x|}{x}$ is equal to

(a) 1 (b) −1 (c) 0 (d) does not exist

41. $\lim\limits_{x \to 0} \dfrac{|\sin x|}{x}$ is

(a) 1 (b) −1 (c) 0 (d) does not exist

42. If $f(x) = \begin{cases} \dfrac{\sin [x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, where [·] denotes the greatest integer function, then $\lim\limits_{x \to 0} f(x)$ is equal to

(a) 1 (b) 0 (c) −1 (d) none of these

ANSWERS

1. (c)	2. (d)	3. (b)	4. (a)	5. (a)	6. (b)	7. (b)	8. (a)
9. (d)	10. (d)	11. (b)	12. (b)	13. (d)	14. (b)	15. (d)	16. (b)
17. (a)	18. (b)	19. (b)	20. (a)	21. (c)	22. (b)	23. (b)	24. (b)
25. (a)	26. (a)	27. (a)	28. (a)	29. (c)	30. (a)	31. (d)	32. (b)
33. (c)	34. (d)	35. (a)	36. (c)	37. (b)	38. (d)	39. (d)	40. (d)
41. (d)	42. (d)						

SUMMARY

1. $\lim\limits_{x \to a} f(x)$ exists $\Leftrightarrow \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$

   For a function $f(x)$ and a real number a, $\lim\limits_{x \to a} f(x)$ and $f(a)$ may not be same. In fact:

   (i) $\lim\limits_{x \to a} f(x)$ exists but $f(a)$ (the value of $f(x)$ at $x = a$) may not exists

   (ii) The value $f(a)$ exists but $\lim\limits_{x \to a} f(x)$ does not exist

   (iii) $\lim\limits_{x \to a} f(x)$ and $f(a)$ both exist but are unequal

   (iv) $\lim\limits_{x \to a} f(x)$ and $f(a)$ both exist and are equal.

3. Let $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$. If l and m both exist, then

   (i) $\lim\limits_{x \to a} k f(x) = k \lim\limits_{x \to a} f(x)$

   (ii) $\lim\limits_{x \to a} (f \pm g)(x) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = l + m$

   (iii) $\lim\limits_{x \to a} (fg)(x) = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x) = lm$

(iv) $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}$

(v) $\lim_{x \to a} \{f(a)\}^{g(x)} = l^m$

4. Following are some standard limits:

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(iv) $\lim_{x \to a} \frac{\sin(x-a)}{x-a} = 1$

(v) $\lim_{x \to a} \frac{\tan(x-a)}{x-a} = 1$

(vi) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

(vii) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a, a \neq 0, a > 1$

(viii) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

CHAPTER 30

DERIVATIVES

30.1 DERIVATIVE AT A POINT

DEFINITION *Let $f(x)$ be a real valued function defined on an open interval (a, b) and let $c \in (a, b)$. Then, $f(x)$ is said to be differentiable or derivable at $x = c$, iff*

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists finitely.}$$

This limit is called the derivative or differentiation of $f(x)$ at $x = c$ and is denoted by $f'(c)$ or $Df(c)$ or $\left\{\frac{d}{dx} f(x)\right\}_{x=c}$.

That is,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}, \text{ provided that the limit exists.}$$

Throughout this chapter it will be assumed that a given function $f(x)$ is differentiable at every point in its domain i.e. $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists for all c in its domain.

$$\therefore \quad f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

$$\Rightarrow \quad f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \text{ or, } f'(c) = \lim_{h \to 0} \frac{f(c - h) - f(c)}{-h}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the derivative of $f(x) = k$ at $x = 0$ and $x = 5$.* **[NCERT]**

SOLUTION By definition

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{k - k}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

and, $$f'(5) = \lim_{h \to 0} \frac{f(5 + h) - f(5)}{h} = \lim_{h \to 0} \frac{k - k}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

EXAMPLE 2 *Find the derivative of* $\sin x$ *at* $x = 0$. **[NCERT]**

SOLUTION Let $f(x) = \sin x$. Then,

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{\sin h - \sin 0}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1$$

EXAMPLE 3 *Let f be a real valued function defined by $f(x) = x^2 + 1$. Find $f'(2)$.*

SOLUTION We have, $f(x) = x^2 + 1$

$$\therefore \quad f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$\Rightarrow \quad f'(2) = \lim_{h \to 0} \frac{\{(2+h)^2 + 1\} - \{2^2 + 1\}}{h}$$

$$\Rightarrow \quad f'(2) = \lim_{h \to 0} \frac{(h^2 + 4h + 5) - 5}{h} = \lim_{h \to 0} \frac{h^2 + 4h}{h} = \lim_{h \to 0} h + 4 = 4$$

EXAMPLE 4 *If f is a real valued function defined by $f(x) = x^2 + 4x + 3$, then find $f'(1)$ and $f'(3)$.*

SOLUTION We have, $f(x) = x^2 + 4x + 3$

$$\therefore \quad f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \quad f'(1) = \lim_{h \to 0} \frac{\{(1+h)^2 + 4(1+h) + 3\} - \{1^2 + 4 \times 1 + 3\}}{h}$$

$$\Rightarrow \quad f'(1) = \lim_{h \to 0} \frac{(h^2 + 6h + 8) - 8}{h} = \lim_{h \to 0} \frac{h^2 + 6h}{h} = \lim_{h \to 0} h + 6 = 6$$

$$\text{and,} \quad f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$\Rightarrow \quad f'(3) = \lim_{h \to 0} \frac{\{(3+h)^2 + 4(3+h) + 3\} - \{3^2 + 4 \times 3 + 3\}}{h}$$

$$\Rightarrow \quad f'(3) = \lim_{h \to 0} \frac{(h^2 + 10h + 24) - 24}{h} = \lim_{h \to 0} \frac{h^2 + 10h}{h} = \lim_{h \to 0} h + 10 = 10$$

EXAMPLE 5 *Find the derivative of $f(x) = 2x^2 + 3x - 5$ at $x = -1$. Also, prove that*

$f'(0) + 3f'(-1) = 0$. [NCERT]

SOLUTION Let us first find the derivatives of $f(x)$ at $x = 0$ and $x = -1$.

By definition

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$\Rightarrow \quad f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$\Rightarrow \quad f'(0) = \lim_{h \to 0} \frac{(2h^2 + 3h - 5) - \{2 \times (0)^2 + 3 \times (0) - 5\}}{h}$$

$$\Rightarrow \quad f'(0) = \lim_{h \to 0} \frac{(2h^2 + 3h - 5) - (-5)}{h}$$

$$\Rightarrow \quad f'(0) = \lim_{h \to 0} \frac{2h^2 + 3h}{h} = \lim_{h \to 0} (2h + 3) = 2 \times 0 + 3 = 3$$

$$\text{and,} \quad f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$\Rightarrow \quad f'(-1) = \lim_{h \to 0} \frac{\{2(-1+h)^2 + 3(-1+h) - 5\} - \{2(-1)^2 + 3(-1) - 5\}}{h}$$

$$\Rightarrow \quad f'(-1) = \lim_{h \to 0} \frac{2h^2 - h}{h} = \lim_{h \to 0} (2h - 1) = 2 \times 0 - 1 = -1$$

$$\therefore \quad f'(0) + 3f'(-1) = 3 + 3 \times -1 = 3 - 3 = 0$$

EXERCISE 30.1

LEVEL-1

1. Find the derivative of $f(x) = 3x$ at $x = 2$
2. Find the derivative of $f(x) = x^2 - 2$ at $x = 10$ [NCERT]
3. Find the derivative of $f(x) = 99x$ at $x = 100$ [NCERT]
4. Find the derivative of $f(x) = x$ at $x = 1$ [NCERT]
5. Find the derivative of $f(x) = \cos x$ at $x = 0$
6. Find the derivative of $f(x) = \tan x$ at $x = 0$
7. Find the derivatives of the following functions at the indicated points:

   (i) $\sin x$ at $x = \frac{\pi}{2}$ (ii) x at $x = 1$ (iii) $2 \cos x$ at $x = \frac{\pi}{2}$ (iv) $\sin 2x$ at $x = \frac{\pi}{2}$

ANSWERS

1. 3 2. 20 3. 99 4. 1 5. 0 6. 1
7. (i) 0 (ii) 1 (iii) -2 (iv) -2

HINTS TO NCERT & SELECTED PROBLEMS

2. We have, $f(x) = x^2 - 2$.

$$\therefore \quad f'(10) = \lim_{h \to 0} \frac{f(10+h) - f(10)}{h} = \lim_{h \to 0} \frac{\{(10+h)^2 - 2\} - \{10^2 - 2\}}{h}$$

$$= \lim_{h \to 0} \frac{(10+h)^2 - 10^2}{h} = \lim_{h \to 0} \frac{(20+h)\,h}{h} = \lim_{h \to 0} (20 + h) = 20$$

3. We have, $f(x) = 99x$

$$\therefore \quad f'(100) = \lim_{h \to 0} \frac{f(100+h) - f(100)}{h}$$

$$= \lim_{h \to 0} \frac{99(100+h) - 99 \times 100}{h}$$

$$= 99 \lim_{h \to 0} \frac{100 + h - 100}{h} = 99 \times \lim_{h \to 0} 1 = 99$$

4. We have, $f(x) = x$

$$\therefore \quad f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h) - 1}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

30.1.1 PHYSICAL INTEGPRETATION OF DERIVATIVE AT A POINT

Let a particle be moving in a straight line OX starting from point O towards point X as shown in Fig. 30.1.

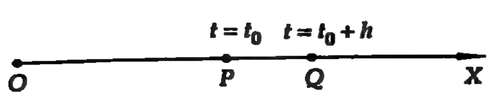


Fig. 30.1

Clearly, the position of the particle at any time t depends upon the time elapsed. In other words, the distance of the particle from O depends upon the time i.e. it is a function f of time t taken by the particle.

Let at any time t_0 i.e. at time $t = t_0$, the particle be at P and after a further time h i.e. at time $t = t_0 + h$, it is at Q.

$\therefore \quad OP = f(t_0)$ and $OQ = f(t_0 + h)$

Distance travelled in time $h = PQ = OQ - OP = f(t_0 + h) - f(t_0)$

Clearly, Average speed of the particle during the journey from P to $Q = \dfrac{PQ}{h} = \dfrac{f(t_0 + h) - f(t_0)}{h}$

As $h \to 0$, we observe that $Q \to P$.

$$\therefore \quad (\text{Instantaneous speed at time } t = t_0) = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} = f'(t_0)$$

Thus, if $f(t)$ gives the distance of a moving particle at time t, then $f'(t_0)$ i.e. the derivative of f at $t = t_0$ represents the instantaneous speed of the particle at time $t = t_0$ or, at the point P.

ILLUSTRATION *The distance $f(t)$ in metres moved by a particle travelling in a straight line in t seconds is given by $f(t) = t^2 + 3t + 4$. Find the speed of the particle at the end of 2 seconds.*

SOLUTION We have, $f(t) = t^2 + 3t + 4$.

The speed of the particle at the end of 2 seconds is given by $f'(2)$ i.e. the derivative of $f(t)$ at $t = 2$. Now,

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$\Rightarrow \quad f'(2) = \lim_{h \to 0} \frac{\{(2+h)^2 + 3(2+h) + 4\} - \{2^2 + 3 \times 2 + 4\}}{h}$$

$$\Rightarrow \quad f'(2) = \lim_{h \to 0} \frac{(h^2 + 7h + 14) - 14}{h} = \lim_{h \to 0} \frac{h^2 + 7h}{h} = \lim_{h \to 0} h + 7 = 7$$

Hence, the speed of the particle at the end of 2 seconds is 7 m/sec.

30.1.2 GEOMETRICAL INTERPETATION OF DERIVATIVE AT A POINT

Let $f(x)$ be a differentiable function. Consider the curve $y = f(x)$. Let $P(c, f(c))$ be a point on the curve $y = f(x)$ as shown in Fig. 30.2 and let $Q(c + h, f(c + h))$ be a neighbouring point on the curve $y = f(x)$. Then,

$$\text{Slope of chord } PQ = \tan \angle QPN = \frac{QN}{PN} = \frac{f(c+h) - f(c)}{h}.$$

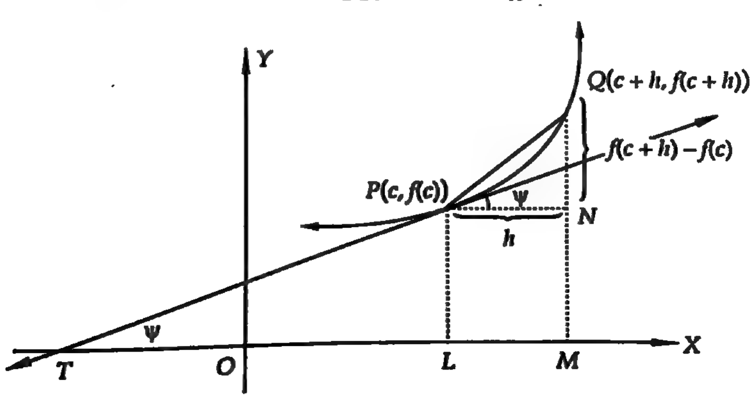


Fig. 30.2

Taking limit as $Q \to P$ i.e. $h \to 0$, we obtain

$$\lim_{Q \to P} (\text{Slope of chord } PQ) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \quad \text{...(i)}$$

As $Q \to P$, chord PQ tends to the tangent to $y = f(x)$ at point P. Therefore, from (i), we get

$$\text{Slope of the tangent at } P = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$\Rightarrow$ Slope of the tangent at $P = f'(c)$ i.e, $\tan \psi = f'(c)$,

where ψ is the inclination of the tangent to the curve $y = f(x)$ at point $(c, f(c))$ with the x-axis. *Thus, the derivative of a function $f(x)$ at a point $x = c$ is the slope of the tangent to the curve $y = f(x)$ at the point $(c, f(c))$.*

ILLUSTRATION *Find the slope of the tangent to the curve $y = x^2$ at $(-1/2, 1/4)$.*

SOLUTION Let $f(x) = x^2$. Then, $y = f(x)$ is the given curve. Clearly, slope of the tangent to the curve at $(-1/2, 1/4)$ is equal to $f'(-1/2)$ i.e. the derivative of $f(x)$ at $x = -1/2$.

Now,
$$f'\left(-\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(-\frac{1}{2} + h\right) - f\left(-\frac{1}{2}\right)}{h}$$

$$\Rightarrow \quad f'\left(-\frac{1}{2}\right) = \lim_{h \to 0} \frac{\left(-\frac{1}{2} + h\right)^2 - \left(-\frac{1}{2}\right)^2}{h}$$

$$\Rightarrow \quad f'\left(-\frac{1}{2}\right) = \lim_{h \to 0} \frac{\left(\frac{1}{4} - h + h^2\right) - \frac{1}{4}}{h} = \lim_{h \to 0} \frac{h^2 - h}{h} = \lim_{h \to 0} h - 1 = -1$$

Hence, slope of the tangent to the curve $y = x^2$ at point $(-1/2, 1/4)$ is equal to -1. This means that the tangent to the curve at point $(-1/2, 1/4)$ makes $135°$ angle with the positive direction of X-axis.

30.2 DERIVATIVE OF A FUNCTION

In the previous section, we have learnt about the derivative of a function at a point in its domain. Let $f(x)$ be a function differentiable at every point in its domain. Then corresponding to every point c in the domain, we obtain a unique real number equal to the derivative $f'(c)$ of $f(x)$ at $x = c$. Thus, there is one-to-one correspondence between points in the domain of the function and the derivatives at these points. This correspondence induces a function such that the image of any point x in the domain is the value of the derivative of f at x i.e. $f'(x)$ or $\frac{d}{dx}(f(x))$. This function is called the derivative or differentiation of $f(x)$ and is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or,} \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The process of finding the derivative of a function by using the above formula is known as the differentiation or derivative from the first principles.

30.2.1 DERIVATIVE AS A RATE MEASURER

Let $f(x)$ be a function of x and let $y = f(x)$. Clearly, the value of y depends upon the value of x and it changes with a change in the value of x. So, x is called the *independent variable* and y the *dependent variable*. Let Δx be a small change (positive or negative) in x and let Δy be the

corresponding change in $y = f(x)$. Then, the value of x changes from x to $x + \Delta x$ and the value of the $f(x)$ changes from $f(x)$ to $f(x + \Delta x)$. So, change in the value of f is

$$f(x + \Delta x) - f(x) \quad \text{or,} \quad \Delta y = f(x + \Delta x) - f(x) \qquad \text{...(i)}$$

Thus, we observe that due to change Δx in x, there is change Δy in y. Therefore, due to one unit change in x, change in y is equal to $\frac{\Delta y}{\Delta x}$. This is known as the average rate of charge of y with respect to x.

As $\Delta x \to 0$, we observe that Δy also tends to zero.

$\therefore$ Instantaneous rate of change of y with respect to $x = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

If we use the phrase rate of change instead of instantaneous rate of change, we have

$$\begin{aligned}\text{Rate of change in } y \text{ with respect to } x &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{[Using (i)]} \\ &= \frac{d}{dx}(f(x)) && \text{[Using def. of derivative]} \\ &= \frac{dy}{dx}\end{aligned}$$

Thus, $\frac{dy}{dx}$ or, $\frac{d}{dx}(f(x))$ measures the rate of change of $y = f(x)$ with respect to x.

i.e., $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

<u>REMARK 1</u> *The meaning of the term "rate of change of y with respect to x" is that if x is increased by an additional unit the change in y is given by* $\frac{dy}{dx}$*. For example, the rate of change of displacement of a particle is defined as its velocity, so if we say that a particle is moving with the velocity v km/hr then it means that when time is increased by one hour the displacement changes by v km.*

<u>REMARK 2</u> *Some authors also define* $\frac{dy}{dx}$ *as* $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ *or,* $\lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ *which are exactly identical to the definition given in this book.*

<u>REMARK 3</u> *We have seen that* $\frac{dy}{dx}$ *or,* $\frac{d}{dx}(f(x))$ *is the derivative or differentiation of* $y = f(x)$*. Also,* $\frac{dy}{dx}$ *or,* $\frac{d}{dx}(f(x))$ *measures the rate of change of y with respect to x. So, we can say that the derivative of a function* $y = f(x)$ *is same as the rate of change of* $f(x)$ *with respect to x. Consequently, phrases such as "differentiation of a function* $f(x)$*" and "differentiation of a function* $f(x)$ *with respect to x" convey the same meaning and are used invariably.*

30.3 DIFFERENTIATION FROM FIRST PRINCIPLES

In the previous sections, we have learnt that the derivative of a function $f(x)$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The process of finding the derivative of a function by using the above definition is called the differentiation from first principles or by *ab*-initio method or, by delta method.

In this section, we will find the derivatives of some standard functions viz. x^n, e^x, a^x,

$\log x$, $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\operatorname{cosec} x$ and $\sec x$ by first principles. Following results will be very helpful in finding the same.

(i) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(ii) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(iii) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ (iv) $\tan A \pm \tan B = \dfrac{\sin(A \pm B)}{\cos A \cos B}$

(v) $\tan A - \tan B = \tan(A - B)\ \{1 + \tan A \tan B\}$

(vi) $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

(vii) $\sin C - \sin D = 2 \sin \dfrac{C-D}{2} \cos \dfrac{C+D}{2}$

(viii) $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

(ix) $\cos C - \cos D = -2 \sin \dfrac{C+D}{2} \sin \dfrac{C-D}{2} = 2 \sin \dfrac{C+D}{2} \sin \dfrac{D-C}{2}$

(x) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

(xi) $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

(xii) $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

(xiii) $2 \sin A \sin B = \cos(A-B) - \cos(A-B)$

(xiv) $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

(xv) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1, \quad \lim\limits_{x \to a} \dfrac{\sin(x-a)}{x-a} = 1$

(xvi) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1, \quad \lim\limits_{x \to a} \dfrac{\tan(x-a)}{x-a} = 1$

(xvii) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a, a > 0, a \neq 1$

(xviii) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$

(xix) $\lim\limits_{x \to 0} \dfrac{\log_e(1+x)}{x} = 1$

THEOREM 1 *If $f(x) = x^n$, where $n \in R$, then, the differentiation of x^n with respect to x is nx^{n-1}.*

i.e. $\dfrac{d}{dx}(x^n) = nx^{n-1}$ **[NCERT]**

PROOF Let $f(x) = x^n$. Then, $f(x+h) = (x+h)^n$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{(x+h) - x}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{z \to x} \frac{z^n - x^n}{z - x}, \text{ where } z = x + h \text{ and } z \to x \text{ as } h \to 0$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = nx^{n-1} \qquad \left[\text{Using: } \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

Hence, $\frac{d}{dx}(x^n) = nx^{n-1}$

Q.E.D.

ILLUSTRATIONS *Using the above formula, we obtain*

(i) $\frac{d}{dx}(x^5) = 5x^{5-1} = 5x^4$

(ii) $\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-3-1} = \frac{-3}{x^4}$

(iii) $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{1/2-1} = \frac{1}{2\sqrt{x}}$

(iv) $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-1/2-1} = -\frac{1}{2}x^{\frac{-3}{2}}$

(v) $\frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1 \times x^{1-1} = 1 \times x^0 = 1$

(vi) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1 \times x^{-1-1} = -\frac{1}{x^2}$

Q.E.D.

THEOREM 2 *The differentiation of e^x with respect to x is e^x.*

i.e. $\frac{d}{dx}(e^x) = e^x$

<u>PROOF</u> Let $f(x) = e^x$. Then, $f(x+h) = e^{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h \to 0} e^x\left(\frac{e^h - 1}{h}\right)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^x \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) = e^x \times 1 = e^x \qquad \left[\because \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) = 1\right]$$

Hence, $\frac{d}{dx}(e^x) = e^x$

Q.E.D.

THEOREM 3 *The differentiation of a^x $(a > 0, a \neq 1)$ with respect to x is $a^x \log_e a$.*

i.e. $\frac{d}{dx}(a^x) = a^x \log_e a$

<u>PROOF</u> Let $f(x) = a^x$. Then, $f(x+h) = a^{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = a^x \lim_{h \to 0}\left(\frac{a^h - 1}{h}\right) = a^x \log_e a \qquad \left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a\right]$$

Q.E.D.

Hence, $\frac{d}{dx}(a^x) = a^x \log_e a$

ILLUSTRATIONS *Using the above formula, we get*

(i) $\frac{d}{dx}(5^x) = 5^x \log_e 5$

(ii) $\frac{d}{dx}(10^x) = 10^x \log_e 10.$

(iii) $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}\left((e^2)^x\right) = (e^2)^x \log e^2 = e^{2x} \cdot 2 \log e = 2e^{2x}$

Q.E.D.

THEOREM 4 *The differentiation of* $\log_e x$, $x > 0$ is $\frac{1}{x}$.

i.e. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

PROOF Let $f(x) = \log_e x$. Then, $f(x+h) = \log_e (x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log_e (x+h) - \log_e x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log_e (1+h/x)}{h} = \lim_{h \to 0} \frac{\log_e (1+h/x)}{h/x} \cdot \frac{1}{x}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{1}{x} \qquad \left[\because \lim_{x \to 0} \frac{\log_e (1+x)}{x} = 1\right]$$

Hence, $\frac{d}{dx}(\log_e x) = \frac{1}{x}$ Q.E.D.

THEOREM 5 *The differentiation of* $\log_a x$ $(a > 0, a \neq 1)$ *with respect to* x is $\frac{1}{x \log_e a}$

i.e. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

PROOF Let $f(x) = \log_a x$. Then, $f(x+h) = \log_a (x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log_a (x+h) - \log_a x}{h} = \lim_{h \to 0} \frac{\log_a \left(\frac{x+h}{x}\right)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log_a (1+h/x)}{h} = \lim_{h \to 0} \frac{\log_e (1+h/x)}{(\log_e a) \cdot h} \qquad \left[\because \log_a \lambda = \frac{\log_e \lambda}{\log_e a}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{1}{\log_e a} \lim_{h \to 0} \frac{\log_e (1+h/x)}{x(h/x)} = \frac{1}{x \log_e a} \qquad \left[\because \lim_{h \to 0} \frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = 1\right]$$

Hence, $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$ Q.E.D.

ILLUSTRATIONS *We have,*

(i) $\frac{d}{dx}(\log_3 x) = \frac{1}{x \log_e 3}$

(ii) $\frac{d}{dx}\left(\frac{1}{\log_x 5}\right) = \frac{d}{dx}(\log_5 x) = \frac{1}{x \log_e 5}$

THEOREM 6 *The differentiation of* sin x *with respect to* x is cos x.

i.e. $\frac{d}{dx}(\sin x) = \cos x$ [NCERT]

PROOF Let $f(x) = \sin x$. Then, $f(x+h) = \sin(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{h} \quad \left[\because \sin C - \sin D = 2\sin\frac{C-D}{2}\cos\frac{C+D}{2}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(2\sin h/2)\cos(x + h/2)}{2(h/2)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right) \times \lim_{h \to 0} \frac{\sin(h/2)}{(h/2)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = (\cos x) \times 1 = \cos x \quad \left[\because \lim_{h \to 0} \frac{\sin(h/2)}{(h/2)} = 1\right]$$

Hence, $\frac{d}{dx}(\sin x) = \cos x$ Q.E.D.

THEOREM 7 *The differentiation of* cos x *with respect to* x *is* $-\sin x$. [NCERT]

i.e. $\frac{d}{dx}(\cos x) = -\sin x$

PROOF Let $f(x) = \cos x$. Then, $f(x+h) = \cos(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2} \times \sin\frac{C-D}{2}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h \to 0} \sin\left(x + \frac{h}{2}\right) \cdot \lim_{h \to 0} \frac{\sin(h/2)}{(h/2)} = (-\sin x) \times 1 \quad \left[\because \lim_{h \to 0} \frac{\sin(h/2)}{(h/2)} = 1\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\sin x$$

Hence, $\frac{d}{dx}(\cos x) = -\sin x$

THEOREM 8 *The differentiation of* tan x *with respect to* x *is* $\sec^2 x$. Q.E.D.

i.e. $\frac{d}{dx}(\tan x) = \sec^2 x$ [NCERT]

PROOF Let $f(x) = \tan x$. Then, $f(x+h) = \tan(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\dfrac{\sin(x+h)}{\cos(x+h)} - \dfrac{\sin x}{\cos x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h \cos x \cos(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin h}{h} \times \frac{1}{\cos x \cos(x+h)} \qquad [\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin h}{h} \times \lim_{h \to 0} \frac{1}{\cos x \cos(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 1 \times \frac{1}{\cos x \cos x} = \sec^2 x \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

Hence, $\frac{d}{dx}(\tan x) = \sec^2 x$ **Q.E.D.**

THEOREM 9 The differentiation of cot x with respect to x is $-\operatorname{cosec}^2 x$.

i.e. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

PROOF Let $f(x) = \cot x$. Then, $f(x+h) = \cot(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\cot(x+h) - \cot x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\dfrac{\cos(x+h)}{\sin(x+h)} - \dfrac{\cos x}{\sin x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{h \sin x \sin(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin(x-(x+h))}{h \sin x \sin(x+h)} \qquad [\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h \to 0} \frac{\sin h}{h} \times \lim_{h \to 0} \frac{1}{\sin x \sin(x+h)} \qquad [\because \sin(-h) = -\sin h]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = (-1)\frac{1}{\sin x \sin x} = -\operatorname{cosec}^2 x \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

Hence, $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ **Q.E.D.**

THEOREM 10 *The differentiation of* sec x *with respect to x is* sec x tan x.

i.e. $\frac{d}{dx}(\sec x) = \sec x \tan x$

PROOF Let $f(x) = \sec x$. Then, $f(x+h) = \sec(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\cos x - \cos(x+h)}{h \cos x \cos(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x+h-x}{2}\right)}{h \cos x \cos(x+h)}$$

$$\left[\because \cos C - \cos D = 2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{D-C}{2}\right)\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right)}{h \cos x \cos(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x \cos(x+h)} \times \lim_{h \to 0} \frac{\sin(h/2)}{(h/2)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{\sin x}{\cos x \cos x} \times 1 = \tan x \sec x. \qquad \left[\because \lim_{h \to 0} \frac{\sin(h/2)}{(h/2)} = 1\right]$$

Hence, $\frac{d}{dx}(\sec x) = \sec x \tan x.$ Q.E.D.

THEOREM 11 *The differentiation of* $\operatorname{cosec} x$ *with respect to* x *is* $-\operatorname{cosec} x \cot x$.

i.e. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

PROOF *Let* $f(x) = \operatorname{cosec} x$. *Then,* $f(x+h) = \operatorname{cosec}(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\operatorname{cosec}(x+h) - \operatorname{cosec} x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{1}{\sin(x+h)} - \frac{1}{\sin x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin x - \sin(x+h)}{h \sin x \sin(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin\left(\frac{x-x-h}{2}\right) \cos\left(\frac{x+x+h}{2}\right)}{h \sin x \sin(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin(-h/2) \cos(x+h/2)}{h \sin x \sin(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h \to 0} \frac{\sin (h/2)}{h/2} \times \lim_{h \to 0} \frac{\cos (x + h/2)}{\sin x \sin (x + h)} \qquad \left[\because \sin\left(-\frac{h}{2}\right) = -\sin\frac{h}{2}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = (-1) \times \frac{\cos x}{\sin x \sin x} = -\cot x \operatorname{cosec} x.$$

Hence, $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$ **Q.E.D.**

The above results can be summarized as under:

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$ (ii) $\frac{d}{dx}(e^x) = e^x$ (iii) $\frac{d}{dx}(a^x) = a^x \log_e a.$

(iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$ (v) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$ (vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$ (viii) $\frac{d}{dx}(\tan x) = \sec^2 x$ (ix) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(x) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (xi) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the derivatives of the following functions from first principles:*

(i) $x^3 - 27$ (ii) $(x-1)(x-2)$ (iii) $\frac{1}{x^2}$ [NCERT]

SOLUTION (i) Let $f(x) = x^3 - 27$. Then, $f(x+h) = (x+h)^3 - 27$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left\{(x+h)^3 - 27\right\} - \left(x^3 - 27\right)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x^3 + 3x^2 h + 3xh^2 + h^3 - 27) - (x^3 - 27)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2 + 3x \times 0 + 0 = 3x^2$$

(ii) Let $f(x) = (x-1)(x-2)$. Then, $f(x+h) = (x+h-1)(x+h-2)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\{(x-1)+h\}\{(x-2)+h\} - (x-1)(x-2)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x-1)(x-2) + h(x-1) + h(x-2) + h^2 - (x-1)(x-2)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{h(x-1) + h(x-2) + h^2}{h} = \lim_{h \to 0} \{(x-1) + (x-2)\} + h = 2x - 3$$

(iii) Let $f(x) = \frac{1}{x^2}$. Then, $f(x+h) = \frac{1}{(x+h)^2}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{h\, x^2 (x+h)^2}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-2hx - h^2}{hx^2 (x+h)^2} = \lim_{h \to 0} \frac{-2x - h}{x^2 (x+h)^2} = \frac{-2x - 0}{x^2 \times x^2} = \frac{-2}{x^3}$$

EXAMPLE 2 *Differentiate the following functions with respect to x from first principles:*

(i) $\sqrt{x}$ (ii) $\sqrt{ax+b}$ (iii) $\frac{1}{x}$ (iv) $\frac{1}{ax+b}$

SOLUTION (i) Let $f(x) = \sqrt{x}$. Then, $f(x+h) = \sqrt{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left(\sqrt{x+h} + \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\therefore \quad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

(ii) Let $f(x) = \sqrt{ax+b}$. Then, $f(x+h) = \sqrt{a(x+h)+b} = \sqrt{(ax+b)+ah}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sqrt{(ax+b)+ah} - \sqrt{ax+b}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sqrt{(ax+b)+ah} - \sqrt{ax+b}}{h} \times \frac{\sqrt{(ax+b)+ah} + \sqrt{ax+b}}{\sqrt{(ax+b)+ah} + \sqrt{ax+b}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(ax+b) + ah - (ax+b)}{h\left\{\sqrt{(ax+b)+ah} + \sqrt{ax+b}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{ah}{h\left\{\sqrt{(ax+b)+ah} + \sqrt{ax+b}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{a}{\sqrt{(ax+b)+ah} + \sqrt{ax+b}} = \frac{a}{\sqrt{ax+b} + \sqrt{ax+b}}$$

Hence, $\frac{d}{dx}(\sqrt{ax+b}) = \frac{a}{2\sqrt{ax+b}}$

(iii) Let $f(x) = \frac{1}{x}$. Then, $f(x+h) = \frac{1}{x+h}$.

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{x+h-x}{h\left\{\sqrt{x+h} + \sqrt{x}\right\}} = \lim_{h \to 0} \frac{x-(x+h)}{hx(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

Hence, $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

(iv) Let $f(x) = \frac{1}{ax+b}$. Then, $f(x+h) = \frac{1}{a(x+h)+b} = \frac{1}{(ax+b)+ah}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{1}{(ax+b)+ah} - \frac{1}{ax+b}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\{ax+b\} - \{(ax+b)+ah\}}{h(ax+b)\{(ax+b)+ah\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-ah}{h(ax+b)\{(ax+b)+ah\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-a}{(ax+b)\{(ax+b)+ah\}} = -\frac{a}{(ax+b)^2}$$

Hence, $\frac{d}{dx}\left(\frac{1}{ax+b}\right) = \frac{-a}{(ax+b)^2}$

EXAMPLE 3 *Differentiate the following functions with respect to x from first principles:*

(i) $\sqrt{2x+3}$ (ii) $\sqrt{4-x}$ (iii) $ax^2 + \frac{b}{x}$ (iv) $\frac{2x+3}{3x+2}$ (v) $x^{-3/2}$

SOLUTION (i) Let $f(x)=\sqrt{2x+3}$. Then, $f(x+h)=\sqrt{2(x+h)+3}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sqrt{2(x+h)+3}-\sqrt{2x+3}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\left\{\sqrt{2(x+h)+3}-\sqrt{2x+3}\right\}\left\{\sqrt{2(x+h)+3}+\sqrt{2x+3}\right\}}{h\left\{\sqrt{2(x+h)+3}+\sqrt{2x+3}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{(2x+2h+3-2x-3)}{h}\times\frac{1}{\left\{\sqrt{2x+2h+3}+\sqrt{2x+3}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{2h}{h}\times\lim_{h\to 0}\frac{1}{\left\{\sqrt{2x+2h+3}+\sqrt{2x+3}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 2\times\frac{1}{\sqrt{2x+3}+\sqrt{2x+3}} = \frac{2}{2(\sqrt{2x+3})} = \frac{1}{\sqrt{2x+3}}$$

(ii) Let $f(x)=\sqrt{4-x}$. Then, $f(x+h) = \sqrt{4-(x+h)}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sqrt{4-(x+h)}-\sqrt{4-x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\left\{\sqrt{4-(x+h)}-\sqrt{4-x}\right\}\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}{h\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{4-(x+h)-(4-x)}{h\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{-h}{h\left\{\sqrt{4-x-h}+\sqrt{4-x}\right\}} = \frac{-1}{2\sqrt{4-x}}$$

(iii) Let $f(x)=ax^2+\dfrac{b}{x}$. Then, $f(x+h) = a(x+h)^2+\dfrac{b}{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\left\{a(x+h)^2+\frac{b}{x+h}\right\}-\left\{ax^2+\frac{b}{x}\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{a\{(x+h)^2 - x^2\} + b\left\{\frac{1}{x+h} - \frac{1}{x}\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{a(2hx + h^2) + b\left\{\frac{x-x-h}{x(x+h)}\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \left\{\frac{a(2hx+h^2)}{h} + \frac{b(-h)}{hx(x+h)}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \left\{a(2x+h) - \frac{b}{x(x+h)}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 2ax - \frac{b}{x^2}$$

(iv) Let $f(x) = \frac{2x+3}{3x+2}$. Then, $f(x+h) = \frac{2(x+h)+3}{3(x+h)+2} = \frac{2x+3+2h}{3x+2+3h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\frac{2x+3+2h}{3x+2+3h} - \frac{2x+3}{3x+2}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{(2x+3+2h)(3x+2) - (2x+3)(3x+2+3h)}{h(3x+2)(3x+2+3h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{(2x+3)(3x+2) + 2h(3x+2) - (2x+3)(3x+2) - 3h(2x+3)}{h(3x+2)(3x+2+3h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{h(6x+4-6x-9)}{h(3x+2)(3x+2+3h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{-5}{(3x+2)(3x+2+3h)} = -\frac{5}{(3x+2)^2}.$$

(v) Let $f(x) = x^{-3/2}$. Then, $f(x+h) = (x+h)^{-3/2}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{(x+h)^{-3/2} - x^{-3/2}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{(x+h)^{-3/2} - x^{-3/2}}{(x+h)-x}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{z\to x} \frac{z^{-3/2} - x^{-3/2}}{z-x}, \text{ where } z = x+h \text{ and } z \to x \text{ as } h \to 0$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = (-3/2)\, x^{-3/2-1} \qquad \left[\text{Using: } \lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\frac{3}{2} x^{-5/2}$$

EXAMPLE 4 *Differentiate the following functions with respect to x from first principles:*

(i) $\sin 2x$ (ii) $\sin 2x$ (iii) $\sin x^2$ (iv) $\sin (x^2+1)$

SOLUTION (i) Let $f(x) = \sin 2x$. Then, $f(x+h) = \sin 2(x+h)$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin 2(x+h) - \sin 2x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin h \cos (2x+h)}{h} \quad \left[\because \sin C - \sin D = 2 \sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 2 \lim_{h \to 0} \frac{\sin h}{h} \times \lim_{h \to 0} \cos (2x+h) = 2(\cos 2x)(1) = 2 \cos 2x$$

$$\therefore \quad \frac{d}{dx}(\sin 2x) = 2 \cos 2x$$

(ii) Let $f(x) = \sin^2 x$. Then, $f(x+h) = \sin^2 (x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin^2 (x+h) - \sin^2 x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin (x+h+x) \sin (x+h-x)}{h}$$

$$[\because \sin^2 A - \sin^2 B = \sin (A+B) \sin (A-B)]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin h}{h} \times \lim_{h \to 0} \sin (2x+h) = 1(\sin 2x) = \sin 2x$$

$$\therefore \quad \frac{d}{dx}(\sin^2 x) = \sin 2x$$

(iii) Let $f(x) = \sin x^2$. Then, $f(x+h) = \sin (x+h)^2$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin (x+h)^2 - \sin x^2}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin\left(\frac{2hx + h^2}{2}\right)\cos\left(\frac{2x^2 + 2hx + h^2}{2}\right)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin\left(\frac{2hx + h^2}{2}\right)}{h\left(\frac{2x+h}{2}\right)}\left(\frac{2x+h}{2}\right) \cos\left(\frac{2x^2 + 2hx + h^2}{2}\right)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\frac{2hx + h^2}{2}\right)}{\left(\frac{2hx + h^2}{2}\right)} \times \lim_{h \to 0} (2x+h) \times \lim_{h \to 0} \cos\left(\frac{2x^2 + 2hx + h^2}{2}\right)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \times \lim_{h \to 0} (2x + h) \times \lim_{h \to 0} \cos\left(\frac{2x^2 + 2h\,x + h^2}{2}\right),$$

where $\theta = \dfrac{2h\,x + h^2}{2}$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = (1) \times (2x) \cos x^2 = 2x \cos x^2$$

$$\therefore \quad \frac{d}{dx}(\sin x^2) = 2x \cos x^2$$

(iv) Let $f(x) = \sin(x^2 + 1)$. Then, $f(x + h) = \sin\left\{(x + h)^2 + 1\right\}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x + h)) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left\{(x + h)^2 + 1\right\} - \sin(x^2 + 1)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin\left(\frac{2h\,x + h^2}{2}\right) \cos\left\{\frac{(x + h)^2 + 1 + x^2 + 1}{2}\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin\left(\frac{2h\,x + h^2}{2}\right)}{h\left(\frac{2h\,x + h^2}{2}\right)} \times \left(\frac{2h\,x + h^2}{2}\right) \times \cos\left\{\frac{(x + h)^2 + 1 + x^2 + 1}{2}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\frac{2hx + h^2}{2}\right)}{\left(\frac{2hx + h^2}{2}\right)} \times (2x + h) \times \cos\left\{\frac{(x + h)^2 + 1 + x^2 + 1}{2}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \times \lim_{h \to 0} (2x + h) \times \lim_{h \to 0} \cos\left\{\frac{(x + h)^2 + 1 + x^2 + 1}{2}\right\},$$

where $\theta = \dfrac{2h\,x + h^2}{2}$. Clearly $\theta \to 0$, as $h \to 0$.

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 1 \times (2x) \times \cos(x^2 + 1) = 2x \cos(x^2 + 1)$$

LEVEL-2

EXAMPLE 5 *Differentiate xe^x from first principles.*

SOLUTION Let $f(x) = xe^x$. Then, $f(x + h) = (x + h)\,e^{(x + h)}$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x + h)\,e^{x + h} - xe^x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(xe^{x+h} - xe^x) + he^{x+h}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \left\{ xe^x \left(\frac{e^h - 1}{h} \right) + e^{x+h} \right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = xe^x \lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) + \lim_{h \to 0} e^{x+h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = xe^x + e^x = (x+1)e^x$$

EXAMPLE 6 *Differentiate the following functions with respect to x from first principles:*
(i) $\tan \sqrt{x}$ (ii) $\cot \sqrt{x}$ (iii) $\sqrt{\sin x}$ (iv) $\sin \sqrt{x}$

SOLUTION (i) Let $f(x) = \tan \sqrt{x}$. Then, $f(x+h) = \tan \sqrt{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\tan \sqrt{x+h} - \tan \sqrt{x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{h \cos \sqrt{x+h} \cos \sqrt{x}} \qquad \left[\because \tan A - \tan B = \frac{\sin(A-B)}{\cos A \cos B}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{(x+h-x) \cos \sqrt{x} \cos \sqrt{x+h}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right) \cos \sqrt{x} \cos \sqrt{x+h}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{\left(\sqrt{x+h} - \sqrt{x}\right)} \cdot \lim_{h \to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right) \cos \sqrt{x+h} \cos \sqrt{x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 1 \times \frac{1}{2\sqrt{x} \cos \sqrt{x} \cos \sqrt{x}} \qquad \left[\because \lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{\left(\sqrt{x+h} - \sqrt{x}\right)} = 1\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{1}{2\sqrt{x}} \sec^2 \sqrt{x}$$

(ii) Let $f(x) = \cot \sqrt{x}$. Then, $f(x+h) = \cot \sqrt{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\cot\sqrt{x+h} - \cot\sqrt{x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-\sin\left(\sqrt{x+h}-\sqrt{x}\right)}{h \sin\sqrt{x+h}\sin\sqrt{x}} \qquad \left[\because \cot A - \cot B = \frac{-\sin(A-B)}{\sin A \sin B}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-\sin\left(\sqrt{x+h}-\sqrt{x}\right)}{\{(x+h)-x\}\sin\sqrt{x+h}\sin\sqrt{x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h \to 0} \frac{\sin\left(\sqrt{x+h}-\sqrt{x}\right)}{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)\sin\sqrt{x+h}\sin\sqrt{x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h \to 0} \frac{\sin\left(\sqrt{x+h}-\sqrt{x}\right)}{\sqrt{x+h}-\sqrt{x}} \times \lim_{h \to 0} \frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)\sin\sqrt{x+h}\sin\sqrt{x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\frac{1}{2\sqrt{x}\sin\sqrt{x}\sin\sqrt{x}} = \frac{-\operatorname{cosec}^2\sqrt{x}}{2\sqrt{x}}$$

(iii) Let $f(x) = \sqrt{\sin x}$. Then, $f(x+h) = \sqrt{\sin(x+h)}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sqrt{\sin(x+h)} - \sqrt{\sin x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}}{h\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2\sin\left\{\frac{h}{2}\right\}\cos\left\{\frac{2x+h}{2}\right\}}{h\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}} \qquad \left[\because \sin C - \sin D = 2\sin\frac{C-D}{2}\cos\frac{C+D}{2}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(\sin h/2)}{(h/2)} \times \lim_{h \to 0} \frac{\cos(x+h/2)}{\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{\cos x}{\sqrt{\sin x} + \sqrt{\sin x}} = \frac{\cos x}{2\sqrt{\sin x}}$$

(iv) Let $f(x) = \sin\sqrt{x}$. Then, $f(x+h) = \sin\sqrt{x+h}$.

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\sqrt{x+h} - \sin\sqrt{x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)\cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)} \times \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)h} \cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)} \times \lim_{h \to 0} \frac{x+h-x}{(\sqrt{x+h}+\sqrt{x})\,h} \times \lim_{h \to 0} \cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \times \lim_{h \to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \times \lim_{h \to 0} \cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right),$$

$$\text{where } \theta = \frac{\sqrt{x+h}-\sqrt{x}}{2}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 1 \times \frac{1}{2\sqrt{x}} \times (\cos\sqrt{x}) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

$$\therefore \quad \frac{d}{dx}(\sin\sqrt{x}) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

EXAMPLE 7 *Differentiate* $x^2 \cos x$ *from first principles.*

SOLUTION Let $f(x) = x^2 \cos x$. Then, $f(x+h) = (x+h)^2 \cos(x+h)$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x+h)^2 \cos(x+h) - x^2 \cos x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x^2 + 2hx + h^2)\cos(x+h) - x^2 \cos x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left\{x^2 \cos(x+h) - x^2 \cos x\right\} + 2hx\cos(x+h) + h^2\cos(x+h)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \left[x^2 \left\{\frac{\cos(x+h) - \cos x}{h}\right\} + 2x\cos(x+h) + h\cos(x+h)\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} -2x^2 \frac{\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} + \lim_{h \to 0} 2x\cos(x+h) + \lim_{h \to 0} h\cos(x+h)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -x^2 \lim_{h \to 0} \frac{\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h/2} + \lim_{h \to 0} 2x\cos(x+h) + \lim_{h \to 0} h\cos(x+h)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -x^2 \sin x + 2x\cos x + 0 \times \cos x = -x^2 \sin x + 2x \cos x.$$

EXAMPLE 8 *Using first principles, prove that* $\frac{d}{dx}\left\{\frac{1}{f(x)}\right\} = -\frac{f'(x)}{\{f(x)\}^2}$.

SOLUTION Let $\phi(x) = \frac{1}{f(x)}$. Then, $\phi(x+h) = \frac{1}{f(x+h)}$

$$\therefore \quad \frac{d}{dx}\{\phi(x)\} = \lim_{h \to 0} \frac{\phi(x+h)-\phi(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = \lim_{h \to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = \lim_{h \to 0} \frac{f(x)-f(x+h)}{hf(x)\,f(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = \lim_{h \to 0} \frac{f(x)-f(x+h)}{h} \times \lim_{h \to 0} \frac{1}{f(x)\,f(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = -\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \times \lim \frac{1}{f(x)\,f(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = -f'(x) \times \frac{1}{f(x)\,f(x)} \qquad \left[\begin{array}{l} \because f(x) \text{ is differentiable} \\ \therefore f(x) \text{ is continuous} \Rightarrow \lim_{h \to 0} f(x+h) = f(x) \end{array}\right]$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = -\frac{f'(x)}{\{f(x)\}^2}$$

EXAMPLE 9 *Find the derivative of* $\sqrt[3]{\sin x}$ *from first principles.*

SOLUTION Let $f(x) = \sqrt[3]{\sin x}$. Then, $f(x+h) = \sqrt[3]{\sin(x+h)}$

$$\therefore \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sqrt[3]{\sin(x+h)} - \sqrt[3]{\sin x}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left\{\sqrt[3]{\sin(x+h)}\right\}^3 - \left\{\sqrt[3]{\sin x}\right\}^3}{h\left[\sin^{2/3}(x+h) + \sin^{1/3}(x+h)\sin^{1/3}x + \sin^{2/3}x\right]}$$

$$\left[\because a-b = \frac{a^3-b^3}{a^2+ab+b^2}\right]$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin(x+h)-\sin x}{h} \times \frac{1}{\sin^{2/3}(x+h) + \sin^{1/3}(x+h)\sin^{1/3}x + \sin^{2/3}x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)}{h} \times \frac{1}{\sin^{2/3}(x+h) + \sin^{1/3}(x+h)\sin^{1/3}x + \sin^{2/3}x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{\frac{h}{2}} \times \lim_{h \to 0} \frac{1}{\sin^{2/3}(x+h)+\sin^{1/3}(x+h)\sin^{1/3}x+\sin^{2/3}x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \cos x \times \frac{1}{\sin^{2/3}x+\sin^{2/3}x+\sin^{1/3}x\sin^{1/3}x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \frac{\cos x}{3\sin^{2/3}x}$$

EXAMPLE 10 *Differentiate* $\log \sin x$ *from first principles.*

SOLUTION Let $f(x) = \log \sin x$. Then, $f(x+h) = \log \sin (x+h)$

$$\therefore \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log \sin (x+h) - \log \sin x}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{\frac{\sin (x+h)}{\sin x}\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{1+\frac{\sin (x+h)}{\sin x}-1\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{1+\frac{\sin (x+h)-\sin x}{\sin x}\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{1+\frac{\sin (x+h)-\sin x}{\sin x}\right\}}{h\left\{\frac{\sin (x+h)-\sin x}{\sin x}\right\}} \times \left\{\frac{\sin (x+h)-\sin x}{\sin x}\right\}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{1+\frac{\sin (x+h)-\sin x}{h}\right\}}{\left\{\frac{\sin (x+h)-\sin x}{h}\right\}} \times \frac{\sin (x+h)-\sin x}{h} \times \frac{1}{\sin x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{1+\frac{\sin (x+h)-\sin x}{h}\right\}}{\left\{\frac{\sin (x+h)-\sin x}{h}\right\}} \times \lim_{h \to 0} \frac{2\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)}{h} \times \frac{1}{\sin x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{1+\frac{\sin (x+h)-\sin x}{h}\right\}}{\left\{\frac{\sin (x+h)-\sin x}{h}\right\}} \times \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{\frac{h}{2}} \times \frac{1}{\sin x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = 1 \times \cos x \times \frac{1}{\sin x} = \cot x.$$

EXAMPLE 11 *Differentiate* $e^{\sqrt{\tan x}}$ *from first principles.*

SOLUTION Let $f(x) = e^{\sqrt{\tan x}}$. Then, $f(x+h) = e^{\sqrt{\tan(x+h)}}$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{e^{\sqrt{\tan(x+h)}} - e^{\sqrt{\tan x}}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} e^{\sqrt{\tan x}} \left\{ \frac{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1}{h} \right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \lim_{h \to 0} \left\{ \frac{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1}{\sqrt{\tan(x+h)} - \sqrt{\tan x}} \times \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h} \right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times \lim_{h \to 0} \left\{ \frac{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1}{\sqrt{\tan(x+h)} - \sqrt{\tan x}} \right\} \times \lim_{h \to 0} \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times 1 \times \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h} \times \frac{1}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times \lim_{h \to 0} \frac{\sin h}{h \cos(x+h) \cos x} \times \frac{1}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times \frac{1}{\cos^2 x} \times \frac{1}{2\sqrt{\tan x}} = \frac{e^{\sqrt{\tan x}}}{2\sqrt{\tan x}} \sec^2 x$$

EXERCISE 30.2

LEVEL-1

Differentiate each of the following from first principles:

1. (i) $\frac{2}{x}$ (ii) $\frac{1}{\sqrt{x}}$ (iii) $\frac{1}{x^3}$

(iv) $\frac{x^2+1}{x}$ [NCERT] (v) $\frac{x^2-1}{x}$ (vi) $\frac{x+1}{x+2}$

(vii) $\frac{x+2}{3x+5}$ (viii) $k\,x^n$ (ix) $\frac{1}{\sqrt{3-x}}$

(x) $x^2 + x + 3$ (xi) $(x+2)^3$ (xii) $x^3 + 4x^2 + 3x + 2$

(xiii) $(x^2+1)(x-5)$ (xiv) $\sqrt{2x^2+1}$ (xv) $\frac{2x+3}{x-2}$

2. (i) e^{-x} (ii) e^{3x} (iii) e^{ax+b}

(iv) $x\,e^x$ (v) $-x$ [NCERT] (vi) $(-x)^{-1}$ [NCERT]

(vii) $\sin(x+1)$ [NCERT] (viii) $\cos\left(x - \frac{\pi}{8}\right)$ [NCERT] (ix) $x \sin x$

(x) $x \cos x$ (xi) $\sin(2x-3)$

LEVEL-2

3. (i) $\sqrt{\sin 2x}$ (ii) $\frac{\sin x}{x}$ (iii) $\frac{\cos x}{x}$
(iv) $x^2 \sin x$ (v) $\sqrt{\sin(3x+1)}$ (vi) $\sin x + \cos x$
(vii) $x^2 e^x$ (viii) e^{x^2+1} (ix) $e^{\sqrt{2x}}$
(x) $e^{\sqrt{ax+b}}$ (xi) $a^{\sqrt{x}}$ (xii) 3^{x^2}

4. (i) $\tan^2 x$ (ii) $\tan(2x+1)$ (iii) $\tan 2x$ (iv) $\sqrt{\tan x}$

5. (i) $\sin\sqrt{2x}$ (ii) $\cos\sqrt{x}$ (iii) $\tan\sqrt{x}$ (iv) $\tan x^2$

ANSWERS

1. (i) $-2x^{-2}$ (ii) $-\frac{1}{2}x^{-3/2}$ (iii) $-3x^{-4}$ (iv) $1-\frac{1}{x^2}$
(v) $1+\frac{1}{x^2}$ (vi) $\frac{1}{(x+2)^2}$ (vii) $\frac{-1}{(3x+5)^2}$ (viii) nkx^{n-1}
(ix) $\frac{1}{2(3-x)^{3/2}}$ (x) $2x+1$ (xi) $3(x+2)^2$ (xii) $3x^2+8x+3$
(xiii) $3x^2-10x+1$ (xiv) $\frac{2x}{\sqrt{2x^2+1}}$ (xv) $\frac{-7}{(x-2)^2}$

2. (i) $-e^{-x}$ (ii) $3e^{3x}$ (iii) $a\,e^{ax+b}$ (iv) $(x+1)e^x$
(v) -1 (vi) $\frac{1}{x^2}$ (vii) $\cos(x+1)$ (viii) $-\sin\left(x-\frac{\pi}{8}\right)$
(ix) $\sin x + x\cos x$ (x) $\cos x - x\sin x$ (xi) $2\cos(2x-3)$

3. (i) $\frac{\cos 2x}{\sqrt{\sin 2x}}$ (ii) $\frac{x\cos x-\sin x}{x^2}$ (iii) $\frac{-x\sin x-\cos x}{x^2}$ (iv) $x^2\cos x + 2x\sin x$
(v) $\frac{3\cos(3x+1)}{2\sqrt{\sin(3x+1)}}$ (vi) $\cos x-\sin x$ (vii) $(x^2+2x)e^x$ (viii) $2x\,e^{x^2+1}$
(ix) $\frac{e^{\sqrt{2x}}}{\sqrt{2x}}$ (x) $\frac{a\,e^{\sqrt{ax+b}}}{2\sqrt{ax+b}}$ (xi) $\frac{1}{2\sqrt{x}}a^{\sqrt{x}}\log_e a$ (xii) $2x\,3^{x^2}\log 3$

4. (i) $2\tan x\sec^2 x$ (ii) $2\sec^2(2x+1)$ (iii) $2\sec^2 2x$ (iv) $\frac{\sec^2 x}{2\sqrt{\tan x}}$

5. (i) $\frac{\cos\sqrt{2x}}{\sqrt{2x}}$ (ii) $-\frac{\sin\sqrt{x}}{2\sqrt{x}}$ (iii) $\frac{\sec^2\sqrt{x}}{2\sqrt{x}}$ (iv) $2x\sec^2 x^2$

HINTS TO NCERT & SELECTED PROBLEMS

1. (iv) Let $f(x)=\frac{x^2+1}{x}=x+\frac{1}{x}$. Then, $f(x+h)=(x+h)+\frac{1}{x+h}$

$$\therefore \quad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad f'(x)=\lim_{h\to 0}\frac{\left\{(x+h)+\frac{1}{x+h}\right\}-\left\{x+\frac{1}{x}\right\}}{h}=\lim_{h\to 0}\frac{\left\{(x+h)-x\right\}+\left\{\frac{1}{x+h}-\frac{1}{x}\right\}}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{h + \frac{x - x - h}{x(x+h)}}{h} = \lim_{h \to 0} \frac{h - \frac{h}{x(x+h)}}{h} = \lim_{h \to 0} 1 - \frac{1}{x(x+h)} = 1 - \frac{1}{x^2}$$

2. (v) Let $f(x) = -x$. Then, $f(x+h) = -(x+h)$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-(x+h) + x}{h} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

(vi) Let $f(x) = -\frac{1}{x}$. Then, $f(x+h) = -\frac{1}{x+h}$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{-\frac{1}{x+h} + \frac{1}{x}}{h} = \lim_{h \to 0} \frac{-x + (x+h)}{x(x+h)h} = \lim_{h \to 0} \frac{h}{hx(x+h)} = \lim_{h \to 0} \frac{1}{x(x+h)} = \frac{1}{x^2}$$

(vii) Let $f(x) = \sin(x+1)$. Then, $f(x+h) = \sin(x+h+1) = \sin\{(x+1)+h\}$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{\sin((x+1)+h) - \sin(x+1)}{h} = \lim_{h \to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left\{(x+1) + \frac{h}{2}\right\}}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)\cos\left\{(x+1) + \frac{h}{2}\right\}}{\left(\frac{h}{2}\right)} = \cos(x+1)$$

(viii) We have, $f(x) = \cos\left(x - \frac{\pi}{8}\right)$

$$\therefore \quad f(x+h) = \cos\left(x + h - \frac{\pi}{8}\right) = \cos\left\{\left(x - \frac{\pi}{8}\right) + h\right\}$$

So, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{\cos\left\{\left(x - \frac{\pi}{8}\right) + h\right\} - \cos\left(x - \frac{\pi}{8}\right)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{-2\sin\left\{\left(x - \frac{\pi}{8}\right) + \frac{h}{2}\right\}\sin\left(\frac{h}{2}\right)}{h}$$

$$\Rightarrow \quad f'(x) = -\lim_{h \to 0} \sin\left\{\left(x - \frac{\pi}{8}\right) + \frac{h}{2}\right\} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = -\sin\left(x - \frac{\pi}{8}\right)$$

30.4 FUNDAMENTAL RULES FOR DIFFERENTIATION

In the previous section, we have used the definition of derivative to find derivatives. This section is mainly devoted to develop several rules that allow us to find derivatives without using definition directly.

THEOREM 1 *Differentiation of a constant function is zero i.e.,* $\frac{d}{dx}(c) = 0$.

PROOF Let $f(x) = c$ be a constant function. Then,

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h} = 0.$$

Hence, $\frac{d}{dx}(c) = 0$, *where c is a constant.*

REMARK *Geometrically the graph of a constant function is a straight line parallel is x-axis. So, tangent at every point is parallel to x-axis. Consequently, the slope of the tangent at every point is zero, i.e.* $\frac{dy}{dx} = 0$.

THEOREM 2 *Let* $f(x)$ *be a differentiable function and let c be a constant. Then,* $c\,f(x)$ *is also differentiable such that* $\frac{d}{dx}\{c\,f(x)\} = c\frac{d}{dx}(f(x))$.

i.e. the derivative of a constant times a function is the constant times the derivative of the function.

PROOF Since $f(x)$ is differentiable. Therefore,

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists finitely and is equal to $\frac{d}{dx}(f(x))$.

i.e. $$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad ...(i)$$

Let $g(x) = c\,f(x)$. Then,

$$\Rightarrow \quad \frac{d}{dx}(g(x)) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(g(x)) = \lim_{h \to 0} \frac{c\,f(x+h) - c\,f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(g(x)) = c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c\frac{d}{dx}(f(x)) \qquad \text{[Using (i)]}$$

Hence, $g(x) = c\,f(x)$ is differentiable such that $\frac{d}{dx}(c\,f(x)) = c\frac{d}{dx}(f(x))$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Differentiate the following functions with respect to x:*

(i) $\log_x x$ (ii) $e^{3 \log x}$ (iii) $2^{\log_2 x}$ (iv) $5(2^{3 \log_2 x})$ (v) $5e^x$ (vi) $9(3^x)$

SOLUTION (i) We know that $\log_x x = 1$.

$$\therefore \quad \frac{d}{dx}(\log_x x) = \frac{d}{dx}(1) = 0.$$

(ii) We know that $e^{\log k} = k$.

$$\therefore \quad \frac{d}{dx}(e^{3 \log x}) = \frac{d}{dx}(e^{\log x^3}) = \frac{d}{dx}(x^3) = 3x^2$$

(iii) We know that $a^{\log_a n} = n$

$$\therefore \quad \frac{d}{dx}(2^{\log_2 x}) = \frac{d}{dx}(x) = 1.$$

(iv) Clearly,

$$\frac{d}{dx}(5\cdot 2^{3\log_2 x}) = 5\frac{d}{dx}(2^{3\log_2 x}) = 5\frac{d}{dx}(2^{\log_2 x^3}) = 5\frac{d}{dx}(x^3) \qquad [\because a^{\log_a n} = n]$$

$$= 5(3x^2) = 15x^2$$

(v) Clearly,

$$\frac{d}{dx}(5e^x) = 5\frac{d}{dx}(e^x) = 5e^x$$

(vi) Clearly,

$$\frac{d}{dx}(9.3^x) = 9\frac{d}{dx}(3^x) = 9(3^x \log_e 3).$$

THEOREM 3 *If $f(x)$ and $g(x)$ are differentiable functions, then show that $f(x) \pm g(x)$ are also differentiable such that*

$$\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$$

i.e. the derivative of the sum or difference of two functions is the sum or difference of their derivatives.

PROOF Since $f(x)$ and $g(x)$ both are differentiable functions. Therefore,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \text{ and } \frac{d}{dx}(g(x)) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \text{ both exist.} \qquad ...(i)$$

Now,

$$\frac{d}{dx}\left\{f(x)+g(x)\right\} = \lim_{h\to 0}\frac{\{f(x+h)\pm g(x+h)\} - \{f(x)\pm g(x)\}}{h}$$

$$= \lim_{h\to 0}\frac{\{f(x+h)-f(x)\} \pm \{g(x+h)-g(x)\}}{h}$$

$$= \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\} \pm \lim_{h\to 0}\left\{\frac{g(x+h)-g(x)}{h}\right\}$$

$$= \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\} \qquad \text{[Using (i)]}$$

Hence, $f(x) \pm g(x)$ is differentiable and $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$.

REMARK *The above result can be extended to a finite number of differentiable functions. Thus, we have*

$$\frac{d}{dx}\{f_1(x) \pm f_2(x) + ... \pm f_n(x)\} = \frac{d}{dx}\{f_1(x)\} \pm \frac{d}{dx}\{f_2(x)\} \pm ... \pm \frac{d}{dx}\{f_n(x)\}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Differentiate the following functions with respect to x.*

(i) $x^2 + \sin x + \frac{1}{x^2}$ (ii) $\frac{ax^2+bx+c}{\sqrt{x}}$ (iii) $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$

SOLUTION (i) Clearly,

$$\frac{d}{dx}\left\{x^2 + \sin x + \frac{1}{x^2}\right\}$$

$$= \frac{d}{dx}(x^2 + \sin x + x^{-2})$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin x) + \frac{d}{dx}(x^{-2}) = 2x + \cos x + (-2)x^{-3} = 2x + \cos x - \frac{2}{x^3}$$

(ii) Clearly,

$$\frac{d}{dx}\left\{\frac{ax^2+bx+c}{\sqrt{x}}\right\}$$

$$=\frac{d}{dx}\left\{\frac{ax^2}{\sqrt{x}}+\frac{bx}{\sqrt{x}}+\frac{c}{\sqrt{x}}\right\}$$

$$=\frac{d}{dx}\left\{ax^{3/2}+bx^{1/2}+cx^{-1/2}\right\}$$

$$=\frac{d}{dx}\left\{ax^{3/2}\right\}+\frac{d}{dx}\left\{bx^{1/2}\right\}+\frac{d}{dx}\left\{cx^{-1/2}\right\}$$

$$=a\frac{d}{dx}\left\{x^{3/2}\right\}+b\frac{d}{dx}\left\{x^{1/2}\right\}+c\frac{d}{dx}\left\{x^{-1/2}\right\}$$

$$=a\left(\frac{3}{2}x^{1/2}\right)+b\left(\frac{1}{2}x^{-1/2}\right)+c\left(-\frac{1}{2}x^{-3/2}\right)=\frac{3a}{2}x^{1/2}+\frac{b}{2}x^{-1/2}-\frac{c}{2}x^{-3/2}$$

(iii) We have,

$$\frac{d}{dx}\left\{\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\right\}$$

$$=\frac{d}{dx}\left\{x+\frac{1}{x}+2\right\}=\frac{d}{dx}(x)+\frac{d}{dx}(x^{-1})+\frac{d}{dx}(2)=1+(-1)x^{-2}+0=1-\frac{1}{x^2}$$

EXAMPLE 2 *If* $y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$, *show that* $\frac{dy}{dx}=y$.

SOLUTION We have,

$$y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$\therefore \quad \frac{dy}{dx}=\frac{d}{dx}(1)+\frac{d}{dx}\left(\frac{x}{1!}\right)+\frac{d}{dx}\left(\frac{x^2}{2!}\right)+\frac{d}{dx}\left(\frac{x^3}{3!}\right)+\ldots$$

or, $$\frac{dy}{dx}=\frac{d}{dx}(1)+\frac{1}{1!}\frac{d}{dx}(x)+\frac{1}{2!}\frac{d}{dx}(x^2)+\frac{1}{3!}\frac{d}{dx}(x^3)+\ldots$$

or, $$\frac{dy}{dx}=0+1+\frac{1}{2!}(2x)+\frac{1}{3!}(3x^2)+\ldots$$

or, $$\frac{dy}{dx}=1+x+\frac{x^2}{2!}+\ldots$$

or, $$\frac{dy}{dx}=y.$$

ALITER We have,

$$y=e^x \Rightarrow \frac{dy}{dx}=\frac{d}{dx}(e^x)=e^x=y.$$

EXAMPLE 3 *If* $y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}$, *show that* $\frac{dy}{dx}-y+\frac{x^n}{n!}=0$.

SOLUTION We have,

$$y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(1) + \frac{d}{dx}\left(\frac{x}{1!}\right) + \frac{d}{dx}\left(\frac{x^2}{2!}\right) + \frac{d}{dx}\left(\frac{x^3}{3!}\right) + \dots + \frac{d}{dx}\left(\frac{x^n}{n!}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}(1) + \frac{1}{1!}\frac{d}{dx}(x) + \frac{1}{2!}\frac{d}{dx}(x^2) + \frac{1}{3!}\frac{d}{dx}(x^3) + \dots + \frac{1}{n!}\frac{d}{dx}(x^n)$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 + \frac{1}{1!} + \frac{1}{2!}(2x) + \frac{1}{3!}(3x^2) + \dots + \frac{1}{n!}(n\,x^{n-1})$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!}.$$

$$\Rightarrow \quad \frac{dy}{dx} = \left\{1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}\right\} - \frac{x^n}{n!}$$

$$\Rightarrow \quad \frac{dy}{dx} = y - \frac{x^n}{n!}$$

$$\Rightarrow \quad \frac{dy}{dx} - y + \frac{x^n}{n!} = 0.$$

EXAMPLE 4 *If* $y = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$, $x \in \left(0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right)$, *then find* $\dfrac{dy}{dx}$.

SOLUTION We have,

$$y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\frac{2\sin^2 x}{2\cos^2 x}} = \sqrt{\tan^2 x}$$

$$\Rightarrow \quad y = |\tan x|, \text{ where } x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$\Rightarrow \quad y = \begin{cases} \tan x, & x \in \left(0, \frac{\pi}{2}\right) \\ -\tan x, & x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

$$\Rightarrow \quad \frac{dy}{dx} = \begin{cases} \sec^2 x, & \text{if } x \in \left(0, \frac{\pi}{2}\right) \\ -\sec^2 x, & \text{if } x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

EXAMPLE 5 *Differentiate the following functions with respect to x:*

(i) $(x^2 - 3x + 2)(x + 2)$ (ii) $\left(x^2 + \dfrac{1}{x^2}\right)^3$

SOLUTION (i) Clearly,

$$\frac{d}{dx}\left\{(x^2 - 3x + 2)(x + 2)\right\}$$

$$= \frac{d}{dx}(x^3 - x^2 - 4x + 4)$$

$$= \frac{d}{dx}(x^3) - \frac{d}{dx}(x^2) - \frac{d}{dx}(4x) + \frac{d}{dx}(4)$$

$$= \frac{d}{dx}(x^3) - \frac{d}{dx}(x^2) - 4\frac{d}{dx}(x) + \frac{d}{dx}(4) = 3x^2 - 2x - 4 + 0 = 3x^2 - 2x - 4$$

(ii) Clearly,

$$\frac{d}{dx}\left\{\left(x^2 + \frac{1}{x^2}\right)^3\right\}$$

$$= \frac{d}{dx}\left\{x^6 + 3x^2 + \frac{3}{x^2} + \frac{1}{x^6}\right\}$$

$$= \frac{d}{dx}(x^6) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(3x^{-2}) + \frac{d}{dx}(x^{-6})$$

$$= \frac{d}{dx}(x^6) + 3\frac{d}{dx}(x^2) + 3\frac{d}{dx}(x^{-2}) + \frac{d}{dx}(x^{-6})$$

$$= 6x^5 + 6x + 3(-2)x^{-3} + (-6)x^{-7} = 6x^5 + 6x - \frac{6}{x^3} - \frac{6}{x^7}$$

EXAMPLE 6 *If $f(x) = \alpha x^n$, prove that $\alpha = \frac{f'(1)}{n}$.*

SOLUTION We have, $f(x) = \alpha x^n$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(\alpha x^n)$$

$$\Rightarrow \quad f'(x) = \alpha \frac{d}{dx}(x^n)$$

$$\Rightarrow \quad f'(x) = \alpha n x^{n-1}$$

Putting $x = 1$ on both sides, we get

$$f'(1) = \alpha n \Rightarrow \alpha = \frac{f'(1)}{n}$$

EXAMPLE 7 *If $f(x) = x^n$ and if $f'(1) = 10$, find the value of n.*

SOLUTION We have, $f(x) = x^n$.

Differentiating both sides with respect to x, we get $f'(x) = nx^{n-1}$.

Putting $x = 1$, we get

$$f'(1) = n \Rightarrow 10 = n \qquad [\because f'(1) = 10]$$

EXAMPLE 8 *If $f(x) = mx + c$ and $f(0) = f'(0) = 1$. What is $f(2)$?*

SOLUTION We have,

$$f(x) = mx + c \qquad \text{...(i)}$$

Differentiating with respect to we get

$$f'(x) = m \cdot 1 + 0 \Rightarrow f'(x) = m \qquad \text{...(ii)}$$

Putting $x = 0$ in (i) and (ii), we get

$f(0) = c$ and $f'(0) = m$

$$\Rightarrow \quad 1 = c \text{ and } 1 = m \qquad [\because f(0) = f'(0) = 1]$$

Putting the values of m and c in $f(x) = mx + c$, we get $f(x) = x + 1$.

$$\therefore \quad f(2) = 2 + 1 = 3. \qquad [\text{Putting } x = 2 \text{ in } f(x) = x + 1]$$

EXAMPLE 9 *Find* $\frac{dy}{dx}$, *when* $y = 3\tan x + 5\log_a x + \sqrt{x} - 3e^x + \frac{1}{x}$.

SOLUTION We have,

$$y = 3\tan x + 5\log_a x + \sqrt{x} - 3e^x + \frac{1}{x}$$

$$\Rightarrow \quad y = 3\tan x + 5\log_a x + x^{1/2} - 3e^x + x^{-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}(3\tan x) + \frac{d}{dx}(5\log_a x) + \frac{d}{dx}(x^{1/2}) - \frac{d}{dx}(3e^x) + \frac{d}{dx}(x^{-1})$$

$$\Rightarrow \quad \frac{dy}{dx} = 3\frac{d}{dx}(\tan x) + 5\frac{d}{dx}(\log_a x) + \frac{d}{dx}(x^{1/2}) - 3\frac{d}{dx}(e^x) + \frac{d}{dx}(x^{-1})$$

$$\Rightarrow \quad \frac{dy}{dx} = 3\sec^2 x + \frac{5}{x\log_e a} + \frac{1}{2}x^{-1/2} - 3e^x + (-1)x^{-2}.$$

EXAMPLE 10 *Differentiate the following functions with respect to x:*

(i) $\sin(x+a)$ [NCERT] (ii) $\frac{\sin(x+a)}{\cos x}$ [NCERT]

SOLUTION (i) Clearly,

$$\frac{d}{dx}\{\sin(x+a)\}$$

$$= \frac{d}{dx}\{\sin x\cos a + \cos x\sin a\}$$

$$= \frac{d}{dx}(\sin x\cos a) + \frac{d}{dx}(\cos x\sin a)$$

$$= \cos a\frac{d}{dx}(\sin x) + \sin a\frac{d}{dx}(\cos x)$$

$$= \cos a\cos x + \sin a(-\sin x) = \cos x\cos a - \sin x\sin a = \cos(x+a)$$

(ii) Clearly,

$$\frac{d}{dx}\left\{\frac{\sin(x+a)}{\cos x}\right\}$$

$$= \frac{d}{dx}\left\{\frac{\sin x\cos a + \cos x\sin a}{\cos x}\right\}$$

$$= \frac{d}{dx}\{\tan x\cos x + \sin a\}$$

$$= \frac{d}{dx}(\tan x\cos a) + \frac{d}{dx}(\sin a)$$

$$= \cos a\frac{d}{dx}(\tan x) + \frac{d}{dx}(\sin a) = \cos a \times \sec^2 x + 0 = \sec^2 x\cos a$$

EXERCISE 30.3

LEVEL-1

Differentiate the following functions with respect to x: (1-18)

1. $x^4 - 2\sin x + 3\cos x$
2. $3^x + x^3 + 3^3$
3. $\frac{x^3}{3} - 2\sqrt{x} + \frac{5}{x^2}$
4. $e^{x\log a} + e^{a\log x} + e^{a\log a}$
5. $(2x^2+1)(3x+2)$
6. $\log_3 x + 3\log_e x + 2\tan x$

7. $\left(x+\frac{1}{x}\right)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$
8. $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^3$
9. $\frac{2x^2+3x+4}{x}$
10. $\frac{(x^3+1)(x-2)}{x^2}$
11. $\frac{a\cos x+b\sin x+c}{\sin x}$
12. $2\sec x+3\cot x-4\tan x$
13. $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n.$
14. $\frac{1}{\sin x}+2^{x+3}+\frac{4}{\log_x 3}$
15. $\frac{(x+5)(2x^2-1)}{x}$
16. $\log\left(\frac{1}{\sqrt{x}}\right)+5x^a-3a^x+3\sqrt{x^2}+6\,4\sqrt{x^{-3}}$
17. $\cos(x+a)$
18. $\frac{\cos(x-2)}{\sin x}$
19. If $y=\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2$, find $\frac{dy}{dx}$ at $x=\frac{\pi}{6}$.
20. If $y=\left(\frac{2-3\cos x}{\sin x}\right)$, find $\frac{dy}{dx}$ at $x=\frac{\pi}{4}$
21. Find the slope of the tangent to the curve $f(x)=2x^6+x^4-1$ at $x=1$.
22. If $y=\sqrt{\frac{x}{a}}+\sqrt{\frac{a}{x}}$, prove that $2xy\frac{dy}{dx}=\left(\frac{x}{a}-\frac{a}{x}\right)$
23. Find the rate at which the function $f(x)=x^4-2x^3+3x^2+x+5$ changes with respect to x.
24. If $y=\frac{2x^9}{3}-\frac{5}{7}x^7+6x^3-x$, find $\frac{dy}{dx}$ at $x=1$.
25. If for $f(x)=\lambda x^2+\mu x+12$, $f'(4)=15$ and $f'(2)=11$, then find λ and μ.
26. For the function $f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$. Prove that $f'(1)=100f'(0)$. [NCERT]

ANSWERS

1. $4x^3-2\cos x-3\sin x.$
2. $3^x\log 3+3x^2$
3. $x^2-x^{-1/2}-10x^{-3}$
4. $a^x\log a+ax^{a-1}$
5. $18x^2+8x+3$
6. $\frac{1}{x\log 3}+\frac{3}{x}+2\sec^2 x$
7. $\frac{3}{2}x^{1/2}+\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{-3/2}-\frac{3}{2}x^{-5/2}$
8. $\frac{3}{2}x^{1/2}-\frac{3}{2}x^{-5/2}+\frac{3}{2}x^{-1/2}-\frac{3}{2}x^{-3/2}$
9. $2-\frac{4}{x^2}$
10. $2x-2-\frac{1}{x^2}+\frac{4}{x^3}$
11. $-a\operatorname{cosec}^2 x-c\operatorname{cosec} x\cot x.$
12. $2\sec x\tan x-3\operatorname{cosec}^2 x-4\sec^2 x$
13. $na_0x^{n-1}+(n-1)a_1x^{n-2}+\ldots+a_{n-1}$
14. $-\operatorname{cosec} x\cot x+2^{x+3}\log 2+\frac{4}{x\log 3}$
15. $4x+10+\frac{5}{x^2}$
16. $-\frac{1}{2x}+5ax^{a-1}-3a^x\log a+\frac{2}{3}x^{-1/3}-\frac{9}{2}x^{-7/4}$
17. $-\sin(x+a)$
18. $-\operatorname{cosec}^2 x\cos 2$
19. $\frac{\sqrt{3}}{2}$
20. $6-2\sqrt{2}$
21. 16
23. $4x^3-6x^2+6x+1$
24. 18
25. $\lambda=1$, $\mu=7$

HINTS TO NCERT & SELECTED PROBLEMS

26. We have,

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + ... + \frac{x^2}{2} + x + 1$$

$$\Rightarrow \quad f'(x) = x^{99} + x^{98} + ... + x + 1$$

$$\therefore \quad f'(1) = 1^{99} + 1^{98} + ... + 1^1 + 1 = 100 \text{ and } f'(0) = 1.$$

30.4.1 PRODUCT RULE FOR DIFFERENTIATION

THEOREM 1 *If $f(x)$ and $g(x)$ are two differentiable functions, show that $f(x)\,g(x)$ is also differentiable such that*

$$\frac{d}{dx}[f(x)\,g(x)] = f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}$$

i.e. Derivative of the product of two functions

= [*(First function)* × *(Derivative of 2nd function)* + *(Second function)* × *(Derivative of first function)*]

<u>PROOF</u> Since $f(x)$ and $g(x)$ are differentiable functions. Therefore,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \text{ and } \frac{d}{dx}(g(x)) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \qquad ...(i)$$

Let $\phi(x) = f(x)\,g(x)$. Then,

$$\frac{d}{dx}(\phi(x)) = \lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = \lim_{h\to 0}\frac{f(x+h)\,g(x+h)-f(x)\,g(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = \lim_{h\to 0}\frac{f(x+h)\,g(x+h)-f(x+h)\,g(x)+f(x+h)\,g(x)-f(x)\,g(x)}{h}$$

[Adding and subtracting $f(x+h)\,g(x)$ in numerator]

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = \lim_{h\to 0}\frac{f(x+h)\{g(x+h)-g(x)\}+g(x)\{f(x+h)-f(x)\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = \lim_{h\to 0}\frac{f(x+h)\{g(x+h)-g(x)\}}{h} + \lim_{h\to 0}\frac{g(x)\{f(x+h)-f(x)\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}$$

$$\left[\begin{array}{l}\because f(x) \text{ is differentiable.} \\ \therefore \text{ It is continuous, and hence } \lim_{h\to 0} f(x+h) = f(x)\end{array}\right]$$

Hence, $\phi(x) = f(x)\,g(x)$ is differentiable and $\frac{d}{dx}\{f(x)\,g(x)\} = f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}$.

<u>REMARK</u> The above result may also be expressed as

$$(fg)' = f'g + fg' \quad \text{or,} \quad (fg)' = (fg)\left(\frac{f'}{f} + \frac{g'}{g}\right)$$

It can be generalized for the derivative of the product of more than two functions as given below

$$(fgh)' = (fgh)\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)$$

THEOREM 2 *(Generalization of the product rule) Let $f(x), g(x), h(x)$ be three differentiable functions. Then,*

$$\frac{d}{dx}\{f(x)\, g(x)\, h(x)\}$$
$$= \left\{\frac{d}{dx}(f(x))\right\} g(x)\, h(x) + f(x)\left\{\frac{d}{dx}(g(x))\right\} h(x) + f(x)\, g(x)\left\{\frac{d}{dx}(h(x))\right\}$$

PROOF We have,

$$\frac{d}{dx}\{f(x)\, g(x)\, h(x)\} = \frac{d}{dx}[\{f(x)\, g(x)\}\, h(x)]$$
$$= \{f(x)\, g(x)\}\frac{d}{dx}\{h(x)\} + h(x)\frac{d}{dx}\{f(x)\, g(x)\}$$
$$= \{f(x)\, g(x)\}\frac{d}{dx}\{h(x)\} + h(x)\left[f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}\right]$$
$$= \left\{\frac{d}{dx}(f(x))\right\} g(x)\, h(x) + f(x)\left\{\frac{d}{dx}(g(x))\right\} h(x) + f(x)\, g(x)\left\{\frac{d}{dx}(h(x))\right\}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Differentiate the following functions with respect to x:*

(i) $x \sin x$ (ii) $\dfrac{x^3 \sin x}{\cos x}$

(iii) $e^x \sin x + x^n \cos x$ (iv) $e^x (x + \log x)$

(v) $(x + \sec x)(x - \tan x)$ [NCERT] (vi) $(x + \cos x)(x - \tan x)$ [NCERT]

(vii) $(x^2 + 1)\cos x$ [NCERT] (viii) $(ax^2 + \sin x)(p + q \cos x)$ [NCERT]

SOLUTION (i) $\frac{d}{dx}(x \sin x)$

$$= x\frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x) = x\cos x + \sin x \cdot 1 = x \cos x + \sin x.$$

(ii) We have,

$$\frac{d}{dx}\left(x^3 \frac{\sin x}{\cos x}\right)$$
$$= \frac{d}{dx}(x^3 \tan x)$$
$$= x^3 \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(x^3) = x^3 \sec^2 x + (\tan x)\, 3x^2 = x^3 \sec^2 x + 3x^2 \tan x.$$

(iii) We have,

$$\frac{d}{dx}(e^x \sin x + x^n \cos x)$$
$$= \frac{d}{dx}(e^x \sin x) + \frac{d}{dx}(x^n \cos x)$$

$$= e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x) + x^n \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^n)$$

$$= e^x \cos x + (\sin x)\, e^x + x^n(-\sin x) + \cos x\,(nx^{n-1})$$

$$= e^x \cos x + e^x \sin x - x^n \sin x + nx^{n-1} \cos x$$

(iv) We have,

$$\frac{d}{dx}\{e^x (x + \log x)\}$$

$$= e^x \frac{d}{dx}(x + \log x) + (x + \log x)\frac{d}{dx}(e^x)$$

$$= e^x \left\{ \frac{d}{dx}(x) + \frac{d}{dx}(\log x) \right\} + (x + \log x)\, e^x$$

$$= e^x \left\{1 + \frac{1}{x}\right\} + (x + \log x)\, e^x = e^x \left(1 + \frac{1}{x} + x + \log x\right)$$

(v) We have,

$$\frac{d}{dx} \{(x + \sec x)(x - \tan x)\}$$

$$= \frac{d}{dx}(x + \sec x) \times (x - \tan x) + (x + \sec x)\frac{d}{dx}(x - \tan x)$$

$$= \left\{\frac{d}{dx}(x) + \frac{d}{dx}(\sec x)\right\}(x - \tan x) + (x + \sec x)\left\{\frac{d}{dx}(x) - \frac{d}{dx}(\tan x)\right\}$$

$$= (1 + \sec x \tan x)(x - \tan x) + (x + \sec x)(1 - \sec^2 x)$$

(vi) We have,

$$\frac{d}{dx}\left\{(x + \cos x)\ (x - \tan x)\right\}$$

$$= (x - \tan x)\frac{d}{dx}(x + \cos x) + (x + \cos x)\frac{d}{dx}(x - \tan x)$$

$$= (x - \tan x)\left\{\frac{d}{dx}(x) + \frac{d}{dx}(\cos x)\right\} + (x + \cos x)\left\{\frac{d}{dx}(x) - \frac{d}{dx}(\tan x)\right\}$$

$$= (x - \tan x)(1 - \sin x) + (x + \cos x)(1 - \sec^2 x)$$

(vii) We have,

$$\frac{d}{dx}\left\{(x^2 + 1)\cos x\right\}$$

$$= \cos x \frac{d}{dx}(x^2 + 1) + (x^2 + 1)\frac{d}{dx}(\cos x)$$

$$= (2x + 0)\cos x + (x^2 + 1)(-\sin x)$$

$$= 2x \cos x + (x^2 + 1)(-\sin x) = 2x \cos x - (x^2 + 1)\sin x$$

(viii) We have,

$$\frac{d}{dx}\left\{(ax^2 + \sin x)(p + q\cos x)\right\}$$

$$= (p + q\cos x)\frac{d}{dx}(ax^2 + \sin x) + (ax^2 + \sin x)\frac{d}{dx}(p + q\cos x)$$

$$=(p+q\cos x)\left\{\frac{d}{dx}(ax^2)+\frac{d}{dx}(\sin x)\right\}+(ax^2+\sin x)\left\{\frac{d}{dx}(p)+\frac{d}{dx}(q\cos x)\right\}$$

$$=(p+q\cos x)\{a(2x)+\cos x\}+(ax^2+\sin x)(0-q\sin x)$$

$$=(p+q\cos x)(2ax+\cos x)-(ax^2+\sin x)\,q\sin x.$$

EXAMPLE 2 *Differentiate the following functions with respect to x:*

(i) $x^3 e^x \sin x$ (ii) $x\sin x\log x$ (iii) $x^n \log_a x\, e^x$

SOLUTION (i) We have,

$$\frac{d}{dx}(x^3 e^x \sin x)$$

$$=\left\{\frac{d}{dx}(x^3)\right\}e^x\sin x+x^3\left\{\frac{d}{dx}(e^x)\right\}\sin x+x^3e^x\left\{\frac{d}{dx}(\sin x)\right\}$$

$$=3x^2e^x\sin x+x^3e^x\sin x+x^3e^x\cos x = x^2e^x(3\sin x+x\sin x+x\cos x)$$

(ii) We have,

$$\frac{d}{dx}(x\sin x\log x)$$

$$=\left\{\frac{d}{dx}(x)\right\}\sin x\log x+x\frac{d}{dx}(\sin x)\log x+x\sin x\frac{d}{dx}(\log x)$$

$$=1(\sin x)\cdot\log x+x\cdot(\cos x)\cdot\log x+x\cdot(\sin x)\frac{1}{x}=\sin x\log x+x\cos x\cdot\log x+\sin x.$$

(iii) We have,

$$\frac{d}{dx}\left(x^n\cdot\log_a x\cdot e^x\right)$$

$$=\left\{\frac{d}{dx}(x^n)\right\}(\log_a x)e^x+x^n\left\{\frac{d}{dx}(\log_a x)\right\}e^x+x^n\log_a x\left\{\frac{d}{dx}(e^x)\right\}$$

$$=n\,x^{n-1}\log_a x\,e^x+x^n\left(\frac{1}{x\log_e a}\right)e^x+x^n(\log_a x)\,e^x$$

$$=e^x x^{n-1}\left\{n\log_a x+\frac{1}{\log_e a}+x\log_a x\right\}$$

EXAMPLE 3 *Using mathematical induction prove that:* $\frac{d}{dx}(x^n)=nx^{n-1}$ *for all* $n\in N$

SOLUTION Let $P(n)$ be the statement given by

$$P(n):\frac{d}{dx}(x^n)=nx^{n-1}$$

STEP I *We have,*

$\because$ $$\frac{d}{dx}(x^1)=\frac{d}{dx}(x)=1=1\times x^{1-1}$$

$\therefore$ $P(1)$ is true.

STEP II *Let the statement be true for n = m. Then,*

$$\frac{d}{dx}(x^m)=mx^{m-1} \qquad ...(i)$$

Now, $\frac{d}{dx}(x^{m+1}) = \frac{d}{dx}(x \times x^m)$

$\Rightarrow \quad \frac{d}{dx}(x^{m+1}) = x^m \frac{d}{dx}(x) + x\frac{d}{dx}(x^m)$ [Using product rule]

$\Rightarrow \quad \frac{d}{dx}(x^{m+1}) = x^m + x \times mx^{m-1}$ [Using (i)]

$\Rightarrow \quad \frac{d}{dx}(x^{m+1}) = (m+1)\, x^m$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$

i.e. $\frac{d}{dx}(x^n) = nx^{n-1}$ for all $n \in N$.

EXERCISE 30.4

LEVEL-1

Differentiate the following functions with respect to x: (1-22)

1. $x^3 \sin x$
2. $x^3 e^x$
3. $x^2 e^x \log x$
4. $x^n \tan x$
5. $x^n \log_a x$
6. $(x^3 + x^2 + 1) \sin x$
7. $\sin x \cos x$
8. $\frac{2^x \cot x}{\sqrt{x}}$
9. $x^2 \sin x \log x$
10. $x^5 e^x + x^6 \log x$
11. $(x \sin x + \cos x)(x \cos x - \sin x)$
12. $(x \sin x + \cos x)(e^x + x^2 \log x)$
13. $(1 - 2\tan x)(5 + 4\sin x)$
14. $(1 + x^2)\cos x$
15. $\sin^2 x$
16. $\log_{x^2} x$
17. $e^x \log \sqrt{x} \tan x$
18. $x^3 e^x \cos x$
19. $\frac{x^2 \cos \frac{\pi}{4}}{\sin x}$ [NCERT]
20. $x^4 (5 \sin x - 3 \cos x)$ [NCERT]
21. $(2x^2 - 3)\sin x$
22. $x^5 (3 - 6x^{-9})$
23. $x^{-4}(3 - 4x^{-5})$
24. $x^{-3}(5 + 3x)$
25. $(ax + b)/(cx + d)$
26. $(ax + b)^n (cx + d)^m$
27. Differentiate in two ways, using product rule and otherwise, the function $(1 + 2\tan x)(5 + 4\cos x)$. Verify that the answers are the same.
28. Differentiate each of the following functions by the product rule and the other method and verify that answer from both the methods is the same.

    (i) $(3x^2 + 2)^2$ (ii) $(x + 2)(x + 3)$ (iii) $(3\sec x - 4\,\text{cosec}\, x)(-2\sin x + 5\cos x)$

ANSWERS

1. $x^2(x\cos x + 3\sin x)$ 2. $x^2 e^x (3 + x)$ 3. $xe^x(1 + x\log x + 2\log x)$

4. $x^{n-1}(n\tan x + x\sec^2 x)$ 5. $x^{n-1}\left(n\log_a x + \frac{1}{\log a}\right)$

6. $(x^3 + x^2 + 1)\cos x + (3x^2 + 2x)\sin x$ 7. $\cos 2x$

8. $\frac{2^x}{\sqrt{x}}\left\{\log 2 \cdot \cot x - \operatorname{cosec}^2 x - \frac{\cot x}{2x}\right\}$ 9. $2x \sin x \log x + x^2 \cos x \cdot \log x + x \sin x$

10. $x^4 (5e^x + xe^x + x + 6x \log x)$ 11. $x\{x \cos 2x - \sin 2x\}$

12. $x \cos x \{e^x + x^2 \log x\} + (x \sin x + \cos x)(e^x + x + 2x \log x)$

13. $4(\cos x - 2 \sin x - 2 \tan x \sec x - 5/2 \sec^2 x)$

14. $2x \cos x - (1 + x^2)\sin x$ 15. $\sin 2x$ 16. 0

17. $\frac{1}{2} e^x \left\{\log x . \tan x + \frac{\tan x}{x} + \log x . \sec^2 x\right\}$

18. $x^2 e^x (x \cos x + 3 \cos x - x \sin x)$ 19. $\left(\frac{2x}{\sin x} - \frac{x^2 \cot x}{\sin x}\right)\cos\frac{\pi}{4}$

20. $20x^3 \sin x + 5x^4 \cos x - 12x^3 \cos x + 3x^4 \sin x$

21. $4x \sin x + (2x^2 - 3)\cos x$ 22. $15x^4 + 24x^{-5}$ 23. $-12x^{-5} + 36x^{-10}$

24. $-15x^{-4} - 6x^{-3}$ 25. $\frac{ad - bc}{(cx + d)^2}$

26. $(ax + b)^{n-1}(cx + d)^{m-1}\{mc(ax + b) + na(cx + d)\}$

HINTS TO NCERT & SELECTED PROBLEMS

19. $\frac{d}{dx}\left(\frac{x^2 \cos\frac{\pi}{4}}{\sin x}\right) = \frac{1}{\sqrt{2}}\frac{d}{dx}(x^2 \operatorname{cosec} x)$

$$= \frac{1}{\sqrt{2}}\left\{\operatorname{cosec} x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(\operatorname{cosec} x)\right\} = \frac{1}{\sqrt{2}}(2x \operatorname{cosec} x - x^2 \operatorname{cosec} x \cot x)$$

20. $\frac{d}{dx}\left\{x^4(5 \sin x - 3 \cos x)\right\} = (5 \sin x - 3 \cos x)\frac{d}{dx}(x^4) + x^4 \frac{d}{dx}(5 \sin x - 3 \cos x)$

$$= 4x^3(5 \sin x - 3 \cos x) + x^4(5 \cos x + 3 \sin x)$$

30.4.2 QUOTIENT RULE FOR DIFFERENTIATION

THEOREM *If $f(x)$ and $g(x)$ are two differentiable functions and $g(x) \neq 0$, then show that $\frac{f(x)}{g(x)}$ is also differentiable and*

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}\{f(x)\} - f(x)\frac{d}{dx}\{g(x)\}}{[g(x)]^2}$$

PROOF Since $f(x)$ and $g(x)$ are differentiable functions. Therefore,

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0}\frac{f(x+h) - f(x)}{h} \text{ and, } \frac{d}{dx}(g(x)) = \lim_{h \to 0}\frac{g(x+h) - g(x)}{h} \quad \text{...(i)}$$

Let $\phi(x) = \frac{f(x)}{g(x)}$. Then,

$$\frac{d}{dx}(\phi(x))$$

$$= \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x) - f(x)\, g(x+h)}{g(x)\, g(x+h)\, h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x) - f(x)\, g(x) + f(x)\, g(x) - f(x)\, g(x+h)}{h g(x)\, g(x+h)}$$

[On subtracting and adding $f(x)\, g(x)$ in numerator]

$$= \lim_{h \to 0} \frac{g(x)\,[f(x+h) - f(x)] - f(x)\,[g(x+h) - g(x)]}{h g(x)\, g(x+h)}$$

$$= \left[\lim_{h \to 0} g(x) \left\{\frac{f(x+h) - f(x)}{h}\right\} - \lim_{h \to 0} f(x) \left\{\frac{g(x+h) - g(x)}{h}\right\}\right] \times \left[\lim_{h \to 0} \frac{1}{g(x)\, g(x+h)}\right]$$

$$= \left[g(x) \lim_{h \to 0} \left\{\frac{f(x+h) - f(x)}{h}\right\} - f(x) \lim_{h \to 0} \left\{\frac{g(x+h) - g(x)}{h}\right\}\right] \times \left[\lim_{h \to 0} \frac{1}{g(x)\, g(x+h)}\right]$$

$$= \left[g(x) \frac{d}{dx}\{f(x)\} - f(x) \frac{d}{dx}\{g(x)\}\right] \times \frac{1}{[g(x)]^2}$$

$\left[\begin{array}{l} \because\ g(x) \text{ is differentiable.} \\ \therefore\ \text{It is continuous, and hence} \\ \lim_{h \to 0} g(x+h) = g(x) \end{array}\right]$

$$= \frac{g(x) \frac{d}{dx}\{f(x)\} - f(x) \frac{d}{dx}\{g(x)\}}{[g(x)]^2}$$

Hence, $\frac{f(x)}{g(x)}$ is differentiable and $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \frac{d}{dx}\{f(x)\} - f(x) \frac{d}{dx}\{g(x)\}}{[g(x)]^2}$

REMARK *It is advisable to remember this result in the following form:*

$$\frac{d}{dx}\left\{\frac{N^r}{D^r}\right\} = \frac{D^r \frac{d}{dx}(N^r) - N^r \frac{d}{dx}(D^r)}{(D^r)^2}, \text{ where } N^r = \text{Numerator}, D^r = \text{Denominator}.$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Differentiate the following functions with respect to x:*

(i) $\frac{e^x}{1 + \sin x}$ (ii) $\frac{x + \sin x}{x + \cos x}$

(iii) $\frac{\sin x + \cos x}{\sin x - \cos x}$ [NCERT] (iv) $\frac{\sec x - 1}{\sec x + 1}$ [NCERT]

SOLUTION (i) Using quotient rule, we have

$$\frac{d}{dx}\left(\frac{e^x}{1+\sin x}\right)$$

$$= \frac{(1+\sin x)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)\,e^x - e^x(0+\cos x)}{(1+\sin x)^2} = \frac{e^x(1+\sin x-\cos x)}{(1+\sin x)^2}.$$

(ii) Using quotient rule, we have

$$\frac{d}{dx}\left(\frac{x+\sin x}{x+\cos x}\right)$$

$$= \frac{(x+\cos x)\frac{d}{dx}(x+\sin x) - (x+\sin x)\frac{d}{dx}(x+\cos x)}{(x+\cos x)^2}$$

$$= \frac{(x+\cos x)(1+\cos x) - (x+\sin x)(1-\sin x)}{(x+\cos x)^2}$$

$$= \frac{x+\cos x + x\cos x + \cos^2 x - x - \sin x + x\sin x + \sin^2 x}{(x+\cos x)^2}$$

$$= \frac{\cos x - \sin x + x\cos x + x\sin x + \cos^2 x + \sin^2 x}{(x+\cos x)^2}$$

$$= \frac{\cos x - \sin x + x(\cos x + \sin x) + 1}{(x+\cos x)^2}.$$

(iii) Using quotient rule, we have

$$\frac{d}{dx}\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)$$

$$= \frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x) - (\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$= \frac{(\sin x-\cos x)(\cos x-\sin x) - (\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$= \frac{-(\sin x-\cos x)^2 - (\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$= \frac{-\left\{(\sin x-\cos x)^2 + (\sin x+\cos x)^2\right\}}{(\sin x-\cos x)^2} = \frac{-2\left(\sin^2 x+\cos^2 x\right)}{\sin^2 x+\cos^2 x - 2\sin x\cos x} = \frac{-2}{1-\sin 2x}$$

(iv) We have,

$$\frac{d}{dx}\left(\frac{\sec x-1}{\sec x+1}\right) = \frac{(\sec x+1)\frac{d}{dx}(\sec x-1) - (\sec x-1)\frac{d}{dx}(\sec x+1)}{(\sec x+1)^2}$$

$$= \frac{(\sec x + 1)(\sec x \tan x - 0) - (\sec x - 1)(\sec x \tan x + 0)}{(\sec x + 1)^2}$$

$$= \frac{\sec^2 x \tan x + \sec x \tan x - \sec^2 x \tan x + \sec x \tan x}{(\sec x + 1)^2}$$

$$= \frac{2 \sec x \tan x}{(\sec x + 1)^2} = \frac{2 \sin x}{(1 + \cos x)^2}$$

EXAMPLE 2 *Differentiate the following functions with respect to x:*

(i) $\frac{2x + 3}{x^2 - 5}$ (ii) $\frac{x + 3}{x^2 + 1}$ (iii) $\frac{1 + \tan x}{1 - \tan x}$ (iv) $\frac{\sec x + \tan x}{\sec x - \tan x}$

SOLUTION (i) Using quotient rule, we have

$$\frac{d}{dx}\left(\frac{2x + 3}{x^2 - 5}\right) = \frac{(x^2 - 5)\frac{d}{dx}(2x + 3) - (2x + 3)\frac{d}{dx}(x^2 - 5)}{(x^2 - 5)^2}$$

$$= \frac{(x^2 - 5)(2) - (2x + 3)(2x)}{(x^2 - 5)^2} = \frac{-2(x^2 + 3x + 5)}{(x^2 - 5)^2}.$$

(ii) $$\frac{d}{dx}\left(\frac{x + 3}{x^2 + 1}\right) = \frac{(x^2 + 1)\frac{d}{dx}(x + 3) - (x + 3)\frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2}$$

$$= \frac{(x^2 + 1)1 - (x + 3)(2x)}{(x^2 + 1)^2} = \frac{-x^2 - 6x + 1}{(x^2 + 1)^2} = \frac{1 - 6x - x^2}{(x^2 + 1)^2}.$$

(iii) $$\frac{d}{dx}\left(\frac{1 + \tan x}{1 - \tan x}\right) = \frac{(1 - \tan x)\frac{d}{dx}(1 + \tan x) - (1 + \tan x)\frac{d}{dx}(1 - \tan x)}{(1 - \tan x)^2}$$

$$= \frac{(1 - \tan x)(0 + \sec^2 x) - (1 + \tan x)(0 - \sec^2 x)}{(1 - \tan x)^2}$$

$$= \frac{2 \sec^2 x}{(1 - \tan x)^2} = \frac{2}{(\cos x - \sin x)^2} = \frac{2}{1 - \sin 2x}.$$

(iv) We have,

$$\frac{\sec x + \tan x}{\sec x - \tan x} = \frac{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}} = \frac{1 + \sin x}{1 - \sin x}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{\sec x + \tan x}{\sec x - \tan x}\right) = \frac{d}{dx}\left(\frac{1 + \sin x}{1 - \sin x}\right)$$

$$= \frac{(1 - \sin x)\frac{d}{dx}(1 + \sin x) - (1 + \sin x)\frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$

$$= \frac{(1 - \sin x)(0 + \cos x) - (1 + \sin x)(0 - \cos x)}{(1 - \sin x)^2} = \frac{2 \cos x}{(1 - \sin x)^2}$$

EXERCISE 30.5

LEVEL-1

Differentiate the following functions with respect to x :

1. $\dfrac{x^2+1}{x+1}$
2. $\dfrac{2x-1}{x^2+1}$
3. $\dfrac{x+e^x}{1+\log x}$
4. $\dfrac{e^x-\tan x}{\cot x-x^n}$
5. $\dfrac{ax^2+bx+c}{px^2+qx+r}$
6. $\dfrac{x}{1+\tan x}$ [NCERT]
7. $\dfrac{1}{ax^2+bx+c}$
8. $\dfrac{e^x}{1+x^2}$
9. $\dfrac{e^x+\sin x}{1+\log x}$
10. $\dfrac{x\tan x}{\sec x+\tan x}$
11. $\dfrac{x\sin x}{1+\cos x}$
12. $\dfrac{2^x\cot x}{\sqrt{x}}$
13. $\dfrac{\sin x-x\cos x}{x\sin x+\cos x}$
14. $\dfrac{x^2-x+1}{x^2+x+1}$
15. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$
16. $\dfrac{a+\sin x}{1+a\sin x}$
17. $\dfrac{10^x}{\sin x}$
18. $\dfrac{1+3^x}{1-3^x}$
19. $\dfrac{3^x}{x+\tan x}$
20. $\dfrac{1+\log x}{1-\log x}$
21. $\dfrac{4x+5\sin x}{3x+7\cos x}$ [NCERT]
22. $\dfrac{x}{1+\tan x}$ [NCERT]
23. $\dfrac{a+b\sin x}{c+d\cos x}$ [NCERT]
24. $\dfrac{px^2+qx+r}{ax+b}$ [NCERT]
25. $\dfrac{\sec x-1}{\sec x+1}$ [NCERT]
26. $\dfrac{x^5-\cos x}{\sin x}$ [NCERT]
27. $\dfrac{x+\cos x}{\tan x}$ [NCERT]
28. $\dfrac{x^n}{\sin x}$ [NCERT]
29. $\dfrac{ax+b}{px^2+qx+r}$ [NCERT]
30. $\dfrac{1}{ax^2+bx+c}$ [NCERT]

ANSWERS

1. $\dfrac{x^2+2x-1}{(x+1)^2}$
2. $\dfrac{2(1+x-x^2)}{(1+x^2)^2}$
3. $\dfrac{x\log x\cdot(1+e^x)-e^x(1-x)}{x(1+\log x)^2}$
4. $\dfrac{(\cot x-x^n)(e^x-\sec^2 x)+(e^x-\tan x)(\operatorname{cosec}^2 x+nx^{n-1})}{(\cot x-x^n)^2}$
5. $\dfrac{(aq-bp)x^2+2(ar-cp)x+br-cq}{(px^2+qx+r)^2}$
6. $\dfrac{1+\tan x-x\sec^2 x}{(1+\tan x)^2}$
7. $\dfrac{-(2ax+b)}{(ax^2+bx+c)^2}$
8. $\dfrac{e^x(1-x)^2}{(1+x^2)^2}$
9. $\dfrac{x(1+\log x)(e^x+\cos x)-(e^x+\sin x)}{x(1+\log x)^2}$
10. $\dfrac{x\sec x(\sec x-\tan x)+\tan x}{(\sec x+\tan x)}$
11. $\dfrac{(x+\sin x)}{(1+\cos x)}$
12. $\dfrac{2^x\left[-x\operatorname{cosec}^2 x+x\cot x\cdot\log 2-\left(\frac{1}{2}\right)\cot x\right]}{x^{3/2}}$

13. $\dfrac{x^2}{(x\sin x+\cos x)^2}$ 14. $\dfrac{2(x^2-1)}{(x^2+x+1)^2}$ 15. $\dfrac{\sqrt{a}}{\sqrt{x}(\sqrt{a}-\sqrt{x})^2}$ 16. $\dfrac{(1-a^2)\cos x}{(1+a\sin x)^2}$

17. $10^x \operatorname{cosec} x\,[\log 10-\cot x]$ 18. $\dfrac{2\cdot 3^x\log 3}{(1-3^x)^2}$

19. $\dfrac{3^x\{(x+\tan x)\log 3-(1+\sec^2 x)\}}{(x+\tan x)^2}$ 20. $\dfrac{2}{x(1-\log x)^2}$

21. $\dfrac{15x\cos x+28\,x\sin x+28\cos x-15\sin x+35}{(3x+7\cos x)^2}$

22. $\dfrac{1+\tan x-x\sec^2 x}{(1+\tan x)^2}$ 23. $\dfrac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$

24. $\dfrac{apx^2+2bpx+bq-ar}{(ax+b)^2}$ 25. $\dfrac{2\sec x\tan x}{(\sec x+1)^2}$

26. $\dfrac{-x^5\cos x+5x^4\sin x-1}{\sin^2 x}$ 27. $\dfrac{(1-\sin x)\tan x-(x+\cos x)\sec^2 x}{\tan^2 x}$

28. $\dfrac{n\,x^{n-1}\sin x-x^n\cos x}{\sin^2 x}$ 29. $\dfrac{-apx^2-2bpx+ar-bq}{(px^2+qx+r)^2}$ 30. $\dfrac{-(2ax+b)}{(ax^2+bx+c)^2}$

HINTS TO NCERT & SELECTED PROBLEMS

6. $$\frac{d}{dx}\left(\frac{x}{1+\tan x}\right)=\frac{(1+\tan x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}=\frac{(1+\tan x)-x\sec^2 x}{(1+\tan x)^2}$$

21. $$\frac{d}{dx}\left(\frac{4x+5\sin x}{3x+7\cos x}\right)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)(4+5\cos x)-(4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+28x\sin x+28\cos x-15\sin x+35}{(3x+7\cos x)^2}$$

22. $$\frac{d}{dx}\left(\frac{1}{ax^2+bx+c}\right)=\frac{(ax^2+bx+c)\cdot\frac{d}{dx}(1)-\frac{d}{dx}(ax^2+bx+c)\times 1}{(ax^2+bx+c)^2}$$

$$=\frac{(ax^2+bx+c)\times 0-(2ax+b)}{(ax^2+bx+c)^2}=\frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

23. $$\frac{d}{dx}\left(\frac{a+b\sin x}{c+d\cos x}\right)=\frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x)-(a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$=\frac{(c+d\cos x)(0+b\cos x)-(a+b\sin x)(0-d\sin x)}{(c+d\cos x)^2}$$

$$= \frac{bc \cos x + ad \sin x + bd}{(c + d \cos x)^2}$$

24. $\frac{d}{dx}\left(\frac{px^2 + qx + r}{ax + b}\right) = \frac{(ax+b)\frac{d}{dx}(px^2 + qx + r) - (px^2 + qx + r)\frac{d}{dx}(ax+b)}{(ax+b)^2}$

$$= \frac{(ax+b)(2\,px + q) - a\,(p\,x^2 + qx + r)}{(ax+b)^2} = \frac{ap\,x^2 + 2b\,px + bq - ar}{(ax+b)^2}$$

25. $\frac{d}{dx}\left(\frac{\sec x - 1}{\sec x + 1}\right) = \frac{(\sec x + 1)\frac{d}{dx}(\sec x - 1) - (\sec x - 1)\frac{d}{dx}(\sec x + 1)}{(\sec x + 1)^2}$

$$= \frac{(\sec x + 1)\sec x \tan x - (\sec x - 1)(\sec x \tan x)}{(\sec x + 1)^2} = \frac{2 \sec x \tan x}{(\sec x + 1)^2}$$

26. $\frac{d}{dx}\left(\frac{x^5 - \cos x}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(x^5 - \cos x) - (x^5 - \cos x)\frac{d}{dx}(\sin x)}{(\sin x)^2}$

$$= \frac{\sin x\,(5\,x^4 + \sin x) - (x^5 - \cos x)\cos x}{\sin^2 x} = \frac{-x^5 \cos x + 5\,x^4 \sin x + 1}{\sin^2 x}$$

27. $\frac{d}{dx}\left(\frac{x + \cos x}{\tan x}\right) = \frac{\tan x \frac{d}{dx}(x + \cos x) - (x + \cos x)\frac{d}{dx}(\tan x)}{\tan^2 x}$

$$= \frac{\tan x\,(1 - \sin x) - (x + \cos x)\sec^2 x}{\tan^2 x}$$

28. $\frac{d}{dx}\left(\frac{x^n}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(x^n) - x^n \frac{d}{dx}(\sin x)}{(\sin x)^2} = \frac{(\sin x)(nx^{n-1}) - x^n \cos x}{\sin^2 x}$

29. $\frac{d}{dx}\left(\frac{ax + b}{px^2 + qx + r}\right) = \frac{(px^2 + qx + r)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(px^2 + qx + r)}{(px^2 + qx + r)^2}$

$$= \frac{a\,(px^2 + qx + r) - (ax+b)(2\,px + q)}{(px^2 + qx + r)^2} = \frac{-apx^2 - 2\,bpx + ar - bq}{(px^2 + qx + r)^2}$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the value of $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$.
2. Write the value of $\lim_{x \to a} \frac{x f(a) - a f(x)}{x - a}$.

3. If $x < 2$, then write the value of $\frac{d}{dx}(\sqrt{x^2 - 4x + 4})$.

4. If $\frac{\pi}{2} < x < \pi$, then find $\frac{d}{dx}\left(\sqrt{\frac{1+\cos 2x}{2}}\right)$.

5. Write the value of $\frac{d}{dx}(x|x|)$.

6. Write the value of $\frac{d}{dx}\{(x+|x|)|x|\}$.

7. If $f(x) = |x| + |x-1|$, write the value of $\frac{d}{dx}(f(x))$.

8. Write the value of the derivative of $f(x) = |x-1| + |x-3|$ at $x = 2$.

9. If $f(x) = \frac{x^2}{|x|}$, write $\frac{d}{dx}(f(x))$.

10. Write the value of $\frac{d}{dx}(\log|x|)$.

11. If $f(1) = 1, f'(1) = 2$, then write the value of $\lim_{x\to 1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1}$.

12. Write the derivative of $f(x) = 3|2+x|$ at $x = -3$.

13. If $|x| < 1$ and $y = 1 + x + x^2 + x^3 + \ldots$, then write the value of $\frac{dy}{dx}$.

14. If $f(x) = \log_{x^2} x^3$, write the value of $f'(x)$.

ANSWERS

1. $f'(c)$
2. $f(a) - a f'(a)$
3. -1
4. $\sin x$
5. $\begin{cases} 2x, & x > 0 \\ -2x, & x < 0 \end{cases}$
6. $\begin{cases} 0, & x < 0 \\ 4x, & x > 0 \end{cases}$
7. $\frac{d}{dx}\{f(x)\} = \begin{cases} 2, & x > 1 \\ 0, & 0 < x < 1 \\ -2, & x < 1 \end{cases}$
8. 0
9. $\frac{d}{dx}(f(x)) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$
10. $\frac{1}{|x|}, x \neq 0$
11. 2
12. -3
13. $\frac{1}{(1-x)^2}$
14. 0

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. Let $f(x) = x - [x], x \in R$, then $f'\left(\frac{1}{2}\right)$ is

(a) $\frac{3}{2}$ (b) 1 (c) 0 (d) -1

2. If $f(x) = \frac{x-4}{2\sqrt{x}}$, then $f'(1)$ is

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$ (c) 1 (d) 0

3. If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, then $\frac{dy}{dx} =$

(a) $y+1$ (b) $y-1$ (c) y (d) y^2

4. If $f(x) = 1 - x + x^2 - x^3 + \ldots - x^{99} + x^{100}$, then $f'(1)$ equals

(a) 150 (b) -50 (c) -150 (d) 50

5. If $y = \dfrac{1 + \dfrac{1}{x^2}}{1 - \dfrac{1}{x^2}}$, then $\dfrac{dy}{dx} =$

(a) $-\dfrac{4x}{(x^2-1)^2}$ (b) $-\dfrac{4x}{x^2-1}$ (c) $\dfrac{1-x^2}{4x}$ (d) $\dfrac{4x}{x^2-1}$

6. If $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, then $\dfrac{dy}{dx}$ at $x = 1$ is

(a) 1 (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{\sqrt{2}}$ (d) 0

7. If $f(x) = x^{100} + x^{99} + \ldots + x + 1$, then $f'(1)$ is equal to

(a) 5050 (b) 5049 (c) 5051 (d) 50051

8. If $f(x) = 1 + x + \dfrac{x^2}{2} + \ldots + \dfrac{x^{100}}{100}$, then $f'(1)$ is equal to

(a) $\dfrac{1}{100}$ (b) 100 (c) 50 (d) 0

9. If $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is

(a) -2 (b) 0 (c) 1/2 (d) does not exist

10. If $y = \dfrac{\sin(x+9)}{\cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is

(a) $\cos 9$ (b) $\sin 9$ (c) 0 (d) 1

11. If $f(x) = \dfrac{x^n - a^n}{x - a}$, then $f'(a)$ is

(a) 1 (b) 0 (c) $\dfrac{1}{2}$ (d) does not exist

12. If $f(x) = x \sin x$, then $f'(\pi/2) =$

(a) 0 (b) 1 (c) -1 (d) $\dfrac{1}{2}$

ANSWERS

1. (b) 2. (a) 3. (c) 4. (d) 5. (a) 6. (d) 7. (a) 8. (b)
9. (a) 10. (a) 11. (d) 12. (b)

SUMMARY

1. A function $f(x)$ is differentiable at $x = c$ iff $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists finitely.

   This limit is called the derivative or differentiation of $f(x)$ at $x = c$ and is denoted by $f'(c)$.
2. Geometrically the derivative of a function $f(x)$ at a point $x = c$ is the slope of the tangent to the curve $y = f(x)$ at the point $(c, f(c))$.
3. If $f(x)$ is a differentiable function, then $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ is called the differentiation of $f(x)$ or differentiation of $f(x)$ with respect to x.

4. Mechanically, $\frac{d}{dx}(f(x))$ measures the rate of change of $f(x)$ with respect to x.

5. Following are some standard derivatives:

(i) $\frac{d}{dx}(x^n) = n\,x^{n-1}$

(ii) $\frac{d}{dx}(a^x) = a^x \log_e a, a > 0, a \neq 1$

(iii) $\frac{d}{dx}(e^x) = e^x$

(iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(v) $\frac{d}{dx}(\sin x) = \cos x$

(vi) $\frac{d}{dx}(\cos x) = -\sin x$

(vii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(viii) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(ix) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(x) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

6. Following are the fundamental rules for differentiation:

(i) Differentiation of a constant function is zero i.e., $\frac{d}{dx}(c) = 0$

(ii) Differentiation of a constant and a function is equal to constant times the differentiation of the function.

(iii) If $f(x)$ and $g(x)$ are differentiable functions, then

(a) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$

(b) $\frac{d}{dx}\{f(x) \times g(x)\} = g(x) \times \frac{d}{dx}(f(x)) + f(x) \times \frac{d}{dx}(g(x))$

(c) $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \times \frac{d}{dx}(f(x)) - f(x) \times \frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$

CHAPTER 31

MATHEMATICAL REASONING

31.1 INTRODUCTION

In this chapter, we shall learn about some basics of mathematical reasoning. As all of us know that the main asset that makes humans far more superior than the other species is the ability to reasoning. The ability of reasoning varies from person to person. Also, it is the ability of reasoning which makes one person superior than the other. In this chapter, we shall discuss the process of reasoning especially in the context of mathematics. In mathematical language, there are two kinds of reasoning.

(i) Inductive reasoning. (ii) Deductive reasoning.

In the chapter on Mathematical induction, we have already discussed the inductive reasoning. In this, chapter, we shall discuss some basics of deductive reasoning.

31.2 STATEMENTS

In reasoning we communicate our ideas or thoughts with the help of sentences in a particular language. The following types of sentences are normally used in our every day communication.

ASSERTIVE SENTENCE *A sentence that makes an assertion is called an assertive sentence or a declarative sentence.*

For example, "Mars supports life" is an assertive or a declarative sentence. "Any two individuals are always related" is also a declarative sentence.

IMPERATIVE SENTENCE *A sentence that expresses a request or a command is called an imperative sentence.*

For example, "Please bring me a cup of tea" is an imperative sentence.

EXCLAMATORY SENTENCE *A sentence that expresses some strong feeling is called an exclamatory sentence.*

For example, "How big is the whale fish !" is an exclamatory sentence.

INTERROGATIVE SENTENCE *A sentence that asks some question is called an interrogative sentence.*

For example, "What is your age ?" is an interrogative sentence.

In this chapter, we shall be discussing about a specific type of sentences which will be called as statements or propositions.

STATEMENT *A statement or a proposition is an assertive (or a declarative) sentence which is either true or false but not both.*

A statement is assumed to be either true or false. A true statement is also known as a valid statement. If a statement is false, we say that it is an invalid statement. A statement cannot be both true and false at the same time. This fact is known as the law of the excluded middle.

A sentence which is both true and false simultaneously is not a statement, rather it is a paradox.

ILLUSTRATION 1 *Consider the following sentences:*

(i) *Washington D.C. is in America.* (ii) *Moon revolves around the Earth.*
(iii) *Two plus three is five.* (iv) *Every square is a rectangle.*
(v) *The sun is a star.*

Each of these sentences is a true declarative sentence and so, each of them is a statement.

ILLUSTRATION 2 *Consider the following sentences:*

(i) *Three plus four is 6.*
(ii) *The earth is a star.*
(iii) *Every rectangle is a square.*
(iv) *New Delhi is in Nepal.*
(v) *Every relation is a function.*

Each of these sentences is a false declarative sentence and hence each of them is a statement.

ILLUSTRATION 3 *Consider the following sentences:*

(i) *Give me a glass of water.*
(ii) *Switch on the light.*
(iii) *Bring some fruits from the fruit shop.*
(iv) *Do your home work.*
(v) *Please do me a favour.*

We observe that each of these sentences is an imperative sentence. In other words, each of them either expresses a request or a command. So, they are not statements.

ILLUSTRATION 4 *Consider the following sentences:*

(i) *How are you ?*
(ii) *Where are you going ?*
(iii) *Is every set finite ?*
(iv) *Have you ever seen Taj Mahal ?*
(v) *Where is your pen ?*

Clearly, each of these sentences is asking a question. So, they cannot be assigned, true or false. Hence, none of the them is a statement.

ILLUSTRATION 5 *Consider the following sentences:*

(i) *May God bless you !*
(ii) *May you live long !*

Each of these sentences is an optive. So, we cannot assign true or false to them and hence none of them is a statement.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Which of the following sentences are statements or propositions? Justify your answer.*

(i) *The set of prime integers is infinite.*
(ii) *Paris is in England.*
(iii) *The moon is made of green cheese.*
(iv) *May god bless you !*
(v) *Who are you ?*
(vi) *The number x is a positive integer.*

SOLUTION (i) The "set of prime integers is infinite" is a true declarative sentence. So, it is a true statement.

(ii) "Paris is in England" is a false declarative sentence. So, it is a false statement.

(iii) The sentence " The moon is made of green cheese" is a false declarative sentence. So, it is a false statement.

(iv) The sentence "May god bless you!" is an exclamatory sentence. So, it is not a statement.

(v) The sentence "who are you?" is an interrogative sentence. So, it is not a statement.

(vi) The sentence "The number x is a positive integer" is not a statement unless the variable x is assigned a specific value.

REMARK *The sentence "This sentence is false" cannot be assigned a truth value of either true or false, because either assignment contradicts the sense of the sentence. Although it is a declarative sentence, but it is not a proposition.*

EXAMPLE 2 *Which of the following is a statement (or proposition) ?*

(i) $x + 2 = 9.$
(ii) *6 has three prime factors.*
(iii) $x^2 + 5x + 6 = 0.$

SOLUTION (i) The sentence : $x + 2 = 9$ is an open sentence. Its truth value cannot be confirmed unless we are given the value of x. So, it is not a statement.

(ii) The sentence "6 has three prime factors" is a false statement, because 6 has two prime factors, viz. 2 and 3.

(iii) $x^2 + 5x + 6 = 0$ is not a statement, because its truth or falsity cannot be confirmed without knowing the value of x.

EXAMPLE 3 *Check whether the following sentences are statements. Give reasons for your answer.*

(i) 18 *is less than* 16.
(ii) *Every set is a finite set.*
(iii) *The sun is a star.*
(iv) *Mathematics is fun.*
(v) There is no rain without clouds.
(vi) *How far is Chennai from here?*

SOLUTION (i) This sentence is always false, because $18 > 16$. Hence, it is a statement.

(ii) This sentence is always false, because there are sets which are not finite. Hence, it is a statement.

(iii) Since the sun is a star (it is a scientific fact). So, the given sentence is a always true. Hence, it is a statement.

(iv) Mathematics is a fun is true for those who like mathematics. But, for others, it may not be true. So, the given sentence may or may not be true. Hence, it is not a statement.

(v) It is scientifically established natural phenomenon that cloud is formed before it rains. Therefore, this sentence is always true. Hence, it is a statement.

(vi) It is an interrogative sentence. Hence, it is not a statement.

EXERCISE 31.1

LEVEL-1

1. Find out which of the following sentences are statements and which are not. Justify your answer.
   (i) Listen to me, Ravi !
   (ii) Every set is a finite set.
   (iii) Two non-empty sets have always a non-empty intersection.
   (iv) The cat pussy is black.
   (v) Are all circles round ?
   (vi) All triangles have three sides.
   (vii) Every rhombus is a square.
   (viii) $x^2 + 5|x| + 6 = 0$ has no real roots.
   (ix) This sentence is a statement.
   (x) Is the earth round ?
   (xi) Go !
   (xii) The real number x is less than 2.
   (xiii) There are 35 days in a month.
   (xiv) Mathematics is difficult.
   (xv) All real numbers are complex numbers.
   (xvi) The product of (-1) and 8 is 8.
2. Give three examples of sentences which are not statements. Give reasons for the answers.

ANSWERS

1. Statements : (ii) (iii) (vi) (vii) (viii) (xiii) (xv) (xvi)

31.3 NEGATION OF A STATEMENT

The denial of a statement p is called its negation and is written as $\sim p$, and read as 'not p'.

Negation of any statement p is formed by writing "It is not the case that ....." or "It is false that ...." before p or, if possible by inserting in p the word "not".

Let us consider the statement:

p : *All integers are rational numbers.*

The negation of this statement is:

$\sim p$: *It is not the case that all integers are rational numbers.*

or

$\sim p$: *It is false that all integers are rational numbers.*

or

$\sim p$: *At least one integer is not a rational numbers.*

Consider now the statement: $p: 7 > 9$

The negation of this statement is: $\sim p: 7 \ngtr 9$ or $\sim p: 7 \leq 9$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Write the negation of the following statements:*

(i) *New Delhi is a city.*
(ii) *I went to my class yesterday.*
(iii) $\sqrt{7}$ *is rational.*
(iv) *The number 2 is greater than 7.*
(v) $\sqrt{2}$ *is not a complex number.*

SOLUTION (i) The given statement is:

p: *New Delhi is a city.*

The negation of this statement is:

$\sim p$: *It is not the case that New Delhi is a city.*

or

$\sim p$: *It is false that New Delhi is a city.*

or

$\sim p$: *New Delhi is not a city.*

(ii) Let q denote the given statement i.e. q: *I went to my class yesterday.*

The negation of this statement is:

$\sim q$: *I did not go to my class yesterday.*

or

$\sim q$: *It is not the case that I went to my class yesterday.*

or

$\sim q$: *It is false that I went to my class yesterday.*

or

$\sim q$: *I was absent from my class yesterday.*

(iii) Let r denote the given statement i.e. $r: \sqrt{7}$ is rational

The negation $\sim r$ of this statement is given by:

$\sim r$: *It is not the case that* $\sqrt{7}$ *is rational.*

or

$\sim r$: $\sqrt{7}$ *is not rational.*

or

$\sim r$: *It is false that* $\sqrt{7}$ *is rational.*

(iv) Let the given statement be denoted by s i.e. s: *The number 2 is greater than 7.*

The negative $\sim s$ of this statement is given by

$\sim s$: *The number 2 is not greater than 7.*

or

$\sim s$: *The number 2 is less than or equal to 7.*

or

$\sim s$: *It is false that the number 2 is greater than 7.*

(v) Let the given statement be denoted by u i.e. u: $\sqrt{2}$ *is not a complex number.*

The negation $\sim v$ of this statement is given by:

$\sim u$: *$\sqrt{2}$ is a complex number.*

or

$\sim u$: *It is false that $\sqrt{2}$ is not a complex number.*

EXAMPLE 2 *Write the negation of the following statements and check whether the resulting statements are true:*

(i) *The sum of 2 and 5 is 9.*

(ii) *Every natural number is greater than 0.*

(iii) *Australia is a continent.*

(iv) *There does not exist a quadrilateral which has all its sides equal.*

SOLUTION (i) The negation of the given statement is

It is false that the sum of 2 and 5 is 9.

or

The sum of 2 and 5 is not equal to 9.

We know that $2+5 = 7 \neq 9$. So, this statement is true.

(ii) The negation of the given statement is:

It is false that every natural number is greater than 0.

or

There exists a natural number which is not greater than 0.

We know that all natural numbers are greater than 0. So, this statement is false.

(iii) The negation of the given statement is:

It is false that Australia is a continent.

or

Australia is not a continent.

We know that Australia is a continent. So, this statement is false.

(iv) The negation of the given statement is:

It is not the case that there does not exist a quadrilateral which has all its sides equal.

or

There exists a quadrilateral which has all its sides equal.

We know that square and rhombus are quadrilaterals having all sides equal. So, this statement is true.

NOTE *It should be noted that the negation of "Every or For all" is "there exists" and vice-versa.*

EXAMPLE 3 *Write the negation of the following statements:*

(i) *Everyone in Germany speaks German.*

(ii) *All primes are odd.*

(iii) *All mathematicians are man.*

(iv) *All triangles are not equilateral triangles.*

(v) *All complex numbers are real numbers.*

(vi) *Every natural number is an integer.*

(vii) *All cats scratch.*

SOLUTION (i) The negation of the given statement is:

It is false that everyone in Germany speaks German.

or

There exists a person in Germany who does not speak German.

or

At least one person in Germany does not speak German.

(ii) The negation of the given statement is:

There exists a prime which is not odd.

or

Some primes are not odd.

or

At least one prime is not odd.

(iii) The negation of the given statement is:

Some mathematicians are not man

or

There exists a mathematician who is not man.

or

At least one mathematician is not man.

or

It is false that all mathematicians are man.

(iv) The negation of the given statement is:

Some triangles are equilateral.

or

There exists a triangle which is equilateral.

or

At least one triangle is equilateral.

(v) The negation of the given statement is:

Some complex numbers are not real numbers.

or

There exists a complex number which is not a real number.

or

At least one complex number is not a real number.

(vi) The negation of the given statement is:

There exists a natural number which is not an integer.

or

At least one natural number is not an integer.

or

Some natural numbers are not integers.

(vii) The negation of the given statement is:

Some cats do not scratch.

or

There exists a cat which does not scratch.

or

At least one cat does not scratch.

EXERCISE 31.2

LEVEL-1

1. Write the negation of the following statements:
   (i) Banglore is the capital of Karnataka.
   (ii) It rained on July 4, 2005.
   (iii) Ravish is honest.
   (iv) The earth is round.
   (v) The sun is cold.

2. (i) All birds sing. (ii) Some even integers are prime.
   (iii) There is a complex number which is not a real number.
   (iv) I will not go to school.
   (v) Both the diagonals of a rectangle have the same length.
   (vi) All policemen are thieves.
3. Are the following pairs of statements are negation of each other:
   (i) The number x is not a rational number.
   The number x is not an irrational number.
   (ii) The number x is not a rational number.
   The number x is an irrational number.
4. Write the negation of the following statements:
   (i) p : For every positive real number x, the number $(x-1)$ is also positive.
   (ii) q : For every real number x, either $x > 1$ or $x < 1$.
   (iii) r : There exists a number x such that $0 < x < 1$.
5. Check whether the following pair of statements are negation of each other. Give reasons for your answer.
   (i) $a + b = b + a$ is true for every real number a and b.
   (ii) There exist real numbers a and b for which $a + b = b + a$.

ANSWERS

1. (i) Banglore is not the capital of Karnataka.
   (ii) It did not rain on July 4, 2005. (iii) Ravish is not honest.
   (iv) The earth is not round. (v) The sun is not cold.
2. (i) Some birds do not sing.
   or
   At least one bird does not sing.
   or
   There exists a bird which does not sing.
   (ii) No even integer is prime. (iii) All complex numbers are real numbers.
   (iv) I will go to school.
   (v) There is at least one rectangle whose both diagonals do not have the same length.
   (vi) No policemen is a thief.
3. (i) Yes (ii) No
4. (i) $\sim p$: There exists a positive real number x such that the number $(x-1)$ is not positive.
   (ii) $\sim q$: There exists a real number such that neither $x > 1$ nor $x < 1$.
   (iii) $\sim r$: For every real number x, either $x \leq 0$ or $x \geq 1$.
5. (i) No. The negation of first statement is
   (ii) There exist real numbers a and b for which $a + b \neq b + a$.

31.4 COMPOUND STATEMENTS

In Mathematical reasoning, we generally come across two types of statements namely, simple statements and compound statements as defined below.

SIMPLE STATEMENTS *Any statement whose truth value does not explicity depend on another statement is said to be a simple statement.*

In other words, a statement is said to be simple if it cannot be broken down into simpler statements, that is, if it is not composed of simpler statements.

ILLUSTRATION 1 *Consider the following statements :*

(i) $\sqrt{2}$ *is an irrational number.* (ii) *The set of real numbers is an infinite set.* (iii) $2 + 5 < 4$.

All these statements are simple statements.

COMPOUND STATEMENTS *If a statement is combination of two or more simple statements, then it is said to be a compound statement or a compound proposition.*

ILLUSTRATION 2 *Each of the following statements is a compound statement:*

(i) "Roses are red and violets are blue" is a compound statement as it is composed of the statements ; "Roses are red" and "Violets are blue".

(ii) "The school works or a holiday is declared" is a compound statement as it is a combination of the statements : "The school works" and " A holiday is declared".

(iii) "John is intelligent or studies every night" is also a compound statement as it is composed of the statements : "John is intelligent" and "John studies every night".

(iv) "If it rains, then the school may be closed" is a compound statement as it is obtained by connecting two simple statements:

"It rains" and "The school may be closed" by using the phrase 'if .... then'.

(v) "A quadrilateral is a rhombus if and only if its diagonals are at right angles" is a compound statement obtained by connecting two simple statements : "A quadrilateral is a rhombus" and "Diagonals of a quadrilateral intersect at right angles" by using the phrase 'if and only if'.

The simple statements which form a compound statement are known as its sub-statements or component statements.

The fundamental property of a compound statement is that its truth value is completely determined by the truth values of the sub-statements together with the way in which they are connected to form the compound statement.

31.5 BASIC CONNECTIVES

In the previous section, we have learnt that the words 'or' & 'and' connect two or more simple statements to form a compound statement. These are called sentenial connectives or simply connectives. In this section, we shall learn how the truth and falsity of a compound statement depends upon the truth value of the component statements.

31.5.1 THE WORD "AND"

Any two simple statements can be connected by the word "and" to form a compound statement. For example, consider the statement "The earth is round and the sun is cold". This statement can be broken into two component statements given by

p : *The earth is round.*

q : *The sun is cold.*

Let us now consider the statement "84 is divisible by 4, 7 and 12". The component statements of this statement are

p : *84 is divisible by 4.*

q : *84 is divisible by 7.*

r : *84 is divisible by 12.*

NOTE 1 *It should be noted that the word "and" is used as a connective as we use in the English language. But, 'and' is also used with other meanings. For example, in the statement "Ravish and Ravi are good friends" the word 'and' is not a connective. Similarly, in the statement "Mohan opened the door and ran away", the word 'and' is used in the sense of 'and then' because the action described in "Mohan ran away" occurs after action described in "Mohan opened the door".*

NOTE 2 *In our day-to-day life, the word 'and' is used between two statements which have some kind of relation. But, in reasoning it can be used even for the statements which are not related to each other. For example, "it is raining and 5 is a prime number" is a compound statement whose component statements are*

p : *It is raining.*

q: *5 is a prime number.*

Clearly, these two statements are not related to each other.

We shall now see how the truth or falsity of a compound statement with "and" depends upon the truth or falsity of its component statements.

Consider the statement:

$p: 9>4$ *and* $2<7$

The component statements of this statement are:

$q: 9>4$

$r: 2<7$

Clearly, q and r are true statements. Also, p is a true statement.

Thus, if two statements are true, then their compound statement with "and" is also true.

Consider the statement:

p: *The earth is round and the sun is cold.*

Its component statements are:

q: *The earth is round.*

r: *The sun is cold.*

Clearly, statement q is true and r is false. Also, p is false.

Thus, if one of the two statements is true and the other is false, then the compound statement with "and" is false.

Let us now consider the statement:

$p: 5<12$ *and* $15<7$

Its component statements are :

$q: 5<12$

$r: 15<7$

Clearly, p, q and r are false statements.

Thus, the compound statement with "and" is a false, if the component statements are false.

The above discussion suggests us the following rules:

RULE 1 *The compound statement with "and" is true if all its component statements are true.*

RULE 2 *The compound statement with "and" is false if any or all of its compound statements is false.*

31.5.2 THE WORD "OR"

Any two statements can be connected by the word "OR" to form a compound statement. For example, consider the statement "The sun shines or it rains". This statement can be broken into two component statements given by:

p: *The sun shines.*

q: *It rains.*

Consider now the statement "Two lines in a plane either intersect at one point or they are parallel". The component statements of this statement are:

p: *Two lines in a plane intersect at a point.*

q: *Two lines in a plane are parallel.*

It should be noted that in addition to the connective the word "OR" is also used with other meanings in English language. For example, in the statement *"five or six children are playing in the playground"* the word "or" is used for indicating an approximate number of children. It is not used as a connective. As a connective also the word "OR" is used in two distinct ways in English language. Sometimes it is used in the sense of "p or q or both", i.e. at least one of the two alternatives occurs and sometimes it is used in the sense of "p or q but not both", i.e. exactly one

of the two alternatives occurs. When it used for at least one of the two alternatives, we call it *inclusive* "OR". In case of exactly one of the two alternatives, it is called exclusive "OR".

Let us consider the statement given by:

p : *The school is closed if it is a holiday or Sunday.*

This means that the school remains closed on a holiday. It also remains closed on sunday. If a holiday falls on Sunday, then also the school remains closed. So, in this case, we are using the word "OR" as an inclusive "OR".

Consider now the following statement:

q : *An ice-cream or Coca-cola is available with a piza in piza-hut.*

This means that a person who does not want ice-cream can have a coca-cola with a piza or one does not want coca-cola can have an ice-cream along with a piza. That is who do not want an ice-cream can have coca-cola and vice-versa. A person cannot have both ice-cream and coca-cola with a piza. So, the "OR" used is an exclusive "OR".

NOTE *Throughout this chapter we will be using the word "OR" as an inclusive "OR" unless it is stated otherwise.*

We shall now see how the truth and falsity of the compound statement with an "OR" depends upon the truth and falsity of its component statements.

Consider the compound statement:

p : *Two lines intersect at a point or they are parallel.*

The component statements of this statement are :

q : *Two lines intersect at a point.*

r : *Two lines are parallel.*

We observe that when q is true r is false and when r is true q is false. Also, p is always true.

Thus, if one of the component statements is true, then the compound statement connected with "OR" is always true.

Consider another statement

p : 45 *is a multiple of* 4 *or* 6.

Its component statements are:

q : 45 *is multiple of* 4

r : 45 *is a multiple of* 6

We observe that both q and r are false. Also, p is a false.

Thus, if both the component statements are false, then the compound statement connected with "OR" is always false.

Again, consider the following statement:

p : *The earth is round or the sun is hot.*

Its component statements are:

q : *The earth is round.*

r : *The sun is hot.*

We observe that both q and r are true. Also, p is true.

Thus, if both the component statements are true, then the compound statement with "OR" is always true.

The above discussion suggests us the following rules for the compound statements with an "OR".

RULE 1 ***A compound statement with an "OR" is true when one component statement is true or both the component statements are true.***

RULE 2 *A compound statement with an "OR" is false when both the component statements are false.*

NOTE *If p and q are two simple statements, then the negation of the compound statement*

(i) p *or* q is $\sim p$ *and* $\sim q$ i.e., $\sim (p \text{ or } q) \cong \sim p$ *and* $\sim q$.

(ii) p *and* q is $\sim p$ *or* $\sim q$ i.e., $\sim (p$ *and* $q) \cong \sim p$ *or* $\sim q$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the component statements of the following compound statements:*

(i) *There is somethings wrong with the bulb or with wiring.*

(ii) *It is raining and it is cold.*

(iii) *The roof is red and the wall is white.*

(iv) *The sun shines or it rains.*

(v) *0 is a positive number or a negative number.*

SOLUTION (i) The component statements are:

p: *There is some thing wrong with the bulb.*

q: *There is some thing wrong with the wiring.*

(ii) The component statements are:

p: *It is raining.*

q: *It is cold.*

(iii) The component statements are:

p: *The roof is red.*

q: *The wall is white..*

(iv) The component statements are:

p: *The sun shines.*

q: *It rains.*

(v) The component statements are:

p: *0 is a positive number.*

q: *0 is a negative number.*

EXAMPLE 2 *Find the component statement of the following and check whether they are true or not:*

(i) $\sqrt{2}$ *is a rational number or an irrational number.*

(ii) *All integers are positive or negative.*

(iii) *All primes are either even or odd.*

(iv) *24 is a multiple of 2, 4 and 8.*

SOLUTION (i) The component statements are:

p: $\sqrt{2}$ *is a rational number.*

q: $\sqrt{2}$ *is an irrational number.*

Clearly, p is false and q is true. The connecting word is 'or'

(ii) The component statements are:

p: *All integers are positive.*

q: *All integers are negative.*

Clearly, p and q both are false. The connecting word is 'or'.

(iii) The component statemetns are:

p: *All primes are even.*

q: *All primes are odd.*

Clearly, p and q both are false. Then correcting word is 'or'.

(iv) The component statements are:

p: *24 is a multiple of 2.*

q: *24 is a multiple of 4.*

r: *24 is a multiple of 8.*

Here, p, q and r are true statements. The connecting words are 'and'.

EXAMPLE 3 *For each of the following statements, determine whether an inclusive "OR" or exclusive "OR" is used. Give reasons for your answer.*

(i) *Sun rises or Moon sets.*
(ii) *All integers are positive or negative.*
(iii) *Two lines intersect at a point or are parallel.*
(iv) *The school is closed if it is a holiday or a Sunday.*

SOLUTION (i) Here "OR" is exclusive since sun rises and moon sets during day time.

(ii) Since all integers cannot be both positive as well as negative. Therefore, "OR" is an exclusive "OR".

(iii) Here "OR" is exclusive because it is not possible for two lines to intersect and parallel together.

(iv) Here "OR" is inclusive since school is closed on holiday as well as on Sunday.

EXAMPLE 4 *Write the component statements of the following compound statements and check whether the compound statement is true or false:*

(i) 50 *is a multiple of both* 2 *and* 5.
(ii) 0 *is less than every positive integer and every negative integer.*
(iii) *A line is straight and extends indefinitely in both directions.*
(iv) *All living things have two legs and two eyes.*

SOLUTION (i) The component statements of the given statement are:

p : 50 *is multiple of* 2.
q : 50 *is a multiple of* 5.

We observe that both p and q are true statements. Therefore, the compound statement is true.

(ii) The component statements of the given statements are:

p : 0 *is less than every positive integer.*
q : 0 *is less than every negative integer.*

We observe that p is true and q is false. Therefore, the compound statement is false.

(iii) The component statements of the given statement are:

p : *A line is straight.*
q : *A line extends indefinitely in both directions.*

We observe that both p and q are true. Therefore, the compound statement is true.

(iv) The component statements of the given statement are:

p : *All living things have two legs.*
q : *All living things have two eyes.*

We find that both p and q are false statements. Therefore, the compound statement is false.

EXAMPLE 5 *Write the component statements of the following compound statements and check whether the compound statement is true or false:*

(i) *125 is a multiple of 7 or 8.*
(ii) *Mumbai is the capital of Gujrat or Maharashtra.*
(iii) $\sqrt{2}$ *is a rational number or an irrational number.*
(iv) *The school is closed, if there is a holiday or Sunday.*
(v) *A rectangle is a quadrilateral or a 5-sided polygon.*

SOLUTION (i) The component statements of the given statement are:

p : *125 is a multiple of 7.*
q : *125 is a multiple of 8.*

We observe that both p and q are false statements. Therefore, the compound statement is also false.

(ii) The component statements of the given statement are:

p : *Mumbai is the Capital of Gujrat.*

q : *Mumbai is the Capital of Maharashtra.*

We find that p is false and q is true. Therefore, the compound statement is true.

(iii) The component statements are:

$p : \sqrt{2}$ *is a rational number.*

$q : \sqrt{2}$ *is an irrational number.*

Clearly, p is false and q is true. Therefore, the compound statement is true.

(iv) The component statements are:

p : *The school is closed if there is a holiday.*

q : *The school is closed if there is a Sunday.*

Both, p and q are true statements. Therefore, the compound statement is true.

(v) The component statements are :

p : *A rectangle is a quadrilateral.*

q : *A rectangle is a 5-sided polygon.*

We observe that p is true and q is false. Therefore, the compound statements is true.

EXAMPLE 6 *Write the negation of the following compound statements:*

(i) *It is daylight and all the people have arisen.*

(ii) *All the students completed their homework and the teacher is present.*

(iii) *All rational numbers are real and all real numbers are complex.*

(iv) *Square of an integer is positive or negative.*

(v) *The sand heats up quickly in the sun and does not cool down fast at night.*

SOLUTION (i) In writing down the negations of the above statements, we will be using the following results:

(a) $\sim (p \text{ or } q) \cong \sim p \text{ and } \sim q$.

(b) $\sim (p \text{ and } q) \cong \sim p \text{ or } \sim q$.

(c) $\sim$ (*For all*) $\cong$ *There exists or some or atleast one.*

(d) $\sim$ (*At least one or There exists or Some*) $\cong$ *For all or for every.*

(ii) The component statements of the given statement are:

p : *All the students completed their homework.*

q : *The teacher is present.*

The given statement is (p and q). So, its negation is :

$\sim p$ or $\sim q$ = *Some of the students did not complete their homework or the teacher is not present.*

(ii) The component statements of the given statement are :

p : *It is daylight.*

q : *All the people have risen.*

The given statement is (p and q). So, its negation is:

$\sim p$ or $\sim q$ $\equiv$ *It is not daylight or it is false that all the people have arisen.*

$\equiv$ *It is night or someone has not arisen.*

(iii) The component statements of the given statement are:

p : *All rational numbers are real.*

q : *All real numbers are complex.*

The given statement is (p and q). So, its negation is:

$\sim p$ or $\sim q$: *Some rational are not real or some reals are not complex.*

(iv) The component statements of the given statements are:

p : *Square of an integer is positive.*

q : *Square of an integer is negative.*

The given statement is (p or q). So, its negation is:

$\sim p$ and $\sim q$ = *There exists an integer whose square is neither positive nor negative..*

(v) The component statements of the given statement are:

p : *The sand heats up quickly in the sun.*

q : *The sand does not cool down fast at night.*

The given statement is (p and q). So, its negation is:

$\sim p$ or $\sim q$ = *Either the sand does not heat up quickly in the sun or it cools down fast at night.*

EXERCISE 31.3

LEVEL-1

1. Find the component statements of the following compound statements:
   (i) The sky is blue and the grass is green.
   (ii) The earth is round or the sun is cold.
   (iii) All rational numbers are real and all real numbers are complex.
   (iv) 25 is a multiple of 5 and 8.
2. For each of the following statements, determine whether an inclusive "OR" or exclusive "OR" is used. Give reasons for your answer.
   (i) Students can take Hindi or Sanskrit as their third language.
   (ii) To entry a country, you need a passport or a voter registration card.
   (iii) A lady gives birth to a baby boy or a baby girl.
   (iv) To apply for a driving licence, you should have a ration card or a passport.
3. Write the component statements of the following compound statements and check whether the compound statement is true or false:
   (i) To enter into a public library children need an identity card from the school or a letter from the school authorities.
   (ii) All rational numbers are real and all real numbers are not complex.
   (iii) Square of an integer is positive or negative.
   (iv) $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.
   (v) The sand heats up quickly in the sun and does not cool down fast at night.
4. Determine whether the following compound statements are true or false:
   (i) Delhi is in India and $2 + 2 = 4$.
   (ii) Delhi is in England and $2 + 2 = 4$.
   (iii) Delhi is in India and $2 + 2 = 5$.
   (iv) Delhi is in England and $2 + 2 = 5$.

ANSWERS

1. (i) p : The sky is blue
   q : The grass is green
   (ii) p : The earth is round
   q : The sun is cold
   (iii) p : All rational numbers are real
   q : All real numbers are complex
   (iv) p : 25 is a multiple of 5
   q : 25 is a multiple of 8.
2. (i) Exclusive "OR". A student cannot take both Hindi and Sanskrit.
   (ii) Inclusive "OR". Since a person can have both a passport and a voter registration card to enter a country.
   (iii) Exclusive "OR". A lady cannot give birth to a baby who is both a boy and a girl.
   (iv) Inclusive "OR". Since a person can have both a ration card and a passport to apply for a driving licence.
3. (i) p : To get into a public library children need an identity card.
   q : To get into a public library children need a letter from the school authorities.
   True.

(ii) p : All rational numbers are real.
q : All real numbers are not complex.

(iii) p : Square of an integer is positive.
q : Squre of an integer is negative. True.

(iv) $p : x = 2$ is a root of the equation $3x^2 - x - 10 = 0$.
$q : x = 3$ is a root of the equation $3x^2 - x - 10 = 0$. False.

(v) p : The sand heats up quickly in the sun.
q : The sand does not cool down fast at night . False.

4. (i) True (ii) False (iii) False (iv) False

31.6 QUANTIFIERS

In Mathematics we come across many mathematical statements containing phrases "There exists" and "For every". These two phrases are called quantifiers. Depending upon the context the phrase "There exists" can also be replaced by the equivalent phrases "There is" or, "There is at least one" or, "It is possible to find" or, "some".

Consider the following statements:

$p : x + 4 > 3$ *for all* $x \in N$.

q : *For every prime number* x, $\sqrt{x}$ *is an irrational number.*

r : *There exists a rectangle whose all sides are equal.*

s : *There exists* $x \in N$ *such that* $x + 4 < 7$ or, *For some* $x \in N$, $x + 4 < 7$.

The statement p means that for every natural number x, $x + 4 > 3$.

The statement q means that if S denotes the set of all prime numbers, then for all the members x of the set S, $\sqrt{x}$ is an irrational number.

The statement r means that there is at least one rectangle whose all sides are equal.

The statement s means that there is at least one natural number x such that $x + 4 < 7$.

Phrase "for every (or for all)" is called the *universal quantifier* and the phrase "There exists" is known as the *existential quantifier.*

Consider the statement: *For every* $x \in N$, $x + 5 > 4$

If $p(x)$ denotes $x + 5 > 4$, then the above statement can be written as

For every $x \in N$, $p(x)$

Consider the statement:

All Math Majors are male ...(i)

If M denotes the set of all Math majors, and $p(x)$ denotes 'x is male' then the above statement can be written as:

For every $x \in M$, *x is male* ...(ii)

or *For every* $x \in M$, $p(x)$ ...(iii)

The negation of statement (i) is

There exists at least one Math major who is female (not male).

or *There exists* $x \in M$ *such that x is not male*

or *There exists* $x \in M, \sim p(x)$.

Thus, we have

$$\sim (\text{For every } x \in M, p(x)) \cong (\text{There exists } x \in M, \sim p(x))$$

This is true for any M and any $p(x)$.

Similarly, we have

$$\sim (\text{There exists } x \in M, p(x)) \cong (\text{For every } x \in M, \sim p(x))$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Identify the quantifier in each of the following statements:*

(i) *For every real number x, $x + 4$ is greater than x.*
(ii) *There exists a real number which is twice of itself.*
(iii) *There exists a (living) person who is 200 years old.*
(iv) *For every $x \in N$, $x + 1 > x$.*

SOLUTION (i) For every (ii) There exists (iii) There exists (iv) For every.

EXAMPLE 2 *Write the negation of the following statements:*

(i) *For all positive integers x, we have $x + 2 > 8$.*
(ii) *Every living person is not 150 years old.*
(iii) *All students live in the dormitories.*
(iv) *Some students are 25 (years) or older.*

SOLUTION (i) There existsa positive integer $x$such that $x + 2 \ngtr 8$.
or There exists a positive integer x such that $x + 2 \leq 8$.
(ii) There exists a (living) person who is 150 years old.
(iii) Some students do not live in the dormitories.
or At least one student does not live in the dormitories.
or There exists a student who does not live in the dormitories.
(iv) None of the students is 25 or older
or All the students are under 25.

EXAMPLE 3 *Write the negation of each of the following statements:*

(i) *For every real number x, $x + 0 = x = 0 + x$.*
(ii) *For every real number x, x is less than $x + 1$*
(iii) *There exists a capital for every state in India.*
(iv) *There exists a number which is equal to its square.*

SOLUTION (i) There existsa real number $x$such that $x + 0 \neq x = 0 + x$
(ii) There exists a real number x such that x is not less than $x + 1$.
(iii) There exists a state in India which does not have its capital.
(iv) For every real number x, $x^2 \neq x$.

EXERCISE 31.4

LEVEL-1

1. Write the negation of each of the following statements:
   (i) For every $x \in N$, $x + 3 < 10$ (ii) There exists $x \in N$, $x + 3 = 10$
2. Negate each of the following statements:
   (i) All the students completed their homework.
   (ii) There exists a number which is equal to its square.

ANSWERS

1. (i) There exists $x \in N$ such that $x + 3 \geq 10$.
   (ii) For every $x \in N$, $x + 3 \neq 10$
2. (i) Some of the students did not complete their home work
   or There exists a student who did not complete his home work.
   (ii) For every real number x, $x^2 \neq x$.

31.7 IMPLICATIONS

In Mathematics we come across many statements of the form "if p then q", "p only if q" and "if and only if" such statements are called implications. In this section, we shall discuss about such statements.

IF-THEN IMPLICATION

Two statements connected by the connective phrase "if - then" give rise to a compound statement which is known as if-then implication.

For example,

If it rains, then the atmospheric humidity increases.

If $x = 4$, *then* $x^2 = 16$.

If ABCD is a parallelogram, then $AB = CD$

are implications.

If p and q are two statements forming the implication "if p then q", then we denote this implication by "$p \Rightarrow q$".

In the implication "$p \Rightarrow q$" p is called the antecedent and q the consequent.

We shall now see how the truth and falsity of an implication "$p \Rightarrow q$" depends upon the truth and falsity of its antecedent p and consequent q.

(i) *If both p and q are true, then* $p \Rightarrow q$ *is also true.*

Verification Let p denote the statement : "The number $N = 43221$ is divisible by 3"

and q denote the statement "The sum of the digits forming N is divisible by 3".

Clearly, p and q both are true.

Now, $p \Rightarrow q$: *If the number N is divisible by 3, then the sum of the digits forming N is divisible by* 3.

Clearly, $p \Rightarrow q$ is also true.

Thus, if p and q are true, then $p \Rightarrow q$ is also true.

(ii) *If p is true and q is false, then* $p \Rightarrow q$ *is false.*

Verification Consider the following statements:

p : *The number* $N = 43221$ *is divisible by* 3.

q : *The sum of the digits forming N is not divisible by* 3.

Clearly, p is true and q is false.

Now,

$p \Rightarrow q$: *If the number N is divisible by* 3, *then the sum of the digits forming N is not divisible by* 3.

Clearly, $p \Rightarrow q$ is false.

Thus, if p is true and q is false, then $p \Rightarrow q$ is false.

(iii) *If p is false and q is true, then* $p \Rightarrow q$ *is true.*

If p is false and q is true, then $p \Rightarrow q$ is assumed to be true. This assumption is made to be consistent with the other assumptions.

(iv) *If both p and q are false, then* $p \Rightarrow q$ *is true.*

Verification Consider the following statements:

p : *The number* $N = 43211$ *is divisible by* 3

q : *The sum of the digits forming N is divisible by* 3.

Clearly, p and q are false statements.

Now,

$p \Rightarrow q$: *If the number* $N = 43211$ *is divisible by* 3, *then the sum of the digits forming N is divisible by* 3.

Clearly, $p \Rightarrow q$ has truth value T.

Thus, if both p and q are false, then $p \Rightarrow q$ is true.

The above discussion suggests us the following rule for the implication "if-then".

RULE *The implication "if p, then q" is always true except when the antecedent p is true and the consequent q is false.*

The implication $p \Rightarrow q$ is same as each of the following:

(i) p is sufficient condition for q.

(ii) p only if q.

(iii) q is necessary condition for p.

(iv) $\sim q \Rightarrow \sim p$.

Consider the statement:

If a number is a multiple of 9, then it is a multiple of 3.

Clearly, it is an implication having antecedent (p) and consequent (q) as follows:

p : *a number is a multiple of 9.*

q : *a number is a multiple of 3.*

The above statement says that knowing that a number is a multiple of 9 is sufficient to conclude that it is a multiple of 3 i.e. $p \Rightarrow q$ is same as p is sufficient codition for q.

Also, the given statement says that a number is a multiple of 9 only if it is a multiple of 3 i.e. $p \Rightarrow q$ is same as p only if q.

The above statement also means that when a number is a multiple of 9, it is necessarily a multiple of 3 i.e. $p \Rightarrow q$ is same as q is necessary condition for p.

The above statement also says that if a number is not a multiple of 3, then it is not a multiple of 9 i.e. $p \Rightarrow q$ is same as $\sim q \Rightarrow \sim p$".

CONTRA POSITIVE *If p and q are two statements, then the contrapositive of the implication "if p then q" is "if $\sim$ q, then $\sim$ p ."*

CONVERSE *If p and q are two statements, then the converse of the implication "if p, then q" is "if q, then p".*

INVERSE *If p and q are two statements, then the inverse of "If p, then q" is "If $\sim$ p, then $\sim$ q".*

Consider the statement:

If a number is divisible by 9, then it is divisible by 3

It is an implication with "if ... then" having antecedent (p) and consequent (q) as given below:

p : *a number is divisible by 9.*

q : *a number is divisible by 3.*

The given statement is "if p, then q." Its contrapositive is: If $\sim q$, then $\sim p$

i.e., *If a number is not divisible by 3, then it is not divisible by 9.*

The converse of the given statement is: If q, then p

i.e., *If a number is divisible by 3, then it is divisible by 9.*

IF AND ONLY IF IMPLICATION

If p and q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called if and only if implication and is denoted by $p \Leftrightarrow q$.

Consider the statement:

A triangle is equilateral if and only if it is equiangular.

This is if and only if implication with the component statements:

p : *A triangle is equilateral.*

q : *A triangle is equiangular.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Rewrite the following statement with "if–then" in five different ways conveying the same meaning:*

If a natural number is odd, then its square is also odd.

SOLUTION The component statement of the given statement are:

p : *A natural number is odd.*

q : *Square of a natural number is odd.*

The given statement is: "If p, then q."

It is same as each of the following statements:

(i) $p \Rightarrow q$ i.e., x is an odd natural number $\Rightarrow x^2$ is an odd natural number

(ii) p is a sufficient condition for q.
i.e. Knowing that a natural number is odd is sufficient to conclude that its square is odd.

(iii) p only if q i.e., A natural number is odd only if its square is odd.

(iv) q is necessary condition for p
i.e. When a natural number is odd, its square is necessarily odd.

(v) $\sim q \Rightarrow \sim p$
i.e. If the square of a natural number is not odd, then the natural number is not odd.

EXAMPLE 2 *Write each of the following statements in the form "if–then".*

(i) *You get job implies that your credentials are good.*

(ii) *You can access the website only if you pay a subscription fee.*

(iii) *The Banana trees will bloom if it stays warm for a month.*

(iv) *A quadrilateral is a parallelogram if its diagonals bisect each other.*

(v) *To get A^+ in the class, it is necessary that you do all the exercises of the book.*

SOLUTION (i) We know that "If p, then q" is equivalent to "$p \Rightarrow q$".
Therefore, the given statement can be written as
"If you get a job, then your credentials are good"

(ii) We know that "p only if q" is equivalent to "If p, then q".
Therefore, the given statement can be written as:
"If you can access the website, then you pay a subscription fee".

(iii) The given statement can be written as:
"If it stays warm for a month, then the Banana trees will bloom".

(iv) The given statement can be written as:
"If the diagonals of aquadrilateral bisect each other, then it is a parallelogram".

(v) The given statement can be written as:
"If you get A^+ in the class, then you do all the exercise of the book."

EXAMPLE 3 *Write the contrapositive of the following statements:*

(i) *If a number is divisible by 9, then it is divisible by 3.*

(ii) *If you are born in India, then you are a citizen of India.*

(iii) *If a triangle is equilateral, it is isosceles.*

(iv) *If x is prime number, then x is odd.*

(v) *If two lines are parallel, then they do not intersect in the same plane.*

(vi) *x is an even number implies that x is divisible by 4.*

(vii) *Something is cold implies that it has low temperature.*

(viii) *You cannot comprehend geometry if you do not know how to reason deductively.*

SOLUTION We know that the contrapositive of the statement "If p, then q" is "if $\sim q$, then $\sim p$". Therefore contrapositive of the given statements are:

(i) If a number is not divisible by 3, it is not divisible by 9.
(ii) If you are not a citizen of India, then you were not born in India.
(iii) If a triangle is not isosceles, then it is not equilateral.
(iv) If a number x is not odd, then x is not prime.
(v) If two lines do not intersect in the same plane, then they are not parallel.
(vi) If x is not divisible by 4, then x is not an even number.
(vii) If something does not have low temperature, then it is not cold.
(viii) If you can comprehend geometry, then you know how to reason deductively.

EXAMPLE 4 *Write the converse of the following statements:*

(i) *If a number n is even, then n^2 is even.*
(ii) *If you do all the exercises in the book, you get an A grade in the class.*
(iii) *If two integers a and b are such that $a > b$, then $a - b$ is always a positive integer.*
(iv) *If x is prime number, then x is odd.*
(v) *If two lines are parallel, then they do not intersect in the same plane.*

SOLUTION (i) If a number n^2 is even, then is even.
(ii) If you get an A grade in the class, then you have done all the exercises of the book.
(iii) If two integers a and b are such that $a - b$ is always a positive integer, then $a > b$.
(iv) If x is an odd number, then x is a prime number.
(v) If two lines do not intersect in the same plane, then they are parallel.

EXAMPLE 5 *Write the component statements of each of the following statements. Also, check whether the statements are true or not.*

(i) *If a triangle ABC is equilateral, then it is isosceles.*
(ii) *If a and b are integers, then ab is a rational number.*

SOLUTION (i) The component statements of the given statement are:

p : *The triangle ABC is equilateral.*
q : *The triangle ABC is isosceles.*

Since an equilateral triangle is isosceles, so the given statement is true.

(ii) The component statements are:

p : a and b *are intergers.*
q : ab *is a rational number.*

Since the product of two integers is an integer and therefore a rational number. So, the compound statement is true.

EXAMPLE 6 *Given below are two pairs of statements. Combine these two statements using "if and only if".*

(i) p : *If a rectangle is a square, then all its four sides are equal.*
q : *If all the four sides of a rectangle are equal, then the rectangle is a square.*
(ii) p : *If the sum of the digits of a number is divisible by 3, then the number is divisible by 3.*
q : *If a number is divisible by 3, then the sum of its divisible by 3.*

SOLUTION (i) A rectangle is a square if and only if all its four sides are equal.
(ii) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

EXAMPLE 7 *For the given statement identify the necessary and sufficient conditions:*
p : *"If you drive over 80 km per hour, then you will get a fine".*

SOLUTION Let p and q denote the statements:
p : *You drive over 80 km per hour.*

q : *You will get a fine.*

We know that the implications of "if p, then q" indicates that p is sufficient for q. It also indicates that q is necessary for p. Therefore, sufficient condition is "Driving over 80 km per hour" and the necessary conditions is "getting a fine".

EXERCISE 31.5

LEVEL-1

1. Write each of the following statements in the form "if p, then q".
   (i) You can access the website only if you pay a subscription fee.
   (ii) There is traffic jam whenever it rains.
   (iii) It is necessary to have a passport to log on to the server.
   (iv) It is necessary to be rich in order to be happy.
   (v) The game is cancelled only if it is raining.
   (vi) It rains only if it is cold.
   (vii) Whenever it rains it is cold.
   (viii) It never rains when it is cold.
2. State the converse and contrapositive of each of the following statements:
   (i) If it is hot outside, then you feel thirsty.
   (ii) I go to a beach whenever it is a sunny day.
   (iii) A positive integer is prime only if it has no divisors other than 1 and itself.
   (iv) If you live in Delhi, then you have winter clothes.
   (v) If a quadrilateral is a parallelogram, then its diagonals bisect each other.
3. Rewrite each of the following statements in the form "p if and only if q".
   (i) p : If you watch television, then your mind is free and if your mind is free, then you watch television.
   (ii) q : If a quadrilateral is equiangular, then it is a rectangle and if a quadrilateral is a you rectangle, then it is equiangular.
   (iii) r : For you to get an A grade, it is necessary and sufficient that you do all the homework you regularly.
   (iv) s : If a tumbler is half empty, then it is half full and if a tumbler is half full, then it is half empty.
4. Determine the contrapositive of each of the following statements:
   (i) If Mohan is a poet, then he is poor.
   (ii) Only if Max studies will he pass the test.
   (iii) If she works, she will earn money.
   (iv) If it snows, then they do not drive the car.
   (v) It never rains when it is cold.
   (vi) If Ravish skis, then it snowed.
   (vii) If x is less than zero, then x is not positive.
   (viii) If he has courage he will win.
   (ix) It is necessary to be strong in order to be a sailor.
   (x) Only if he does not tire will he win.
   (xi) If x is an integer and x^2 is odd, then x is odd.

ANSWERS

1. (i) If you access the website, then you pay a subscription fee.
   (ii) If it rains, then there is traffic jam.
   (iii) If you log on to the server, then you must have a passport.
   (iv) If he is happy, then he is rich.
   (v) If it is raining, then the game is cancelled.
   (vi) If it rains, then it is cold.
   (vii) If it rains, then it is cold.

(viii) If it is cold, then it never rains.

2. (i) Converse: If you feel thirsty, then it is hot outside.
Contrapositive: If you do not feel thirsty, then it is not hot outside.
(ii) Converse: If I go to a beach, then it is a sunny day.
Contrapositive: If I do not go to a beach, then it is not a sunny day.
(iii) Converse: If an integer has no divisors other than 1 and itself, then it is prime.
Contrapositive: If an integer has some divisors other than 1 and itself, then it is not prime.
(iv) Converse: If you have winter clothes, then you live in Delhi.
Contrapositive: If you do not have winter clothes, then you do not live in Delhi.
(v) Converse: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.
Contrapositive: If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.

3. (i) You watch television if and only if your mind is free.
(ii) A quadrilateral is a rectangle if and only if it is equiangular.
(iii) You get an A grade if and only if you do all the homework regularly.
(iv) A tumbler is half empty if and only if it is half full.

4. (i) If Mohan is not poor, then he is not a poet.
(ii) If Max does not study, then he will not pass the test.
(iii) If she does not earn money, then she does not work.
(iv) If they do not drive the car, then there is no snow.
(v) If it rains, then it is not cold.
(vi) If it did not snow, then Ravish will not ski.
(vii) If x is positive, then x is not less than zero.
(viii) If he does not win, then he does not have courage.
(ix) If he is not strong, then he is not a sailor.
(x) If he tires, then he will not win.
(xi) If x is even, then x^2 is even.

31.8 VALIDITY OF STATEMENTS

In this section, we will study validity of statements. Checking the validity of a statement means checking when it is true and when it is not true. This depends upon which of the special words "and", "or", and which of the implications "if-then", "if and only if", and which of the quantifiers "for every", "there exists", appear in the given statement.

Let us now discuss some techniques or rules to find when a statement is valid or true.

31.8.1 VALIDITY OF STATEMENTS WITH "AND"

If p and q are mathematical statements, then in order to show that the statement "p and q" is true, we follow the following steps:

STEP I *Show that the statement p is true.*
STEP II *Show that the statement q is true.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Given below are two statements:*

p : *80 is a multiple of* 5.
q : *80 is a multiple of* 4.

Write the compound statement connecting these two statements with "and" and checks its validity.

SOLUTION The compound statement is: "*80 is multiple of* 5 *and* 4."

We know that 80 is a multiple of 5 as well as 4. So, p and q are true statements.

Hence, the compound statement is also true i.e. the compound statement "p and q" is a valid statement.

EXAMPLE 2 *If p and q are two statements given by:*

p : 25 *is multiple of* 5.

q : 25 *is a multiple of* 8.

Write the compound statement connecting these two statements with "and" and check its validity.

SOLUTION The compound statement is : "25 *is a multiple of* 5 *and* 8"

Since 25 is a multiple of 5 but it is not a multiple of 8. Therefore, p is true but q is not true.

Hence, the compound statements is not true i.e., the statement "p and q" is not a valid statement.

31.8.2 VALIDITY OF STATEMENTS WITH "OR"

If p and q are mathematical statements, then in order to show that the compound statement "p or q" is true, we proceed as follows:

Assuming that p is false, show that q must be true.

or Assuming that q is false, show that p must be true.

Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Given below are two statements:*

p : 25 *is a multiple of* 5.

q : 25 *is a multiple of* 8.

Write the compound statement connecting these two statements with "OR" and check its validity.

SOLUTION The compound statement is: "25 *is a multiple of* 5 *or* 8".

Let us assume that the statement q is false i.e. 25 is not a multiple of 8. Clearly, p is true.

Thus, if we assume that q is false, then p is true. Hence, the compound statement is true i.e. valid.

EXAMPLE 2 *Check the validity of the following statement:*

"Square of an integer is positive or negative "

SOLUTION The given statement is a compound statement with "OR" whose component statements are:

p : *Square of an integer is positive.*

q : *Square of an integer is negative.*

Let us assume that p is false i.e. square of an integer is not positive. Then, for any integer x, we have

$$x^2 \ngtr 0 \Rightarrow x^2 < 0 \Rightarrow q \text{ is true.}$$

Thus, if we assume that p is false, then q is true.

Hence, "p or q" is a valid statement. In other words, the given statement is true.

31.8.3 VALIDITY OF STATEMENTS WITH "IF-THEN"

If p and q are two mathematical statements, then to prove the validity of the statement "if p, then q", we may use any one of the following methods.

(i) DIRECT METHOD

STEP I *Assume that p is true.*

STEP II *Prove that q is true.*

(ii) CONTRAPOSITIVE METHOD

STEP I *Assume that q is false.*

STEP II *Prove that p is false.*

(iii) CONTRADICTION METHOD

STEP I *Assume that p is true and q is false.*

STEP II *Obtain a contradication from step I.*

Following examples will illustrate the above methods.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Check whether the following statement is true or not:*
"If x and y are odd integers, then xy is an odd integer"

SOLUTION Let p and q be the statements given by

p : x and y are odd integers.

q : xy is an odd integer.

Then, the given statement is: If p, then q.

Direct Method: Let p be true. Then,

p is true

$\Rightarrow$ x and y are odd integers

$\Rightarrow$ $x = 2m+1, y = 2n+1$ for some integers m, n.

$\Rightarrow$ $xy = (2m+1)(2n+1)$

$\Rightarrow$ $xy = 2(2mn+m+n)+1$

$\Rightarrow$ xy is an odd integer

$\Rightarrow$ q is true.

Thus, p is true $\Rightarrow$ q is true.

Hence, "If p, then q" is a true statement.

Contrapositive Method: Let q be not true. Then,

q is not true

$\Rightarrow$ xy is an even integer

$\Rightarrow$ either x is even or y is even or both x and y are even

$\Rightarrow$ p is not true.

Thus, q is false $\Rightarrow$ p is false

Hence, "If p, then q" is a true statement.

EXAMPLE 2 *Check whether the following statement is true or false by proving its contrapositive.*
"If x, y are integers such that xy is odd, then both x and y are odd integers"

SOLUTION Let p and q be two statements given by

p : xy is an odd integer

q : Both x and y are odd integers.

Let q be not true. Then,

q is not true

$\Rightarrow$ It is false that both x and y are odd integers

$\Rightarrow$ At least one of x and y is an even integer.

Let $x = 2n$ for some integer n. Then,

$xy = 2ny$ for some integer n.

$\Rightarrow$ xy is an even integer

$\Rightarrow$ xy is not an odd integer

$\Rightarrow$ $\sim p$ is true

Thus, $\sim q$ is true $\Rightarrow \sim p$ is true

Hence, the given statement is true.

EXAMPLE 3 *Show that the statement*

p : If x is a real number such that $x^3 + 4x = 0$, then x is 0" is true by

(i) *direct method* (ii) *method of contradiction* (iii) *method of contrapositive*

SOLUTION Let q and r be the statements given by

$q : x$ is a real number such that $x^3 + 4x = 0$.

$r : x$ is 0.

Then, p : If q, then r.

(i) *Direct method*: Let q be true. Then,

q is true

$\Rightarrow$ x is a real number such that $x^3 + 4x = 0$

$\Rightarrow$ x is a real number such that $x(x^2 + 4) = 0$

$\Rightarrow$ $x = 0$ $[\because x \in R \ \therefore x^2 + 4 \neq 0]$

$\Rightarrow$ r is true.

Thus, q is true $\Rightarrow$ r is true.

Hence, p is true.

(ii) *Method of contradiction*: If possible, let p be not true. Then,

p is not true

$\Rightarrow$ $\sim p$ is true

$\Rightarrow$ $\sim (q \Rightarrow r)$ is true $[\because p : q \Rightarrow r]$

$\Rightarrow$ q and $\sim r$ is true $[\because \sim (q \Rightarrow r) \cong q \text{ and } \sim r]$

$\Rightarrow$ x is a real number such that $x^3 + 4x = 0$ and $x \neq 0$

$\Rightarrow$ $x = 0$ and $x \neq 0$

This a contradiction.

Hence, p is true.

(iii) *Method of contrapositive:* Let r be not true. Then,

r is not true

$\Rightarrow$ $x \neq 0, \ x \in R$

$\Rightarrow$ $x(x^2 + 4) \neq 0, \ x \in R$

$\Rightarrow$ q is not true

Thus, $\sim r \Rightarrow \sim q$.

Hence, $p : q \Rightarrow r$ is true.

EXAMPLE 4 *Show that the following statement is true by the method of contrapositive.*

p : If x is an integer and x^2 is even, then x is also even.

SOLUTION Let q and r be the statements given by

q : If x is an integer and x^2 is even

r : x is an even integer.

Then, *p : "If q, then r".*

If possible, let r be false. Then,

r is false

$\Rightarrow$ x is not an even integer

$\Rightarrow$ x is an odd integer

$\Rightarrow$ $x = (2n+1)$ for some integer n

$\Rightarrow$ $x^2 = 4n^2 + 4n + 1$

$\Rightarrow$ $x^2 = 4n(n+1) + 1$

$\Rightarrow$ x^2 is an odd integer [$\because$ $4n(n+1)$ is even]

$\Rightarrow$ q is false.

Thus, r is false $\Rightarrow$ q is false.

Hence, p: "If q, then r" is a true statement.

31.8.4 VALIDITY OF STATEMENTS WITH "IF AND ONLY IF"

In order to prove the validity of the statement "p if and only if q", we proceed as follows:

STEP I Show that: If p is true, then q is true.

STEP II Show that: If q is true, then p is true.

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Using the words "necessary and sufficient" rewrite the statement*

"The integer n is odd if and only if n^2 is odd"

Also check whether the statement is true.

SOLUTION The given statement can be re-written as

"The necessary and sufficient condition that the integer n is odd is n^2 must be odd"

Let p and q be the statements given by

p : *the integer n is odd.*

q : *n^2 is odd.*

The given statement is

"p if and only if q"

In order to check its validity, we have to check the validity of the following statements.

(i) *"If p, then q"* (ii) *"If q, then p"*

Checking the validity of "If p, then q":

The statement "if p, then q" is given by:

"If the integer n is odd, then n^2 is odd"

Let us assume that n is odd. Then,

$n = 2m + 1$, where m is an integer

$\Rightarrow$ $n^2 = (2m+1)^2$

$\Rightarrow$ $n^2 = 4m(m+1) + 1$

$\Rightarrow$ n^2 is an odd integer [$\because$ $4m(m+1)$ is even]

$\Rightarrow$ n^2 is odd.

Thus, n is odd $\Rightarrow$ n^2 is odd

$\therefore$ *"If p, then q" is true.*

Checking the validity of "If q, then p":

The statement "If q, then p" is given by

"*If n is an integer and n^2 is odd, then n is odd*"

To check the validity of this statements, we will use contrapositive method. So, let n be an even integer. Then,

n is even

$\Rightarrow$ $n = 2k$ for some integer k

$\Rightarrow$ $n^2 = 4k^2$

$\Rightarrow$ n^2 is an even integer

$\Rightarrow$ n^2 is not an odd integer.

Thus, n is not odd $\Rightarrow$ n^2 is not odd

$\therefore$ "If q, then p" is true.

Hence, "p if and only if q" is true.

31.8.5 VALIDITY OF STATEMENTS BY CONTRADICTION

Sometimes to check whether a statement p is true or not, we assume that p is not true i.e. $\sim p$ is true. Then, we arrive at some result which contradicts our supposition. Therefore, we conclude that p is true. This method is known as contradiction method.

Following examples will illustrate this method.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Verify by the method of contradiction that $\sqrt{7}$ is irrational.*

SOLUTION Let p be the statement given by $p : \sqrt{7}$ *is irrational.*

If possible, let p be not true i.e. let p be false. Then,

p is false

$\Rightarrow$ $\sqrt{7}$ is rational

$\Rightarrow$ $\sqrt{7} = \frac{a}{b}$, where a and b are integers having no common factor.

$\Rightarrow$ $7 = \frac{a^2}{b^2}$

$\Rightarrow$ $a^2 = 7b^2$

$\Rightarrow$ 7 divides a^2

$\Rightarrow$ 7 divides a

$\Rightarrow$ $a = 7c$ for some integer c

$\Rightarrow$ $a^2 = 49c^2$

$\Rightarrow$ $7b^2 = 49c^2$ $\quad [\because a^2 = 7b^2]$

$\Rightarrow$ $b^2 = 7c^2$

$\Rightarrow$ 7 divides b^2

$\Rightarrow$ 7 divides b

Thus, 7 is a common factor of both a and b. This contradicts that a and b have no common factor. So, the supposition $\sqrt{7}$ is rational is wrong. Hence, the statement "$\sqrt{7}$ is irrational" is true.

EXAMPLE 2 *Check the validity of the statement given below by contradiction method.*

"*p : The sum of an irrational number and a rational number is irrational*"

SOLUTION If possible, let p be not true. Then,

p is false

$\Rightarrow$ The sum of an irrational number and a rational number is not irrational

$\Rightarrow$ There exists an irrational number x (say) and a rational number y (say) such that $x + y$ is not irrational.

$\Rightarrow$ $x + y$ is rational say, z

$\Rightarrow$ $x + y = z$

$\Rightarrow$ $x = z - y$

$\Rightarrow$ x is rational [$\because$ $z - y$ is rational]

But, x is irrational. So, we arrive at a contradiction.

Thus, our supposition is wrong.

Hence, p is true.

31.8.6 INVALIDITY OF STATEMENTS BY COUNTER EXAMPLES

In order to show that a statement is false, we may give an example of a situation where the statement is not valid. Such an example is called a counter example. The name itself suggests that this is an example to counter the statement. Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *By giving an example, show that the following statement is false.*

"If n is an odd integer, then n is prime"

SOLUTION We observe that 9 is an odd integer which is not prime. Similarly, 21, 25 etc are odd integers which are not primes.

Hence, the given statement is false.

EXAMPLE 2 *Show that the statement*

"For any real numbers a and b, $a^2 = b^2$ implies that $a = b$"

is not true by giving a counter example.

SOLUTION We observe that $(-2)^2 = 2^2$ but $-2 \neq 2$.

So, the given statement is not true.

EXAMPLE 3 *By giving a counter example, show that the following statement is not true:*

p : "The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2"

SOLUTION We observe that $x = 1$ is a root of $x^2 - 1 = 0$ and $x = 1$ lies between 0 and 2.

So, the given statement is not true.

EXERCISE 31.6

LEVEL-1

1. Check the validity of the following statements:
   (i) p : 100 is a multiple of 4 and 5. (ii) q : 125 is a multiple of 5 and 7.
   (iii) r : 60 is a multiple of 3 or 5.
2. Check whether the following statement are true or not:
   (i) p : If x and y are odd integers, then $x + y$ is an even integer.
   (ii) q : If x, y are integers such that xy is even, then at least one of x and y is an even integer.

3. Show that the statement

p : *"If x is a real number such that $x^3 + x = 0$, then x is 0"*

is true by

(i) direct method (ii) method of contrapositive (iii) method of contradition.

4. Show that the following statement is true by the method of contrapositive

p : *"If x is an integer and x^2 is odd, then x is also odd"*

5. Show that the following statement is true

"The integer n is even if and only if n^2 is even"

6. By giving a counter example, show that the following statement is not true.

p : *"If all the angles of a triangle are equal, then the triangle is an obtuse angled triangle".*

7. Which of the following statements are true and which are false? In each case give a valid reason for saying so

(i) p : Each radius of a circle is a chord of the circle.

(ii) q : The centre of a circle bisects each chord of the circle.

(iii) r : Circle is a particular case of an ellipse.

(iv) s : If x and y are integers such that $x > y$, then $-x < -y$.

(v) t : $\sqrt{11}$ is a rational number.

8. Determine whether the argument used to check the validity of the following statement is correct:

p : "If x^2 is irrational, then x is rational"

The statement is true because the number $x^2 = \pi^2$ is irrational, therefore $x = \pi$ is irrational.

ANSWERS

1. (i) True	(ii) False	(iii) True	2. (i) True	(ii) True
7. (i) False	(ii) False	(iii) True	(iv) True	(v) False

SUMMARY

1. A sentence is called a mathematically acceptable statement or simply a statement if it is either true or false but not both.
2. The denial of a statement is called negation of the statement. The negation of a statement p is denoted by $\sim p$ and is read as "not p".
3. A statement is called a compound statement if it is made up of two or more simple statements. The simple statements are called component statements of the compound statement.
4. Compound statements are obtained by using connecting words like "and", "or" etc and phrases "If-then", "Only if", "if and only if", "There exists", "For all" etc.
5. The compound statement with "AND" is
   (i) true if all its component statements are true.
   (ii) false if any of its component statements is false.
6. The compound statement with "OR" is
   (i) true when one component statement is true or both the component statements are true.
   (ii) false when both the component statements are false.

7. A sentence with "If p, then q" can be written in the following ways:
   (i) p implies q (denoted by $p \Rightarrow q$)
   (ii) p is sufficient condition for q
   (iii) q is necessary condition for p
   (iv) p only if q
   (v) $\sim q$ implies $\sim p$

8. (i) The contrapositive of the statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$.
   (ii) The converse of the statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.
   (iii) The inverse of the statement $p \Rightarrow q$ is the statement $\sim p \Rightarrow \sim q$.

9. For all or for every is called universal quantifier. There exists is called existential quantifier.

10. A statement is said to valid or invalid according as it is true or false.

11. If p and q are two mathematical statements, then the statement
   (i) "p and q" is true if both p and q are true.
   (ii) "p or q" is true if p is false $\Rightarrow$ q is true or, q is false $\Rightarrow$ p is true.
   (iii) "If p, then q" is true if
   (a) p is true $\Rightarrow$ q is true
   or (b) q is false $\Rightarrow$ p is false
   or (c) p is true and q is false leads us to a contradiction.
   (iv) "p if and only if q" is true, if
   (a) p is true $\Rightarrow$ q is true
   and (b) q is true $\Rightarrow$ p is true.

CHAPTER 32

STATISTICS

32.1 INTRODUCTION

In earlier classes, we have learnt about methods of representing data graphically and in tabular form. Such representations exhibit certain characteristics or salient features of the data. We have also studied various methods of finding a representative value of the given data. This value is called the central value for the given data and various methods for finding the central value are known as the measures of central tendency. The measures of central tendency are: mean (arithmetic mean), median and mode. We have learnt that the measures of central tendency give us one single figure that represents the entire data i.e., they give us one single figure around which the observations are concentrated. In other words, measures of central tendency give us a rough idea where observations are centred. But the central values are inadequate to give us a complete idea of the distribution as they do not tell us the extent to which the observations vary from the central value. In order to make better interpretation from the data, we should also have an idea how the observations are scattered or how much they are bunched around a central value. There can be two or more distributions having the same central value but still there can be wide disparities in the formation of the distribution as discussed below.

Consider following three distributions:

(i) 1, 5, 9, 13, 17 (ii) 3, 6, 9, 12, 15 (iii) 7, 8, 9, 10, 11

In all these distributions we have the same number of observations and the same mean and median both equal to 9. Therefore, if we are given that the mean of 5 observations is 9, we are unable to say whether it is the average of first distribution or second distribution or third distribution.

Let us now plot these distributions on a number line as shown below:

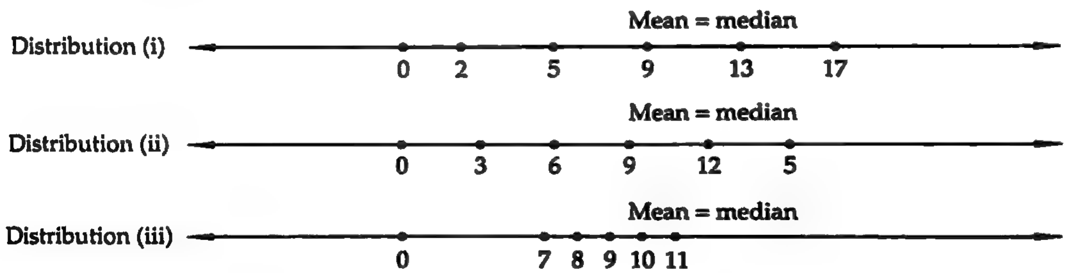


Fig. 32.1

We observe that the dots representing observations in distribution (iii) are more close to each other and are clustering around the mean and median (central value). So, we say that there is more variability in the values of observations in distribution (i) in comparison to distributions (ii) and (iii). We can also say that the distribution (iii) is more consistent than distributions (i) and (ii).

Let us now consider the runs scored by two batsmen B_1 and B_2 in their last ten matches as given below:

Match:	1	2	3	4	5	6	7	8	9	10
Batsman B_1:	30	91	0	64	42	80	30	5	117	71
Batsman B_2:	53	46	48	50	53	53	58	60	57	52

The mean and median of the scores are as under:

	Mean	Median
Batsman B_1	53	53
Batsman B_2	53	53

We observe that the mean and median of the runs scored by both the batsmen B_1 and B_2 are same. On the basis of this a natural question arises : Is the performance of two players same? The answer is of course not in affirmative. Because the variability in the scores of batsman B_1 is more as he has scored runs from 0 (minimum) to 117 (maximum), where as the batsman B_2 has scored runs more consistently as the runs scored by him vary from 46 (minimum) to 60 (maximum). If the scores of batsmen B_1 and B_2 are plotted on a number line, we find that the points representing scores of batsman B_2 cluster around the central value (mean = median) while those corresponding to batsman B_1 are scattered or more spread out.

It follows from the above discussion that the central values (mean, mode, median) are not sufficient to give complete information about a distribution. Variability in the values of the observations of given data gives us better information about the data. So, variability is another factor which is required to be studied in statistics. Like central value, we have a single number to describe variability of a distribution. This single number is called the dispersion of the distribution and various methods of determining or measuring dispersion are called the measures of dispersion. In this chapter, we shall learn some of the important measures of dispersion.

32.2 MEASURES OF DISPERSION

As discussed above that the dispersion is the measure of variations in the values of the variable. It measures the degree of scatteredness of the observations in a distribution around the central value.

Following are commonly used measures of dispersion:

(i) Range (ii) Quartile deviation (iii) Mean deviation (iv) Standard deviation.

In this chapter, we shall study all of these measures of dispersion except the quartile deviation.

32.3 RANGE

RANGE *The range is the difference between two extreme observations of the distribution.*

If A and B are the greatest and smallest values respectively of observations in a distribution, then its rang is $A - B$.

Thus,

Range of a distribution = Maximum value − Minimum value

In section 32.1, we have

Range of scores of batsman $B_1 = 117 - 0 = 117$

Range of scores of batsman $B_2 = 60 - 46 = 14$.

Clearly, the range of scores of batsman B_1 is more than that of B_2. Therefore, the scores of batsman B_1 are more scattered or dispersed while the scores are more close to each other for batsman B_2.

Range is the simplest but a crude measure of dispersion. As it is based upon two extreme observations so it does not measure the dispersion of the data from a central value. Therefore, we require some other measures of variability which depend upon the difference (or deviation) of the values from the central value. Such measures of dispersion are mean deviation and standard deviation. Let us discuss them in detail.

32.4 MEAN DEVIATION

In this section, we will learn how to calculate mean deviation about mean and median for various types of data.

32.4.1 MEAN DEVIATION FOR UNGROUPED DATA OR INDIVIDUAL OBSERVATIONS

If $x_1, x_2, \ldots, x_n$ are n values of a variable X, then the mean deviation from an average A (median or AM) is given by

$$M.D. = \frac{1}{n} \sum_{i=1}^{n} |x_i - A| = \frac{1}{n} \Sigma |d_i|, \text{ where } d_i = x_i - A$$

We may use the following algorithm to find mean deviation of individual observations:

ALGORITHM

STEP I *Compute the central value or average 'A' about which mean deviation is to be calculated.*

STEP II *Take deviations of the observations about the central value 'A' obtained in step I ignoring ± signs and denote these deviations by $|d_i|$.*

STEP III *Obtain the total of these deviations i.e. $\Sigma |d_i|$.*

STEP IV *Divide the total obtained in step III by the number of observations.*

Following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Calculate the mean deviation about median from the following data: 340, 150, 210, 240, 300, 310, 320.*

SOLUTION Arranging the observations in ascending order of magnitude, we obtain 150, 210, 240, 300, 310, 320, 340.

Clearly, the middle observation is 300. So, median = 300.

Calculation of Mean Deviation

x_i	$\|d_i\| = \|x_i - 300\|$
340	40
150	150
210	90
240	60
300	0
310	10
320	20
Total	$d_i = \Sigma \|x_i - 300\| = 370$

$$\therefore \quad M.D. = \frac{1}{n} \Sigma |d_i| = \frac{1}{7} \Sigma |x_i - 300| = \frac{370}{7} = 52.8$$

EXAMPLE 2 *The scores of a batsman in ten innings are : 38, 70, 48, 34, 42, 55, 63, 46, 54, 44. Find the mean deviation about the median.*

SOLUTION Arranging the data in ascending order, we obtain

34, 38, 42, 44, 46, 48, 54, 55, 63, 70

Here $n = 10$. So, median is the $A.M.$ of 5th and 6th observations.

$\therefore \quad \text{Median} = \dfrac{46+48}{2} = 47$

Calculation of Mean Deviation

x_i	$\lvert d_i \rvert = \lvert x_i - 47 \rvert$
38	9
70	23
48	1
34	13
42	5
55	8
63	16
46	1
54	7
44	3
Total	$\Sigma \lvert d_i \rvert = 86$

$\therefore \quad M.D. = \dfrac{1}{n}\,\Sigma \lvert d_i \rvert = \dfrac{86}{10} = 8.6$

EXAMPLE 3 *Find the mean deviation from the mean for the data:* 6, 7, 10, 12, 13, 4, 8, 20 [NCERT]

SOLUTION Let $\overline{X}$ be the mean of the given data. Then,

$$\overline{X} = \frac{6+7+10+12+13+4+8+20}{8} = 10$$

Computation of Mean Deviation

x_i	$\lvert d_i \rvert = \lvert x_i - \overline{X} \rvert = \lvert x_i - 10 \rvert$
6	4
7	3
10	0
12	2
13	3
4	6
8	2
20	10
Total	$\Sigma\, d_i = 30$

Now, $\quad \Sigma \lvert d_i \rvert = 30$ and $n = 8$

$\therefore \quad M.D. = \dfrac{1}{n}\,\Sigma \lvert d_i \rvert = \dfrac{30}{8} = 3.75$

Thus, the mean deviation from the mean for the given data is 3.75.

LEVEL-2

EXAMPLE 4 *Calculate the mean deviation about the mean of the set of first n natural numbers when n is odd natural number.* [NCERT, NCERT EXEMPLAR]

SOLUTION Since n is an odd natural number. Therefore, $n = 2m+1$ for some natural number m. Let $\overline{X}$ be the mean of first n natural numbers. Then,

$$\overline{X} = \frac{1+2+3+\ldots+(n-1)+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\Rightarrow \quad \overline{X}=\frac{2m+1+1}{2}=m+1$$

The mean deviation (M.D.) about mean is given by

$$M.D.=\frac{1}{n}\sum_{r=1}^{n}|r-\overline{X}|$$

$$\Rightarrow \quad M.D.=\frac{1}{2m+1}\sum_{r=1}^{2m+1}|r-(m+1)| \qquad [\because n=2m+1]$$

$$\Rightarrow \quad M.D.=\frac{1}{2m+1}\left\{\sum_{r=1}^{m}|r-(m+1)|+\sum_{r=m+1}^{2m+1}|r-(m+1)|\right\}$$

$$\Rightarrow \quad M.D.=\frac{1}{2m+1}\left\{\sum_{r=1}^{m}-\{r-(m+1\}+\sum_{r=m+1}^{2m+1}\{r-(m+1)\}\right\}$$

$$\Rightarrow \quad M.D.=\frac{1}{2m+1}\left\{-\sum_{r=1}^{m}r+(m+1)\sum_{r=1}^{m}1+\sum_{r=m+1}^{2m+1}r-(m+1)\sum_{r=m+1}^{2m+1}1\right\}$$

$$\Rightarrow \quad M.D.=\frac{1}{2m+1}\left\{-\frac{m(m+1)}{2}+m(m+1)+\left(\frac{m+1}{2}\right)\{(m+1)+(2m+1)\}-(m+1)(m+1)\right\}$$

$$\Rightarrow \quad M.D.=\frac{1}{2m+1}\left\{-\frac{m(m+1)}{2}+m(m+1)+\frac{1}{2}(m+1)(3m+2)-(m+1)^2\right\}$$

$$\Rightarrow \quad M.D.=\frac{1}{2m+1}\left\{\frac{m(m+1)}{2}+\frac{1}{2}(m+1)(3m+2)-(m+1)^2\right\}$$

$$\Rightarrow \quad M.D.=\frac{m+1}{2(2m+1)}\{m+(3m+2)-2(m+1)\}$$

$$\Rightarrow \quad M.D.=\frac{m+1}{2(2m+1)}(2m)=\frac{m(m+1)}{2m+1}=\frac{\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)}{n} \qquad [\because n=2m+1]$$

$$\Rightarrow \quad M.D.=\frac{1}{n}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)=\frac{n^2-1}{4n}$$

EXAMPLE 5 *Calculate the mean deviation about the mean of the set of first n natural numbers when n is even natural number.* **[NCERT EXEMPLAR]**

SOLUTION Since n is an even natural number. Therefore, $n=2m$ for some natural number m. Let $\overline{X}$ be the mean of first n natural numbers. Then,

$$\overline{X}=\frac{1+2+3+\ldots+n}{n}=\frac{n(n+1)}{2n}=\frac{n+1}{2}$$

$$\Rightarrow \quad \overline{X}=\frac{2m+1}{2}=m+\frac{1}{2} \qquad [\because n=2m]$$

The mean deviation (M.D.) about mean is given by

$$M.D.=\frac{1}{n}\sum_{r=1}^{n}|r-\overline{X}|$$

$$\Rightarrow \quad M.D.=\frac{1}{2m}\sum_{r=1}^{2m}\left|r-\left(m+\frac{1}{2}\right)\right| \qquad [\because n=2m+1]$$

$$\Rightarrow \quad M.D.=\frac{1}{2m}\left[\sum_{r=1}^{m}\left|r-\left(m+\frac{1}{2}\right)\right|+\sum_{r=m+1}^{2m}\left|r-\left(m+\frac{1}{2}\right)\right|\right]$$

$$\Rightarrow \quad M.D. = \frac{1}{2m}\left[\sum_{r=1}^{m} -\left\{r-\left(m+\frac{1}{2}\right)\right\} + \sum_{r=m+1}^{2m}\left\{r-\left(m+\frac{1}{2}\right)\right\}\right]$$

$$\Rightarrow \quad M.D. = \frac{1}{2m}\left\{-\sum_{r=1}^{m} r + \sum_{r=1}^{m}\left(m+\frac{1}{2}\right) + \sum_{r=m+1}^{2m} r - \sum_{r=m+1}^{2m}\left(m+\frac{1}{2}\right)\right\}$$

$$\Rightarrow \quad M.D = \frac{1}{2m}\left\{-\frac{m(m+1)}{2} + m\left(m+\frac{1}{2}\right) + \frac{m}{2}\{(m+1)+2m\} - \left(m+\frac{1}{2}\right)m\right\}$$

$$\Rightarrow \quad M.D = \frac{1}{2m}\left\{-\frac{m(m+1)}{2} + \frac{m(2m+1)}{2} + \frac{m(3m+1)}{2} - \frac{m(2m+1)}{2}\right\}$$

$$\Rightarrow \quad M.D = \frac{1}{2m}\left\{\frac{-m(m+1)}{2} + \frac{m(3m+1)}{2}\right\} = \frac{m}{4m}(-m-1+3m+1) = \frac{m}{2} = \frac{n}{4} \qquad [\because n = 2m]$$

EXERCISE 32.1

LEVEL-1

1. Calculate the mean deviation about the median of the following observations:
   (i) 3011, 2780, 3020, 2354, 3541, 4150, 5000
   (ii) 38, 70, 48, 34, 42, 55, 63, 46, 54, 44
   (iii) 34, 66, 30, 38, 44, 50, 40, 60, 42, 51
   (iv) 22, 24, 30, 27, 29, 31, 25, 28, 41, 42
   (v) 38, 70, 48, 34, 63, 42, 55, 44, 53, 47 [NCERT]
2. Calculate the mean deviation from the mean for the following data:
   (i) 4, 7, 8, 9, 10, 12, 13, 17 [NCERT]
   (ii) 13, 17, 16, 14, 11, 13, 10, 16, 11, 18, 12, 17 [NCERT]
   (iii) 38, 70, 48, 40, 42, 55, 63, 46, 54, 44 [NCERT]
   (iv) 36, 72, 46, 42, 60, 45, 53, 46, 51, 49 [NCERT]
   (v) 57, 64, 43, 67, 49, 59, 44, 47, 61, 59
3. Calculate the mean deviation of the following income groups of five and seven members from their medians:

I *Income in ₹*	*II* *Income in ₹*
4000	3800
4200	4000
4400	4200
4600	4400
4800	4600
	4800
	5800

4. The lengths (in cm) of 10 rods in a shop are given below:
   40.0, 52.3, 55.2, 72.9, 52.8, 79.0, 32.5, 15.2, 27.9, 30.2
   (i) Find mean deviation from median
   (ii) Find mean deviation from the mean also.
5. In question 1 (iii), (iv), (v) find the number of observations lying between $\overline{X} - M.D.$ and $\overline{X} + M.D$, where M.D. is the mean deviation from the mean.

ANSWERS

1. (i) 649.4	(ii) 8.6	(iii) 8.7	(iv) 4.7	(v) 8.4
2. (i) 3	(ii) 2.33	(iii) 8.4	(iv) 7.2	(v) 74
3. 320, 457.14	4. (i) 16.4	(ii) 16.44	5. 6, 5 and 6	

HINTS TO NCERT & SELECTED PROBLEMS

1. (v) Arranging the observations in ascending order of magnitudes, we obtain

34, 38, 42, 44, 47, 48, 53, 55, 63, 70

These are 10 in number. Therefore,

$$\text{Median} = \text{AM of 5th and 6th observation} = \frac{47+48}{2} = 47.5$$

Calculation of Mean deviation about median

x_i	$\lvert d_i \rvert = \lvert x_i - 47.5 \rvert$
34	13.5
38	9.5
42	5.5
44	3.5
47	0.5
48	0.5
53	5.5
55	7.5
63	15.5
70	22.5
Total	$\Sigma \lvert d_i \rvert = 84$

Clearly, $n = 10$ and $\Sigma|d_i| = 84$.

$\therefore$ $$\text{Mean Deviation} = \frac{1}{n}\Sigma|d_i| = \frac{84}{10} = 8.4$$

2. (i) Let $\overline{X}$ be the mean of the given observations. Then,

$$\overline{X} = \frac{4+7+8+9+10+12+13+17}{8} = 10$$

Computation of Mean deviation about mean

x_i	$\lvert d_i \rvert = \lvert x_i - \overline{X} \rvert = \lvert x_i - 10 \rvert$
4	6
7	3
8	2
9	1
10	0
12	2
13	3
17	7
Total	$\Sigma \lvert d_i \rvert = 24$

Clearly, $\Sigma |d_i| = 24$ and $n = 8$.

$\therefore$ Mean deviation $= \frac{1}{n} \Sigma |d_i| = \frac{24}{8} = 3$

(ii) Let $\overline{X}$ be the mean of the given data. Then,

$$\overline{X} = \frac{13+17+16+14+11+13+10+16+11+18+12+17}{12} = \frac{168}{12} = 14$$

Computation of Mean deviation about mean

x_i	$\lvert d_i \rvert = \lvert x_i - 14 \rvert$
13	1
17	3
16	2
14	0
11	3
13	1
10	4
16	2
11	3
18	4
12	2
17	3
Total	$\Sigma \lvert d_i \rvert = 28$

Thus, we have

$n = 12$ and $\Sigma |d_i| = 28$

$\therefore$ Mean deviation $= \frac{1}{n} \Sigma |d_i| = \frac{28}{12} = 2.33$

(iii) Let $\overline{X}$ be the mean of given observations. Then,

$$\overline{X} = \frac{38+70+48+40+42+55+63+46+54+44}{10} = \frac{500}{10} = 50$$

Computation of Mean deviation about mean

x_i	$\lvert d_i \rvert = \lvert x_i - 50 \rvert$
38	12
70	20
48	2
40	10
42	8
55	5
63	13
46	4
54	4
44	6
Total	$\Sigma \lvert d_i \rvert = 84$

Thus, we have

$n = 10$ and $\Sigma |d_i| = 104$

$\therefore$ Mean deviation $= \frac{1}{n} \Sigma |d_i| = \frac{84}{10} = 8.4$

(iv) Let $\overline{X}$ be the mean of given data. Then,

$$\overline{X} = \frac{36 + 72 + 46 + 42 + 60 + 45 + 53 + 46 + 51 + 49}{10} = \frac{500}{10} = 50$$

Computation of Mean deviation about mean

x_i	$\lvert d_i \rvert = \lvert x_i - 50 \rvert$
36	14
72	22
46	4
42	8
60	10
45	5
53	3
46	4
51	1
49	1
Total	$\Sigma \lvert d_i \rvert = 72$

Thus, we have, $n = 10$ and $\Sigma |d_i| = 72$.

$\therefore$ Mean deviation $= \frac{1}{n} \Sigma d_i = \frac{72}{10} = 7.2$

32.4.2 MEAN DEVIATION OF A DISCRETE FREQUENCY DISTRIBUTION

If x_i / f_i, $i = 1, 2, \ldots, n$ is the frequency distribution, then mean deviation from an average A (median or AM) is given by

$$M.D. = \frac{1}{N} \Sigma f_i |x_i - A|, \text{ where } \sum_{i=1}^{n} f_i = N$$

We may use the following algorithm to find the mean deviation of a discrete frequency distribution.

ALGORITHM

<u>STEP I</u> *Calculate the central value or average 'A' of the given frequency distribution about which mean deviation is to be calculated.*

<u>STEP II</u> *Take deviations of the observations from the central value in step I ignoring signs and denote them by* $|d_i|$.

<u>STEP III</u> *Multiply these deviations by respective frequencies and obtain the total* $\Sigma f_i |d_i|$.

<u>STEP IV</u> *Divide the total obtained in step III by the number of observations i.e.* $N = \Sigma f_i$ *to obtain the mean deviation.*

Following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Calculate mean deviation about mean from the following data:*

x_i :	3	9	17	23	27
f_i :	8	10	12	9	5

SOLUTION Calculation of mean deviation about mean.

x_i	f_i	$f_i x_i$	$\lvert x_i - 15 \rvert$	$f_i \lvert x_i - 15 \rvert$
3	8	24	12	96
9	10	90	6	60
17	12	204	2	24
23	9	207	8	72
27	5	135	12	60
	$N = \Sigma f_i = 44$	$\Sigma f_i x_i = 660$		$\Sigma f_i \lvert x_i - 15 \rvert = 312$

$$\text{Mean} = \overline{X} = \frac{1}{N}(\Sigma f_i x_i) = \frac{660}{44} = 15$$

$$\text{Mean deviation} = M.D. = \frac{1}{N} \Sigma f_i \lvert x_i - 15 \rvert = \frac{312}{44} = 7.09.$$

EXAMPLE 2 *Calculate the mean deviation from the median for the following distribution:*

x_i	10	15	20	25	30	35	40	45
f_i	7	3	8	5	6	8	4	9

SOLUTION We have to calculate mean deviation about median. So, first we calculate median.

x_i	f_i	*Cumulative frequency*	$\lvert d_i \rvert = \lvert x_i - 30 \rvert$	$f_i \lvert d_i \rvert$
10	7	7	20	140
15	3	10	15	45
20	8	18	10	80
25	5	23	5	25
30	6	29	0	0
35	8	37	5	40
40	4	41	10	40
45	9	50	15	135
	$N = \Sigma f_i = 50$			$\Sigma f_i \lvert d_i \rvert = 505$

Clearly, $N = 50 \Rightarrow N/2 = 25$.

The cumulative frequency just greater than $N/2$ is 29 and the corresponding value of x is 30.

Therefore, median = 30.

Clearly, $\Sigma f_i \lvert x_i - 30 \rvert = \Sigma f_i d_i = 505$ and $N = 50$.

$\therefore$ $$\text{Mean deviation} = \frac{1}{N} \Sigma f_i \lvert d_i \rvert = \frac{505}{50} = 10.1$$

EXERCISE 32.2

LEVEL-1

1. Calculate the mean deviation from the median of the following frequency distribution:

Heights in inches	58	59	60	61	62	63	64	65	66
No. of students	15	20	32	35	35	22	20	10	8

2. The number of telephone calls received at an exchange in 245 successive one-minute intervals are shown in the following frequency distribution:

Number of calls	0	1	2	3	4	5	6	7
Frequency	14	21	25	43	51	40	39	12

Compute the mean deviation about median.

3. Calculate the mean deviation about the median of the following frequency distribution:

x_i	5	7	9	11	13	15	17
f_i	2	4	6	8	10	12	8

4. Find the mean deviation from the mean for the following data:

(i)

x_i	5	7	9	10	12	15
f_i	8	6	2	2	2	6

[NCERT]

(ii)

x_i	5	10	15	20	25
f_i	7	4	6	3	5

[NCERT]

(iii)

x_i	10	30	50	70	90
f_i	4	24	28	16	8

[NCERT]

(iv)

Size:	20	21	22	23	24
Frequency:	6	4	5	1	4

[NCERT EXEMPLAR]

(v)

Size:	1	3	5	7	9	11	13	15
Frequency:	3	3	4	14	7	4	3	4

[NCERT EXEMPLAR]

5. Find the mean deviation from the median for the following data:

(i)

x_i	15	21	27	30
f_i	3	5	6	7

[NCERT]

(ii)

x_i	74	89	42	54	91	94	35
f_i	20	12	2	4	5	3	4

(iii)

Mark obtained	10	11	12	14	15
No. of students	2	3	8	3	4

[NCERT EXEMPLAR]

ANSWERS

1. 1.703 2. 1.49 3. 2.72
4. (i) 3.38 (ii) 6.32 (iii) 15.3 (iv) 0.32 (v) 2.95
5. (i) 5.93 (ii) 12.5 (iii) 1.25

HINTS TO NCERT & SELECTED PROBLEMS

4. (i) Computation of mean deviation about mean

x_i	f_i	$f_i\, x_i$	$\lvert x_i - 9 \rvert$	$\Sigma f_i \lvert x_i - 9 \rvert$
5	8	40	4	32
7	6	42	2	12
9	2	18	0	0
10	2	20	1	2
12	2	24	3	6
15	6	90	6	36
	$N = \Sigma f_i = 26$	$\Sigma f_i\, x_i = 234$		$\Sigma f_i \lvert x_i - 9 \rvert = 88$

We have, $N = \Sigma f_i = 26$, and $\Sigma f_i\, x_i = 234$

$\therefore$ $\quad$ Mean $= \overline{X} = \dfrac{1}{N} \Sigma f_i\, x_i = \dfrac{234}{26} = 9$

Mean deviation $= \dfrac{1}{N} \Sigma f_i \lvert x_i - 9 \rvert = \dfrac{88}{26} = 3.38$

4. (ii) Computation of Mean deviation about mean

x_i	f_i	$f_i\, x_i$	$\lvert x_i - 14 \rvert$	$f_i \lvert x_i - 14 \rvert$
5	7	35	9	63
10	4	40	4	16
15	6	90	1	6
20	3	60	6	18
25	5	125	11	55
	$N = \Sigma f_i = 25$	$\Sigma f_i\, x_i = 350$		$\Sigma f_i \lvert x_i - 14 \rvert = 158$

$$\overline{X} = \frac{1}{N} \Sigma f_i\, x_i = \frac{350}{25} = 14$$

Mean deviation $= \dfrac{1}{N} \Sigma f_i \lvert x_i - 14 \rvert = \dfrac{158}{25} = 6.32$

(iii) Computation of mean deviation about mean

x_i	f_i	$f_i\, x_i$	$\lvert x_i - 49 \rvert$	$f_i \lvert x_i - 49 \rvert$
10	4	40	39	156
30	24	720	19	456
50	28	1400	1	28
70	16	1120	21	336
80	8	640	31	248
	$N = \Sigma f_i = 80$	$\Sigma f_i\, x_i = 3920$		$\Sigma f_i \lvert x_i - 49 \rvert = 1224$

$$\therefore \quad \overline{X} = \frac{1}{N} \Sigma f_i x_i = \frac{3920}{80} = 49$$

and, $\quad$ Mean deviation $= \frac{1}{N} \Sigma f_i | x_i - 49 | = \frac{1224}{80} = 15.3$

5. (i) Computation of mean deviation from median

x_i	f_i	$c.f.$	$\lvert x_i - 30 \rvert$	$f_i \lvert x_i - 30 \rvert$
15	3	3	15	45
21	5	8	9	45
27	6	14	7	42
30	7	21	0	0
35	8	29	5	40
		$N = \Sigma f_i = 29$		$\Sigma f_i \lvert x_i - 30 \rvert = 172$

We have, $N = 29 \Rightarrow \frac{N}{2} = 14.5$

The cumulative frequency just greater than $\frac{N}{2}$ i.e. 14.5 is 21. The corresponding value of the variable is 30. So, Median = 30.

$$\text{Mean deviation} = \frac{1}{N} \Sigma f_i | x_i - 30 | = \frac{172}{29} = 5.93$$

32.4.3 MEAN DEVIATION OF A GROUPED OR CONTINUOUS FREQUENCY DISTRIBUTION

For calculating mean deviation of a continuous frequency distribution the procedure is same as for a discrete frequency distribution. The only difference is that here we have to obtain the mid-points of the various classes and take the deviations of the these mid-points from the given central value (median or mean).

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the mean deviation about the median of the following frequency distribution:*

Class:	0-6	6-12	12-18	18-24	24-30
Frequency:	8	10	12	9	5

[NCERT EXEMPLAR]

SOLUTION *Calculation of Mean Deviation about the Median*

Class	*Mid-Values* (x_i)	*Frequency* (f_i)	*Cumulative Frequency (c.f.)*	$\lvert x_i - 14 \rvert$	$f_i \lvert x_i - 14 \rvert$
0-6	3	8	8	11	88
6-12	9	10	18	5	50
12-18	15	12	30	1	12
18-24	21	9	39	7	63
24-30	27	5	44	13	65
			$N = \Sigma f_i = 44$		$\Sigma f_i \lvert x_i - 14 \rvert = 278$

Here $N = 44$, so $\frac{N}{2} = 22$ and the cumulative frequency just greater than $\frac{N}{2}$ is 30. Thus 12-18 is the median class.

$\therefore$ Median $= l + \frac{N/2 - F}{f} \times h$, where $l = 12, h = 6, f = 12, F = 18$.

$\Rightarrow$ Median $= 12 + \frac{22-18}{12} \times 6 = 12 + \frac{4 \times 6}{12} = 14$.

Clearly, $\Sigma f_i\,|x_i - 14| = 278$

$\therefore$ Mean deviation about median $= \frac{1}{N}\,\Sigma f_i\,|x_i - 14| = \frac{278}{44} = 6.318$

EXAMPLE 2 *Calculate the mean deviation from the median of the following data:*

Wages per week (in ₹)	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of workers	4	6	10	20	10	6	4

SOLUTION *Calculation of Mean Deviation from Median*

Wages per week (in ₹)	*Mid-Values* (x_i)	*Frequency* f_i	*Cumulative frequency*	$\lvert d_i\rvert = \lvert x_i - 45\rvert$	$f_i\,\lvert d_i\rvert$
10-20	15	4	4	30	120
20-30	25	6	10	20	120
30-40	35	10	20	10	100
40-50	45	20	40	0	0
50-60	55	10	50	10	100
60-70	65	6	56	20	120
70-80	75	4	60	30	120
		$N = \Sigma f_i = 60$			$\Sigma f_i\,\lvert d_i\rvert = 680$

Here, $N = 60$. So, $\frac{N}{2} = 30$. The cumulative frequency just greater than $\frac{N}{2} = 30$ is 40 and the corresponding class is 40-50. So, 40-50 is the median class.

$\therefore$ $l = 40,\ f = 20,\ h = 10,\ F = 20$.

So, Median $= l + \frac{N/2 - F}{f} \times h = 40 + \frac{30-20}{20} \times 10 = 45$.

Thus, we have

$$\Sigma f_i\,|x_i - 45| = \Sigma f_i\,|d_i| = 680 \text{ and } N = 60.$$

$\therefore$ Mean deviation from median $= \frac{\Sigma f_i\,|d_i|}{N} = \frac{680}{60} = 11.33$.

EXAMPLE 3 *Find the mean deviation from the mean for the following data:*

Classes :	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequencies :	2	3	8	14	8	3	2

SOLUTION *Computation of Mean Deviation from Mean*

Classes	*Mid-values* x_i	*frequencies* f_i	$f_i\ x_i$	$\lvert x_i - \overline{X} \rvert = \lvert x_i - 45 \rvert$	$f_i \lvert x_i - \overline{X} \rvert$
10-20	15	2	30	30	60
20-30	25	3	75	20	60
30-40	35	8	280	10	80
40-50	45	14	630	0	0
50-60	55	8	440	10	80
60-70	65	3	195	20	60
70-80	75	2	150	30	60
		$N = \Sigma f_i = 40$	$\Sigma f_i\ x_i = 1800$		$\Sigma f_i \lvert x_i - \overline{X} \rvert = 400$

Clearly, $N = 40$ and $\Sigma f_i\ x_i = 1800$

$$\therefore \quad \overline{X} = \frac{\Sigma f_i\ x_i}{N} = \frac{1800}{40} = 45$$

From the above table, we get

$$\Sigma f_i \lvert x_i - \overline{X} \rvert = 400 \text{ and } N = \Sigma f_i = 40$$

$$\therefore \quad M.D. = \frac{1}{N} \Sigma f_i \lvert x_i - \overline{X} \rvert = \frac{400}{40} = 10.$$

EXAMPLE 4 *Find the mean deviation about the mean for the following data:*

Marks obtained:	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of students:	2	3	8	14	8	3	2

[NCERT]

SOLUTION In order to avoid the tedious calculations of computing mean ($\overline{X}$), let us compute $\overline{X}$ by step-deviation method. The formula for the same is

$$\overline{X} = a + h \left(\frac{1}{N} \sum_{i=1}^{n} f_i\ d_i \right), \text{ where } d_i = \frac{x_i - a}{h}, a = \text{assumed mean and, } h = \text{common factor.}$$

Let us take the assumed mean $a = 45$ and $h = 10$ and form the following table:

Marks obtained	*Number of students* f_i	*Mid-points* x_i	$d_i = \frac{x_i - 45}{10}$	$f_i\ d_i$	$\lvert x_i - \overline{X} \rvert = \lvert x_i - 45 \rvert$	$f_i \lvert x_i - \overline{X} \rvert$
10-20	2	15	−3	−6	30	60
20-30	3	25	−2	−6	20	60
30-40	8	35	−1	−8	10	80
40-50	14	45	0	0	0	0
50-60	8	55	1	8	10	80
60-70	3	65	2	6	20	60
70-80	2	75	3	6	30	60
	$N = 40$			$\Sigma f_i\ d_i = 0$		$\Sigma f_i \lvert x_i - \overline{X} \rvert = 400$

Clearly, $N = 40$, $\Sigma f_i\ d_i = 0$.

$\therefore \quad \overline{X} = a + h\left(\frac{1}{N}\Sigma f_i d_i\right) = 45 + 10 \times \frac{0}{40} = 45.$

It is evident from the table that $\Sigma f_i |x_i - \overline{X}| = 400$

$\therefore \quad \text{M.D.} = \frac{1}{N}\ \Sigma f_i\ |x_i - \overline{X}| = \frac{400}{40} = 10.$

EXERCISE 32.3

LEVEL-1

1. Compute the mean deviation from the median of the following distribution:

Class	0-10	10-20	20-30	30-40	40-50
Frequency	5	10	20	5	10

2. Find the mean deviation from the mean for the following data :

(i)

Classes	0-100	100-200	200-300	300-400	400-500	500-600	600-700	700-800
Frequencies	4	8	9	10	7	5	4	3

[NCERT]

(ii)

Classes	95-105	105-115	115-125	125-135	135-145	145-155
Frequencies	9	13	16	26	30	12

[NCERT]

(iii)

Classes	0-10	10-20	20-30	30-40	40-50	50-60
Frequencies	6	8	14	16	4	2

[NCERT]

3. Compute mean deviation from mean of the following distribution:

Marks	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of students	8	10	15	25	20	18	9	5

4. The age distribution of 100 life-insuance policy holders is as follows:

Age (on nearest birth day)	17-19.5	20-25.5	26-35.5	36-40.5	41-50.5	51-55.5	56-60.5	61-70.5
No. of persons	5	16	12	26	14	12	6	5

Calculate the mean deviation from the median age.

5. Find the mean deviation from the mean and from median of the following distribution:

Marks	0-10	10-20	20-30	30-40	40-50
No. of students	5	8	15	16	6

6. Calculate mean deviation about median age for the age distribution of 100 persons given below:

Age:	16-20	21-25	26-30	31-35	36-40	41-45	46-50	51-55
Number of persons	5	6	12	14	26	12	16	9

7. Calculate the mean deviation about the mean for the following frequency distribution:

Class interval:	0-4	4-8	8-12	12-16	16-20
Frequency:	4	6	8	5	2

8. Calculate mean deviation from the median of the following data: [NCERT EXEMPLAR]

Class interval:	0-6	6-12	12-18	18-24	24-30
Frequency:	4	5	3	6	2

ANSWERS

1. 9 2. (i) 157.92 (ii) 12.005 (iii) 10.576 3. 14.218
4. 10.605 5. 9.44, 9.56 7. 0.99 8. 7.08

HINTS TO NCERT & SELECTED PROBLEMS

2. (i) Computation of mean deviation from the mean

Classes	f_i	Mid-points x_i	$d_i = \frac{x_i - 450}{100}$	$f_i\, d_i$	$\lvert x_i - \overline{X} \rvert$ $= \lvert x_i - 358 \rvert$	$f_i \lvert x_i - \overline{X} \rvert$
$0 - 100$	4	50	-4	-16	308	1232
$100 - 200$	8	150	-3	-24	208	1664
$200 - 300$	9	250	-2	-18	108	972
$300 - 400$	10	350	-1	-10	8	80
$400 - 500$	7	450	0	0	92	644
$500 - 600$	5	550	1	5	192	960
$600 - 700$	4	650	2	8	292	1168
$700 - 800$	3	750	3	9	392	1176
	$\Sigma f_i = 50$			$\Sigma f_i\, d_i = -46$		$\Sigma f_i \lvert x_i - \overline{X} \rvert = 7896$

Thus, we have $a = 450$, $h = 100$, $N = 50$, $\Sigma f_i\, d_i = -46$ and $\Sigma f_i \lvert x_i - \overline{X} \rvert = 7896$

$$\therefore \quad \overline{X} = a + h\left(\frac{1}{N} \Sigma f_i\, d_i\right) = 450 + 100 \times \left(-\frac{46}{50}\right) = 358$$

and, $$\text{Mean deviation} = \frac{1}{N} \Sigma f_i \lvert x_i - \overline{X} \rvert = \frac{7896}{50} = 157.92$$

(ii) Computation of mean deviation about mean

Classes	f_i	Mid-values x_i	$d_i = \frac{x_i - 130}{10}$	$f_i d_i$	$\|x_i - \overline{X}\|$	$f_i \|x_i - \overline{X}\|$
95 – 105	9	100	– 3	– 27	28.58	257.22
105 – 115	13	110	– 2	– 26	18.58	241.54
115 – 125	16	120	– 1	– 16	8.58	137.28
125 – 135	26	130	0	0	1.42	36.92
135 – 145	30	140	1	30	11.42	342.6
145 – 155	12	150	2	24	21.42	257.04
	$N = \Sigma f_i = 106$			$\Sigma f_i d_i = -15$		$\Sigma f_i \|x_i - \overline{X}\| = 1272.60$

Clearly, $N = \Sigma f_i = 106$, $a = 130$, $h = 10$ and Σf_i

$$\therefore \quad \overline{X} = a + h\left(\frac{1}{N}\Sigma f_i d_i\right) = 130 + 10 \times \frac{-15}{106} = 128.58$$

Also, $\Sigma f_i |x_i - \overline{X}| = 1272.60$ and $N = 106$

$$\therefore \quad \text{Mean deviation} = \frac{1}{N}\Sigma f_i |x_i - \overline{X}| = \frac{1272.60}{106} = 12.005$$

(iii) Computation of mean deviation about mean

Classes	f_i	Mid-values x_i	$d_i = \frac{x_i - 25}{10}$	$f_i d_i$	$\|x_i - \overline{X}\| = \|x_i - 29.8\|$	$f_i \|x_i - \overline{X}\|$
0 – 10	6	5	– 2	– 12	24.8	148.8
10 – 20	8	15	– 1	– 8	14.8	118.4
20 – 30	14	25	0	14	4.8	67.2
30 – 40	16	35	1	16	5.2	83.2
40 – 50	4	45	2	8	15.2	60.8
50 – 60	2	55	3	6	25.2	50.4
	$N = 50$			$\Sigma f_i d_i = 24$		$\Sigma f_i \|x_i - \overline{X}\| = 528.8$

Clearly, $N = 50$, $a = 25$, $h = 10$ and $\Sigma f_i d_i = 24$.

$$\therefore \quad \overline{X} = a + h\left(\frac{\Sigma f_i d_i}{N}\right) = 25 + \frac{24}{50} \times 10 = 29.8$$

Also, $\Sigma f_i |x_i - \overline{X}| = 528.8$ and $N = 50$

$$\therefore \quad \text{Mean deviation} = \frac{1}{N}\sum f_i |x_i - \overline{X}| = \frac{528.8}{50} = 10.576$$

6. Convert the given data into continous frequency distribution by subtracting 0.5 from the lower limit and adding 0.5 to the upper limit of each class interval as given below:

Age	15.5-20.5	20.5-25.5	25.5-30.5	30.5-35.5	35.5-40.5	40.5-45.5	45.5-50.5	50.5-55.5
Number of persons:	5	6	12	14	26	12	16	9

32.4.4 LIMITATIONS OF MEAN DEVIATION

Following are some limitations or demerits of mean deviation.

(i) In a frequency distribution the sum of absolute values of deviations from the mean is always more than the sum of the deviations from median. Therefore, mean deviation about mean is not very scientific. Thus, in many cases, mean deviation may give unsatisfactory results.

(ii) In a distribution, where the degree of variability is very high, the median is not a representative central value. Thus, the mean deviation about median calculated for such series can not be fully relied.

(iii) In the computation of mean deviation we use absolute values of deviations. Therefore, it cannot be subjected to further algebraic treatment.

32.5 VARIANCE AND STANDARD DEVIATON

VARIANCE *The variance of a variate X is the arithmetic mean of the squares of all deviations of X from the arithmetic mean of the observations and is denoted by Var(X) or* σ^2.

The positive square root of the variance of a variate X is known as its standard deviation and is denoted by σ.

Thus, Standard deviation $= +\sqrt{\text{Var}(X)}$

Similar to the mean deviation, we shall discuss the calculation of variance and standard deviation in the following three cases:

(i) Individual observations (ii) Discrete frequency distribution

(iii) Continuous or grouped frequency distribution.

32.5.1 VARIANCE OF INDIVIDUAL OBSERVATIONS

If $x_1, x_2, \ldots, x_n$ are n values of a variable X, then

$$\text{Var}(X) = \frac{1}{n}\left\{\sum_{i=1}^{n}(x_i - \overline{X})^2\right\} \text{ or, } \sigma^2 = \frac{1}{n}\left\{\sum_{i=1}^{n}(x_i - \overline{X})^2\right\} \quad \ldots(i)$$

Now, $$\text{Var}(X) = \frac{1}{n}\left\{\sum_{i=1}^{n}(x_i - \overline{X})^2\right\} = \frac{1}{n}\left\{\sum_{i=1}^{n}(x_i^2 - 2x_i\overline{X} + \overline{X}^2)\right\}$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \frac{1}{n}\sum_{i=1}^{n}2x_i\overline{X} + \frac{1}{n}\sum_{i=1}^{n}\overline{X}^2$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - 2\overline{X}\left\{\frac{1}{n}\sum_{i=1}^{n}x_i\right\} + \frac{n\overline{X}^2}{n}$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - 2\overline{X}^2 + \overline{X}^2 \quad \left[\because \frac{1}{n}\sum_{i=1}^{n}x_i = \overline{X}\right]$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \overline{X}^2$$

$$\therefore \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \left\{\frac{1}{n}\sum_{i=1}^{n}x_i\right\}^2 \quad \ldots(ii)$$

If the values of variable X are large, the calculation of variance from the above formulae is quite tedious and time consuming. In that case we take deviations from an arbitrary point A (say).

If $d_i = x_i - A$; $i = 1, 2, \ldots, n$, then

$$\sum_{i=1}^{n}d_i = \sum_{i=1}^{n}(x_i - A) = \sum_{i=1}^{n}x_i - nA$$

$$\Rightarrow \quad \frac{1}{n}\left\{\sum_{i=1}^{n} d_i\right\} = \frac{1}{n}\sum_{i=1}^{n} x_i - A$$

$$\Rightarrow \quad \frac{1}{n}\left\{\sum_{i=1}^{n} d_i\right\} = \overline{X} - A$$

$$\Rightarrow \quad \overline{d} = \overline{X} - A, \text{ where } \overline{d} = \frac{1}{N}\sum_{i=1}^{n} d_i$$

$$\therefore \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{X})^2$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n}(x_i - A + A - \overline{X})^2$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n}(d_i - \overline{d}\,)^2$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n}(d_i^2 - 2\,d_i\,\overline{d} + \overline{d}^2) = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - \frac{1}{n}\sum_{i=1}^{n} 2d_i\,\overline{d} + \frac{1}{n}\sum_{i=1}^{n}\overline{d}^2$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - 2\left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)\overline{d} + \frac{n\,\overline{d}^2}{n}$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - 2\,\overline{d}^2 + \overline{d}^2$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - \overline{d}^2 = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)^2$$

$$\text{Thus,} \quad \text{Var}(X) = \frac{1}{n}\sum_{i=1}^{n} di^2 - \left(\frac{1}{n}\sum_{i=1}^{n} di\right)^2 \qquad \text{...(iii)}$$

It follows from the above discussion that in case of individual observations, variance and standard deviation may be computed by applying any of the above three formulas. Following algorithm is useful for finding the variance when deviations are taken from the actual mean.

ALGORITHM

STEP I *Compute the mean* $\overline{X}$ *of the given observations* $x_1, x_2, \ldots, x_n$.

STEP II *Take the deviations of the observations from the mean i.e. find* $x_i - \overline{X}$; $i = 1, 2, \ldots, n$.

STEP III *Square the deviations obtained in step II and obtain the sum* $\sum_{i=1}^{n}(x_i - \overline{X})^2$.

STEP IV *Divide the sum* $\sum_{i=1}^{n}(x_i - \overline{X})^2$ *obtained in step III by n. This gives the value of variance of X.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING VARIANCE AND STANDARD DEVIATION OF INDIVIDUAL OBSERVATIONS**

EXAMPLE 1 *Compute the variance and standard deviation of the following observations of marks of 5 students of a tutorial group:*

Marks out of 25 : 8, 12, 13, 15, 22

SOLUTION Clearly,

$$\overline{X} = \frac{8+12+13+15+22}{5} = 14$$

Calculation of variance

x_i	$x_i - \overline{X}$	$(x_i - \overline{X})^2$
8	−6	36
12	−2	4
13	−1	1
15	1	1
22	8	64
		$\Sigma(x_i - \overline{X})^2 = 106$

Here, $n = 5$ and $\Sigma(x_i - \overline{X})^2 = 106$

$\therefore$ $\text{Var}(X) = \frac{1}{n}\Sigma(x_i - \overline{X})^2 = \frac{106}{5} = 21.2$

$\Rightarrow$ $\text{S.D.} = \sqrt{\text{Var}(X)} = \sqrt{21.2} = 4.604.$

EXAMPLE 2 *Find the variance and standard deviation for the following data:*

65, 68, 58, 44, 48, 45, 60, 62, 60, 50

SOLUTION Let $\overline{X}$ be the mean of the given set of observations. Then,

$$\overline{X} = \frac{65+68+58+44+48+45+60+62+60+50}{10} = \frac{560}{10} = 56$$

Computation of Variance

x_i	$x_i - \overline{X} = x_i - 56$	$(x_i - \overline{X})^2$
65	9	81
58	2	4
68	12	144
44	−12	144
48	−8	64
45	−11	121
60	4	16
62	6	36
60	4	16
50	−6	36
		$\Sigma(x_i - \overline{X})^2 = 662$

Clearly, $n = 10$ and $\Sigma(x_i - \overline{X})^2 = 662.$

$\therefore$ $\text{Variance} = \frac{1}{n}\Sigma(x_i - \overline{X})^2 = \frac{662}{10} = 66.2$

Hence, Standard deviation $(\sigma) = \sqrt{\text{Variance}} = \sqrt{66.2} = 8.13$

Type II **ON PROVING RESULTS ON VARIANCE**

EXAMPLE 3 *Let $x_1, x_2, x_3, \dots, x_n$ be n values of a variable X. If these values are changed to $x_1 + a, x_2 + a, \dots, x_n + a$, where $a \in R$, show that the variance remains unchanged.* [NCERT]

SOLUTION Let $u_i = x_i + a, i = 1, 2, \dots, n$ be the n values of variable U. Then,

$$\overline{U} = \frac{1}{n}\sum_{i=1}^{n} u_i = \frac{1}{n}\sum_{i=1}^{n}(x_i + a) = \frac{1}{n}\left\{\sum_{i=1}^{n} x_i + na\right\} = \frac{1}{n}\sum_{i=1}^{n} x_i + a = \overline{X} + a$$

$\therefore \quad u_i - \overline{U} = (x_i + a) - (\overline{X} + a) = x_i - \overline{X}, \; i = 1, 2, ..., n.$

$$\Rightarrow \quad \sum_{i=1}^{n} (u_i - \overline{U})^2 = \sum_{i=1}^{n} (x_i - \overline{X})^2$$

$$\Rightarrow \quad \frac{1}{n} \sum_{i=1}^{n} (u_i - \overline{U})^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{X})^2$$

$$\Rightarrow \quad \text{Var}(U) = \text{Var}(X).$$

EXAMPLE 4 *Let $x_1, x_2, ..., x_n$ values of a variable X and let 'a' be a non-zero real number. Then, prove that the variance of the observations $ax_1, ax_2, ..., ax_n$ is a^2 Var (X). Also, find their standard deviation.* [NCERT]

SOLUTION Let $u_1, u_2, ..., u_n$ be the n values of variable U such that $u_i = ax_i$, $i = 1, 2, ..., n$. Then,

$$\overline{U} = \frac{1}{n} \sum_{i=1}^{n} u_i = \frac{1}{n} \sum_{i=1}^{n} (ax_i) = a \left\{ \frac{1}{n} \sum_{i=1}^{n} x_i \right\} = a\overline{X}$$

$\therefore \quad u_i - \overline{U} = ax_i - a\overline{X}$ for all $i = 1, 2, ..., n$

$\Rightarrow \quad u_i - \overline{U} = a(x_i - \overline{X})$

$\Rightarrow \quad (u_i - \overline{U})^2 = a^2 (x_i - \overline{X})^2$

$$\Rightarrow \quad \sum_{i=1}^{n} (u_i - \overline{U})^2 = a^2 \sum_{i=1}^{n} (x_i - \overline{X})^2$$

$$\Rightarrow \quad \frac{1}{n} \sum_{i=1}^{n} (u_i - \overline{U})^2 = a^2 \left\{ \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{X})^2 \right\}$$

$\Rightarrow \quad \text{Var}(U) = a^2 \; \text{Var}(X)$

$\therefore \quad \sigma_U = \sqrt{\text{Var}(U)} = \sqrt{a^2 \; \text{Var}(X)} = |a| \sqrt{\text{Var}(X)} = |a| \, \sigma_X.$

<u>REMARK</u> *The variance of 20 observations is 5. If each observation is multiplied by 2, then from the above example,*
New variance of the resulting observations $= 2^2 \times 5 = 20$

EXAMPLE 5 *Let $x_1, x_2, x_3, ..., x_n$ be n values of a variable X, and let $x_i = a + hu_i$, $i = 1, 2, ..., n$, where $u_1, u_2, ..., u_n$ are the values of variable U. Then, prove that* Var (X) $= h^2$ Var (U), $h \neq 0$.

SOLUTION We have,

$x_i = a + h u_i, \; i = 1, 2, ..., n$

$$\Rightarrow \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (a + h u_i)$$

$$\Rightarrow \quad \sum_{i=1}^{n} x_i = na + h \sum_{i=1}^{n} u_i$$

$$\Rightarrow \quad \frac{1}{n} \sum_{i=1}^{n} x_i = a + h \left(\frac{1}{n} \sum_{i=1}^{n} u_i \right)$$

$$\Rightarrow \quad \overline{X} = a + h\overline{u} \qquad \left[\because \overline{X} = \frac{1}{n} \sum_{i=1}^{n} x_i \text{ and } \overline{U} = \frac{1}{n} \sum_{i=1}^{n} u_i \right]$$

$\therefore \quad x_i - \overline{X} = (a + h u_i) - (a + h \overline{U}), i = 1, 2, ..., n$

$\Rightarrow \quad x_i - \overline{X} = h(u_i - \overline{U}), \; i = 1, 2, ..., n$

$$\Rightarrow \quad (x_i - \overline{X})^2 = h^2 (u_i - \overline{U})^2, i = 1, 2, \dots, n$$

$$\Rightarrow \quad \sum_{i=1}^{n} (x_i - \overline{X})^2 = h^2 \sum_{i=1}^{n} (u_i - \overline{U})^2$$

$$\Rightarrow \quad \frac{1}{n} \Sigma (x_i - \overline{X})^2 = h^2 \left\{ \frac{1}{n} \sum_{i=1}^{n} (u_i - \overline{U})^2 \right\} \qquad \text{[Dividing both sides by } n\text{]}$$

$$\Rightarrow \quad \text{Var}(X) = h^2 \text{ Var}(U).$$

Type III **ON FINDING THE DESIRED VALUES BY USING THE FORMULAS FOR MEAN AND VARIANCE OF INDIVIDUAL OBSERVATIONS**

EXAMPLE 6 *If the mean and standard deviation of 100 observations are 50 and 4 respectively. Find the sum of all the observations and the sum of their squares.* [NCERT EXEMPLAR]

SOLUTION Let $x_1, x_2, \dots, x_{100}$ be 100 observations and their mean and standard deviation be $\overline{X}$ and σ respectively. Then,

$$\overline{X} = \frac{1}{100} \sum_{i=1}^{100} x_i \text{ and, } \sigma^2 = \frac{1}{100} \sum_{i=1}^{100} x_i^2 - \overline{X}^2$$

$$\Rightarrow \quad 50 = \frac{1}{100} \sum_{i=1}^{100} x_i \text{ and, } 16 = \frac{1}{100} \sum_{i=1}^{100} x_i^2 - (50)^2 \qquad [\because \overline{X} = 50 \text{ and } \sigma = 4]$$

$$\Rightarrow \quad 5000 = \sum_{i=1}^{100} x_i \text{ and, } 1600 = \sum_{i=1}^{100} x_i^2 - 250000$$

$$\Rightarrow \quad \sum_{i=1}^{100} x_i = 5000 \text{ and, } \sum_{i=1}^{100} x_i^2 = 251600$$

EXAMPLE 7 *If for a distribution of 18 observations $\Sigma(x_i - 5) = 3$ and $\Sigma(x_i - 5)^2 = 43$, find the mean and standard deviation.* [NCERT EXEMPLAR]

SOLUTION We have

$$\sum_{i=1}^{18} (x_i - 5) = 3 \text{ and, } \sum_{i=1}^{18} (x_i - 5)^2 = 43$$

$$\Rightarrow \quad \sum_{i=1}^{18} x_i - \sum_{i=1}^{18} 5 = 3 \text{ and, } \sum_{i=1}^{18} x_i^2 - 10 \sum_{i=1}^{18} x_i + \sum_{i=1}^{18} 25 = 43$$

$$\Rightarrow \quad \sum_{i=1}^{18} x_i - 18 \times 5 = 3 \text{ and, } \sum_{i=1}^{18} x_i^2 - 10 \sum_{i=1}^{18} x_i + 18 \times 25 = 43$$

$$\Rightarrow \quad \sum_{i=1}^{18} x_i = 93 \text{ and, } \sum_{i=1}^{18} x_i^2 - 10 \times 93 + 18 \times 25 = 43$$

$$\Rightarrow \quad \sum_{i=1}^{18} x_i = 93 \text{ and, } \sum_{i=1}^{18} x_i^2 = 523$$

$$\therefore \quad \text{Mean} = \frac{1}{18} \sum_{i=1}^{18} x_i = \frac{93}{18} = 5.17$$

$$\text{S.D.} = \sqrt{\frac{1}{18} \sum_{i=1}^{18} x_i^2 - \left(\frac{1}{18} \sum_{i=1}^{18} x_i \right)^2} = \sqrt{\frac{523}{18} - \left(\frac{93}{18} \right)^2} = \sqrt{\frac{9414 - 8649}{324}} = \frac{\sqrt{765}}{18} = \frac{27.6586}{18} = 1.536$$

Type IV ON FINDING CORRECTED MEAN AND CORRECTED VARIANCE OR S.D.

EXAMPLE 8 *For a group of 200 candidates the mean and S.D. were found to be 40 and 15 respectively. Later on it was found that the score 43 was misread as 34. Find the correct mean and correct S.D.*

SOLUTION We have, $n = 200$, $\overline{X} = 40$, $\sigma = 15$.

$\therefore \quad \overline{X} = \frac{1}{n} \Sigma x_i \Rightarrow \Sigma x_i = n\overline{X} = 200 \times 40 = 8000.$

Now, Corrected Σx_i = Incorrect Σx_i − (Sum of incorrect values) + (Sum of correct values)
$= 8000 - 34 + 43 = 8009.$

$\therefore \quad \text{Corrected mean} = \frac{\text{Corrected } \Sigma x_i}{n} = \frac{8009}{200} = 40.045$

and, $\sigma = 15$

$\Rightarrow \quad 15^2 = \text{Variance}$

$\Rightarrow \quad 15^2 = \frac{1}{200}(\Sigma x_i^2) - \left(\frac{1}{200}\Sigma x_i\right)^2$

$\Rightarrow \quad 225 = \frac{1}{200}(\Sigma x_i^2) - \left(\frac{8000}{200}\right)^2$

$\Rightarrow \quad 225 = \frac{1}{200}(\Sigma x_i^2) - 1600$

$\Rightarrow \quad \Sigma x_i^2 = 200 \times 1825 = 365000$

$\Rightarrow \quad \text{Incorrect } \Sigma x_i^2 = 365000.$

$\therefore \quad \text{Corrected } \Sigma x_i^2 = (\text{Incorrect } \Sigma x_i^2) - (\text{Sum of squares of incorrect values})$
$+ (\text{Sum of squares of correct values})$

$\Rightarrow \quad \text{Corrected } \Sigma x_i^2 = 365000 - (34)^2 + (43)^2 = 365693$

So, $\quad \text{Corrected } \sigma = \sqrt{\frac{1}{n}\text{Corrected } \Sigma x_i^2 - \left(\frac{1}{n}\text{Corrected } \Sigma x_i\right)^2} = \sqrt{\frac{365693}{200} - \left(\frac{8009}{200}\right)^2}$

$= \sqrt{1828.465 - 1603.602} = 14.995.$

EXAMPLE 9 *The mean and standard deviation of 20 observations are found to be 10 and 2 respectively. On rechecking, it was found that an observation 8 was incorrect. Calculate the correct mean and standard deviation in each of the following cases:* (i) *If the wrong item is omitted.* (ii) *If it is replaced by 12.*

SOLUTION We have, $n = 20$, $\overline{X} = 10$ and $\sigma = 2$.

$\therefore \quad \overline{X} = \frac{1}{n}\Sigma x_i$

$\Rightarrow \quad \Sigma x_i = n\overline{X} = 20 \times 10 = 200$

$\Rightarrow \quad \text{Incorrected } \Sigma x_i = 200.$

and, $\sigma = 2$

$\Rightarrow \quad \sigma^2 = 4$

$\Rightarrow \quad \frac{1}{n}\Sigma x_i^2 - (\text{Mean})^2 = 4$

$\Rightarrow \quad \frac{1}{20}\Sigma x_i^2 - 100 = 4 \qquad [\because \text{Mean} = 10]$

$\Rightarrow \quad \Sigma x_i^2 = 104 \times 20$

$\Rightarrow \quad \text{Incorrected } \Sigma x_i^2 = 2080.$

(i) ***When 8 is omitted from the data:*** If 8 is omitted from the data, then 19 observations are left.

Now, $\text{Incorrected } \Sigma x_i = 200$

$\Rightarrow \quad \text{Corrected } \Sigma x_i + 8 = 200$

$\Rightarrow$ Corrected $\Sigma x_i = 192$

and,

Incorrected $\Sigma x_i^2 = 2080$

$\Rightarrow$ Corrected $\Sigma x_i^2 + 8^2 = 2080$

$\Rightarrow$ Corrected $\Sigma x_i^2 = 2080 - 64$

$\Rightarrow$ Corrected $\Sigma x_i^2 = 2016$

$\therefore$ Corrected mean $= \dfrac{\text{Corrected } \Sigma x_i}{19} = \dfrac{192}{19} = 10.10$

$\Rightarrow$ Corrected variance $= \dfrac{1}{19}$ (Corrected Σx_i^2) $-$ (Corrected mean)^2

$\Rightarrow$ Corrected variance $= \dfrac{2016}{19} - \left(\dfrac{192}{19}\right)^2 = \dfrac{38304 - 36864}{361} = \dfrac{1440}{361}$

$\therefore$ Corrected standard deviation $= \sqrt{\dfrac{1440}{361}} = \dfrac{12\sqrt{10}}{19} = 1.997.$

(ii) *When the incorrect observation 8 is replaced by 12:*

Now, Incorrected $\Sigma x_i = 200$

$\Rightarrow$ Corrected $\Sigma x_i - 12 + 8 = 200$

$\Rightarrow$ Corrected $\Sigma x_i = 200 - 8 + 12 = 204$

and, Incorrected $\Sigma x_i^2 = 2080$

$\therefore$ Corrected $\Sigma x_i^2 = 2080 - 8^2 + 12^2 = 2160.$

Now, Corrected mean $= \dfrac{204}{20} = 10.2$

Corrected Variance $= \dfrac{1}{20}$ (Corrected Σx_i^2) $-$ (Corrected mean)^2

$\Rightarrow$ Corrected Variance $= \dfrac{2160}{20} - \left(\dfrac{204}{20}\right)^2$

$\Rightarrow$ Corrected Variance $= \dfrac{2160 \times 20 - (204)^2}{(20)^2} = \dfrac{43200 - 41616}{400} = \dfrac{1584}{400}$

$\therefore$ Corrected standard deviation $= \sqrt{\dfrac{1584}{400}} = \dfrac{\sqrt{396}}{10} = \dfrac{19.899}{10} = 1.9899$

EXAMPLE 10 *The mean and variance of 7 observations are 8 and 16 respectively. If 5 of the observations are 2, 4, 10, 12, 14, find the remaining two observations.*

SOLUTION Let x and y be the remaining two observations. Then,

$$\text{Mean} = 8$$

$$\Rightarrow \quad \frac{2 + 4 + 10 + 12 + 14 + x + y}{7} = 8$$

$$\Rightarrow \quad 42 + x + y = 56$$

$$\Rightarrow \quad x + y = 14 \qquad \text{...(i)}$$

and,

$$\text{Variance} = 16$$

$$\Rightarrow \quad \frac{1}{7}(2^2 + 4^2 + 10^2 + 12^2 + 14^2 + x^2 + y^2) - (\text{Mean})^2 = 16$$

$\Rightarrow \quad \frac{1}{7}(4+16+100+144+196+x^2+y^2)-64 = 16$

$\Rightarrow \quad 460+x^2+y^2 = 7\times 80$

$\Rightarrow \quad x^2+y^2 = 100 \quad$ ...(ii)

Now, $(x+y)^2+(x-y)^2 = 2(x^2+y^2)$

$\Rightarrow \quad 196+(x-y)^2 = 2\times 100$

$\Rightarrow \quad (x-y)^2 = 4$

$\Rightarrow \quad x-y = \pm 2$

If $x-y = 2$, then $x+y = 14$ and $x-y = 2$ give $x = 8, y = 6$

If $x-y = -2$, then $x+y = 14$ and $x-y = -2$ give $x = 6, y = 8$.

Hence, the remaining two observations are 6 and 8.

Type V **ON FINDING THE VARIANCE WHEN DEVIATIONS ARE TAKEN FROM AN ASSUMED MEAN**

Following algorithm is helpful for finding the variances when deviations are taken from an assumed mean.

ALGORITHM

STEP I *Choose an assumed mean A (say).*

STEP II *Take the deviations d_i of the observations from an assumed mean i.e. obtain $d_i = x_i - A,\ i=1,2,\dots,n$. Take the total of these deviations i.e. $\sum_{i=1}^{n} d_i$.*

STEP III *Square the deviations obtained in step II and obtain the total $\sum_{i=1}^{n} d_i^2$.*

STEP IV *Substitute the values of $\sum_{i=1}^{n} d_i^2$, $\sum_{i=1}^{n} d_i$ and n in the formula*

$$\text{Var}(X) = \frac{1}{n}\left(\sum_{i=1}^{n} d_i^2\right) - \left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)^2$$

EXAMPLE 11 *The scores of a batsman in 10 matches were as follows:* 38, 70, 48, 34, 42, 55, 63, 46, 54, 44 *Compute the variance and standard deviation.*

SOLUTION Let the assumed mean be $A = 48$.

Calculation of Variance

x_i	$d_i = x_i - A$	d_i^2
38	–10	100
70	22	484
48	0	0
34	–14	196
42	–6	36
55	7	49
63	15	225
46	–2	4
54	6	36
44	–4	16
	$\Sigma d_i = 14$	$\Sigma d_i^2 = 1146$

Here, $n = 10$, $\Sigma d_i = 14$ and $\Sigma d_i^2 = 1146$

$\therefore \quad \text{Var}(X) = \frac{1}{n}\left(\Sigma d_i^2\right) - \left(\frac{1}{n}\Sigma d_i\right)^2 = \frac{1146}{10} - \left(\frac{14}{10}\right)^2 = 112.64$

Hence, $S.D. = \sqrt{\text{Var}(X)} = \sqrt{112.64} = 10.61$

LEVEL-2

Type VI **ON FINDING VARIANCE AND STANDARD DEVIATION OF INDIVIDUAL OBSERVATIONS**

EXAMPLE 12 *Calculate the mean and standard deviation of first n natural numbers.* [NCERT]

SOLUTION Here $x_i = i$; $i = 1, 2, ..., n$.

Let $\overline{X}$ be the mean and σ be the S.D. Then,

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}(1+2+3+\ldots+n) = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

Now, $\sigma^2 = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right) - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right) - (\overline{X})^2$

$$\Rightarrow \quad \sigma^2 = \frac{1}{n}(1^2 + 2^2 + \ldots + n^2) - \left(\frac{n+1}{2}\right)^2$$

$$\Rightarrow \quad \sigma^2 = \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$$

$$\therefore \quad \text{Mean} = \frac{n+1}{2} \text{ and S.D.} = \sqrt{\frac{n^2-1}{12}}.$$

EXAMPLE 13 *Find the mean and standard deviation of first n terms of an A.P. whose first term is a and common difference is d.* [NCERT EXEMPLAR]

SOLUTION The terms of the A.P. are: $a, a+d, a+2d, a+3d, \ldots, a+(r-1)d, \ldots, a+(n-1)d$.

Let $\overline{X}$ be the mean of these terms. Then,

$$\overline{X} = \frac{1}{n}\{a + (a+d) + (a+2d) + \ldots + (a+(n-1)d)\} = \frac{1}{n}\left[\frac{n}{2}\left\{2a + (n-1)d\right\}\right] = a + (n-1)\frac{d}{2}$$

Let σ be the standard deviation of n terms of the A.P. then,

$$\sigma^2 = \frac{1}{n}\sum_{r=1}^{n}\left[\left\{a+(r-1)d\right\} - \overline{X}\right]^2 \qquad \left[\text{Using: } \sigma^2 = \frac{1}{n}\sum_{r=1}^{n}(x_i - \overline{X})^2\right]$$

$$\Rightarrow \quad \sigma^2 = \frac{1}{n}\sum_{r=1}^{n}\left[\left\{a+(r-1)d\right\} - \left\{a+(n-1)\frac{d}{2}\right\}\right]^2$$

$$\Rightarrow \quad \sigma^2 = \frac{d^2}{4n}\left[\sum_{r=1}^{n}(2r-2-n+1)^2\right]$$

$$\Rightarrow \quad \sigma^2 = \frac{d^2}{4n}\left[\sum_{r=1}^{n}\{2r-(n+1)\}^2\right]$$

$$\Rightarrow \quad \sigma^2 = \frac{d^2}{4n}\left[\sum_{r=1}^{n}\left\{4r^2 - 4(n+1)r + (n+1)^2\right\}\right]$$

$$\Rightarrow \quad \sigma^2 = \frac{d^2}{4n}\left[4\left(\sum_{r=1}^{n} r^2\right) - 4(n+1)\left(\sum_{r=1}^{n} r\right) + \sum_{r=1}^{n}(n+1)^2\right]$$

$$\Rightarrow \quad \sigma^2 = \frac{d^2}{4n}\left\{\frac{4n(n+1)(2n+1)}{6} - \frac{4(n+1)n(n+1)}{2} + n(n+1)^2\right\}$$

$$\Rightarrow \quad \sigma^2 = \frac{d^2}{4n}\left\{\frac{2n(n+1)(2n+1)}{3} - n(n+1)^2\right\}$$

$$\Rightarrow \quad \sigma^2 = \frac{d^2}{12n} n(n+1)\{2(2n+1) - 3(n+1)\} = \frac{(n^2-1)d^2}{12}$$

$$\Rightarrow \quad \sigma = d\sqrt{\frac{n^2-1}{12}}.$$

EXERCISE 32.4

LEVEL-1

1. Find the mean, variance and standard deviation for the following data:
   (i) 2, 4, 5, 6, 8, 17. (ii) 6, 7, 10, 12, 13, 4, 8, 12 [NCERT]
   (iii) 227, 235, 255, 269, 292, 299, 312, 321, 333, 348. (iv) 15, 22, 27, 11, 9, 21, 14, 9.
2. The variance of 20 observations is 5. If each observation is multiplied by 2, find the variance of the resulting observations. [NCERT]
3. The variance of 15 observations is 4. If each observation is increased by 9, find the variance of the resulting observations.
4. The mean of 5 observations is 4.4 and their variance is 8.24. If three of the observations are 1, 2 and 6, find the other two observations. [NCERT]
5. The mean and standard deviation of 6 observations are 8 and 4 respectively. If each observation is multiplied by 3, find the new mean and new standard deviation of the resulting observations. [NCERT]
6. The mean and variance of 8 observations are 9 and 9.25 respectively. If six of the observations are 6, 7, 10, 12, 12 and 13, find the remaining two observations. [NCERT]
7. For a group of 200 candidates, the mean and standard deviations of scores were found to be 40 and 15 respectively. Later on it was discovered that the scores of 43 and 35 were misread as 34 and 53 respectively. Find the correct mean and standard deviation.
8. The mean and standard deviation of 100 observations were calculated as 40 and 5.1 respectively by a student who took by mistake 50 instead of 40 for one observation. What are the correct mean and standard deviation? [NCERT]
9. The mean and standard deviation of 20 observations are found to be 10 and 2 respectively. On rechecking it was found that an observation 8 was incorrect. Calculate the correct mean and standard deviation in each of the following cases:
   (i) If wrong item is omitted [NCERT] (ii) if it is replaced by 12.
10. The mean and standard deviation of a group of 100 observations were found to be 20 and 3 respectively. Later on it was found that three observations were incorrect, which were recorded as 21, 21 and 18. Find the mean and standard deviation if the incorrect observations were omitted. [NCERT]

LEVEL-2

11. Show that the two formulae for the standard deviation of ungrouped data

$$\sigma = \sqrt{\frac{1}{n}\sum(x_i - \overline{X})^2} \text{ and } \sigma' = \sqrt{\frac{1}{n}\sum x_i^2 - \overline{X}^2} \text{ are equivalent, where } \overline{X} = \frac{1}{n}\sum x_i.$$

ANSWERS

1. (i) 7, 23.33, 4.83 (ii) 9, 9.25, 3.04 (iii) 289.10, 1539.77, 39.24 (iv) 16, 38.68, 6.22
2. 20
3. 4
4. 9, 4
5. 18, 12
6. 4, 8
7. 39.955, 14.9
8. Mean = 39.9 S.D. = 5
9. (i) 1.997 (ii) 1.98
10. 20, 3.035

HINTS TO NCERT & SELECTED PROBLEMS

1. (iii) Let the assumed mean be $A = 9$

Calculation of Variance

x_i	$d_i = (x_i - 9)$	d_i^2
6	-3	9
7	-2	4
10	1	1
12	3	9
13	4	16
4	-5	25
8	-1	1
12	3	9
	$\Sigma d_i = 0$	$\Sigma d_i^2 = 74$

Hence, $n = 8, a = 9, \Sigma d_i = 0$ and $\Sigma d_i^2 = 74$

$\therefore \quad \overline{X} = a + \frac{\Sigma d_i}{n} = 9 + 0 = 9$

and, $\quad \text{Var}(X) = \frac{1}{n}\left(\Sigma d_i^2\right) - \left(\frac{1}{n}\Sigma d_i\right)^2 = \frac{74}{8} - \left(\frac{0}{8}\right)^2 = 9.25$

So, $\quad \text{S.D.} = \sqrt{\text{Var}(X)} = \sqrt{9.25} = 3.041$

2. We know that if the observations $x_1, x_2, x_3, ..., x_n$ has variance Var (X). Then, observations $ax_1, ax_2, ..., ax_n$ has variance a^2 Var (X).

Thus, if variance of 20 observations is 5 and each observation is multiplied by 2, then variance of resulting observations is $2^2(5) = 20$.

4. Let the other two observations be x and y. Then,

$$\text{Mean} = 4.4 \Rightarrow \frac{1+2+6+x+y}{5} = 4.4 \Rightarrow x + y = 13 \quad ...(i)$$

Variance = 8.24

$$\Rightarrow \quad \frac{1}{5}\left(1^2 + 2^2 + 6^2 + x^2 + y^2\right) - (4.4)^2 = 8.24$$

$$\Rightarrow \quad \frac{41 + x^2 + y^2}{5} - 19.36 = 8.24$$

$$\Rightarrow \quad x^2 + y^2 + 41 = 138 \Rightarrow x^2 + y^2 = 97 \quad ...(ii)$$

Now, $\quad (x-y)^2 + (x+y)^2$

$$\Rightarrow \quad (x-y)^2 + 169 = 2 \times 97 \quad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad (x-y)^2 = 25$$

$$\Rightarrow \quad x - y = 5 \quad ...(iii)$$

Solving (i) and (iii), we get $x = 9$ and $y = 4$.

5. If the mean and standard deviation of observations $x_1, x_2 \ldots, x_n$ are $\overline{X}$ and σ respectively, then the mean and standard deviations of $ax_1, ax_2 \ldots, ax_n$ are $a\overline{X}$ and $|a|\sigma$ respectively.

$\therefore$ New mean $= 8 \times 3 = 24$ and, New standard deviation $= 3 \times 4 = 12$.

6. Let the remaining two observations be x and y. Then,

$$\text{Mean} = 9 \Rightarrow \frac{6+7+10+12+12+13+x+y}{8} = 9 \Rightarrow x+y = 12 \quad \ldots(i)$$

Variance $= 9.25$

$$\therefore \quad \text{Variance} = \frac{1}{n}\Sigma x_i^2 - (\text{Mean})^2$$

$$\Rightarrow \quad \frac{1}{8}\left(36+49+100+144+144+169+x^2+y^2\right) - 9^2 = 9.25$$

$$\Rightarrow \quad 642 + x^2 + y^2 = 722$$

$$\Rightarrow \quad x^2 + y^2 = 80 \quad \ldots(ii)$$

$$\text{Now,} \quad (x-y)^2 + (x+y)^2 = 2(x^2+y^2)$$

$$\Rightarrow \quad (x-y)^2 + 144 = 2 \times 80$$

$$\Rightarrow \quad (x-y)^2 = 16$$

$$\Rightarrow \quad x - y = 4 \quad \ldots(iii)$$

Solving (i) and (iii), we obtain $x = 8$, $y = 4$.

7. We have,

$n = 200$, Incorrected mean $= 40$ and, Incorrected S.D. $= 5.1$

Now, Incorrected mean $= 40$

$$\Rightarrow \quad \frac{\text{Incorrected } \Sigma x_i}{200} = 40 \quad \left[\because \overline{X} = \frac{1}{n}\Sigma x_i\right]$$

$\Rightarrow$ Incorrected $\Sigma x_i = 8000$

$\therefore$ Corrected $\Sigma x_i = 8000 - (34+53) + (43+35) = 7991$

So, Corrected mean $= \dfrac{7991}{200} = 39.955$

Now, Incorrected S.D. $= 15$

$\Rightarrow$ Incorrected variance $= 225$

$$\Rightarrow \quad \frac{1}{200}(\text{Incorrected } \Sigma x_i^2) - (40)^2 = 225$$

$\Rightarrow$ Incorrected $\Sigma x_i^2 = 365000$

$$\therefore \quad \text{Corrected } \Sigma x_i^2 = \text{Incorrected } \Sigma x_i^2 - (34^2+53^2) + (43^2+35^2)$$

$$= 365000 - (1156+2809) + (1849+1225) = 364109$$

$$\text{So, Corrected variance} = \frac{1}{200}(\text{Corrected } \Sigma x_i^2) - (\text{Corrected mean})^2$$

$$= \frac{364109}{200} - (39.955)^2 = 1820.545 - 1596.402 = 224.143$$

$$\therefore \quad \text{Corrected S.D.} = \sqrt{224.143} = 14.971$$

8. We have,

$n = 100$, Incorrected mean $= 40$, Incorrected S.D. $= 5.1$

Now,

$$\text{Incorrected mean} = 40$$

$$\Rightarrow \quad \frac{\text{Incorrected } \Sigma x_i}{100} = 40$$

$$\Rightarrow \quad \text{Incorrected } \Sigma x_i = 4000$$

$$\Rightarrow \quad \text{Corrected } \Sigma x_i = 4000 - 50 + 40 = 3990$$

$$\therefore \quad \text{Corrected mean} = \frac{3990}{100} = 39.90$$

Now,

$$\text{Incorrected S.D.} = 5.1$$

$$\Rightarrow \quad \text{Incorrected variance} = 26.01$$

$$\Rightarrow \quad \frac{1}{100}(\text{Incorrected } \Sigma x_i^2) - (\text{Incorrected mean})^2 = 26.01$$

$$\Rightarrow \quad \frac{1}{100}(\text{Incorrected } \Sigma x_i^2) - 40^2 = 26.01$$

$$\Rightarrow \quad \text{Incorrected } \Sigma x_i^2 = 162601$$

$$\Rightarrow \quad \text{Corrected } \Sigma x_i^2 = 162601 - 50^2 + 40^2 = 161701$$

$$\therefore \quad \text{Corrected Variance} = \frac{161701}{100} - (39.9)^2 = 1617.01 - 1592.01 = 25$$

Hence, Corrected S.D. $= \sqrt{25} = 5$

9. We have, $n = 20$, Incorrected mean $= 10$, Incorrected S.D. $= 2$.

Now,

$$\text{Incorrected Mean} = 10$$

$$\Rightarrow \quad \frac{\text{Incorrected } \Sigma x_i}{20} = 10$$

$$\Rightarrow \quad \text{Incorrected } \Sigma x_i = 200$$

and, Incorrected S.D. $= 2$

$$\Rightarrow \quad \text{Incorrected Variance} = 4$$

$$\Rightarrow \quad \frac{\text{Incorrected } \Sigma x_i^2}{20} - (10)^2 = 4$$

$$\Rightarrow \quad \text{Incorrected } \Sigma x_i^2 = 2080$$

(i) When wrong item is omitted: In this case, $n = 19$.

$$\text{Corrected } \Sigma x_i = \text{Incorrected } \Sigma x_i - 8 = 200 - 8 = 192$$

$$\text{Corrected } \Sigma x_i^2 = \text{Incorrected } \Sigma x_i^2 - 8^2 = 2080 - 64 = 2016$$

$$\therefore \quad \text{Corrected mean} = \frac{\text{Corrected } \Sigma x_i}{19} = \frac{192}{19} = 10.105$$

$$\text{Corrected Variance} = \frac{\text{Corrected } \Sigma x_i^2}{19} - (\text{Corrected mean})^2$$

$$= \frac{2016}{19} - \left(\frac{192}{19}\right)^2 = \frac{38304 - 36864}{361} = \frac{1440}{361} = 3.9889$$

$$\therefore \quad \text{Corrected S.D.} = \sqrt{3.9889} = 1.997$$

(ii) When wrong observations 8 is replaced by 12: It this case, $n = 20$.

$$\text{Corrected } \Sigma x_i = \text{Incorrected } \Sigma x_i - 8 + 12 = 200 + 4 = 204$$

$$\text{Corrected } \Sigma x_i^2 = \text{Incorrected } \Sigma x_i^2 - 8^2 + 12^2 = 2080 - 64 + 144 = 2160$$

$$\therefore \quad \text{Corrected mean} = \frac{\text{Corrected } \Sigma x_i}{20} = \frac{204}{20} = 10.2$$

$$\text{Corrected Variance} = \frac{\text{Corrected } \Sigma x_i^2}{20} - (\text{Corrected mean})^2$$

$$= \frac{2160}{20} - \left(\frac{204}{20}\right)^2 = 108 - 104.04 = 3.96$$

$$\therefore \quad \text{Corrected S.D.} = \sqrt{3.96} = 1.98$$

10. We have,

$$n = 100, \text{ Incorrected mean} = 20, \text{ Incorrected S.D.} = 3$$

$$\therefore \quad \text{Incorrected mean} = 20$$

$$\Rightarrow \quad \text{Incorrected } \Sigma x_i = 20 \times 100$$

$$\Rightarrow \quad \text{Corrected } \Sigma x_i = 2000 - 21 - 21 - 18 = 1940$$

$$\therefore \quad \text{Corrected mean} = \frac{1940}{97} = 20$$

Now,

$$\text{Incorrected S.D.} = 3$$

$$\Rightarrow \quad \text{Incorrected Variance} = 9$$

$$\Rightarrow \quad \frac{1}{100}(\text{Incorrected } \Sigma x_i^2) - (\text{Incorrected Mean})^2 = 9$$

$$\Rightarrow \quad \frac{1}{100}(\text{Incorrected } \Sigma x_i^2) - 400 = 9$$

$$\Rightarrow \quad \text{Incorrected } x_i^2 = 40900$$

$$\Rightarrow \quad \text{Corrected } \Sigma x_i^2 = 40900 - 21^2 - 21^2 - 18^2 = 39694$$

$$\therefore \quad \text{Variance of the remaining observations} = \frac{39694}{97} - (20)^2 = 409.216 - 400 = 9.216$$

$$\therefore \quad \text{Corrected S.D.} = \sqrt{9.216} = 3.035$$

32.5.2 VARIANCE OF A DISCRETE FREQUENCY DISTRIBUTION

If $x_i / f_i\,;\ i = 1, 2, \ldots, n$ is a discrete frequency distribution of a variate X, then

$$\text{Var}(X) = \frac{1}{N}\left\{\sum_{i=1}^{n} f_i (x_i - \overline{X})^2\right\} \qquad \ldots(i)$$

Also, $$\text{Var}(X) = \frac{1}{N}\left[\sum_{i=1}^{n} f_i (x_i^2 - 2x_i\overline{X} + \overline{X}^2)\right]$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{N}\left(\sum_{i=1}^{n} f_i x_i^2\right) - 2\overline{X}\left(\frac{1}{N}\sum_{i=1}^{n} f_i x_i\right) + \frac{N\overline{X}^2}{N}$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{N}\left(\sum_{i=1}^{n} f_i x_i^2\right) - 2\overline{X}^2 + \overline{X}^2 \qquad \left[\because \frac{1}{N}\sum_{i=1}^{n} f_i x_i = \overline{X}\right]$$

$$\Rightarrow \quad \text{Var}(X) = \frac{1}{N}\left(\sum_{i=1}^{n} f_i x_i^2\right) - \overline{X}^2$$

or, $$\text{Var}(X) = \frac{1}{N}\left(\sum_{i=1}^{n} f_i x_i^2\right) - \left(\frac{1}{N}\sum_{i=1}^{n} f_i x_i\right)^2 \qquad \ldots(ii)$$

If the values x_i of variable X or (and) frequencies f_i are large the calculation of variance from the above formulae is quite tedious and time consuming. In such a case, we take deviations of the values of variable X from an arbitrary point A(say). If $d_i = x_i - A,\ i = 1, 2, \ldots, n$, then the above formula reduces to

$$\text{Var}(X) = \frac{1}{N}\left(\Sigma f_i d_i^2\right) - \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)^2 \qquad \text{...(iii)}$$

Sometimes $d_i = x_i - A$ are divisible by a common number h(say). If we define $u_i = \frac{x_i - A}{h} = \frac{d_i}{h}$, $i = 1, 2, \ldots, n$, then we obtain the following formula for variance.

$$\text{Var}(X) = h^2\left[\left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2\right) - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2\right] \qquad \text{...(iv)}$$

In order to compute variance by using the following formula

$\text{Var}(X) = \frac{1}{N}\left[\sum_{i=1}^{n} f_i (x_i - \overline{X})^2\right]$, we may use the following algorithm.

ALGORITHM

STEP I *Obtain the given frequency distribution.*

STEP II *Find the mean $\overline{X}$ of the given frequency distribution.*

STEP III *Compute deviations $(x_i - \overline{X})$ from the mean $\overline{X}$.*

STEP IV *Find the squares of deviations obtained in step III.*

STEP V *Multiply the squared deviations by respective frequencies and obtain the total $\Sigma f_i (x_i - \overline{X})^2$.*

STEP VI *Divide the total obtained in step V by $N = \Sigma f_i$ to obtain the variance.*

Following example illustrates the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the variance and standard deviation of the following frequency distribution:*

Variable (x_i)	2	4	6	8	10	12	14	16
Frequency (f_i)	4	4	5	15	8	5	4	5

SOLUTION *Calculation of Variance and Standard Deviation*

Variable x_i	Frequency f_i	$f_i x_i$	$x_i - \overline{X} = x_i - 9$	$(x_i - \overline{X})^2$	$f_i (x_i - \overline{X})^2$
2	4	8	−7	49	196
4	4	16	−5	25	100
6	5	30	−3	9	45
8	15	120	−1	1	15
10	8	80	1	1	8
12	5	60	3	9	45
14	4	56	5	25	100
16	5	80	7	49	245
	$N = \Sigma f_i = 50$	$\Sigma f_i x_i = 450$			$\Sigma f_i (x_i - \overline{X})^2 = 754$

Here, $N = 50$, $\Sigma f_i x_i = 450$ and, $\Sigma f_i (x_i - \overline{X})^2 = 754$

$\therefore$ $\overline{X} = \frac{1}{N} \Sigma f_i x_i = \frac{450}{50} = 9$

and, $\text{Var}(X) = \frac{1}{N}\left\{\Sigma f_i (x_i - \overline{X})^2\right\} = \frac{754}{50} = 15.08$

Hence, $S.D. = \sqrt{\text{Var}(X)} = \sqrt{15.08} = 3.88$

NOTE: *In practice the calculation of S.D. and variance by the above algorithm is rarely used, because if the actual mean is in fractions the calculation is quite tedious and time consuming.*

In order to compute the variance by using the following formula

$\text{Var}(X) = \left[\left(\frac{1}{N}\Sigma f_i d_i^2\right) - \left(\frac{1}{N}\Sigma f_i d_i\right)^2\right]$, where $d_i = x_i - A$, we may use the following algorithm.

ALGORITHM

STE I *Take the deviations of observations from an assumed mean A(say) and denote these deviations by d_i.*

STEP II *Multiply the deviations by the respective frequencies and obtain the total $\Sigma f_i d_i$.*

STEP III *Obtain the squares of deviations obtained in step I i.e. d_i^2.*

STEP IV *Multiply the squared deviations by respective frequencies and obtain the total $\Sigma f_i d_i^2$.*

STEP V *Substitute the values in the formula*

$$\text{Var}(X) = \left(\frac{1}{N}\Sigma f_i d_i^2\right) - \left(\frac{1}{N}\Sigma f_i d_i\right)^2$$

Following examples illustrate the above algorithm.

EXAMPLE 2 *Calculate the variance and standard deviation from the data given below:*

Size of item	3.5	4.5	5.5	6.5	7.5	8.5	9.5
Frequency	3	7	22	60	85	32	8

SOLUTION Let the assumed mean be $A = 6.5$

Calculation of Variance and Standard Deviation

Size of item x_i	f_i	$d_i = x_i - 6.5$	d_i^2	$f_i d_i$	$f_i d_i^2$
3.5	3	−3	9	−9	27
4.5	7	−2	4	−14	28
5.5	22	−1	1	−22	22
6.5	60	0	0	0	0
7.5	85	1	1	85	85
8.5	32	2	4	64	128
9.5	8	3	9	24	72
	$N = \Sigma f_i = 217$			$\Sigma f_i d_i = 128$	$\Sigma f_i d_i^2 = 362$

Here, $N = 217$, $\Sigma f_i d_i = 128$ and $\Sigma f_i d_i^2 = 362$

$$\therefore \quad \text{Var}(X) = \left(\frac{1}{N}\Sigma f_i d_i^2\right) - \left(\frac{1}{N}\Sigma f_i d_i\right)^2 = \frac{362}{217} - \left(\frac{128}{217}\right)^2 = 1.668 - 0.347 = 1.321$$

Hence, $\text{S.D.} = \sqrt{\text{Var}(X)} = \sqrt{1.321} = 1.149$

REMARK *Sometimes deviations d_i in the algorithm given above are divisible by a common number h. In such a case, we define $u_i = \frac{x_i - a}{h} = \frac{d_i}{h}$, $i = 1, 2, \ldots, n$ and the formula for computing variance is*

$$\text{Var}(X) = h^2\left[\frac{1}{N}\left(\sum_{i=1}^{n} f_i u_i^2\right) - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2\right].$$

EXAMPLE 3 *Find the variance and standard deviation for the following distribution:*

X:	4.5	14.5	24.5	34.5	44.5	54.5	64.5
f:	1	5	12	22	17	9	4

SOLUTION *Calculation of Variance and Standard Deviation*

x_i	f_i	$d_i = x_i - 34.5$	$u_i = \frac{x_i - 34.5}{10}$	$f_i u_i$	u_i^2	$f_i u_i^2$
4.5	1	–30	–3	–3	9	9
14.5	5	–20	–2	–10	4	20
24.5	12	–10	–1	–12	1	12
34.5	22	0	0	0	0	0
44.5	17	10	1	17	1	17
54.5	9	20	2	18	4	36
64.5	4	30	3	12	9	36
	$N = \Sigma f_i = 70$			$\Sigma f_i u_i = 22$		$\Sigma f_i u_i^2 = 130$

Here, $N = 70$, $\Sigma f_i u_i = 20$, $\Sigma f_i u_i^2 = 130$ and $h = 10$

$$\therefore \quad \text{Var}(X) = h^2\left[\left(\frac{1}{N}\Sigma f_i u_i^2\right) - \left(\frac{1}{N}\Sigma f_i u_i\right)^2\right]$$

$$\Rightarrow \quad \text{Var}(X) = 100\left[\frac{130}{70} - \left(\frac{22}{70}\right)^2\right] = 100\left[\frac{13}{7} - \left(\frac{11}{35}\right)^2\right] = 100\,[1.857 - 0.098] = 175.822$$

Hence, $\text{S.D.} = \sqrt{Var\,(X)} = \sqrt{175.822} = 13.259$

EXAMPLE 4 ***The following table gives the number of finished articles turned out per day by different number of workers in a factory. Find the standard deviation of the daily output of finished articles.***

Number of articles:	18	19	20	21	22	23	24	25	26	27
No. of workers:	3	7	11	14	18	17	13	8	5	4

SOLUTION *Calculation of Standard Deviation*

x	f	$d_i = x_i - 23$	d_i^2	$f_i\, d_i$	$f_i\, d_i^2$
18	3	−5	25	−15	75
19	7	−4	16	−28	112
20	11	−3	9	−33	99
21	14	−2	4	−28	56
22	18	−1	1	−18	18
23	17	0	0	0	0
24	13	1	1	13	13
25	8	2	4	16	32
26	5	3	9	15	45
27	4	4	16	16	64
	$N = \Sigma f_i = 100$			$\Sigma f_i d_i = -62$	$\Sigma f_i\, d_i^2 = 514$

Clearly, $N = 100$, $\Sigma f_i d_i = -62$ and $\Sigma f_i d_i^2 = 514$

$$\therefore \quad \sigma^2 = \frac{1}{N}\left(\Sigma f_i\, d_i^2\right) - \left(\frac{1}{N}\ \Sigma f_i\, d_i\right)^2 = \frac{514}{100} - \left(-\frac{62}{100}\right)^2 = \frac{47556}{10000}$$

Hence, $\sigma = \sqrt{\dfrac{47556}{10000}} = \dfrac{218.07}{100} = 2.1807$

LEVEL-2

EXAMPLE 5 *If a is a positive integer and the frequency distribution:*

$x:$	a	$2a$	$3a$	$4a$	$5a$	$6a$
$f:$	2	1	1	1	1	1

has a variance of 160. Determine the value of a. [NCERT EXEMPLAR]

SOLUTION *Computation of Variance*

x_i	f_i	$f_i x_i$	$f_i x_i^2$
a	2	$2a$	$2a^2$
$2a$	1	$2a$	$4a^2$
$3a$	1	$3a$	$9a^2$
$4a$	1	$4a$	$16a^2$
$5a$	1	$5a$	$25a^2$
$6a$	1	$6a$	$36a^2$
	$N = \Sigma f_i = 7$	$\Sigma f_i x_i = 22a$	$\Sigma f_i x_i^2 = 92a^2$

Here, $N = 7$, $\Sigma f_i x_i = 22a$, $\Sigma f_i x_i^2 = 92a^2$ and Variance $= 160$

Now,

Variance $= 160$

$$\Rightarrow \quad 160 = \left(\frac{1}{N}\Sigma f_i x_i^2\right) - \left(\frac{1}{N}\Sigma f_i x_i\right)^2$$

$$\Rightarrow \quad 160 = \frac{92a^2}{7} - \left(\frac{22a}{7}\right)^2 \Rightarrow 160 = \frac{644a^2 - 484a^2}{49} \Rightarrow 160 = \frac{160a^2}{49} \Rightarrow a^2 = 49 \Rightarrow a = 7$$

EXAMPLE 6 *There are 60 students in a class. The following is the frequency distribution of marks obtained by the students in a test:*

Marks:	0	1	2	3	4	5
Frequency:	$x-2$	x	x^2	$(x+1)^2$	$2x$	$x+1$

where x is a positive integer. Determine the mean and standard deviation of the marks.

[NCERT EXEMPLAR]

SOLUTION It is given that there are 60 students in the class.

$\therefore \quad (x-2)+x+x^2+(x+1)^2+2x+(x+1)=60$

$\Rightarrow \quad 2x^2+7x-60=0$

$\Rightarrow \quad (2x+15)(x-4)=0$

$\Rightarrow \quad x-4=0 \quad [\because x>0 \therefore 2x+15 \neq 0]$

$\Rightarrow \quad x=4$

Thus, we obtain the following frequency distribution:

Marks:	0	1	2	3	4	5
Frequency:	2	4	16	25	8	5

Computation of mean and standard deviation

Marks (x_i)	*Frequency* (f_i)	$f_i x_i$	$f_i x_i^2$
0	2	0	0
1	4	4	4
2	16	32	64
3	25	75	225
4	8	32	128
5	5	25	125
	$N=\Sigma f_i=60$	$\Sigma f_i x_i=168$	$\Sigma f_i x_i^2=546$

Here, $N=60,\ \Sigma f_i x_i=168,\ \Sigma f_i x_i^2=546$

$\therefore \quad \text{Mean}=\frac{1}{N}\Sigma f_i x_i=\frac{168}{60}=2.8$

and, $\text{Variance}=\left(\frac{1}{N}\Sigma f_i x_i^2\right)-\left(\frac{1}{N}\Sigma f_i x_i\right)^2=\frac{546}{60}-\left(\frac{168}{60}\right)^2=9.1-7.84=1.26$

Hence, $\text{S.D.}=\sqrt{\text{Variance}}=\sqrt{1.26}=1.122$

EXERCISE 32.5

LEVEL-1

1. Find the standard deviation for the following distribution:

x:	4.5	14.5	24.5	34.5	44.5	54.5	64.5
f:	1	5	12	22	17	9	4

2. Table below shows the frequency f with which 'x' alpha particles were radiated from a diskette

x:	0	1	2	3	4	5	6	7	8	9	10	11	12
f:	51	203	383	525	532	408	273	139	43	27	10	4	2

Calculate the mean and variance.

3. Find the mean, and standard deviation for the following data:

(i)

Year render:	10	20	30	40	50	60
No. of persons (cumulative):	15	32	51	78	97	109

(ii)

Marks:	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Frequency:	1	6	6	8	8	2	2	3	0	2	1	0	0	0	1

[NCERT EXEMPLAR]

4. Find the standard deviation for the following data:

(i)

x:	3	8	13	18	23
f:	7	10	15	10	6

(ii)

x:	2	3	4	5	6	7
f:	4	9	16	14	11	6

[NCERT EXEMPLAR]

ANSWERS

1. 13.26 2. $\overline{X} = 3.88$, $\sigma^2 = 3.64$ 3. (i) $\overline{X} = 37.25$ years, S.D. = 15.5 years. (ii) $X = 5.975$, S.D. = 2.85. 4. (i) 6.12 (ii) 1.38

32.5.3 VARIANCE OF A GROUPED OR CONTINUOUS FREQUENCY DISTRIBUTION

In a grouped or continuous frequency distribution any of the methods discussed above for a discrete frequency distribution can be used. We may use the following algorithm for computing variance of a grouped or continuous frequency distribution.

ALGORITHM

STEP I *Find the mid-points of various classes.*

STEP II *Take the deviations of these mid-points from an assumed mean. Denote these deviations by d_i.*

STEP III *Divide the deviations in step II by the class interval h and denote them by u_i., i.e. $u_i = \frac{d_i}{h}$.*

STEP IV *Multiply the frequency of each class with the corresponding u_i and obtain $\Sigma f_i u_i$.*

STEP V *Square the values of u_i and multiply them with the corresponding frequencies and obtain $\Sigma f_i u_i^2$.*

STEP VI *Substitute the values of $\Sigma f_i u_i$, $\Sigma f_i u_i^2$ h and $N = \Sigma f_i$ in the formula*

$$\text{Var}(X) = h^2 \left\{ \frac{1}{N} \Sigma f_i u_i^2 - \left(\frac{1}{N} \Sigma f_i u_i \right)^2 \right\}$$

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Calculate the mean and standard deviation for the following distribution:*

Marks:	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of students:	3	6	13	15	14	5	4

SOLUTION *Calculation of Standard Deviation*

Class-interval	*Frequency* (f_i)	*Mid-values* (x_i)	$u_i = \frac{x_i - 55}{10}$	$f_i u_i$	u_i^2	$f_i u_i^2$
20-30	3	25	−3	−9	9	27
30-40	6	35	−2	−12	4	24
40-50	13	45	−1	−13	1	13
50-60	15	55	0	0	0	0
60-70	14	65	1	14	1	14
70-80	5	75	2	10	4	20
80-90	4	85	3	12	9	36
	$N = \Sigma f_i = 60$			$\Sigma f_i u_i = 2$		$\Sigma f_i u_i^2 = 134$

Here, $N = 60$, $\Sigma f_i u_i = 2$, $\Sigma f_i u_i^2 = 134$ and $h = 10$

$\therefore$ Mean $= \overline{X} = A + h\left(\frac{1}{N} \Sigma f_i u_i\right) = 55 + 10\left(\frac{2}{60}\right) = 55.333$

and, $\text{Var}(X) = h^2 \left\{\left(\frac{1}{N} \Sigma f_i u_i^2\right) - \left(\frac{1}{N} \Sigma f_i u_i\right)^2\right\} = 100\left[\frac{134}{60} - \left(\frac{2}{60}\right)^2\right] = 222.9$

$\therefore$ S.D. $= \sqrt{\text{Var}(X)} = \sqrt{222.9} = 14.94.$

EXAMPLE 2 *The following table gives the distribution of income of 100 families in a village. Calculate the standard deviation:*

Income ₹	0-1000	1000-2000	2000-3000	3000-4000	4000-5000	5000-6000
No. of Families	18	26	30	12	10	4

SOLUTION *Calculation of Standard Deviation*

Income ₹	*Mid-values* x_i	*No. of families (frequencies)* f_i	$u_i = \frac{x_i - 2500}{1000}$	$f_i u_i$	u_i^2	$f_i u_i^2$
0-1000	500	18	−2	−36	4	72
1000-2000	1500	26	−1	−26	1	26
2000-3000	2500	30	0	0	0	0
3000-4000	3500	12	1	12	1	12
4000-5000	4500	10	2	20	4	40
5000-6000	5500	4	3	12	9	36
		$\Sigma f_i = 100$		$\Sigma f_i u_i = -18$		$\Sigma f_i u_i^2 = 186$

Here, $N = 100$, $\Sigma f_i u_i = -18$, $\Sigma f_i u_i^2 = 186$ and, $h = 1000$

$\therefore$ $\text{Var}(X) = h^2 \left\{\frac{1}{N}\left(\Sigma f_i u_i^2\right) - \left(\frac{1}{N} \Sigma f_i u_i\right)^2\right\} = (1000)^2 \left\{\frac{186}{100} - \left(\frac{-18}{100}\right)^2\right\} = 1827600$

Hence, S.D. $= \sqrt{\text{Var}(X)} = \sqrt{1827600} = 1351.88$

EXAMPLE 3 *Calculate the mean and standard deviation for the following data:*

Wages upto (in ₹)	15	30	45	60	75	90	105	120
No. of workers	12	30	65	107	157	202	222	230

SOLUTION We are given the cumulative frequency distribution. So, first we will prepare the frequency distribution as given below:

Class-interval	*Cumulative frequency*	*Mid-values*	*Frequency*	$u_i = \frac{x_i - 67.5}{15}$	$f_i u_i$	$f_i u_i^2$
0-15	12	7.5	12	–4	–48	192
15-30	30	22.5	18	–3	–54	162
30-45	65	37.5	35	–2	–70	140
45-60	107	52.5	42	–1	–42	42
60-75	157	67.5	50	0	0	0
75-90	202	82.5	45	1	45	45
90-105	222	97.5	20	2	40	80
105-120	230	112.5	8	3	24	72
			$\Sigma f_i = 230$		$\Sigma f_i u_i = -105$	$\Sigma f_i u_i^2 = 733$

Here, $A = 67.5,\ h = 15,\ N = 230,\ \Sigma f_i u_i = -105$ and $\Sigma f_i u_i^2 = 733$

$$\therefore \quad \text{Mean} = A + h\left(\frac{1}{N}\,\Sigma f_i u_i\right) = 67.5 + 15\left(\frac{-105}{230}\right) = 67.5 - 6.85 = 60.65$$

and,
$$\text{Var}(X) = h^2\left\{\frac{1}{N}\,\Sigma f_i u_i^2 - \left(\frac{1}{N}\,\Sigma f_i u_i\right)^2\right\}$$

$$\Rightarrow \quad \text{Var}(X) = 225\left\{\frac{733}{230} - \left(\frac{-105}{230}\right)^2\right\} = 225\,(3.18 - 0.2025) = 669.9375$$

$$\therefore \quad \text{S.D.} = \sqrt{\text{Var}(X)} = \sqrt{669.9375} = 25.883$$

EXAMPLE 4 *The measurements of the diameters (in mm) of the heads of 107 screws are given below:*

Diameter (in mm)	33-35	36-38	39-41	42-44	45-47
No. of screws	17	19	23	21	27

Calculate the standard deviation.

SOLUTION Here the class intervals are formed by the inclusive method. But, the mid-points of class-intervals remain same whether they are formed by inclusive method or exclusive method. So there is no need to convert them into an exclusive series.

Calculation of Standard Deviation

Diameter (in mm)	*Mid-values x_i*	*No. of screws f_i*	$u_i = \frac{x_i - 40}{3}$	$f_i u_i$	$f_i u_i^2$
33-35	34	17	–2	–34	68
36-38	37	19	–1	–19	19
39-41	40	23	0	0	0
42-44	43	21	1	21	21
45-47	46	27	2	54	108
		$\Sigma f_i = 107$		$\Sigma f_i u_i = 22$	$\Sigma f_i u_i^2 = 216$

Here $N = \Sigma f_i = 107,\ \Sigma f_i u_i = 22,\ \Sigma f_i u_i^2 = 216,\ A = 40$ and, $h = 3$

$$\therefore \quad \text{Var}(X) = h^2 \left\{ \left(\frac{1}{N} \Sigma f_i u_i^2 \right) - \left(\frac{1}{N} \Sigma f_i u_i \right)^2 \right\} = 9 \left\{ \frac{216}{107} - \left(\frac{22}{107} \right)^2 \right\}$$

$$\Rightarrow \quad \text{Var}(X) = 9(2.0187 - 0.0420) = 9 \times 1.9767 = 17.7903$$

$$\therefore \quad \text{S.D.} = \sqrt{17.7903} = 4.2178.$$

EXAMPLE 5 *Calculate the mean and standard deviation for the following table given the age distribution of a group of people:*

Age:	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of persons:	3	51	122	141	130	51	2

SOLUTION Here $A = 55$, $h = 10$.

Calculation of Mean and Standard Deviation

Age	*Mid-values* (x_i)	*Number of persons* (f_i)	$u_i = \frac{x_i - 55}{10}$	$f_i u_i$	u_i^2	$f_i u_i^2$
20-30	25	3	-3	-9	9	27
30-40	35	51	-2	-102	4	204
40-50	45	122	-1	-122	1	122
50-60	55	141	0	0	0	0
60-70	65	130	1	130	1	130
70-80	75	51	2	102	4	204
80-90	85	2	3	6	9	18
		$N = \Sigma f_i = 500$		$\Sigma f_i u_i = 5$		$\Sigma f_i u_i^2 = 705$

Here, $N = \Sigma f_i = 500,\ \Sigma f_i u_i = 5$ and, $\Sigma f_i u_i^2 = 705$

$$\therefore \quad \overline{X} = A + h\left(\frac{1}{N} \Sigma f_i u_i \right) = 55 + 10\left(\frac{5}{500} \right) = 55.1$$

and, $$\sigma^2 = h^2 \left\{ \left(\frac{1}{N} \Sigma f_i u_i^2 \right) - \left(\frac{1}{N} \Sigma f_i u_i \right)^2 \right\}$$

$$\Rightarrow \quad \sigma^2 = 100 \left\{ \frac{705}{500} - \left(\frac{5}{500} \right)^2 \right\} = \frac{100}{50000}(70500 - 5) = \frac{70495}{500} = \frac{14099}{100}$$

$$\Rightarrow \quad \sigma = \frac{\sqrt{14099}}{10} = \frac{118.739}{10} = 11.8739.$$

EXERCISE 32.6

LEVEL-1

1. Calculate the mean and S.D. for the following data:

Expenditure (in ₹):	0-10	10-20	20-30	30-40	40-50
Frequency:	14	13	27	21	15

2. Calculate the standard deviation for the following data:

Class:	0-30	30-60	60-90	90-120	120-150	150-180	180-210
Frequency:	9	17	43	82	81	44	24

3. Calculate the A.M. and S.D. for the following distribution:

Class:	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency:	18	16	15	12	10	5	2	1

4. A student obtained the mean and standard deviation of 100 observations as 40 and 5.1 respectively. It was later found that one observation was wrongly copied as 50, the correct figure being 40. Find the correct mean and S.D.

5. Calculate the mean, median and standard deviation of the following distribution:

Class-interval:	31-35	36-40	41-45	46-50	51-55	56-60	61-65	66-70
Frequency:	2	3	8	12	16	5	2	3

LEVEL-2

6. Find the mean and variance of frequency distribution given below:

x_i :	$1 \le x < 3$	$3 \le x < 5$	$5 \le x < 7$	$7 \le x < 10$
f_1 :	6	4	5	1

7. The weight of coffee in 70 jars is shown in the following table: [NCERT EXEMPLAR]

Weight (in grams):	200-201	201-202	202-203	203-204	204-205	205-206
Frequency:	13	27	18	10	1	1

Determine the variance and standard deviation of the above distribution. [NCERT EXEMPLAR]

8. Mean and standard deviation of 100 observations were found to be 40 and 10 respectively. If at the time of calculation two observations were wrongly taken as 30 and 70 in place of 3 and 27 respectively, find the correct standard deviation. [NCERT EXEMPLAR]

9. While calculating the mean and variance of 10 readings, a student wrongly used the reading of 52 for the correct reading 25. He obtained the mean and variance as 45 and 16 respectively. Find the correct mean and the variance. [NCERT EXEMPLAR]

10. Calculate mean, variance and standard deviation of the following frequency distribution:

Class :	1-10	10-20	20-30	30-40	40-50	50-60
Frequency :	11	29	18	4	5	3

ANSWERS

1. $\overline{X} = 26.11, \sigma = 12.86.$ 2. $\overline{X} = 118.7, \sigma = 42.51.$ 3. $AM = 26.01,\ S.D. = 17.47$
4. $\overline{X} = 39.9, \sigma = 5.$ 5. $\overline{X} = 50.35,\ \sigma = 7.94,$ median $= 50.65$
6. Mean $= 5.5$, Variance $= 4.26$ 7. Variance = 1.16 gm, S.D. 1.08 gm
8. 10.24 9. Mean = 42.3, Variance = 43.81 10. Mean = 21.5, Variance = 161, S.D. = 12.7

32.7 ANALYSIS OF FREQUENCY DISTRIBUTIONS

In this section, we shall see how we can use various measures of dispersion to compare two or more series. In the earlier sections of this chapter we have seen that the mean deviation and standard deviation have the same units in which the data are given. Therefore, measures of dispersion are unable to compare two or more series which are measured in different units even if they have the same mean. Thus, we require those measures which are independent of the units. The measure of variability which is independent of units is called coefficient of variation (C.V.) and is defined as

$$\text{C.V.} = \frac{\sigma}{\overline{X}} \times 100,$$ where σ and $\overline{X}$ are the standard deviation and mean of the data.

For comparing the variability of two series, we calculate the coefficient of variation for each series. The series having greater C.V. is said to be more variable or conversely less consistent, less uniform, less stable or less homogeneous than the other and the series having lesser C.V. is said to be more consistent (or homogeneous) than the other.

Let there be two frequency distributions with standard deviations σ_1 and σ_2 and equal mean $\overline{X}$. Then,

$$\text{C.V. (1st distribution)} = \frac{\sigma_1}{\overline{X}} \times 100 \text{ and, C.V. (2nd distribution)} = \frac{\sigma_2}{\overline{X}} \times 100$$

$$\therefore \quad \frac{\text{C.V. (1st distribution)}}{\text{C.V. (2nd distribution)}} = \frac{\frac{\sigma_1}{\overline{X}} \times 100}{\frac{\sigma_2}{\overline{X}} \times 100} = \frac{\sigma_1}{\sigma_2}$$

This means that the two distributions can be compared on the basis of the values of their standard deviations σ_1 and σ_2 only.

Thus, if two series have equal means then the series with greater standard deviation (or variance) is said to be more variable or dispersed than the other. Also the series with lesser value of the standard deviation (or variance) is said to be more consistent than the other.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *An analysis of monthly wages paid to the workers of two firms A and B belonging to the same industry gives the following results:*

	Firm A	*Firm B*
Number of workers	1000	1200
Average monthly wages	₹ 2800	₹ 2800
Variance of distribution of wages	100	169

In which firm, A or B is there greater variability in individual wages?

SOLUTION We observe that the average monthly wages in both the firms is same i.e. Rs. 2800. Therefore, the firm with greater variance will have more variability. Thus, firm *B* has greater variability in individual wages.

EXAMPLE 2 *An analysis of monthly wages paid to workers in two firms A and B, belonging to the same industry, gives the following results:* [NCERT]

	Firm A	*Firm B*
No. of wage earners	586	648
Mean of monthly wages	₹ 5253	₹ 5253
Variance of the distribution of wages	100	121

(i) *Which firm A or B pays out larger amount as monthly wages?*

(ii) *Which firm A or B shows greater variability in individual wages?*

SOLUTION (i) <u>Firm A:</u>

Number of wage earners (say) $n_1 = 586$

Mean of monthly wages (say) $\overline{X}_1 = ₹\ 5253$

$$\therefore \quad \text{Mean of monthly wages} = \frac{\text{Total monthly wage}}{\text{Number of workers}}$$

$$\Rightarrow \quad 5253 = \frac{\text{Total monthly wages}}{586}$$

$$\Rightarrow \quad \text{Total monthly wages} = ₹\ (5253 \times 586) = ₹\ 3078258$$

<u>Firm B:</u>

Number of wage earners (say) $n_2 = 648$

Mean of monthly wages $= ₹\ 5253$

$\therefore$ Mean of monthly wages $= \dfrac{\text{Total monthly wages}}{\text{Number of workers}}$

$\Rightarrow$ $5253 = \dfrac{\text{Total monthly wages}}{648}$

$\Rightarrow$ Total monthly wages $=$ ₹ (5253×648) $=$₹ 3403944

Clearly, firm B pays out larger amount as monthly wages.

(ii) Since firms A and B have the same mean. Therefore, the firm with greater variance will have more variability individual wages.

Clearly, Variance of firm $B >$ Variance of firm A.

Hence, firm B will have greater variability in individual wages.

EXAMPLE 3 *The following values are calculated in respect of heights and weights of the students of a section of class XI:*

	Height	*Weight*
Mean	162.6 cm	52.36 kg
Variance	127.69 cm^2	23.1361 kg^2

Can we say that the weights show greater variation than the heights? [NCERT]

SOLUTION In order to compare the variability of height and weight, we have to calculate their coefficients of variation. Let σ_1 and σ_2 denote the standard deviations of height and weight respectively. Further, let $\overline{X}_1$ and $\overline{X}_2$ be the mean height and weight respectively.

We have,

$$\overline{X}_1 = 162.6, \qquad \overline{X}_2 = 52.36$$

$$\sigma_1^2 = 127.69 \quad \text{and} \quad \sigma_2^2 = 23.1361$$

$$\Rightarrow \quad \sigma_1 = \sqrt{127.69} = 11.3 \quad \text{and} \quad \sigma_2 = \sqrt{23.1361} = 4.81$$

Now,

$$\text{Coefficient of variation in heights} = \frac{\sigma_1}{\overline{X}_1} \times 100 = \frac{11.3}{162.6} \times 100 = 6.95$$

and,

$$\text{Coefficient of variation in weights} = \frac{\sigma_2}{\overline{X}_2} \times 100 = \frac{4.81}{52.36} \times 100 = 9.18$$

Clearly, coefficient of variation in weights is greater than the coefficient of variation in heights. So, weights shows more variability than heights.

EXAMPLE 4 *The sum and sum of squares corresponding to length x (in cm) and weight y (in gm) of 50 plant products are given below:*

$$\sum_{i=1}^{50} x_i = 212, \quad \sum_{i=1}^{50} x_i^2 = 902.8, \quad \sum_{i=1}^{50} y_i = 261, \quad \sum_{i=1}^{n} y_i^2 = 1457.6$$ [NCERT]

Which is more varying, the length or weight?

SOLUTION We have,

$$\sum_{i=1}^{50} x_i = 212 \quad \text{and} \quad \sum_{i=1}^{50} x_i^2 = 902.80$$

$$\therefore \quad \overline{X} = \frac{\sum_{i=1}^{50} x_i}{50} \quad \text{and} \quad \sigma_X^2 = \frac{1}{50}\left(\sum_{i=1}^{50} x_i^2\right) - \left(\frac{1}{50}\sum_{i=1}^{50} x_i\right)^2$$

$$\Rightarrow \quad \bar{X} = \frac{212}{50} \quad \text{and} \quad \sigma_X^2 = \frac{902.80}{50} - \left(\frac{212}{50}\right)^2$$

$$\Rightarrow \quad \bar{X} = 4.24 \quad \text{and} \quad \sigma_X^2 = 18.056 - (4.24)^2 = 18.056 - 17.9776 = 0.0784$$

$$\Rightarrow \quad \bar{X} = 4.24 \quad \text{and} \quad \sigma_X = \sqrt{0.0784} = 0.28$$

It is given that

$$\sum_{i=1}^{50} y_i = 261 \quad \text{and} \quad \sum_{i=1}^{50} y_i^2 = 1457.6$$

$$\therefore \quad \bar{Y} = \frac{\sum_{i=1}^{50} y_i}{50} \quad \text{and} \quad \sigma_Y^2 = \frac{1}{50}\left(\sum_{i=1}^{50} y_i^2\right) - \left(\frac{1}{50}\sum_{i=1}^{50} y_i\right)^2$$

$$\Rightarrow \quad \bar{Y} = \frac{261}{50} \quad \text{and} \quad \sigma_y^2 = \frac{1457.6}{50} - \left(\frac{261}{50}\right)^2$$

$$\Rightarrow \quad \bar{Y} = 5.22 \quad \text{and} \quad \sigma_Y^2 = 29.152 - (5.22)^2 = 1.9036$$

$$\Rightarrow \quad \bar{Y} = 5.22 \quad \text{and} \quad \sigma_Y = 1.3797$$

In order to determine the variability of length and weight, we will have to compute the coefficients of variations in lengths and weights.

$$\text{Coefficient of variation in lengths} = \frac{\sigma_X}{\bar{X}} \times 100 = \frac{0.28}{4.24} \times 100 = 6.60$$

$$\text{Coefficient of variation in weights} = \frac{\sigma_Y}{\bar{Y}} \times 100 = \frac{1.3797}{5.22} \times 100 = 26.43$$

Clearly, coefficient of variation is weights is greater than the coefficient of variation in lengths. Hence, weights have more variability than lengths.

EXAMPLE 5 *The following is the record of goals scored by team A in football session.*

Number of goals scored:	0	1	2	3	4
Number of matches:	1	9	7	5	3

For the team B, mean number of goals scored per match was 2 with a standard deviation 1.25 goals. Find which team may be considered more consistent?

SOLUTION In order to determine the consistency of teams we will have to find the coefficients of variations of two teams.

Computation of mean and standard deviation of goals scored by team A.

No. of goals scored x_i	*No. of matches* f_i	$f_i x_i$	$f_i x_i^2$
0	1	0	0
1	9	9	9
2	7	14	28
3	5	15	45
4	3	12	48
	$\Sigma f_i = 25$	$\Sigma f_i x_i = 50$	$\Sigma f_i x_i^2 = 130$

We have,

$$N = \Sigma f_i = 25, \ \Sigma f_i x_i = 50 \text{ and } \Sigma f_i x_i^2 = 130$$

$$\therefore \quad \overline{X}_A = \frac{1}{N}(\Sigma f_i x_i) = \frac{50}{25} = 2$$

and, $$\sigma_A^2 = \left(\frac{1}{N}\Sigma f_i x_i^2\right) - \left(\frac{1}{N}\Sigma f_i x_i\right)^2 = \frac{130}{25} - \left(\frac{50}{25}\right)^2 = 5.2 - 4 = 1.2$$

$$\Rightarrow \quad \sigma_A = \sqrt{1.2} = 1.095$$

It is given that $\overline{X}_B = 2$ and $\sigma_B = 1.25$

Now,

Coefficient of variation in goals scored by team $A = \frac{\sigma_A}{\overline{X}_A} \times 100 = \frac{1.095}{2} \times 100 = 54.75$

Coefficient of variation of goals scored by team $B = \frac{\sigma_B}{\overline{X}_B} \times 100 = \frac{1.25}{2} \times 100 = 62.50$

We observe that the coefficient of variation of goals scored by team A is lesser than that of team B. Hence, team A is more consistent.

EXAMPLE 6 *Suppose that samples of polythene bags from two manufacturers, A and B, are tested by a prospective buyer for bursting pressure, with the following results:*

Bursting Pressure in kg	*Number of bags manufactured by manufacturer* A	B
5-10	2	9
10-15	9	11
15-20	29	18
20-25	54	32
25-30	11	27
30-35	5	13

Which set of the bags has the highest average brusting pressure? Which has more uniform pressure?

SOLUTION For determining the set of bags having higher average bursting pressure, we compute mean and for finding out set of bags having more uniform pressure we compute coefficient of variation.

Manufacturer A:

Computation of mean and standard deviation

Bursting pressure	*Mid-values* x_i	f_i	$u_i = \frac{x_i - 17.5}{5}$	$f_i u_i$	$f_i u_i^2$
5-10	7.5	2	−2	−4	8
10-15	12.5	9	−1	−9	9
15-20	17.5	29	0	0	0
20-25	22.5	54	1	54	54
25-30	27.5	11	2	22	44
30-35	32.5	5	3	15	45
		$N = \Sigma f_i = 110$	$\Sigma u_i = 3$	$\Sigma f_i u_i = 78$	$\Sigma f_i u_i^2 = 160$

$$\overline{X}_A = a + h\left(\frac{\Sigma f_i u_i}{N}\right)$$

$$\Rightarrow \overline{X}_A = 17.5 + 5 \times \frac{78}{110} = 175 + 35 = 21 \qquad [\because h = 5, a = 175]$$

$$\sigma_A^2 = h^2 \left\{\left(\frac{1}{N}\Sigma f_i u_i^2\right) - \left(\frac{1}{N}\Sigma f_i u_i\right)^2\right\} = 25\left\{\frac{160}{110} - \left(\frac{78}{110}\right)^2\right\} = 25\left(\frac{17600 - 6084}{110 \times 110}\right) = 23.79$$

$$\Rightarrow \sigma_A = \sqrt{23.79} = 4.87$$

$$\therefore \text{ Coefficient of variation } = \frac{\sigma_A}{\overline{X}_A} \times 100 = \frac{4.87}{21} \times 100 = 23.19$$

Manufacturer B:

Bursting pressure	Mid-values x_i	f_i	$u_i = \frac{x_i - 17.5}{5}$	$f_i u_i$	$f_i u_i^2$
5-10	7.5	9	-2	-18	36
10-15	12.5	11	-1	-11	11
15-20	17.5	18	0	0	0
20-25	22.5	32	1	32	32
25-30	27.5	27	2	54	108
30-35	32.5	13	3	39	117
		$N = \Sigma f_i = 110$		$\Sigma f_i u_i = 96$	$\Sigma f_i u_i^2 = 304$

Here, $N = 110$, $\Sigma f_i u_i = 96$, $a = 17.5$ and $h = 5$

$$\therefore \quad \overline{X}_B = a + h\left(\frac{\Sigma f_i u_i}{N}\right) = 17.5 + 5 \times \frac{96}{110} = 17.5 + 4.36 = 21.81$$

and, $$\sigma_B^2 = h^2 \left\{\left(\frac{1}{N}\Sigma f_i u_i^2\right) - \left(\frac{1}{N}\Sigma f_i u_i\right)^2\right\} = 25\left\{\frac{304}{110} - \left(\frac{96}{110}\right)^2\right\} = 25\left(\frac{33440 - 9216}{110 \times 110}\right) = 50.04$$

$$\Rightarrow \quad \sigma_B = \sqrt{50.04} = 7.07$$

$$\therefore \quad \text{Coefficient of variation } = \frac{\sigma_B}{\overline{X}_B} \times 100 = \frac{7.07}{21.81} \times 100 = 32.41$$

We observe that the average brusting pressure is higher for manufacturer B. So, bags manufactured by B have higher bursting pressure.
The coefficient of variation is less for manufacturer A. So, bags manufactured by A have more uniform pressure.

EXERCISE 32.7

LEVEL-1

1. Two plants A and B of a factory show following results about the number of workers and the wages paid to them

	Plant A	Plant B
No. of workers	5000	6000
Average monthly wages	₹ 2500	₹ 2500
Variance of distribution of wages	81	100

In which plant A or B is there greater variability in individual wages?

2. The means and standard deviations of heights ans weights of 50 students of a class are as follows:

	Weights	Heights
Mean	63.2 kg	63.2 inch
Standard deviation	5.6 kg	11.5 inch

Which shows more variability, heights or weights?

3. Coefficient of variation of two distributions are 60% and 70% and their standard deviations are 21 and 16 respectively. What are their arithmetic means?
4. Calculate coefficient of variation from the following data:

Income (in ₹):	1000-1700	1700-2400	2400-3100	3100-3800	3800-4500	4500-5200
No. of families:	12	18	20	25	35	10

5. An analysis of the weekly wages paid to workers in two firms A and B, belonging to the same industry gives the following results:

	Firm A	Firm B
No. of wage earners	586	648
Average weekly wages	₹ 52.5	₹ 47.5
Variance of the distribution of wages	100	121

(i) Which firm A or B pays out larger amount as weekly wages?
(ii) Which firm A or B has greater variability in individual wages?

6. The following are some particulars of the distribution of weights of boys and girls in a class:

	Boys	Girls
Number	100	50
Mean weight	60 kg	45 kg
Variance	9	4

Which of the distributions is more variable?

7. The mean and standard deviation of marks obtained by 50 students of a class in three subjects, mathematics, physics and chemistry are given below:

Subject	Mathematics	Physics	Chemistry
Mean	42	32	40.9
Standard Deviation	12	15	20

Which of the three subjects shows the highest variability in marks and which shows the lowest? [NCERT]

8. From the data given below state which group is more variable G_1 or G_2?

Marks	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Group G_1	9	17	32	33	40	10	9
Group G_2	10	20	30	25	43	15	7

9. Find the coefficient of variation for the following data:

Size (in cms):	10-15	15-20	20-25	25-30	30-35	35-40
No. of items:	2	8	20	35	20	15

10. From the prices of shares X and Y given below: find out which is more stable in value:

X:	35	54	52	53	56	58	52	50	51	49
Y:	108	107	105	105	106	107	104	103	104	101

11. Life of bulbs produced by two factories A and B are given below:

Length of life (in hours) :	550-650	650-750	750-850	850-950	950-1050
Factory A : (Number of bulbs)	10	22	52	20	16
Factory B : (Number of bulbs)	8	60	24	16	12

The bulbs of which factory are more consistent from the point of view of length of life? [NCERT EXEMPLAR]

12. Following are the marks obtained, out of 100, by two students Ravi and Hashina in 10 tests :

Ravi:	25	50	45	30	70	42	36	48	35	60
Hashina:	10	70	50	20	95	55	42	60	48	80

Who is more intelligent and who is more consistent? [NCERT EXEMPLAR]

ANSWERS

1. Plant B 2. Heights 3. 35, 22.85 4. 3.21 5. (i) Firm B (ii) Firm B
6. Boys 7. Highest: Chemistry Lowest: Mathematics 8. G_1 9. 21.75 10. Y
11. Factory A 12. Hashina is more intelligent and consistent.

HINTS TO NCERT & SELECTED PROBLEM

1. We observe that S.D. of marks in Mathematics is least and that of Chemistry is highest. Therefore, marks in Mathematics have lowest variability and that in Chemistry have highest variability.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the variance of first n natural numbers.
2. If the sum of the squares of deviations for 10 observations taken from their mean is 2.5, then write the value of standard deviation.
3. If $x_1, x_2, \ldots, x_n$ are n values of a variable X and $y_1, y_2, \ldots, y_n$ are n values of variable Y such that $y_i = ax_i + b, i = 1, 2, \ldots, n$, then write Var(Y) in terms of Var(X).
4. If X and Y are two variates connected by the relation $Y = \frac{aX + b}{c}$ and Var $(X) = \sigma^2$, then write the expression for the standard deviation of Y.
5. In a series of 20 observations, 10 observations are each equal to k and each of the remaining half is equal to $-k$. If the standard deviation of the observations is 2, then write the value of k.
6. If each observation of a raw data whose standard deviation is σ is multiplied by a, then write the S.D. of the new set of observations.
7. If a variable X takes values 0, 1, 2,..., n with frequencies ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$, then write variance X.

ANSWERS

1. $\dfrac{n^2-1}{12}$ 2. 0.5 3. Var (Y) $= a^2$ Var (X)

4. $\left|\dfrac{a}{c}\right|\sigma$ 5. ± 2 6. $|a|\sigma$ 7. $\dfrac{n}{4}$

MULTIPLE CHOICE QUESTIONS (MCQs)

1. For a frequency distribution mean deviation from mean is computed by
   (a) M.D. $= \dfrac{\Sigma f}{\Sigma f|d|}$ (b) M.D. $= \dfrac{\Sigma d}{\Sigma f}$
   (c) M.D. $= \dfrac{\Sigma fd}{\Sigma f}$ (d) M.D. $= \dfrac{\Sigma f|d|}{\Sigma f}$
2. For a frequency distribution standard deviation is computed by applying the formula
   (a) $\sigma = \sqrt{\dfrac{\Sigma fd^2}{\Sigma f} - \left(\dfrac{\Sigma fd}{\Sigma f}\right)^2}$ (b) $\sigma = \sqrt{\left(\dfrac{\Sigma fd}{\Sigma f}\right)^2 - \dfrac{\Sigma fd^2}{\Sigma f}}$
   (c) $\sigma = \sqrt{\dfrac{\Sigma fd^2}{\Sigma f} - \dfrac{\Sigma fd}{\Sigma f}}$ (d) $\sqrt{\left(\dfrac{\Sigma fd}{\Sigma f}\right)^2 - \dfrac{\Sigma fd^2}{\Sigma f}}$
3. If v is the variance and σ is the standard deviation, then
   (a) $v = \dfrac{1}{\sigma^2}$ (b) $v = \dfrac{1}{\sigma}$ (c) $v = \sigma^2$ (d) $v^2 = \sigma$
4. The mean deviation from the median is
   (a) equal to that measured from another value
   (b) maximum if all observations are positive
   (c) greater than that measured from any other value.
   (d) less than that measured from any other value.
5. If $n = 10, \overline{X} = 12$ and $\Sigma x_i^2 = 1530$, then the coefficient of variation is
   (a) 36 % (b) 41 % (c) 25 % (d) none of these
6. The standard deviation of the data:

x:	1	a	a^2	....	a^n
f:	${}^nC_0$	${}^nC_1$	${}^nC_2$	....	${}^nC_n$

   is
   (a) $\left(\dfrac{1+a^2}{2}\right)^n - \left(\dfrac{1+a}{2}\right)^n$ (b) $\left(\dfrac{1+a^2}{2}\right)^{2n} - \left(\dfrac{1+a}{2}\right)^n$
   (c) $\left(\dfrac{1+a}{2}\right)^{2n} - \left(\dfrac{1+a^2}{2}\right)^n$ (d) none of these
7. The mean deviation of the series $a, a+d, a+2d, ..., a+2n$ from its mean is
   (a) $\dfrac{(n+1)d}{2n+1}$ (b) $\dfrac{nd}{2n+1}$ (c) $\dfrac{n(n+1)d}{2n+1}$ (d) $\dfrac{(2n+1)d}{n(n+1)}$
8. A batsman scores runs in 10 innings as 38, 70, 48, 34, 42, 55, 63, 46, 54 and 44. The mean deviation about mean is
   (a) 8.6 (b) 6.4 (c) 10.6 (d) 7.6

9. The mean deviation of the numbers 3, 4, 5, 6, 7 from the mean is
(a) 25 (b) 5 (c) 1.2 (d) 0

10. The sum of the squares deviations for 10 observations taken from their mean 50 is 250. The coefficient of variation is
(a) 10 % (b) 40 % (c) 50 % (d) none of these

11. Let $x_1, x_2, \ldots, x_n$ be values taken by a variable X and $y_1, y_2, \ldots, y_n$ be the values taken by a variable Y such that $y_i = ax_i + b, i = 1, 2, \ldots, n$. Then,
(a) $\text{Var}(Y) = a^2 \text{Var}(X)$ (b) $\text{Var}(X) = a^2 \text{Var}(Y)$
(c) $\text{Var}(X) = \text{Var}(X) + b$ (d) none of these

12. If the standard deviation of a variable X is σ, then the standard deviation of variable $\frac{aX+b}{c}$ is
(a) $a\sigma$ (b) $\frac{a}{c}\sigma$ (c) $\left|\frac{a}{c}\right|\sigma$ (d) $\frac{a\sigma+b}{c}$

13. If the S.D. of a set of observations is 8 and if each observation is divided by -2, the S.D. of the new set of observations will be
(a) -4 (b) -8 (c) 8 (d) 4

14. If two variates X and Y are connected by the relation $Y = \frac{aX+b}{c}$, where a, b, c are constants such that $ac < 0$, then
(a) $\sigma_Y = \frac{a}{c}\sigma_X$ (b) $\sigma_Y = -\frac{a}{c}\sigma_X$ (c) $\sigma_Y = \frac{a}{c}\sigma_X + b$ (d) none of these

15. If for a sample of size 60, we have the following information $\Sigma x_i^2 = 18000$ and $\Sigma x_i = 960$, then the variance is
(a) 6.63 (b) 16 (c) 22 (d) 44

16. Let a, b, c, d, e be the observations with mean m and standard deviation s. The standard deviation of the observations $a+k, b+k, c+k, d+k, e+k$ is
(a) s (b) ks (c) $s+k$ (d) $\frac{s}{k}$

17. The standard deviation of first 10 natural numbers is
(a) 5.5 (b) 3.87 (c) 2.97 (d) 2.87

18. Consider the first 10 positive integers. If we multiply each number by -1 and then add 1 to each number, the variance of the numbers so obtained is
(a) 8.25 (b) 6.5 (c) 3.87 (d) 2.87

19. Consider the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. If 1 is added to each number, the variance of the numbers so obtained is
(a) 6.5 (b) 2.87 (c) 3.87 (d) 8.25

20. The mean of 100 observations is 50 and their standard deviation is 5. The sum of all squares of all the observations is
(a) 50,000 (b) 250,000 (c) 252500 (d) 255000

21. Let $x_1, x_2, \ldots, x_n$ be n observations. Let $y_i = ax_i + b$ for $i = 1, 2, \ldots, n$, where a and b are constants. If the mean of x_i's is 48 and their standard deviation is 12, the mean of y_i's is 55 and standard deviation of y_i's is 15, the values of a and b are
(a) $a = 1.25, b = -5$ (b) $a = -1.25, b = 5$ (c) $a = 2.5, b = -5$ (d) $a = 2.5, b = 5$

22. The mean deviation of the data 3, 10, 10, 4, 7, 10, 5 from the mean is
(a) 2 (b) 2.57 (c) 3 (d) 3.57

23. The mean deviation for n observations $x_1, x_2, \ldots, x_n$ from their mean $\overline{X}$ is given by

(a) $\sum_{i=1}^{n} (x_i - \overline{X})$ (b) $\frac{1}{n}\sum_{i=1}^{n} (x_i - \overline{X})$ (c) $\sum_{i=1}^{n} (x_i - \overline{X})^2$ (d) $\frac{1}{n}\sum_{i=1}^{n} (x_i - \overline{X})^2$

24. Let $x_1, x_2, \ldots, x_n$ be n observations and $\overline{X}$ be their arithmetic mean. The standard deviation is given by

(a) $\sum_{i=1}^{n} (x_i - \overline{X})^2$ (b) $\frac{1}{n}\sum_{i=1}^{n} (x_i - \overline{X})^2$ (c) $\sqrt{\frac{1}{n}\sum_{i=1}^{n} (x_i - \overline{X})^2}$ (d) $\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \overline{X}^2}$

25. The standard deviation of the observations 6, 5, 9, 13, 12, 8, 10 is

(a) 6 (b) $\sqrt{6}$ (c) $\frac{52}{7}$ (d) $\sqrt{\frac{52}{7}}$

ANSWERS

1. (d)	2. (a)	3. (c)	4. (d)	5. (c)	6. (a)	7. (c)	8. (a)
9. (c)	10. (a)	11. (a)	12. (c)	13. (d)	14. (b)	15. (d)	16. (a)
17. (d)	18. (b)	19. (c)	20. (a)	21. (a)	22. (b)	23. (b)	24. (c)
25. (d)							

SUMMARY

1. Dispersion means scatternedness around the central value.
2. Following are the measures of dispersion:
   (i) Range (ii) Quartile diviation (iii) Mean deviation (iv) Standard deviation
3. Range is the difference between the greatest and the least values of the variable.
4. Mean deviation is the arithmetic mean of the absolute values of deviations about some point (mean or median or mode).
   (i) For individual observation, we have

$$\text{M.D.} = \frac{1}{n}\sum_{i=1}^{n} |x_i - a|, \text{ where } a = \text{mean, median, mode}$$

$$\text{Also, M.D.} = a + h\left\{\frac{1}{N}\sum_{i=1}^{n} |u_i|\right\}, \text{ where } u_i = \frac{x_i - a}{h}$$

   (ii) For a discrete frequency distribution, we have

$$\text{M.D.} = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - a|, \; a = \text{mean, median, mode}$$

$$\text{M.D.} = a + h\left\{\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right\}, \text{ where } u_i = \frac{x_i - a}{h}$$

5. Standard deviation is the positive square root of variance.
6. Variance is the arithmetic mean of the squares of deviations about mean $\overline{X}$.
   (i) For individual observations, we have

$$\text{Variance (X)} = \frac{1}{n}\sum_{i=1}^{n} (x_i - \overline{X})^2$$

Also, $$\text{Var (X)} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

and, $$\text{Var (X)} = h^2\left\{\left(\frac{1}{n}\sum_{i=1}^{n} u_i^2\right) - \left(\frac{1}{n}\sum_{i=1}^{n} u_i\right)^2\right\}, \text{ where } u_i = \frac{x_i - a}{h}$$

(ii) For a discrete frequency distribution, we have

$$\text{Var}(X) = \frac{1}{N} \sum_{i=1}^{n} f_i \, (x_i - \overline{X})^2$$

Also, $$\text{Var}(X) = \left(\frac{1}{N} \sum_{i=1}^{n} f_i \, x_i^2 \right) - \left(\frac{1}{N} \sum_{i=1}^{n} f_i \, x_i \right)^2$$

and, $$\text{Var}(X) = h^2 \left\{ \left(\frac{1}{N} \sum_{i=1}^{n} f_i \, u_i^2 \right) - \left(\frac{1}{N} \sum_{i=1}^{n} f_i \, u_i \right)^2 \right\}$$

7. In order to compare two or more frequency distributions we compare their coefficients of variations. The coefficient of variation is defined as

$$\text{C.V.} = \frac{\sigma}{\overline{X}} \times 100$$

8. The distribution having greater coefficient of variation has more variability around the central value than the distribution having smaller value of the coefficient of variation.

CHAPTER 33

PROBABILITY

33.1 INTRODUCTION

In earlier classes, we have learnt about two approaches to the theory of probability, namely, (i) Statistical approach and (ii) Classical approach. The statistical approach has been discussed in class IX. It is also known as repeated experiments and observed frequency approach. In this approach, we have defined the probability of an event as the ratio of observed frequency to the total frequency. The classical approach has been discussed in class X. In this approach, we define the probability of occurrence of an event as the ratio of favourable number of outcomes to the total number of equally likely outcomes. These equally likely outcomes are also known as elementary events associated to the experiment. Both the theories have some serious deficiencies and limitations. For instance, these approaches cannot be applied to the experiments which have large number of outcomes. The classical definition of probability cannot be applied whenever it is not possible to make a simple enumeration of cases which can be considered equally likely. For instance, how does it apply to probability of rain? What are the possible outcomes? We might think that there are two cases 'rain' and 'no rain'. But at any given locality it will not usually be agreed that they are equally like. The classical approach also fails to answer questions like "what is the probability that a male will die before the age of 60", "what is the probability that a bulb will burn in less than 2000 hours? etc. In fact, the classical definition is difficult to apply as soon as we deviate from the experiments pertaining to coins, dice, cards and other simple games of chance.

The statistical definition has difficulties from a mathematical point of view because an actual limiting number may not really exist. For this reason, modern probability theory has been developed axiomatically. This theory of probability was developed by A.N. Kolmogrov (1903-1987) a Russian Mathematician in 1933. He laid down certain axioms to interpret probability, in his book 'Foundation of Probability' published in 1933. The axiomatic definition of probability includes 'both' the classical and statistical approaches as particular cases and overcomes the deficiencies of each of them. In order to understand this approach we must know about some basic terms viz. random experiment, elementary events, sample space, compound events etc. So, let us begin with the term random experiment as discussed in the following section.

33.2 RANDOM EXPERIMENTS

The word experiment means an operation which can produce some well-defined outcome(s). There are two types of experiments viz. (i) Deterministic experiments and (ii) Random or Probability experiments.

DETERMINISTIC EXPERIMENTS In our day-to-day life, we perform many activities/experiments which have a fixed outcome or result no matter any number of times they are repeated. Such experiments are known as deterministic experiments. For example, from the set of all triangles in a plane if a triangle is choosen, then even without knowing the three angles, we can definitely say that the sum of the measures of the angles is 180°. In fact, when experiments in science and engineering are repeated under identical conditions, we get the same result every time.

RANDOM OR PROBABILISTIC EXPERIMENTS If an experiment, when repeated under identical conditions, do not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes then such an experiment is known as a probabilistic experiment or a random experiment. In other words, an experiment whose outcomes cannot be predicted or determined in advance is called a random experiment.

For example, in tossing of a coin one is not sure if a head or a tail will be obtained so it is a random experiment. Similarly, rolling an unbiased die and drawing a card from a well shuffled pack of playing cards are examples of random experiments.

33.3 SAMPLE SPACES

In the previous section, we have learnt about random experiments. Throughout this chapter the term experiment will mean random experiment. Associated to every random experiment there are two basic terms viz. outcomes (or elementary events) and sample space. In this section, we will discuss about these two for different random experiments.

ELEMENTARY EVENT *If a random experiment is performed, then each of its outcomes is known as an elementary event.*

In other words, outcomes of a random experiment are known as elementary events associated to it. Elementary events are also known as simple events.

SAMPLE SPACE *The set of all possible outcomes of a random experiment is called the sample space associated with it and it is generally denoted by S.*

If $E_1, E_2, E_3, \ldots, E_n$ are the possible outcomes (or elementary events) of a random experiment, then $S = \{E_1, E_2, \ldots, E_n\}$ is the sample space associated to it.

ILLUSTRATION 1 *Consider the random experiment of tossing of a coin. The possible outcomes of this experiment are H and T. Thus, if we define*

E_1 = Getting head (H) on the upper face and, E_2 = Getting tail (T) on the upper face.

Then, E_1 and E_2 are elementary events associated to the random experiment of tossing of a coin. The sample space associated to this experiment is given by $S = \{E_1, E_2\}$.

E_1 and E_2 are generally denoted by H and T respectively. Thus, we have $S = \{H, T\}$.

ILLUSTRATION 2 *Consider the experiment of throwing a die. Let the six faces of a die be marked as 1, 2, 3, 4, 5 and 6. If the die is thrown, then any one of the six faces may come upward. So, there are six possible outcomes of this experiment, namely, 1, 2, 3, 4, 5, 6. Thus, if we define*

E_i = *Getting a face marked with number i*, where $i = 1, 2, 3, 4, 5, 6$

Then, $E_1, E_2, \ldots, E_6$ are six elementary events associated to this experiment. The sample space associated to this experiment is $S = \{E_1, E_2, \ldots E_6\}$.

In this experiment, elementary even E_i is denoted by i, where $i = 1, 2, \ldots, 6$. Thus, we have

$$S = \{1, 2, 3, 4, 5, 6\}.$$

ILLUSTRATION 3 *Consider the experiment of tossing two coins together or a coin twice. In this experiment the possible outcomes are:*

Head on first and Head on second,
Head on first and Tail on second,
Tail on first and Head on second,
Tail on first and Tail on second.

If we define

HH = Getting head on both coins,
HT = Getting head on first and tail on second,
TH = Getting tail on first and head on second,
TT = Getting tail on both coins.

Then,

HH, HT, TH and TT are elementary events associated to the random experiment of tossing of two coins. The sample space associated to this experiment is given by $S = \{HH, HT, TH, TT\}$.

Similarly, the sample space associated to the random experiment of tossing three coins simultaneously or tossing a coin three times is given by

$$S = \{HHH, HHT, HTH, THH, TTH, HTT, THT, TTT\}$$

ILLUSTRATION 4 *Consider the random experiment in which two dice are tossed together or a die is tossed twice. If we define*

E_{ij} = Getting number i on the upper face of first die and number j on the upper face of second die,

where $i = 1, 2, ..., 6$ and $j = 1, 2, ..., 6$.

Then, E_{ij} are elementary events associated to this experiment and are generally denoted by (i, j).

Thus, $(1, 1)$, $(1, 2)$, ..., $(1, 6)$, $(2, 1)$ ..., $(2, 6)$, $(3, 1)$, ..., $(3, 6)$, $(4, 1)$..., $(4, 6)$, $(5, 1)$, ..., $(5, 6)$ and $(6, 1)$, ..., $(6, 6)$ are 36 elementary events associated to the random experiment of tossing two dice and the sample space associated to it is given by

$$S = \{(1, 1), ..., (1, 6), (2, 1), ..., (2, 6), (3, 1), ..., (3, 6), ..., (6, 1), ..., (6, 6)\}.$$

ILLUSTRATION 5 *Let there be a bag containing 3 white and 2 black balls. Let the white balls be denoted by W_1, W_2 W_3 and black balls be denoted by B_1, B_2. If we draw two balls from the bag, then there are ${}^5C_2$ elementary events associated to this experiment. These elementary events are:*

$B_1 W_1$, $B_1 W_2$, $B_1 W_3$, $B_2 W_1$, $B_2 W_2$, $B_2 W_3$, $W_1 W_2$, $W_1 W_3$, $W_2 W_3$, and $B_1 B_2$. The set of all these elementary events is the sample space associated to the experiment.

ILLUSTRATION 6 *A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 red and 4 black balls ; if it shows a tail, we throw a die. If we denote three red balls as R_1, R_2 and R_3 and four black balls as B_1, B_2, B_3 and B_4. Then the elementary events associated to this experiment are :*

$$HR_1, HR_2, HR_3, HB_1, HB_2, HB_3, HB_4, T1, T2, T3, T4, T5 \text{ and } T6.$$

The set of these elementary events is the sample space associated to the given random experiment.

<u>REMARK 1</u> *Elementary events associated to a random experiment are also known as indecomposable events.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *From a group of 2 boys and 3 girls, two children are selected. Find the sample space associated to this random experiment.*

SOLUTION Let the two boys be taken as B_1 and B_2 and the three girls be taken as G_1, G_2 and G_3. Clearly, there are 5 children, out of which two children can be chosen in ${}^5C_2$ ways. So, there are ${}^5C_2 = 10$ elementary events associated to this experiments and are given by

$$B_1 B_2, B_1 G_1, B_1 G_2, B_1 G_3, B_2 G_1, B_2 G_2, B_2 G_3, G_1 G_2, G_1 G_3 \text{ and } G_2 G_3$$

Consequently, the sample space S associated to this random experiment is given by

$$S = \{B_1 B_2, B_1 G_1, B_1 G_2, B_1 G_3, B_2 G_1, B_2 G_2, B_2 G_3, G_1 G_2, G_1 G_3, G_2 G_3\}.$$

EXAMPLE 2 *A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 red and 4 black balls; if it shows tail, we throw a die. What is the sample associated to this experiment?* [NCERT]

SOLUTION Let the three red balls be taken as R_1, R_2, R_3 and four black balls be taken as B_1, B_2, B_3 and B_4.

If the coin shows head, we draw a ball which can be any one of the 7 balls. So, possible outcomes are (H, R_1), (H, R_2), (H, R_3), (H, B_1), (H, B_2), (H, B_3), (H, B_4).

If the coin shows tail, then we through a die which may produce any one of the six numbers on its upper face. In this case, possible outcomes are $(T, 1)$, $(T, 2)$, $(T, 3)$, $(T, 4)$, $(T, 5)$, $(T, 6)$.

Thus, all elementary events associated to the experiment are:

$$(H, R_1), (H, R_2), (H, R_3), (H, B_1), (H, B_2), (H, B_3), (H, B_4), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6).$$

Consequently, the sample space S is given by

$$S = \{(H, R_1), (H, R_2), (H, R_3), (H, B_1), (H, B_2), (H, B_3), (H, B_4), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}.$$

EXAMPLE 3 *An experiment consists of rolling a die and then tossing a coin once if the number on the die is even. If the number on the die is odd, the coin is tossed twice. Write the sample space for this experiment.* [NCERT]

SOLUTION If the die is rolled and we get an even number (2 or 4 or 6) on its upper face, then we toss a coin which may result in head (H) or tail (T). So the possible outcomes in this case are :

$$(2, H), (4, H), (6, H), (2, T), (4, T), (6, T)$$

If the die is rolled and we get an odd number (1 or 3 or 5) on its upper face, then the coin is tossed twice which may result in one of the following ways: HH, HT, TH, TT. So, the possible outcomes, in this case, are

$$(1, HH), (3, HH), (5, HH), (1, HT), (3, HT), (5, HT), (1, TH), (3, TH), (5, TH), (1, TT), (3, TT), (5, TT).$$

Thus, all elementary events associated to the experiment are:

$$(2, H), (4, H), (6, H), (2, T), (4, T), (6, T), (1, HH), (1, HT), (1, TH), (1, TT), (3, HH), (3, HT), (3, TH), (3, TT), (5, HH), (5, HT), (5, TH), (5, TT).$$

So, the sample space associated to the random experiment is

$$S = \{(2, H), (4, H), (6, H), (2, T), (4, T), (6, T), (1, HH), (1, HT), (1, TH), (1, TT), (3, HH), (3, HT), (3, TH), (3, TT), (5, HH), (5, HT), (5, TH), (5, TT)\}.$$

EXAMPLE 4 *The numbers 1, 2, 3 and 4 are written separately on four slips of paper. The slips are then put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the sample space for the experiment.* [NCERT]

SOLUTION It is given that two slips are drawn from the box one after the other without replacement.

If the slip drawn in first draw bears number 1, then the slip drawn in second draw may bear any one of the remaining 3 numbers viz. 2, 3 and 4. Possible outcomes in this case are $(1, 2)$, $(1, 3)$ and $(1, 4)$.

If the slip drawn in first draw bears number 2, then the slip drawn in second draw may bear any one of the remaining three numbers viz. 1, 3 and 4.

Thus, possible outcomes, in this case, are $(2, 1)$, $(2, 3)$ and $(2, 4)$.

Similarly, possible outcomes when the slip drawn in first draw bears number 3 and 4 are respectively $(3, 1)$, $(3, 2)$, $(3, 4)$ and $(4, 1)$, $(4, 2)$, $(4, 3)$.

Thus, all elementary events associated to the random experiment are $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 1)$, $(2, 3)$, $(2, 4)$, $(3, 1)$, $(3, 2)$, $(3, 4)$, $(4, 1)$, $(4, 2)$ and $(4, 3)$.

The set of all these elementary events is the required sample space.

EXAMPLE 5 *A coin is tossed. If the result is a head, a die is thrown. If the die shows up an even number, the die is thrown again. What is the sample space for this experiment.* [NCERT]

SOLUTION A coin is tossed, if the outcome is tail (T). The experiment is over. If the outcome is head (H), a die is thrown and if the die shows up an odd number the experiment is stopped. Possible outcomes in this case are:

$(H, 1), (H, 3), (H, 5)$.

If the die shows up an even number it is thrown again. In this case, possible outcomes are:

$(H, 2, 1), (H, 2, 2), (H, 2, 3), (H, 2, 4),$
$(H, 2, 5), (H, 2, 6)$
$(H, 4, 1), (H, 4, 2), (H, 4, 3), (H, 4, 4),$
$(H, 4, 5), (H, 4, 6)$
$(H, 6, 1), (H, 6, 2), (H, 6, 3), (H, 6, 4),$
$(H, 6, 5), (H, 6, 6)$

So, all elementary events associated to the given experiment are

$T, (H, 1), (H, 3), (H, 5), (H, 2, 1), (H, 2, 2),$
$(H, 2, 3), (H, 2, 4), (H, 2, 5), (H, 2, 6)$
$(H, 4, 1), (H, 4, 2), (H, 4, 3), (H, 4, 4),$
$(H, 4, 5), (H, 4, 6)$
$(H, 6, 1), (H, 6, 2), (H, 6, 3), (H, 6, 4),$
$(H, 6, 5), (H, 6, 6)$

The set of all these elementary events is the required sample space.

REMARK *There are three stages in the above experiment. Possible outcomes at various stages can be depicted as shown in Fig. 33.1.*

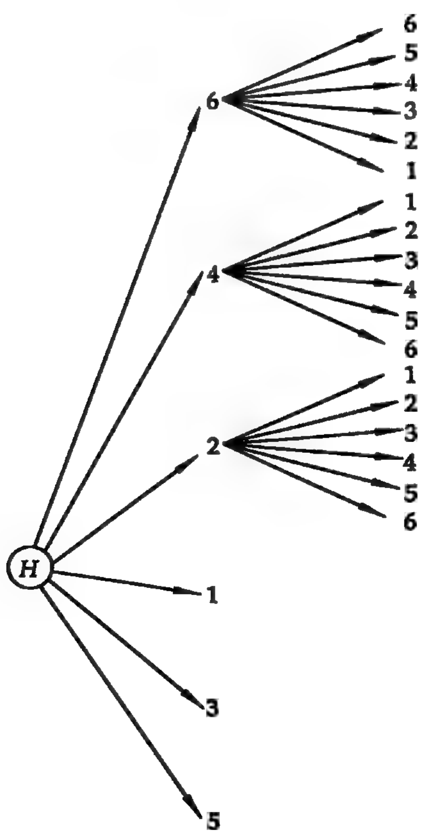


Fig. 33.1

EXAMPLE 6 *A coin is tossed repeatedly until a head comes for the first time. Describe the sample space.*

SOLUTION In this experiment, a coin is tossed. If the outcome is head the experiment is over. Otherwise, the coin is tossed again. In the second toss also if the outcome is head the experiment is over. Otherwise, the coin is tossed again. In the third toss, if the outcome is head the experiment is over, otherwise the coin is tossed again. This process continues indefinitely.

Possible outcomes in various tosses may be exhibited as follows:

Hence, the sample space S associated to this random experiment is

$$S = \{H, TH, TTH, TTTH, TTTTH, ...\}$$

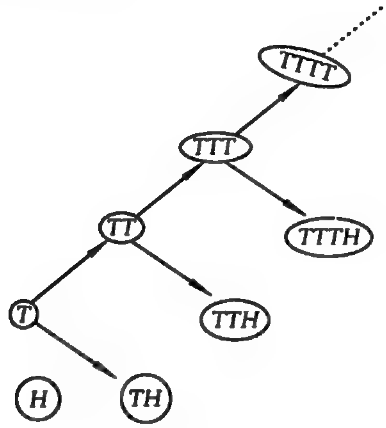


Fig. 33.2

REMARK 1 *In the above example, the sample space is an infinite set.*

REMARK 2 *Let us consider the random experiment of drawing two cards from a well shuffled pack of 52 playing cards . There are* ${}^{52}C_2 = 1326$ *elementary events associated to this experiment. So, the sample space consists of 1326 elements. Clearly, it is not convenient to describe the sample space completely. In the remaining part of this chapter, we will describe the sample space associated to a given random experiment only if it is convenient and does not contain large number of elementary events.*

EXERCISE 33.1

LEVEL-1

1. A coin is tossed once. Write its sample space
2. If a coin is tossed two times, describe the sample space associated to this experiment.
3. If a coin is tossed three times (or three coins are tossed together), then describe the sample space for this experiment. [NCERT]
4. Write the sample space for the experiment of tossing a coin four times. [NCERT]
5. Two dice are thrown. Describe the sample space of this experiment. [NCERT]
6. What is the total number of elementary events associated to the random experiment of throwing three dice together?
7. A coin is tossed and then a die is thrown. Describe the sample space for this experiment.
8. A coin is tossed and then a die is rolled only in case a head is shown on the coin. Describe the sample space for this experiment.
9. A coin is tossed twice. If the second throw results in a tail, a die is thrown. Describe the sample space for this experiment.
10. An experiment consists of tossing a coin and then tossing it second time if head occurs. If a tail occurs on the first toss, then a die is tossed once. Find the sample space. [NCERT]

11. A coin is tossed. If it shows tail, we draw a ball from a box which contains 2 red 3 black balls; if it shows head, we throw a die. Find the sample space of this experiment.
12. A coin is tossed repeatedly until a tail comes up for the first time. Write the sample space for this experiment.
13. A box contains 1 red and 3 black balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment. [NCERT]
14. A pair of dice is rolled. If the outcome is a doublet, a coin is tossed. Determine the total number of elementary events associated to this experiment.
15. A coin is tossed twice. If the second draw results in a head, a die is rolled. Write the sample space for this experiment.
16. A bag contains 4 identical red balls and 3 identical black balls. The experiment consists of drawing one ball, then putting it into the bag and again drawing a ball. What are the possible outcomes of the experiment?
17. In a random sampling three items are selected from a lot. Each item is tested and classified as defective (D) or non-defective (N). Write the sample space of this experiment.
18. An experiment consists of boy-girl composition of families with 2 children.
    (i) What is the sample space if we are interested in knowing whether it is a boy or girl in the order of their births?
    (ii) What is the sample space if we are interested in the number of boys in a family? [NCERT]
19. There are three coloured dice of red, white and black colour. These dice are placed in a bag. One die is drawn at random from the bag and rolled, its colour and the number on its uppermost face is noted. Describe the sample space for this experiment. [NCERT]
20. 2 boys and 2 girls are in room P and 1 boy 3 girls are in room Q. Write the sample space for the experiment in which a room is selected and then a person. [NCERT]
21. A bag contains one white and one red ball. A ball is drawn from the bag. If the ball drawn is white it is replaced in the bag and again a ball is drawn. Otherwise, a die is tossed. Write the sample space for this experiment.
22. A box contains 1 white and 3 identical black balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment.
23. An experiment consists of rolling a die and then tossing a coin once if the number on the die is even. If the number on the die is odd, the coin is tossed twice. Write the sample space for this experiment. [NCERT]
24. A die is thrown repeatedly until a six comes up. What is the sample space for this experiment. [NCERT]

ANSWERS

1. $S = \{H, T\}$
2. $S = \{HH, HT, TH, TT\}$
3. $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
4. $S = \{HHHH, HHHT, HTHH, THHH, HHTH, HHTT, HTTH, TTHH, THHT, HTHT, THTH, TTTH, TTHT, THTT, HTTT, TTTT\}$
5. $S = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$
6. 216
7. $S = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$
8. $S = \{T, (H,1), (H,2), (H,3), (H,4), (H,5), (H,6)\}$
9. $S = \{HH, TH, (HT,1), (HT,2), (HT,3), (HT,4), (HT,5), (HT,6), (TT,1), (TT,2), (TT,3), (TT,4), (TT,5), (TT,6)\}$

10. $S = \{(T,1),(T,2),(T,3),(T,4),(T,5),(T,6),(H,H),(H,T)\}$
11. $S = \{(T,R_1),(T,R_2),(T,B_1),(T,B_2),(T,B_3),(H,1),(H,2),(H,3),(H,4),(H,5),(H,6)\}$
12. $S = \{T, HT, HHT, HHHT, HHHHT, \ldots\}$
13. $S = \{(R,B_1),(R,B_2),(R,B_3),(B_1,R),(B_1,B_2),(B_1,B_3),(B_2,B_1),(B_2,B_3),(B_2,R),$ $(B_3,R),(B_3,B_1),(B_3,B_2)\}$
14. 42
15. $\{TT, HT, (TH,1),(TH,2),(TH,3),(TH,4),(TH,5),(TH,6),(HH,1),(HH,2),(HH,3),$ $(HH,4),(HH,5),(HH,6)\}$
16. RR, RB, BR, BB
17. $S = \{DDD, DDN, DND, NDD, DNN, NDN, NND, NNN\}$
18. (i) $S = \{(B_1,B_2),(B_1,G_2),(G_1,B_2),(G_1,G_2)\}$ (ii) $S = \{0,1,2\}$
19. $S = \{(R,1),(R,2),(R,3),(R,4),(R,5),(R,6),(B,1),(B,2),(B,3),(B,4),(B,5),(B,6)$ $(W,1),(W,2),(W,3),(W,4),(W,5),(W,6)\}$
20. $S = \{(P,B_1),(P,B_2),(P,G_1),(P,G_2),(Q,B_3),(Q,G_3),(Q,G_4),(Q,G_5)\}$
21. $S = \{(W,W),(W,R),(R,1),(R,2),(R,3),(R,4),(R,5),(R,6)\}$
22. $S = \{WB, BW, BB\}$
23. $S = \{(2,H),(2,T),(4,H),(4,T),(6,H),(6,T),(1,HH),(1,HT),(1,TH),(1,TT),(3,HH),$ $(3,HT),(3,TH),(3,TT),(5,HH),(5,HT),(5,TH),(5,TT)\}$
24. $S = \{6,(1,6),(2,6),(3,6),(4,6),(5,6),(1,1,6),(1,2,6),(1,3,6),(1,4,6),(1,5,6),$ $(2,1,6),(2,2,6),(2,3,6),\ldots\ldots\}$

33.4 EVENT

In the previous section, we have learnt about sample spaces associated with several random experiments. In this section, we will introduce an important term associated with a random experiment.

EVENT *A subset of the sample space associated with a random experiment is called an event.*

Consider the random experiment of throwing a die. The sample space associated with this experiment is $S = \{1,2,3,4,5,6\}$. Clearly, S has $2^6 = 64$ subsets.

Each one of these 64 subsets is an event associated with the random experiment of throwing a die.

For Example, $A = \{2,4,6\}$, $B = \{1,3,5\}$, $C = \{3,4,5,6\}$, $D = \{1,2,6\}$ etc. are events as they are subsets of S.

These events A, B and C can also be described in words as follows:

A = Getting an even number, B = Getting an odd number,
C = Getting a number greater than 2

However, there is no general description in words for the event D. Thus, we find that some events associated with a random experiment may be described in words. However, it is not possible for every event.

Consider the experiment of tossing three coins at a time. The sample space S associated with this experiment is $S = \{HHH, HHT, THH, HTH, TTH, THT, HTT, TTT\}$. Let

$A = \{HHT, HTH, THH\}$, $B = \{HHH, HHT, HTH, THH\}$
$C = \{HHH, HHT, HTH, THH, TTH, HTT, THT\}$ and, $D = \{HHH, TTT, HTH\}$

Clearly, A, B, C and D, being subsets of S, are events associated with the random experiment of tossing three coins (or tossing a coin three times). These events can also be described in words as follows:

A = Getting two heads, B = Number of heads exceeds the number of tails,
C = Getting at least one head.

But, event D cannot be described in words.

REMARK *Single element subsets of sample space associated with a random experiment define elementary events associated with the random experiment.*

OCCURRENCE OF AN EVENT *An event A associated to a random experiment is said to occur if any one of the elementary events associated to it is an outcome.*

Thus, if an elementary event E is an outcome of a random experiment and A is an event such that $E \in A$, then we say that the event A has occurred.

Consider the random experiment of throwing an unbiased die. Let A be an event of getting an even number. Then, $A = \{2, 4, 6\}$. Suppose in a trial the outcome is 4. Since $4 \in A$, so we say that the event A has occurred. In another trial, let the outcome be 3, since $3 \notin A$, so we say that in this trial the event A has not occurred.

Suppose a die is thrown and the outcome of the trial is 4. Then, we can say that each of the following events have occurred:

(i) Getting a number greater than or equal to 2, represented by the set $\{2, 3, 4, 5, 6\}$

(ii) Getting a number less than or equal to 5, represented by the set $\{1, 2, 3, 4, 5\}$.

On the basis of the same outcome, we can also say that the following events have not occurred:

(i) Getting an odd number represented by the set $\{1, 3, 5\}$

(ii) Getting a multiple of 3, represented by the set $\{3, 6\}$.

Let us now consider the random experiment of throwing a pair of dice. If (2, 6) is an outcome of a trial, then we can say that each of the following events has occurred:

(i) Getting an even number on first die. (ii) Getting even numbers on both dice.

(iii) Getting 8 as the sum of the numbers on two dice.

However, on the basis of the same outcome, one can also say that following events have not occurred:

(i) Getting a multiple of 3 on first die. (ii) Getting an odd number on first die.

(iii) Getting a doublet.

33.5 ALGEBRA OF EVENTS

In this section, we shall see how new events can be constructed by combining two or more events associated to a random experiment.

Let A and B be two events associated to a random experiment with sample space S. We define the event "A or B" which is said to occur if an elementary event favourable to either A or B or both is an outcome. In other words, the event "A or B" occurs if either A or B or both occur i.e. at least one of A and B occurs. Thus, "A or B" is represented by the subset $A \cup B$ of the sample space S.

For example, in a single throw of a die consider the following events:

A = Getting an even number, B = Getting a multiple of 3.

These two events are described by the sets $\{2, 4, 6\}$ and $\{3, 6\}$ respectively.

Clearly,

$A \cup B$ = Getting a number which is either even or a multiple of 3 or both $= \{2, 3, 4, 6\}$.

Similarly, if A, B and C are three events associated to a random experiment, then $A \cup B \cup C$ denotes the occurrence of at least one of the three events.

The event "A and B" is said to occur if an elementary event favourable to both A and B is an outcome. In other words, the event "A and B" occurs if A and B both occur. The event A and B is denoted by $A \cap B$.

For example, in a single throw of a pair of dice if we define

A = Getting an even number on first-die

and, B = Getting 8 as the sum of the numbers on two dice,

Then,

$A \cap B$ = Getting an even number on first die such that the sum of the numbers is 8
$= \{(2, 6), (6, 2), (4, 4)\}$.

NEGATION OF AN EVENT *Corresponding to every event A associated to a random experiment, we define an event " not A" which is said to occur when and only when A does not occur.*

For example, in a single throw of a die if A denotes the event that the outcome is an odd number. Then $A = \{1, 3, 5\}$ and A does not occur if the outcome is any one of the outcomes 2, 4, 6. Thus, the event "not A" is represented by the set $\overline{A}$ and is called the *complementary event of A or negation of A.*

Sometimes the occurrence of one event implies the occurrence of other. For example, in a single throw of a die if A denotes the event that the outcome is 2 or 4 and B denotes the event that the outcome is even. Then, $A = \{2, 4\}$ and $B = \{2, 4, 6\}$. Clearly, the occurrence of A implies the occurrence of B. For if 2 or 4 occurs, we say that the outcome is an even number.

Thus, if the occurrence of an event A implies the occurrence of event B, then we say that "A implies B". Clearly, if A implies B, then we have $A \subset B$.

Verbal description of the event	*Equivalent set theoretic notation*
Not A	$\overline{A}$
A or B (at least one of A or B)	$A \cup B$
A and B	$A \cap B$
A but not B	$A \cap \overline{B}$
Neither A nor B	$\overline{A} \cap \overline{B}$
At least one of A, B or C	$A \cup B \cup C$
Exactly one of A and B	$(A \cap \overline{B}) \cup (\overline{A} \cap B)$
All three of A, B and C	$A \cap B \cap C$
Exactly two of A, B and C	$(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$

In the above discussion and in the previous sections, we have seen that the events associated to a random experiment are generally described verbally, and it is very important to have the ability of conversion of verbal description to equivalent set theoretical notations. In the following table, we give verbal descriptions of some events and their equivalent set theoretic notations for ready reference.

ILLUSTRATION *If A, B and C are three arbitrary events. Find the expression for the events noted below, in the context of A, B and C.*

(i) *Only A occurs*
(ii) *Both A and B, but not C occur*
(iii) *All the three events occur*
(iv) *At least one occurs*
(v) *At least two occur*
(vi) *One and no more occurs*
(vii) *Two and no more occur*
(viii) *None occurs*
(ix) *Not more than two occur.*

SOLUTION (i) $A \cap \overline{B} \cap \overline{C}$ (ii) $A \cap B \cap \overline{C}$ (iii) $A \cap B \cap C$ (iv) $A \cup B \cup C$

(v) $(A \cap B) \cup (B \cap C) \cup (A \cap C) \cup (A \cap B \cap C)$

(vi) $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$

(vii) $(A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C)$

(viii) $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$

(ix) $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \cup (A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C)$.

33.6 TYPES OF EVENTS

Let there be n elementary events associated with a random experiment. Then the corresponding sample space has n elements and hence 2^n subsets. Each subset of S is an event associated to the random experiment and the sample space is the universal set of these events. These 2^n events are divided into different types on the basis of their nature of occurrence. In this section, we shall learn about such types.

CERTAIN (OR SURE) EVENT *An event associated with a random experiment is called a certain event if it always occurs whenever the experiment is performed.*

For example, associated with the random experiment of rolling a die, the event A "Getting an even number or an odd number" is a certain event. Clearly, this event is represented by the set $\{1, 2, 3, 4, 5, 6\}$ which is the sample space of the experiment.

If S is the sample space associated with a random experiment. Then, S, being subset of itself, defines an event. Also, every outcome of the experiment is an element of S, so the event represented by S always occurs whenever we perform the experiment. Consequently, the event represented by S is a certain event.

Thus, the sample space associated with a random experiment defines a certain event.

IMPOSSIBLE EVENT *An event associated with a random experiment is called an impossible event if it never occurs whenever the experiment is performed.*

Consider the experiment of rolling a die. Let A be the event "The number turns up is divisible by 7". Clearly, none of the possible outcomes 1, 2, 3, 4, 5, 6 is divisible by 7. So, the event A cannot occur at all. In other words, there is no outcome belonging to set representing event A. So, the set A is the null set.

If S is the sample space associated with a random experiment, then the null (empty) set ϕ is a subset of S and no outcome of the experiment is a member of ϕ. So, the event represented by ϕ is an impossible event.

COMPOUND EVENT *An event associated with a random experiment is a compound event, if it is the disjoint union of two or more elementary events.*

In other words, an event having more than one sample point is called a compound event.

In fact, other than elementary events and impossible events associated with a random experiment, all events are compound events as they are obtained by combining two or more elementary events.

For example, in a single throw of an ordinary die there are, 6 elementary events and the total number of events is $2^6 = 64$. So, $2^6 - (6+1) = 57$ is the total number of compound events.

<u>REMARK</u> *If there are n elementary events associated to a random experiment, then the sample space associated to it has n elements and so there are 2^n subsets of it. Out of these 2^n subsets there are n single element subsets. These single element subsets define n elementary events and the remaining $2^n - (n+1)$ subsets (excluding null set) define compound events. Some of these compound events can be described in words whereas for others there may not be any general description.*

If a pair of dice is thrown together, then there are 36 elementary events associated to this experiment. The sample space associated to this experiment is:

$$S = \{(1,1), (1,2) \ldots, (1,6), (2,1), (2,2) \ldots, (2,6), \ldots, (6,1), (6,2), \ldots, (6,6)\}$$

If we define the event A as "Getting a doublet". i.e. $A = \{(1,1), (2,2), \ldots, (6,6)\}$

Clearly, it is a compound event obtained by combining 6 elementary events.

Similarly, the event B given by "Getting 8 as the sum" can be written as

$$B = \{(2, 6), (6, 2), (4, 4), (3, 5), (5, 3)\}$$

It is also a compound event obtained by combining 5 elementary events.

MUTUALLY EXCLUSIVE EVENTS *Two or more events associated with a random experiment are said to be mutually exclusive or incompitable events if the occurrence of any one of them prevents the occurrence of all others i.e., if no two or more of them can occur simultaneously in the same trial.*

Clearly, elementary events associated with a random experiment are always mutually exclusive, because elementary events are outcomes (results) of an experiment when it is performed and at a time only one outcome is possible.

Consider the random experiment of rolling a die. Let A, B, C be three events associated with the experiment as given below:

A = Getting an even number, B = Getting an odd number, C = Getting a multiple of 3.

These events in set theoretical notations are: $A = \{2, 4, 6\}$, $B = \{1, 3, 5\}$ and $C = \{3, 6\}$.
Clearly, $A \cap B = \phi$, $A \cap C \neq \phi$, $B \cap C \neq \phi$ and $A \cap B \cap C = \phi$.

So, A and B are mutually exclusive events but A and C as well as B and C are not mutually exclusive. However, A, B and C taken all the three together are mutually exclusive events.

In the experiment of throwing a pair of dice events A = Getting 8 as the sum and B = Getting an even number on first die are not mutually exclusive, because $A \cap B = \{(2, 6), (6, 2), (4, 4)\} \neq \phi$

Let two cards be drawn from a well-shuffled pack of 52 cards. Consider the following events:

A = Getting both red cards, B = Getting both black cards.

Clearly, A and B are mutually exclusive events because two cards drawn cannot be both red and black at the same time.

EXHAUSTIVE EVENTS *Two or more events associated with a random experiment are exhaustive if their union is the sample space i.e. events $A_1, A_2, \ldots, A_n$ associated with a random experiment with sample space S are exhaustive if $A_1 \cup A_2 \cup \ldots \cup A_n = S$.*

Thus, a set of events associated with a random experiment is an exhaustive set of events if one of them necessarily occurs whenever the experiment is performed.

It is evident from the above definition that all elementary events associated with a random experiment form a set of exhaustive events.

Consider the experiment of drawing a card from a well shuffled deck of playing cards. Let A be the event " card is red", B be the event "card is black." Clearly, A and B are exhaustive events because $A \cup B = S$.

In a single throw of an ordinary die, let us consider the following events:

A_1 = Getting an even number $= \{2, 4, 6\}$, A_2 = Getting an odd number $= \{1, 3, 5\}$,

A_3 = Getting a multiple of 3 $= \{3, 6\}$, A_4 = Getting a number greater than 3 $= \{4, 5, 6\}$

We observe that $A_1 \cup A_2 = S$. Also, $A_1 \cup A_2 \cup A_3 \cup A_4 = S$. But, $A_1 \cup A_3 \neq S$. So, A_1 and A_2 are exhaustive events. Also, A_1, A_2, A_3, A_4 are exhaustive events but A_1 and A_3 are not exhaustive events.

MUTUALLY EXCLUSIVE AND EXHAUSTIVE SYSTEM OF EVENTS *Let S be the sample space associated with a random experiment. A set of events $A_1, A_2, \ldots, A_n$ is said to form a set of mutually exclusive and exhaustive system of events if*

(i) $A_1 \cup A_2 \ldots \cup A_n = S$ *i.e. events $A_1, A_2, \ldots, A_n$ form an exhaustive set of events.*

(ii) $A_i \cap A_j = \phi$ for $i \neq j$ *i.e. events $A_1, A_2, \ldots, A_n$ are mutually exclusive.*

Clearly, elementary events associated with a random experiment always form a system of mutually exclusive and exhaustive events.

In a single throw of a die, the events A = Getting an even number and, B = Getting an odd number are mutually exclusive and exhaustive events.

Consider the experiment of drawing a card from a well-shuffled deck of 52 playing cards. Let A_1, A_2, A_3, A_4 be four events defined as follows:

A_1 = Card drawn is spades, A_2 = Card drawn is clubs,

A_3 = Card drawn is hearts, A_4 = Card drawn is diamonds

Since the card drawn is one of the four types of cards, so one of these events surely occurs whenever the experiment is performed. Also, if one of these events occurs, the others cannot occur. So, A_1, A_2, A_3 and A_4 form a mutually exclusive and exhaustive system of events.

Suppose a die is thrown once. Let A be the event "Getting a number greater than 3", B be the event "Getting a number less than 5". Then, $A = \{4, 5, 6\}$ and $B = \{1, 2, 3, 4\}$. Clearly, $A \cup B = S$ and $A \cap B = \{4\} \neq \phi$. So, events A and B are exhaustive but not mutually exclusive.

FAVOURABLE ELEMENTARY EVENTS *Let S be the sample space associated with a random experiment and A be an event associated with the experiment. Then, elementary events belonging to A are known as favourable elementary events to the event A.*

Thus, an elementary event E is favourable to an event A if the occurrence of E ensures the happening or occurrence of event A.

In a single throw of an ordinary die, let A be the event "Getting a multiple of 3". Clearly, $A = \{3, 6\}$. So, there are two elementary events favourable to A.

Consider the random experiment of throwing a pair of dice. Let A be the event "Getting 8 as the sum". Then, $A = \{(2, 6), (6, 2), (4, 4), (5, 3), (3, 5)\}$. Clearly, A occurs if any one of the elementary events $(2, 6), (6, 2), (4, 4), (5, 3)$, and $(3, 5)$ is an outcome of the experiment. So, all these elementary events are favourable to event A.

Consider a random experiment of drawing 4 cards from a well-shuffled deck of 52 playing cards. There are ${}^{52}C_4$ elementary events associated with this experiment as 4 cards can be drawn out of 52 cards in ${}^{52}C_4$ ways. Let A be the event "Getting all red cards". There are ${}^{26}C_4$ elementary events favourable to A, because 4 red cards can be chosen out of 26 red cards in $26C_4$ ways. In this case, it is not convenient to list all favourable elementary events.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *An experiment consists of rolling die until a 2 appears.*

(i) How many elements of the sample space correspond to the event that the 2 appears on the k^{th} roll of the die?

(ii) How many elements of the sample space correspond to the event that 2 appears not later than the k^{th} roll of the die? **[NCERT EXEMPLAR]**

SOLUTION (i) 2 appears on k^{th} roll of the die means that each one of the first $(k-1)$ rolls have 5 outcomes (1, 3, 4, 5, 6) and k^{th} roll results in 1 outcome i.e. 2.

$\therefore$ Number of elements of the sample space in the event $= \underbrace{5 \times 5 \times \ldots \times 5}_{(k-1) \text{ times}} \times 1 = 5^{k-1}$

(ii) 2 appears not later than k^{th} roll means that 2 may appear in the first roll or in second roll or in third roll, ..., or in k^{th} roll.

From (i), the number of elements of the sample space corresponding to the event that 2 appears on the k^{th} roll of the die is 5^{k-1}.

Hence, The number of elements of the sample space corresponding to the event that 2 appears not later than the k^{th} roll of the die

$$=5^{1-1}+5^{2-1}+5^{3-1}+...+5^{k-1}=1+5+5^2+...+5^{k-1}=\left(\frac{5^k-1}{5-1}\right)=\frac{5^k-1}{4}.$$

EXAMPLE 2 *An experiment involves rolling a pair of dice and recording the numbers that come up. Describe the following events.* [NCERT]

A = the sum is greater than 8, B = 2 occurs on either die., C = the sum is at least 7 and a multiple of 3.

Also, find $A \cap B$, $B \cap C$ and $A \cap C$.

Are: (i) *A and B mutually exclusive?* (ii) *B and C mutually exclusive?*

(iii) *A and C mutually exclusive?*

SOLUTION The sample space associated with the given random experiment is given by

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

We have,

A = The sum is greater than 8

$\Rightarrow$ $A = \{(3,6),(4,5),(5,4),(6,3),(4,6),(6,4),(5,5),(6,5),(5,6),(6,6)\}$

B = 2 occurs on either die

$\Rightarrow$ $B = \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(1,2),(3,2),(4,2),(5,2),(6,2)\}$.

and, C = The sum is atleast 7 and a multiple of 3

$\Rightarrow$ C = The sum is 9 or, 12. $= \{(3,6),(6,3),(4,5),(5,4),(6,6)\}$.

(i) Clearly, $A \cap B = \phi$. So, A and B are mutually exclusive events.

(ii) Clearly, $B \cap C = \phi$. So, B and C are mutually exclusive events.

(iii) Clearly, $A \cap C = \{(3,6),(6,3),(4,5),(5,4),(6,6)\} \neq \phi$. So, A and C are not mutually exclusive events.

EXAMPLE 3 *From a group of 2 boys and 3 girls, two children are selected at random. Describe the events.*

(i) *A = both selected children are girls.* (ii) *B = the selected group consists of one boy and one girl.*

(iii) *C = at least one boy is selected.*

Which pair (s) of events is (are) mutually exclusive?

SOLUTION Let B_1, B_2 be two boys and G_1, G_2, G_3 be three girls. Then, the sample space associated with the random experiment is

$$S = \{B_1B_2, B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3, G_1G_2, G_1G_3, G_2G_3\}$$

(i) We have,

A = Both selected children are girls $= \{G_1G_2, G_1G_3, G_2G_3\}$

(ii) We have,

B = The selected group consists of one boy and one girl.

$\Rightarrow$ $B = \{B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3\}$

(iii) We have,

C = At least one boy is selected $= \{B_1B_2, B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3\}$

Clearly, $A \cap B = \phi$ and $A \cap C = \phi$. So, A and B, A and C are two pairs of mutually exclusive events.

EXAMPLE 4 *Two dice are thrown and the sum of the numbers which come up on the dice is noted. Let us consider the following events:* [NCERT]

$$A = \text{The sum is even}, B = \text{The sum is multiple of } 3, C = \text{The sum is less than } 4,$$
$$D = \text{The sum is greater than } 11$$

Which pairs of these events are mutually exclusive?

SOLUTION The sample space associated with the random experiment is given in example 1.

We have,

A = The sum is even

= The sum is either 2 or 4 or 6, or 8 or 10 or 12

$= \{(1, 1), (2, 2), (1, 3), (1, 5), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)\}$

B = The sum is a multiple of 3

= The sum is either 3 *or* 6 *or* 9 *or* 12

$= \{(1, 2), (2, 1), (1, 5), (5, 1), (2, 4), (4, 2), (3, 3), (3, 6), (6, 3), (4, 5), (5, 4), (6, 6)\}$

C = The sum is less than 4 =(The sum is 2 or 3 = $\{(1, 1), (1, 2), (2, 1)\}$

D = The sum is greater than 11 = The sum is 12 = $\{(6, 6)\}$

We observe that $A \cap B = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 6)\} \neq \phi$. So, A and B are not mutually exclusive events.

Similarly, we observe that $A \cap C \neq \phi, A \cap D \neq \phi, B \cap C \neq \phi, B \cap D \neq \phi$ and, $C \cap D = \phi$.

Hence, C and D are mutually exclusive events.

EXERCISE 33.2

LEVEL-1

1. A coin is tossed. Find the total number of elementary events and also the total number events associated with the random experiment.
2. List all events associated with the random experiment of tossing of two coins. How many of them are elementary events?
3. Three coins are tossed once. Describe the following events associated with this random experiment: [NCERT]

   A = Getting three heads, B = Getting two heads and one tail,

   C = Getting three tails, D = Getting a head on the first coin.

   (i) Which pairs of events are mutually exclusive?

   (ii) Which events are elementary events?

   (iii) Which events are compound events?
4. In a single throw of a die describe the following events:

   (i) A = Getting a number less than 7 (ii) B = Getting a number greater than 7

   (iii) C = Getting a multiple of 3 (iv) D = Getting a number less than 4

   (v) E = Getting an even number greater than 4 (vi) F = Getting a number not less than 3.

   Also, find $A \cup B$, $A \cap B$, $B \cap C$, $E \cap F$, $D \cap F$ and $\overline{F}$.
5. Three coins are tossed. Describe

   (i) two events A and B which are mutually exclusive. [NCERT]

   (ii) three events A, B and C which are mutually exclusive and exhaustive.

   (iii) two events A and B which are not mutually exclusive.

   (iv) two events A and B which are mutually exclusive but not exhaustive.

6. A die is thrown twice. Each time the number appearing on it is recorded. Describe the following events:
   (i) A = Both numbers are odd. (ii) B = Both numbers are even.
   (iii) C = sum of the numbers is less than 6
   Also, find $A \cup B$, $A \cap B$, $A \cup C$, $A \cap C$. Which pairs of events are mutually exclusive?
7. Two dice are thrown. The events A, B, C, D, E and F are described as follows:
   A = Getting an even number on the first die.
   B = Getting an odd number on the first die.
   C = Getting at most 5 as sum of the numbers on the two dice.
   D = Getting the sum of the numbers on the dice greater than 5 but less than 10.
   E = Getting at least 10 as the sum of the numbers on the dice.
   F = Getting an odd number on one of the dice.
   (i) Describe the following events: A and B, B or C, B and C, A and E, A or F, A and F
   (ii) State true or false:
   (a) A and B are mutually exclusive.
   (b) A and B are mutually exclusive and exhaustive events.
   (c) A and C are mutually exclusive events.
   (d) C and D are mutually exclusive and exhaustive events.
   (e) C, D and E are mutually exclusive and exhaustive events.
   (f) A' and B' are mutually exclusive events.
   (g) A, B, F are mutually exclusive and exhaustive events.
8. The numbers 1, 2, 3 and 4 are written separately on four slips of paper. The slips are then put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the following events:
   A = The number on the first slip is larger than the one on the second slip.
   B = The number on the second slip is greater than 2
   C = The sum of the numbers on the two slips is 6 or 7
   D = The number on the second slips is twice that on the first slip.
   Which pair(s) of events is (are) mutually exclusive?
9. A card is picked up from a deck of 52 playing cards.
   (i) What is the sample space of the experiment?
   (ii) What is the event that the chosen card is black faced card?

ANSWERS

1. 2, 4
2. $\{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\}, \{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\}, \{HH, HT, TT\}, \{HH, TH, TT\}, \{HT, TH, TT\}, \{HH, HT, TH, TT\}$ .4
3. $A = \{HHH\}, B = \{HHT, THH, HTH\}, C = \{TTT\}, D = \{HHH, HHT, HTH, HTT\}$
   (i) A, B; A, C; B, C; C, D (ii) A and C (iii) B and D
4. (i) $A = \{1, 2, 3, 4, 5, 6\}$ (ii) ϕ (iii) $C = \{3, 6\}$
   (iv) $D = \{1, 2, 3\}$ (v) $E = \{6\}$ (vi) $F = \{3, 4, 5, 6\}$
   $A \cup B = \{1, 2, 3, 4, 5, 6\}, A \cap B = \phi, B \cap C = \phi, E \cap F = \{6\}, D \cap F = \{3\}, \bar{F} = \{1, 2\}$
5. (i) A = Getting at least two heads, B = Getting at least two tails.
   (ii) A = Getting at most one head, B = Getting exactly two heads,
   C = Getting exactly three heads.
   (iii) A = Getting at most two tails, B = Getting exactly two heads
   (iv) A = Getting exactly one head, B = Getting exactly two heads.

6. (i) $A = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\}$

(ii) $B = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$

(iii) $C = \{(1,1),(1,2),(2,1),(1,3),(3,1),(2,2),(1,4),(4,1),(2,3),(3,2)\}$

$A \cup B = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5),(2,2),(2,4),(2,6),$
$(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$

$A \cap B = \phi$

$A \cup C = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5),(1,2),(1,4),(2,1),$
$(2,2),(2,3),(3,2),(4,1)\}$

$A \cap C = \{(1,1),(1,3),(3,1)\}$ and $B \cap C = \phi$.

A and B, B and C are pairs of mutually exclusive events.

7. (i) $A \cap B = \phi$

$B \cup C = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(3,1),(3,2)$
$(3,3),(3,4),(3,5),(3,6),(4,1),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\}$

$B \cap C = \{(1,1),(1,2),(1,3),(1,4)\,(3,1),(3,2)\}$

$A \cap E = \{(4,6),(6,4),(6,5),(6,6)\}$

$A \cap F = \{(1,2),(1,4),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,2),(3,4),(3,6),$
$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,2),(5,4),(5,6),(6,1),(6,2),(6,3),$
$(6,4),(6,5),(6,6)\}$

$A \cap F = \{(2,1),(2,3),(2,5),(4,1),(4,3),(4,5),(6,1),(6,3),(6,5)\}$

(ii) (a) True, (b) True (c) False (d) False (e) True (f) True (g) False

8. $A = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$, $B = \{(1,3),(2,3),(1,4),(2,4),(3,4),(4,3)\}$
$C = \{(2,4),(3,4),(4,2),(4,3)\}$, $D = \{(1,2),(2,4)\}$

A and D form a pair of mutually exclusive events.

9. (i) The sample space is the set of 52 cards.

(ii) Required event is the set of jack, king and queen of spades and clubs.

HINTS TO NCERT & SELECTED PROBLEMS

3. We have,

$A = \{HHH\}$, $B = \{HHT, HTH, THH\}$, $C = \{TTT\}$, $D = \{HHH, HHT, HTH, HTT\}$

(i) We observe that $A \cap B = \phi$, $A \cap C = \phi$ but $A \cap D \neq \phi$

So, A, B and A, C are pair of simutally exclusive events.

Also, $B \cap C = \phi$ but $B \cap D \neq \phi$. So, B, C is a pair of mutually exclusive events.

Finally, $C \cap D = \phi$. So, C and D form a pair of mutally exclusive events.

(ii) Clearly, HHH and TTT may be outcomes of the random experiment of tossing three coins. So, A and C are elementary events.

(iii) We observe that events B and D are obtained by combining more than one elementary events. So, B and D are compound events.

5. The smple space associated to the random experiment of tossing three coins is

$S = \{HHH, HHT, THH, HTH, HTT, THT, TTH, TTT\}$

(i) Clearly, $A = \{HHT, THH, HHT\}$ and $B = \{TTH, THT, HTT\}$ are mutually exclusive events.

(ii) We observe that $A = \{HHH, TTT\}$, $B = \{HHT, HTH, THH\}$ and $C = \{HTT, THT, TTH\}$ are exhaustive and mutually exclusive events. Because, $A \cap B = \phi = B \cap C = C \cap A$ and $A \cup B \cup C = S$.

(iii) We observe that the events $A = \{HHH, HHT, HTH, THH\}$ and $B = \{HHT, THH, HTH, HTT, THT, TTH, TTT\}$ are not mutuually exclusive, because $A \cap B \neq \phi$.

(iv) Events $A = \{HHT, HTH, THH\}$ $B = \{TTT, TTH, HTT, THT\}$ are mutually exclusive but not exhaustive as $A \cap B = \phi$ but $A \cup B \neq S$.

33.7 AXIOMATIC APPROACH TO PROBABILITY

The axiomatic approach to probability is deduced from the mathematical concepts laid down in the previous sections. It is based upon certain axioms. In this approach, for a given sample space associated to a random experiment, the probability is considered as a function which assigns a non-negative real number $P(A)$ to every event A. This non-negative real number is called the probability of the event A.

PROBABILITY FUNCTION *Let $S = \{w_1, w_2, w_3, ..., w_n\}$ be the sample space associated to a random experiment. Then a function P which assigns every event $A \subset S$ to a unique non-negative real number $P(A)$ is called the probability function, if the following axioms hold:*

Axiom 1: $0 \leq P(w_i) \leq 1$ for all $w_i \in S$

Axiom 2: $P(S) = 1$ i.e., $P(w_1) + P(w_2) + ... + P(w_n) = 1$.

Axiom 3: For any event $A \subset S$, $P(A) = \sum\limits_{w_k \in A} P(w_k)$, the number $P(w_k)$ is called the probability of elementary event w_k.

Consider the experiment 'tossing a coin'. The sample space associated to this random experiment is $S = \{H, T\}$. If we assign the number $\frac{1}{2}$ to each of the outcomes (elementary events) H and T i.e. $P(H) = \frac{1}{2}$ and $P(T) = \frac{1}{2}$. Then this assignment satisfies first two axioms i.e. $0 \leq P(H) \leq 1$, $0 \leq P(T) \leq 1$, and $P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$. So, P is the probability function on S and we can say that the probability of getting head is $\frac{1}{2}$ and the probability of getting tail is also $\frac{1}{2}$. If we assign the number $\frac{1}{4}$ to H and $\frac{3}{4}$ to T i.e. $P(H) = \frac{1}{4}$ and $P(T) = \frac{3}{4}$. This assignment also defines a probability function on $S = \{H, T\}$. However, if we take $P(H) = \frac{1}{8}$ and $P(T) = \frac{3}{8}$. Then this assignment is not a probability function on $S = \{H, T\}$.

ILLUSTRATION *Let $S = \{w_1, w_2\, w_3\, w_4, w_5, w_6\}$ be a sample space. Which of the following assignments of probability to each outcome are valid?* [NCERT]

Outcomes or Elementary events:		w_1	w_2	w_3	w_4	w_5	w_6
Probabilities:	(i)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
	(ii)	1	0	0	0	0	0
	(iii)	$\frac{1}{8}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{4}$
	(iv)	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{2}$
	(v)	0.1	0.2	0.3	0.4	0.5	0.6

SOLUTION (i) We have, $P(w_i) = \frac{1}{6}$, $i = 1, 2, 3, 4, 5, 6$.

$\therefore$ $0 \leq P(w_i) \leq 1$ for all $w_i \in S$

and, $P(w_1) + P(w_2) + ... + P(w_6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$

Hence, the assignment of the probabilities is valid.

(ii) We have,

$$P(w_1) = 1, P(w_2) = P(w_3) = P(w_4) = P(w_5) = P(w_6) = 0$$

$\therefore \quad 0 \le P(w_i) \le 1$ for all $w_i \in S$

Also, $\quad P(w_1) + P(w_2) + P(w_3) + P(w_4) + P(w_5) + P(w_6) = 1$

Hence, the assignment of the probabilities is valid.

(iii) We have,

$$P(w_5) = -\frac{1}{4} < 0 \text{ and } P(w_6) = -\frac{1}{3} < 0.$$

So, axiom 1 is not satisfied. Hence, the assignment of the probabilities is not valid.

(iv) We have, $P(w_6) = \dfrac{3}{2} > 1.$

So, axiom 1 is not satisfied. Hence, the assignment of the probabilities is not valid.

(v) We observe that axiom 1 is satisfied. But,

$$P(w_1) + P(w_2) + P(w_3) + P(w_4) + P(w_5) + P(w_6) = 0.1 + 0.2 + 0.3 + 0.4 + 0.5 + 0.6 = 2.1 \ne 1.$$

So, axiom 2 is not satisfied. Hence, the assignment is not valid.

Let $S = \{w_1, w_2, \dots, w_n\}$ be the sample space associated to a random experiment such that all the outcomes (elementary events) $w_1, w_2, w_3, \dots, w_n$ are equally likely to occur i.e. the chance of occurrence of each elementary event is same. i.e. $P(w_i) = p$ for all $w_i \in S$ where $0 \le p \le 1$.

Using axiom 2, we have

$$\sum_{i=1}^{n} P(w_i) = 1 \Rightarrow \underbrace{p + p + \dots + p}_{(n\text{-times})} = 1 \Rightarrow np = 1 \Rightarrow p = \frac{1}{n}$$

If A is an event such that m elementary events are favourable to A. Then,

$$P(A) = \sum_{w_k \in A} P(w_k)$$

$$\Rightarrow \quad P(A) = \underbrace{\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{m-\text{times}}$$

$$\Rightarrow \quad P(A) = \frac{m}{n} = \frac{\text{Favourable number of elementary events}}{\text{Total number of elementary events}}$$

Thus, we have the following definition of probability of an event when all the elementary events are equally likely to occur.

PROBABILITY OF AN EVENT *If there are n elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of happening or occurrence of A is denoted by P (A) and is defined as the ratio* $\dfrac{m}{n}$.

Thus, $P(A) = \dfrac{m}{n}$

Clearly, $0 \le m \le n$. Therefore,

$$0 \le \frac{m}{n} \le 1 \Rightarrow 0 \le P(A) \le 1$$

If $P(A) = 1$, then A is called certain event and A is called an impossible event, if $P(A) = 0$.

The number of elementary events which will ensure the non-occurrence of A i.e. which ensure the occurrence of $\overline{A}$ is $(n - m)$. Therefore,

$$P(\overline{A}) = \frac{n-m}{n}$$

$\Rightarrow \quad P(\bar{A}) = 1 - \frac{m}{n}$

$\Rightarrow \quad P(\bar{A}) = 1 - P(A)$

$\Rightarrow \quad P(A) + P(\bar{A}) = 1$

The odds in favour of occurrence of the event A are defined by $m : (n - m)$ i.e. $P(A) : P(\bar{A})$ and the odds against the occurrence of A are defined by $n - m : m$ i.e. $P(\bar{A}) : P(A)$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS BASED ON CONSTRUCTION OF SAMPLE SPACE**

EXAMPLE 1 *Find the probability of getting a head in a toss of an unbiased coin.*

SOLUTION The sample space associated with the random experiment is $S = \{H, T\}$.

$\therefore$ Total number of elementary events = 2.

We observe that there are two elementary events viz. H, T associated to the given random experiment. Out of these two elementary events only one is favourable i.e. H.

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= \frac{1}{2}$.

EXAMPLE 2 *In a simultaneous toss of two coins, find the probability of getting:*

(i) 2 *heads* (ii) *exactly one head* (iii) *exactly* 2 *tails*
(iv) *exactly one tail* (v) *no tails.*

SOLUTION The sample space associated to the given random experiment is given by

$S = \{HH, HT, TH, TT\}$

Clearly, there are 4 elements in S.

$\therefore$ Total number of elementary events = 4.

(i) There is only one elementary event i.e. HH favourable to the given event

So, required probability $= \frac{1}{4}$.

(ii) We observe that exactly one head can be obtained in two ways: HT or, TH.

So, favourable number of elementary events = 2.

Hence, required probability $= \frac{2}{4} = \frac{1}{2}$.

(iii) Exactly 2 tails can be obtained in one way i.e. TT. So, favourable number of elementary events = 1.

Hence, required probability $= \frac{1}{4}$.

(iv) Exactly one tail can be obtained in one of the following two ways: HT, TH

$\therefore$ Favourable number of elementary events = 2.

Hence, required probability $= \frac{2}{4} = \frac{1}{2}$.

(v) There is only one elementary event viz. HH favourable to the event "getting no tails".

So, required probability $= \frac{1}{4}$.

EXAMPLE 3 *Three coins are tossed once. Find the probability of getting:*

(i) *all heads* (ii) *at least two heads* (iii) *at most two heads*
(iv) *no heads* (v) *exactly one tail* (vi) *exactly* 2 *tails*
(vii) *a head on first coin.*

SOLUTION Let S be the sample space associated with the random experiment of tossing three coins. Then, $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Clearly, there are 8 elements in S.

$\therefore$ Total number of elementary events = 8.

(i) There is only one elementary event, namely HHH, favourable to the given event.

$\therefore$ Required probability = $\frac{1}{8}$.

(ii) At least two heads can be obtained if we obtain one of the following elementary events as an outcome: HHH, HHT, HTH, THH

$\therefore$ Favourable number of elementary events = 4.

Hence, required probability = $\frac{4}{8} = \frac{1}{2}$.

(iii) At most two heads can be obtained in any one of the following ways:

$HHT, THH, HTH, HTT, THT, TTH, TTT$

$\therefore$ Favourable number of elementary events = $\frac{7}{8}$.

(iv) "Getting no heads" means "Getting all tails". So, there is only one elementary event viz. TTT favourable to the given event.

Hence, required probability = $\frac{1}{8}$.

(v) Elementary events favourable to "Getting exactly one tail" are: HHT, THH, HTH.

$\therefore$ Favourable number of elementary events = 3

Hence, required probability = $\frac{3}{8}$.

(vi) Elementary events favourable to "Exactly 2 tails" are: HTT, THT, TTH.

$\therefore$ Favourable number of elementary events = 3

Hence, required probability = $\frac{3}{8}$.

(vii) A head on first coin can be obtained in one of the following ways: HTT, HHH, HTH, HHT.

$\therefore$ Favourable number of elementary events = 4.

Hence, required probability = $\frac{4}{8} = \frac{1}{2}$.

EXAMPLE 4 *A die is thrown. Find the probability of getting:*

(i) *an even number* (ii) *a prime number*
(iii) *a number greater than or equal to* 3 (iv) *a number less than or equal to* 4
(v) *a number more than* 6 (vi) *a number less than or equal to* 6.

SOLUTION The sample space associated with the random experiment of rolling a die is given by $S = \{1, 2, 3, 4, 5, 6\}$. Clearly, there are 6 elements in S.

$\therefore$ Total number of elementary events = 6.

(i) An even number is obtained, if we obtain any one of 2, 4, 6 as an outcome.

So, favourable number of elementary events = 3.

Hence, required probability = $\frac{3}{6} = \frac{1}{2}$.

(ii) A prime number is obtained, if we get any one of 2, 3, 5 as an outcome.

So, favourable number of elementary events = 3.

Hence, required probability = $\frac{3}{6} = \frac{1}{2}$.

(iii) A number greater than or equal to 3 is obtained, if we get any one of 3, 4, 5, 6 as an outcome.

So, favourable number of elementary events = 4

Hence, required probability $= \frac{4}{6} = \frac{2}{3}$.

(iv) A number less than or equal to 4 is obtained, if we get any one of 1, 2, 3, 4 as an outcome.

So, favourable number of elementary events = 4

Hence, required probability $= \frac{4}{6} = \frac{2}{3}$.

(v) Since no face of the die is marked with a number greater than 6.

So, favourable number of elementary events = 0

Hence, required probability $= \frac{0}{6} = 0$.

In fact, the given event is an impossible event. So, probability of its occurrence is zero.

(vi) Since every face of a die is marked with a number less than or equal to 6.

So, favourable number of elementary events = 6.

Hence, required probability $= \frac{6}{6} = 1$.

In fact, the given event is a certain event. So, probability of its occurrence is 1.

EXAMPLE 5 *Two dice are thrown simultaneously. Find the probability of getting:*

(i) *an even number as the sum* (ii) *the sum as a prime number*
(iii) *a total of at least 10* (iv) *a doublet of even number*
(v) *a multiple of 2 on one dice and a multiple of 3 on the other dice*
(vi) *same number on both dice* (vii) *a multiple of 3 as the sum*

SOLUTION When two dice are thrown together the sample space S associated with the random experiment is given by

$$S = \{(1,1), (1,2), \ldots, (1,6), (2,1), (2,2), \ldots, (2,6), (3,1), (3,2), \ldots, (3,6), (4,1), (4,2), \ldots, (4,6), (5,1), (5,2), \ldots, (5,6), (6,1), (6,2), \ldots, (6,6)\}$$

Clearly, total number of elementary events = 36.

(i) Let A be the event "getting an even number as the sum" i.e., 2, 4, 6, 8, 10, 12 as the sum. Then,

$$A = \{(1,1), (1,3), (3,1), (2,2), (1,5), (5,1), (3,3), (2,4), (4,2), (3,5), (5,3), (4,4), (6,2), (2,6), (5,5), (6,4), (4,6), (6,6)\}$$

$\therefore$ Favourable number of elementary events = 18

So, required probability $= \frac{18}{36} = \frac{1}{2}$.

(ii) Let A be the event "getting the sum as a prime number.. i.e., 2, 3, 5, 7, 11 as the sum. Then,

$$A = \{(1,1), (1,2), (2,1), (1,4), (4,1), (2,3), (3,2), (1,6), (6,1), (2,5), (5,2), (3,4), (4,3), (6,5), (5,6)\}$$

$\therefore$ Favourable number of elementary events = 15

So, required probability $= \frac{15}{36} = \frac{5}{12}$.

(iii) Let A be the event "getting a total of at least 10" *i.e.*, 10, 11, 12 as the sum. Then,

$$A = \{(6,4), (4,6), (5,5), (6,5), (5,6), (6,6)\}$$

$\therefore$ Favourable number of elementary events = 6

So, required probability $= \frac{6}{36} = \frac{1}{6}$.

(iv) Let A be the event "getting a doublet of even number". Then, $A = \{(2, 2), (4, 4), (6, 6)\}$

$\therefore$ Favourable number of elementary events = 3

So, required probability $= \frac{3}{36} = \frac{1}{12}$.

(v) Let A be the event "getting a multiple of 2 on one die and a multiple of 3 on the other". Then,

$$A = \{(2, 3), (2, 6), (4, 3), (4, 6), (6, 3), (6, 6), (3, 2), (3, 4), (3, 6), (6, 2), (6, 4)\}$$

$\therefore$ Favourable number of elementary events = 11

So, required probability $= \frac{11}{36}$.

(vi) Let A be the event "getting the same number on both the dice". Then,

$$A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

$\therefore$ Favourable number of elementary events = 6

So, required probability $= \frac{6}{36} = \frac{1}{6}$.

(vii) Let A be the event " getting a multiple of 3 as the sum" 3, 6, 9, 12 as the sum. Then,

$$A = \{(1, 2), (2, 1), (1, 5), (5, 1), (2, 4), (4, 2), (3, 3), (3, 6), (6, 3), (5, 4), (4, 5), (6, 6)\}$$

$\therefore$ Favourable number of elementary events = 12

So, required probability $= \frac{12}{36} = \frac{1}{3}$.

EXAMPLE 6 *A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed, find the probability that the sum of numbers that turn up is (i) 3 (ii) 12.* [NCERT]

SOLUTION The sample space S associated to the given random experiment is given by

$$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$$

$\therefore$ Total number of elementary events = 12.

(i) Let A be the event that the sum of the number is 3. Then, $A = \{(1, 2)\}$

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= P(A) = \frac{1}{12}$

(ii) Let B denote the event that the sum of the numbers is 12. Then, $B = \{(6,6)\}$.

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= \frac{1}{12}$

EXAMPLE 7 *Suppose each child born is equally likely to be a boy or a girl. Consider the family with exactly three children.*

(a) List the eight elements in the sample space whose outcomes are all possible gender of three children.

(b) Write each of the following events as a set and find its probability:

(i) The event that exactly one child is a girl. *(ii) The event that at least two children are girls.*

(iii) The event that no child is a girl.

SOLUTION (a) All possible genders of three children are:

BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG

So the sample space S is given by $S = \{ BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG \}$.

(b) (i) Let A denote the event "Exactly one child is a girl". Then, $A = \{BBG, BGB, GBB\}$
Clearly, favourable number of elementary events to A is 3.

Hence, $P(A) = \frac{3}{8}$.

(ii) Let B denote the event that at least two children are girls. Then, $B = \{BGG, GBG, GGB, GGG\}$
Clearly, favourable number of elementary events to B is 4.

Hence, $P(B) = \frac{4}{8} = \frac{1}{2}$

(iii) Let C denote the event: "No child is a girl". Then,

$C = \{BBB\}$ and favourable number of elementary events to C is 1.

Hence, $P(C) = \frac{1}{8}$.

EXAMPLE 8 *Find the probability that a leap year, selected at random, will contain 53 Sundays.*

SOLUTION In a leap year there are 366 days.

366 days = 52 weeks and 2 days.

Thus, a leap year has always 52 Sundays. The remaining 2 days can be:

(i) Sunday and Monday, (ii) Monday and Tuesday, (iii) Tuesday and Wednesday, (iv) Wednesday and Thursday, (v) Thursday and Friday, (vi) Friday and Saturday, (vii) Saturday and Sunday.

If S is the sample space associated with this experiment, then S consists of the above seven points.

$\therefore$ Total number of elementary events = 7.

Let A be the event that a leap year has 53 Sundays. In order that a leap year, selected at random, should contain 53 Sundays, one of the 'over' days must be a Sunday. This can be in any one of the following two ways:

(i) Sunday and Monday (ii) Saturday and Sunday

$\therefore$ Favourable number of elementary events = 2.

Hence, required probability = $\frac{2}{7}$.

EXAMPLE 9 *Three dice are thrown together. Find the probability of getting a total of at least 6.*

SOLUTION Since one die can be thrown in six ways to obtain any one of the six numbers marked on its six faces. Therefore, if three dice are thrown, the total number of elementary events $= 6 \times 6 \times 6 = 216$.

Let A be the event of getting a total of at least 6. Then, $\overline{A}$ denotes the event of getting a total of less than 6 i.e., 3, 4, 5.

$\therefore \ \overline{A} = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)\}$
So, favourable number of elementary events = 10

$$\therefore \quad P(\overline{A}) = \frac{10}{216}$$

$$\Rightarrow \quad 1 - P(A) = \frac{10}{216} \Rightarrow P(A) = 1 - \frac{10}{216} = \frac{103}{108}.$$

EXAMPLE 10 *What is the probability that a number selected from the numbers 1, 2, 3, ..., 25, is prime number, when each of the given numbers is equally likely to be selected?*

SOLUTION Let S be the sample space associated with the given experiment and A be the event "selecting a prime number". Then, $S = \{1, 2, 3, ..., 25\}$ and $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$

$\therefore$ Total number of elementary events = 25 and, Favourable number of elementary events = 9

Hence, required probability $= \frac{9}{25}$.

EXAMPLE 11 *Tickets numbered from 1 to 20 are mixed up together and then a ticket is drawn at random. What is the probability that the ticket has a number which is a multiple of 3 or 7?*

SOLUTION Let S be the sample space associated with the given random experiment and A denote the event "getting a ticket bearing a number which is a multiple of 3 or 7". Then,

$S = \{1, 2, ..., 20\}$ and $A = \{3, 6, 9, 12, 15, 18, 7, 14\}$

$\therefore$ Total number of elementary events = 20

Favourable number of elementary events = 8

Hence, required probability $= 8 / 20 = 2 / 5$

EXAMPLE 12 *A coin is tossed. If head comes up, a die is thrown but if tail comes up, the coin is tossed again. Find the probability of obtaining:*

(i) *two tails* (ii) *head and number 6* (iii) *head and an even number,*

SOLUTION The sample space S associated with the given random experiment is

$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, H), (T, T)\}$

Clearly, it has 8 elements.

$\therefore$ Total number of elementary events = 8

(i) If the outcome is (T, T), then we say that two tails are obtained.

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= \frac{1}{8}$

(ii) Head and the number 6 is obtain in only one way i.e. when the outcome is $(H, 6)$

$\therefore$ Favourable number elementary events = 1

Hence, required probability $= \frac{1}{8}$

(iii) Head and an even number can be obtained in any one of the following ways:

$(H, 2), (H, 4), (H, 6)$.

$\therefore$ Favourable number of elementary events = 3

Hence, required probability $= \frac{3}{8}$.

EXAMPLE 13 *One urn contains two black balls (labelled B1 and B2) and one white ball. A second urn contains one black ball and two white balls (labelled W1 and W2). Suppose the following experiment is performed. One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then a second ball is chosen at random from the same urn without replacing the first ball.*

(i) Write the sample space showing all possible outcomes.

(ii) What is the probability that two black balls are chosen?

(iii) What is the probability that two balls of opposite colour are chosen? **[NCERT EXEMPLAR]**

SOLUTION (i) Let the contents of first urn be $W, B1, B2$, and that of second urn be $B, W1, W2$.

When two balls are drawn in succession from first urn, we may get any one of the following outcomes as an outcome:

$WB1, WB2, B1, B2, B1W, B2W, B2B1$

Similarly, when we draw two balls in succession from the second urn, we may obtain any one of the following as an outcome:

$BW1, BW2, W1B, W1W2, W2W1, W2B$

Thus, the sample space S is

$S = \{WB1, WB2, B1B2, B1W, B2W, B2B1, BW1, BW2, W1BW1W2, W2W1, W2B\}$

(ii) We obtain two black balls if the outcome is one of the following outcomes: $B1B2$, $B2B1$.

$\therefore$ Favourable number of elementary events = 2

Hence, required probability $= \frac{2}{12} = \frac{1}{6}$.

(iii) Two balls of opposite colour can be drawn in any one of the following outcomes:

$WB1, WB2, B1W, B2W, BW1, BW2, W1B, W2B$

$\therefore$ Favourable number of elementary events = 8

Hence, required probability $= \frac{8}{12} = \frac{2}{3}$.

<u>NOTE</u> *Consider an experiment of drawing 2 cards from a pack of 52 cards. The sample space associated with this experiment consists of* ${}^{52}C_2 = 1426$ *points and therefore it is not easy to list all the elements of the sample space. So, in future we will not be writing the sample space associated with the given random experiment.*

EXAMPLE 14 *On her vacations Veena visits four cities A, B, C and D in a random order. What is the probability that she visits:*

(i) *A before B?* (ii) *A before B and B berfore C?*
(iii) *A first and B last?* (iv) *A either first or second?*
(v) *A just before B?* [NCERT]

SOLUTION Veena can visit four cities A, B, C and D in any one of the following orders:

$ABCD, ABDC, ACDB, ACBD, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBDA, CBAD, CDAB, CDBA, DABC, DACB, DBCA, DBAC, DCAB, DCBA$

$\therefore$ Total number of arrangements (orders) in which Veena can visit four cities A, B, C and D is $4! = 24$.

(i) Out of these 24 ordered arrangements Veena can visit city A before city B in the following arrangements:

$ABCD, ABDC, ACDB, ACBD, ADBC, ADCB, CABD, CADB, CDAB, DABC, DACB, DCAB$

So, there are 12 ways in which Veena can visit city A before city B.

$\therefore$ Required probability $= \frac{12}{24} = \frac{1}{2}$

(ii) Veena can visit A before B and B before C in any one of the following four ways:

$ABCD, ABDC, DABC, ADBC$

$\therefore$ Required probability $= \frac{4}{24} = \frac{1}{6}$

(iii) Veena can visit city A first and city B last in any one of the following two ways:

$ACDB, ADCB$

$\therefore$ Required probability $= \frac{2}{24} = \frac{1}{12}$

(iv) Veena can visit city A first in $3!$ ways or city A second in $3!$ ways.

$\therefore$ Number of ways in which Veena can visity city A either first or second $= 3! + 3! = 12$

$\therefore$ Required probability $= \frac{12}{24} = \frac{1}{2}$

(v) Taking AB together A, B, C, D can be arranged in $3!$ ways.

So, required probability $= \frac{3!}{24} = \frac{1}{4}$

EXAMPLE 15 *A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once determine:* [NCERT]

(i) $P(2)$ (ii) $P(1 \text{ or } 3)$ (iii) $P(\text{not} 3)$

SOLUTION Total number of elementary events = 6.

(i) Out of six faces, 3 faces are marked with number 2.

$\therefore$ $P(2) = \frac{3}{6} = \frac{1}{2}$

(ii) Two faces are marked with number 1 and one face with number 3. Therefore, a face marked with 1 or 3 can be chosen in 3 ways.

$\therefore$ $P(1 \text{ or } 3) = \frac{3}{6} = \frac{1}{2}$

(iii) There is only one face marked with number 3. Therefore, $P(3) = \frac{1}{6}$.

Hence, $P(\text{not } 3) = 1 - P(3) = 1 - \frac{1}{6} = \frac{5}{6}$

EXAMPLE 16 *What is the probability that a randomly chosen two-digit integer is a multiple of 3?*

SOLUTION 2-digit positive integers are 10, 11, 12, ..., 98, 99. These are 90 numbers out of which one number can be chosen in 90 ways.

$\therefore$ Total number of elementary events = 90.

Out of these 90 numbers, 30 numbers (12, 15, 18, ..., 96, 99) are multiples of 3. One number out of these 30 numbers can be chosen in 30 ways.

$\therefore$ Favourable number of elementary events = 30.

Hence, required probability $= \frac{30}{90} = \frac{1}{3}$.

Type II **PROBLEMS BASED UPON COMBINATIONS OR SELECTIONS**

EXAMPLE 17 *One card is drawn from a pack of 52 cards, each of the 52 cards being equally likely to be drawn. Find the probability that the card drawn is:*

(i) an ace *(ii) red* *(iii) either red or king* *(iv) red and a king.*

SOLUTION Out of 52 cards, one card can be drawn in $^{52}C_1$ ways.

$\therefore$ Total number of elementary events $= {}^{52}C_1 = 52$.

(i) There are four aces in a pack of 52 cards, out of which one ace can be drawn in $^4C_1$ ways.

$\therefore$ Favourable number of elementary events $= {}^4C_1 = 4$.

So, required probability $= \frac{4}{52} = \frac{1}{13}$.

(ii) There are 26 red cards, out of which one red card can be drawn in $^{26}C_1$ ways.

$\therefore$ Favourable number of elementary events $= {}^{26}C_1 = 26$.

So, required probability $= \frac{26}{52} = \frac{1}{2}$.

(iii) There are 26 red cards including 2 red kings and there are 2 more kings. Therefore, there are 28 cards which are either red or king, out of 28 cards, one can be drawn in ${}^{28}C_1$ ways.

$\therefore$ Favourable number of elementary events $= {}^{28}C_1 = 28$.

So, required probability $= \frac{28}{52} = \frac{7}{13}$.

(iv) There are 2 cards which are red and king i.e., red kings.

$\therefore$ Favourable number of elementary events $= {}^{2}C_1 = 2$

So, required probability $= \frac{2}{52} = \frac{1}{26}$.

EXAMPLE 18 *An urn contains 9 red, 7 white and 4 black balls. If two balls are drawn at random, find the probability that:*

(i) *both the balls are red,* (ii) *one ball is white*
(iii) *the balls are of the same colour* (iv) *one is white and other red.*

SOLUTION There are 20 balls in the bag out of which 2 balls can be drawn in ${}^{20}C_2$ ways.

So, total number of elementary events $= {}^{20}C_2 = 190$.

(i) There are 9 red balls out of which 2 balls can be drawn in ${}^{9}C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^{9}C_2 = 36$.

So, required probability $= \frac{36}{190} = \frac{18}{95}$.

(ii) There are 7 white balls out of which one white can be drawn in ${}^{7}C_1$ ways. One ball from the remaining 13 balls can be drawn in ${}^{13}C_1$ ways. Therefore, one white and one other colour ball can be drawn in ${}^{7}C_1 \times {}^{13}C_1$ ways.

So, favourable number of elementary events $= {}^{7}C_1 \times {}^{13}C_1$.

Hence, required probability $= \frac{{}^{7}C_1 \times {}^{13}C_1}{{}^{20}C_2} = \frac{91}{190}$.

(iii) Two balls drawn are of the same colour means that either both are red or both are white or both are black. Out of 9 red balls two red balls can be drawn in ${}^{9}C_2$ ways. Similarly, two white balls can be drawn from 7 white balls in ${}^{7}C_2$ ways and two black balls from 4 black balls in ${}^{4}C_2$ ways. Therefore,

The number of ways of drawing 2 balls of the same colour $= {}^{9}C_2 + {}^{7}C_2 + {}^{4}C_2 = 36 + 21 + 6 = 63$

$\therefore$ Favourable number of elementary events = 63.

So, required probability $= \frac{63}{190}$.

(iv) Out of 7 white balls one white ball can be drawn in ${}^{7}C_1$ ways and out of 9 red balls one red ball can be drawn in ${}^{9}C_1$ ways. Therefore,

One white and one red ball can be drawn in ${}^{7}C_1 \times {}^{9}C_1$ ways.

So, favourable number of elementary events $= {}^{7}C_1 \times {}^{9}C_1 = 63$.

So, required probability $= 63/190$.

EXAMPLE 19 ***A box contains 10 red marbles, 20 blue marbles and 30 green marbles. Five marbles are drawn from the box, what is the probability that (i) all will be blue? (ii) at least one will be green?***

[NCERT]

SOLUTION Out of 60 marbles, 5 marbles can be drawn in ${}^{60}C_5$ ways.

$\therefore$ Total number of elementary events $= {}^{60}C_5$

(i) Out of 20 blue marbles, 5 blue marbles can be chosen in ${}^{20}C_5$ ways.

$\therefore$ Favourable number of ways $= {}^{20}C_5$

Hence, required probability $= \dfrac{{}^{20}C_5}{{}^{60}C_5}$

(ii) Clearly,

Required probability $= 1 -$ Probability that no ball is green

$= 1 -$ Probability that 5 balls drawn are red or blue. $= 1 - \dfrac{{}^{30}C_5}{{}^{60}C_5}$

EXAMPLE 20 *In a lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (i) 1 ticket (ii) two tickets (iii) 10 tickets.* [NCERT]

SOLUTION (i) Out of 10,000 tickets, one ticket can be chosen in ${}^{10000}C_1 = 10000$ ways.

There are 9990 tickets not containing a prize. Out of these 9990 tickets one can be chosen in ${}^{9990}C_1$ ways.

$\therefore$ Probability of not getting a prize $= \dfrac{9990}{10000} = \dfrac{999}{1000}$

(ii) Out of 10,000 tickets, two tickets can be chosen in ${}^{10000}C_2$ ways. As there are 9990 tickets without any prize. Therefore, two drawn tickets will not contain any prize, if they are chosen from the remaining 9990 tickets. This can be done in ${}^{9990}C_2$ ways.

So, required probability $= \dfrac{{}^{9990}C_2}{{}^{10000}C_2}$

(iii) 10 tickets can be drawn out of 10,000 tickets in ${}^{10000}C_{10}$ ways. There are 9990 tickets without any prize. Out of these tickets 10 tickets can be chosen in ${}^{9990}C_{10}$ ways. So, 10 drawn tickets will not contain any prize, if they are chosen from the remaining 9990 tickets.

Hence, required probability $= \dfrac{{}^{9990}C_{10}}{{}^{10000}C_{10}}$

EXAMPLE 21 *The number lock of a suitcase has 4 wheels, each labelled with ten digits i.e. from 0 to 9. The lock opens with a sequence of four digits with no repeats. What is the probability of a person getting the right sequence to open the suitcase.* [NCERT]

SOLUTION There are ${}^{10}C_4 \times 4! = 5040$ sequences of 4 distinct digits out of which there is only one sequence in which the lock opens.

$\therefore$ Required probability $= \dfrac{1}{5040}$

EXAMPLE 22 *Out of 100 students, two sections of 40 and 60 students are formed. If you and your friends are among the 100 students, what is the probability that* [NCERT]

(i) *you both enter the same section?* (ii) *you both enter the different sections?*

SOLUTION Out of 100 students, two sections of 40 and 60 students can be formed in ${}^{100}C_{40} \times {}^{60}C_{60} = \dfrac{100!}{40!\,60!}$ ways.

(i) You and your friend can be in the same section in ${}^{98}C_{38} \times {}^{60}C_{60} + {}^{98}C_{58} \times {}^{40}C_{40}$

$= \left(\dfrac{98!}{60!\,38!} + \dfrac{98!}{58!\,40!}\right)$ ways

$\therefore$ Probability that you and your friend enter the same section $= \dfrac{\dfrac{98!}{60!\,38!} + \dfrac{98!}{58!\,40!}}{\dfrac{100!}{60!\,40!}}$

$$= \frac{40 \times 39}{100 \times 99} + \frac{60 \times 59}{100 \times 99} = \frac{17}{33}$$

(ii) Required probability $= 1 -$ Probability that you and your friend enter the same section

$$= 1 - \frac{17}{33} = \frac{16}{33}$$

EXAMPLE 23 *Four cards are drawn at random from a pack of 52 playing cards. Find the probability of getting*

(i) *all the four cards of the same suit* (ii) *all the four cards of the same number*
(iii) *one card from each suit* (iv) *two red cards and two black cards*
(v) *all cards of the same colour* (vi) *all face cards.*

SOLUTION Four cards can be drawn from a pack of 52 cards in ${}^{52}C_4$ ways.

So, total number of elementary events $= {}^{52}C_4$

(i) There are four suits viz. club, spade, heart and diamond, each of 13 cards. All the four cards are of the same suit means that either four cards drawn are club cards or spade cards or heart cards or diamond cards. So, the total number of ways of getting all the four cards of the same suit is ${}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4\,({}^{13}C_4)$

So, required probability $= \dfrac{4\,({}^{13}C_4)}{{}^{52}C_4} = \dfrac{198}{20825}$

(ii) Four cards drawn can be of the same number in any one of the following ways:

(1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3), ..., (13, 13, 13, 13)

$\therefore$ Favourable number of elementary events = 13.

So, required probability $= \dfrac{13}{{}^{52}C_4} = \dfrac{13}{270725}$

(iii) There are four suits each of 13 cards. One card from each suit means that there is one diamond card, one club card, one spade card and one heart card. There are 13 diamond cards, out of which one can be selected in ${}^{13}C_1$ ways. Similarly, one club, one spade and one heart, each can be selected in ${}^{13}C_1$ ways.

$\therefore$ The number of ways of selecting 4 cards, one from each suit $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$

$$= ({}^{13}C_1)^4$$

So, required probability $= \dfrac{({}^{13}C_1)^4}{{}^{52}C_4} = \dfrac{2197}{20825}$

(iv) There are 26 red cards and 26 black cards. Out of 26 red cards, 2 cards can be drawn in ${}^{26}C_2$ ways. Similarly, 2 black cards can be drawn in ${}^{26}C_2$ ways. Therefore, 2 red and 2 black cards can be drawn in ${}^{26}C_2 \times {}^{26}C_2$ ways.

So, required probability $= \dfrac{{}^{26}C_2 \times {}^{26}C_2}{{}^{52}C_4}$

(v) There are two colours viz. red and black. Out of 26 red colour cards, 4 cards can be drawn in ${}^{26}C_4$ ways. 4 black cards can be drawn in ${}^{26}C_4$ ways. Therefore, 4 red or 4 black cards can be drawn in ${}^{26}C_4 + {}^{26}C_4 = 2\,({}^{26}C_4)$ ways.

So, required probability $= \dfrac{2\,({}^{26}C_4)}{{}^{52}C_4}$

(vi) There are 12 face cards (4 kings, 4 queens and 4 jacks). Out of these 12 face cards, 4 cards can be selected in ${}^{12}C_4$ ways.

$\therefore$ Favourable number of elementary events $= {}^{12}C_4$

So, required probability $= \dfrac{{}^{12}C_4}{{}^{52}C_4}$

EXAMPLE 24 *In a lottery of 50 tickets numbered 1 to 50, two tickets are drawn simultaneously. Find the probability that:*

(i) *both the tickets drawn have prime numbers,* (ii) *none of the tickets drawn has prime number,* (iii) *one ticket has prime number.*

SOLUTION Out of 50 tickets 2 tickets can be drawn in ${}^{50}C_2$ ways.

So, total number of elementary events $= {}^{50}C_2 = 1225$

(i) There are 15 prime numbers between 1 and 50 viz. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. Out of these 15 prime numbers 2 numbers can be selected in ${}^{15}C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^{15}C_2 = 105$

So, required probability $= \dfrac{105}{1225} = \dfrac{21}{245}$

(ii) Number of non-primes from 1 to 50 $= 50 - 15 = 35$. Out of these 35 numbers 2 can be selected in ${}^{35}C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^{35}C_2 = 595$

So, required probability $= \dfrac{595}{1225} = \dfrac{17}{35}$

(iii) Out of 15 primes from 1 to 50, one prime number can be selected in ${}^{15}C_1$ ways. Therefore, one prime and one non-prime can be selected in ${}^{15}C_1 \times {}^{35}C_1$ ways.

$\therefore$ Favourable number of elementary events $= {}^{15}C_1 \times {}^{35}C_1 = 525$

So, required probability $= \dfrac{525}{1225} = \dfrac{3}{7}$

EXAMPLE 25 *Four persons are to be chosen at random from a group of 3 men, 2 women and 4 children. Find the probability of selecting:*

(i) 1 man, 1 woman and 2 children (ii) exactly 2 children (iii) 2 women

SOLUTION There are 9 persons viz. 3 men, 2 women and 4 children. Out of these 9 persons 4 persons can be selected in ${}^{9}C_4 = 126$ ways.

$\therefore$ Total number of elementary events $= 126$

(i) 1 man, 1 woman and 2 children can be selected in ${}^{3}C_1 \times {}^{2}C_1 \times {}^{4}C_2 = 36$ ways.

$\therefore$ Favourable number of elementary events $= 36$

So, required probability $= \dfrac{36}{126} = \dfrac{2}{7}$

(ii) Exactly 2 children means: 2 children out of 4 children and 2 persons from 5 persons consisting of 3 men and 2 women. This can be done in ${}^{4}C_2 \times {}^{5}C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^{4}C_2 \times {}^{5}C_2 = 60$

So, required probability $= \frac{60}{126} = \frac{10}{21}$

(iii) We have to select 4 persons of which 2 are women and the remaining 2 are chosen from 7 persons consisting of 3 men and 4 children. This can be done in ${}^2C_2 \times {}^7C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^2C_2 \times {}^7C_2 = 21$

So, required probability $= \frac{21}{126} = \frac{1}{6}$

EXAMPLE 26 *A box contains 10 bulbs, of which just three are defective. If a random sample of five bulbs is drawn, find the probabilities that the sample contains:*

(i) *exactly one defective bulb,* (ii) *exactlytwo defective bulbs,* (iii) *no defective bulbs.*

SOLUTION Out of 10 bulbs 5 can be chosen in ${}^{10}C_5$ ways.

So, total number of elementary events $= {}^{10}C_5$

(i) There are 3 defective and 7 non-defective bulbs. The number of ways of selecting one defective bulb out of 3 and 4 non-defective out of 7 is ${}^3C_1 \times {}^7C_4$.

$\therefore$ Favourable number of elementary events $= {}^3C_1 \times {}^7C_4$

So, required probability $= \frac{{}^3C_1 \times {}^7C_4}{{}^{10}C_5} = \frac{5}{12}$

(ii) The number of ways of selecting 2 defective bulbs out of 3 defective bulbs and 3 non-defective bulbs out of 7 non defective bulbs is ${}^3C_2 \times {}^7C_3$.

$\therefore$ Favourable number of elementary events $= {}^3C_2 \times {}^7C_3$

So, required probability $= \frac{{}^3C_2 \times {}^7C_3}{{}^{10}C_5} = \frac{5}{12}$

(iii) No defective bulbs means all non-defective bulbs. The number of ways of selecting all 5 non-defective bulbs out of 7 is ${}^7C_5$.

$\therefore$ Favourable number of elementary events $= {}^7C_5$

So, required probability $= \frac{{}^7C_5}{{}^{10}C_5} = \frac{1}{12}$

EXAMPLE 27 *Five marbles are drawn from a bag which contains 7 blue marbles and 4 black marbles. What is the probability that: (i) all will be blue? (ii) 3 will be blue and 2 black?*

SOLUTION There are $7 + 4 = 11$ marbles in the bag out of which 5 marbles can be drawn in ${}^{11}C_5$ ways.

$\therefore$ Total number of elementary events $= {}^{11}C_5$.

(i) There are 7 blue marbles out of which 5 blue marbles can be drawn in ${}^7C_5$ ways.

$\therefore$ Favourable number of elementary events $= {}^7C_5$

Hence, required probability $= \frac{{}^7C_5}{{}^{11}C_5} = \frac{7!}{2!5!} \times \frac{5!6!}{11!} = \frac{1}{22}$

(ii) Three blue out of 7 blue balls and 2 black out of 4 black balls can be drawn in ${}^7C_3 \times {}^4C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^7C_3 \times {}^4C_2$

Hence, required probability $= \frac{{}^7C_3 \times {}^4C_2}{{}^{11}C_5} = \frac{7!}{3!4!} \times \frac{4!}{2!2!} \times \frac{5! \times 6!}{11!} = \frac{5}{11}$

EXAMPLE 28 *Find the probability that when a hand of 7 cards is dealt from a well-shuffled deck of 52 cards, it contains: (i) all 4 kings (ii) exactly 3 kings (iii) at least 3 kings.*

SOLUTION Out of 52 cards from a deck of 52 playing cards, 7 cards can be drawn in $^{52}C_7$ ways.

$\therefore$ Total number of elementary events = $^{52}C_7$

(i) There are 4 kings. Therefore, 4 kings out of 4 kings and 3 other cards from the remaining 48 cards can be chosen in $^4C_4 \times {}^{48}C_3$ ways.

$\therefore$ Favourable number of elementary events = $^4C_4 \times {}^{48}C_3$

Hence, required probability = $\dfrac{^4C_4 \times {}^{48}C_3}{^{52}C_7} = \dfrac{1}{7735}$

(ii) Three kings out of 4 kings and 4 other cards out of remaining 48 cards can be chosen in $^4C_3 \times {}^{48}C_4$ ways.

$\therefore$ Favourable number of elementary events = $^4C_3 \times {}^{48}C_4$

Hence, required probability = $\dfrac{^4C_3 \times {}^{48}C_4}{^{52}C_7} = \dfrac{9}{1547}$

(iii) When 7 cards are drawn from a deck of 52 playing cards, then getting at least 3 kings means: getting 3 kings and 4 other cards or getting 4 kings and 3 other cards. This can be done in $^4C_3 \times {}^{48}C_4 + {}^4C_4 \times {}^{48}C_3$ ways.

$\therefore$ Favourable number of elementary events = $^4C_3 \times {}^{48}C_4 + {}^4C_4 \times {}^{48}C_3$

Hence, required probability = $\dfrac{^4C_3 \times {}^{48}C_4 + {}^4C_4 \times {}^{48}C_3}{^{52}C_7} = \dfrac{46}{7735}$

EXAMPLE 29 *In a single throw of three dice, determine the probability of getting*
(i) *a total of 5* (ii) *a total of at most 5* (iii) *a total of at least 5.*

SOLUTION Total number of elementary events associated with the random experiment of throwing three dice simultaneously is $6 \times 6 \times 6 = 216$.

(i) A total of 5 can be obtain in one of the following ways:

(1, 1, 3), (3, 1, 1), (1, 3, 1), (2, 2, 1), (1, 2, 2), (2, 1, 2)

$\therefore$ Favourable number of elementary events = 6

Hence, required probability $= \dfrac{6}{216} = \dfrac{1}{36}$

(ii) A total of at most 5 can be obtained in any one of the following ways:

(1, 1, 1), (1, 1, 2), (2, 1, 1), (1, 2, 1), (1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)

$\therefore$ Favourable number of elementary events = 10

Hence, required probability = $\dfrac{10}{216} = \dfrac{5}{108}$

(iii) Let A be the event "getting a total of at least 5". Then,

$P(A) = 1 - P(\overline{A}) = 1 - P$ (Getting a total of at most 4)

A total of at most 4 can be obtained in any one of the following ways:

(1, 1, 1), (1, 1, 2), (2, 1, 1), (1, 2, 1).

So, Favourable number of elementary events to $\overline{A}$ is 4.

$\therefore$ $P(\overline{A}) = \dfrac{4}{216} \Rightarrow 1 - P(A) = \dfrac{4}{216} \Rightarrow P(A) = 1 - \dfrac{4}{216} = \dfrac{212}{216} = \dfrac{53}{54}$

Type II **PROBLEMS BASED UPON PERMUTATIONS OR ARRANGEMENTS**

EXAMPLE 30 *If the letters of the word ALGORITHM are arranged at random in a row what is the probability that the letters GOR must remain together as a unit?* [NCERT EXEMPLAR]

SOLUTION There are 9 letters in the word ALGORITHM. These 9 letters can be arranged in a row in 9! ways.

$\therefore$ Total number of elementary events = 9!

Considering GOR as one letter there are 7 letters which can be arranged in a row in 7! ways.

$\therefore$ Favourable number of elementary events = 7!

Hence, required probability $= \frac{7!}{9!} = \frac{1}{72}$

EXAMPLE 31 *If the letters of the word 'ATTRACTION' are written down at random, find the probability that* (i) *all the T's occur together* (ii) *no two T's occur together.*

SOLUTION The total number of arrangements of the letters of the word 'ATTRACTION' is $\frac{10!}{3!\,2!}$.

(i) Considering three T's as one letter there are 8 letters consisting of two identical A's. These 8 letters can be arranged in $\frac{8!}{2!}$ ways.

$$\text{Hence, required probability} = \frac{\frac{8!}{2!}}{\frac{10!}{3!\,2!}} = \frac{3!\,8!}{10!} = \frac{1}{15}$$

(ii) Other than 3 T's there are 7 letters which can be arranged in $\frac{7!}{2!}$ ways. There are 8 places, 6 between the 7 letters and one on extreme left and the other on extreme right. To separate three T's, we arrange them in these 8 places. This can be done in ${}^8C_3$ ways. Therefore,

Number of ways in which no two T's are together $= \frac{7!}{2!} \times {}^8C_3$

$$\text{Hence, required probability} = \frac{\frac{7!}{2!} \times {}^8C_3}{\frac{10!}{3!\,2!}} = \frac{7}{15}$$

EXAMPLE 32 *A five digit number is formed by the digits 1, 2, 3, 4, 5 without repetition. Find the probability that the number is divisible by 4.*

SOLUTION Total number of five digit numbers formed by the digits 1, 2, 3, 4, 5 is 5!.

$\therefore$ Total number of elementary events = 5! = 120.

We know that a number is divisible by 4 if the number formed by last two digits is divisible by 4. Therefore last two digits can be 12, 24, 32, 52 that is, last two digits can be filled in 4 ways. But corresponding to each of these ways there are 3! = 6 ways of filling the remaining three places. Therefore the total number of five digit numbers formed by the digits 1, 2, 3, 4, 5 and divisible by 4 is $4 \times 6 = 24$

$\therefore$ Favourable number of elementary events = 24

So, required probability $= \frac{24}{120} = \frac{1}{5}$

Type III **PROBLEMS BASED UPON COMBINATIONS OR SELECTIONS**

EXAMPLE 33 *Out of 9 outstanding students in a college, there are 4 boys and 5 girls. A team of four students is to be selected for a quiz programme. Find the probability that two are boys and two are girls.*

SOLUTION Out of 9 students 4 students can be selected in $^9C_4$ ways.

So, total number of elementary events $= {}^9C_4$.

There are 4 boys and 5 girls out of which 2 boys and 2 girls can be selected in $^4C_2 \times {}^5C_2$ ways.

So, favourable number of elementary events $= {}^4C_2 \times {}^5C_2$

Hence, required probability $= \dfrac{^4C_2 \times {}^5C_2}{^9C_4} = \dfrac{10}{21}$

EXAMPLE 34 *In a lot of 12 Microwave ovens, there are 3 defective units. A person has ordered 4 of these units and since each is identically packed, the selection will be random. What is the probability that (i) all 4 units are good. (ii) exactly 3 units are good (iii) at least 2 units are good.* [NCERT]

SOLUTION Out of 12 Microwave ovens, 4 can be chosen in $^{12}C_4$ ways.

$\therefore$ Total number of elementary events $= {}^{12}C_4$

(i) There are 9 good units out of which 4 can be chosen in $^9C_4$ ways.

$\therefore$ Favourable number of elementary events $= {}^9C_4$

Hence, required probability $= \dfrac{^9C_4}{^{12}C_4} = \dfrac{14}{55}$

(ii) Exactly 3 good units can be chosen in $^9C_3 \times {}^3C_1$ ways.

$\therefore$ Required probability $= \dfrac{^9C_3 \times {}^3C_1}{^{12}C_4} = \dfrac{28}{55}$

(iii) Required probability $= 1 - P$ (At most one unit is good) $= 1 - \dfrac{^9C_1 \times {}^3C_3}{^{12}C_4} = 1 - \dfrac{1}{55} = \dfrac{54}{55}$

EXAMPLE 35 *In a relay race there are five teams A, B, C, D and E.*

(i) *What is the probability that A, B and C finish first, second and third respectively.*

(ii) *What is the probability that A, B and C are first three to finish (in any order).* [NCERT]

SOLUTION Out of 5 teams first three positions can be occupied by 3 teams in any order in $^5C_3 \times 3!$ ways.

So, total number of elementary events $= {}^5C_3 \times 3! = 60$

(i) Teams A, B and C can finish first, second and third in only one way, because there is only one finishing order.

$\therefore$ Favourable number of elementary events = 1

So, required probability $= \dfrac{1}{60}$

(ii) Teams A, B and C finish at first three places in any order in 3! ways.

$\therefore$ Favourable number of elementary events $= 3! = 6$

So, required probability $= \dfrac{6}{60} = \dfrac{1}{10}$

Type IV **MISCELLANEOUS PROBLEMS**

EXAMPLE 36 ***A card is drawn from an ordinary pack of 52 cards and a gambler bets that, it is a spade or an ace. What are the odds against his winning this bet?***

SOLUTION Let A be the event of getting a spade or an ace from a pack of 52 cards. Then,

Total number of elementary events $= {}^{52}C_1 = 52$

Since there are 13 spade cards including an ace of spade and three aces other than an ace of spade.

$\therefore$ Favourable number of elementary events $= {}^{16}C_1 = 16$

So, $P(A) = \frac{16}{52} = \frac{4}{13}$.

Hence, odds against A are $P(\overline{A}) : P(A) = \frac{9}{13} : \frac{4}{13} = 9:4$

EXAMPLE 37 *The odds in favour of an event are 3 : 5. Find the probability of occurrence of this event.*

SOLUTION It is given that the odds in favour of an event are 3:5. Therefore,

Favourable number of elementary events $= 3x$

Unfavourable number of elementary events $= 5x$.

So, total number of elementary events $= 3x + 5x = 8x$.

Hence, probability of the occurrence of the event $= \frac{3x}{8x} = \frac{3}{8}$

LEVEL-2

Type V MIXED PROBLEMS ON PROBABILITY

EXAMPLE 38 *A letter is chosen at random from the word 'ASSASSINATION'. Find the probability that letter is (i) a vowel (ii) a consonant.* [NCERT]

SOLUTION There are 13 letters in the word 'ASSASSINATION' out of which there are 6 vowels viz., A, A, I, A, I, O and 7 consonants.

Total number of ways of selecting a word from 13 letters is $= 13$.

Number of ways of selecting a vowel out of 6 vowels $= 6$

Number of ways of selecting a consonant out of 7 consonants $= 7$.

$\therefore$ P (Selecting a vowel) $= \frac{6}{13}$ and, P (Selecting a consonant) $= \frac{7}{13}$

EXAMPLE 39 *If the letters of the word ASSASSINATION are arranged at random. Find the probability that*

(i) *Four S's come consecutively in the word.* (ii) *Two I's and two N's come together.*

(iii) *All A's are not coming together.* (iv) *No two A's are coming together.*

[NCERT EXEMPLAR]

SOLUTION There are 13 letters in the word ASSASSINATION out of which there are $3A$'s $4S$'s $2I$'s $2N$'s, one O and one T. These 13 letters can be arranged in a row in $\frac{13!}{3!\,4!\,2!\,2!\,1!\,1!}$ ways.

(i) Considering $4S$'s as one letter there are 10 letters ($3A$'s, $2I$'s, 2 N's, one O; one T and one letter formed by $4S$'s). These 10 letters can be arranged in $\frac{10!}{3!\,2!\,2!\,1!\,1!\,1!}$ ways.

$$\therefore \quad P\,(4S\text{'s come consecutively}) = \frac{\dfrac{10!}{3!\,2!\,2!\,1!\,1!\,1!}}{\dfrac{13!}{3!\,4!\,2!\,2!\,1!\,1!}} = \frac{4! \times 10!}{13!} = \frac{2}{143}$$

(ii) Two I's and two N's can be put together in $\frac{4!}{2!\,2!}$ ways. Considering these 4 letters as one, there are 10 letters which can be arranged in a row in $\frac{10!}{3!\,4!}$ ways.

$\therefore$ Number of arrangements in which two I's and two N's come together $= \dfrac{10!}{3!\,4!} \times \dfrac{4!}{2!\,2!}$

$$= \frac{10!}{3!\,2!\,2!}$$

Hence, P (Two I's and two N's come together) $= \dfrac{\dfrac{10!}{3!\,2!\,2!}}{\dfrac{13!}{3!\,4!\,2!\,2!}} = \dfrac{2}{143}$

(iii) Considering all A's as one letter, there are 11 letters which can be arranged in a row in $\dfrac{11!}{4!\,2!\,2!}$ ways.

$\therefore$ P (All A's come together) $= \dfrac{\dfrac{11!}{4!\,2!\,2!}}{\dfrac{13!}{3!\,4!\,2!\,2!}} = \dfrac{1}{26}$

Hence, P (All A's are not coming together) $= 1 - \dfrac{1}{26} = \dfrac{25}{26}$

(iv) Other than 3 A's there are 10 letters (4S's, 2I's 2N's, one O and one T). These 10 letters can be arranged in a row in $\dfrac{10!}{4!\,2!\,2!}$ ways. In each arrangement of these 10 letters there are 11 places which can be filled by 3 A's in ${}^{11}C_3$ ways.

$\therefore$ Number of arrangements in which no two A's come together $= \dfrac{10!}{4!\,2!\,2!} \times {}^{11}C_3$

$$= \frac{10!}{4!\,2!\,2!} \times \frac{11!}{8!\,3!}$$

Hence, P (No two A's are coming together) $= \dfrac{\dfrac{10!}{4!\,2!\,2!} \times \dfrac{11!}{8!\,3!}}{\dfrac{13!}{3!\,4!\,2!\,2!}} = \dfrac{15}{26}$.

EXAMPLE 40 *In a lottery, a person choses six different natural numebrs at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prize in the game?*

SOLUTION Out of 20 numbers six numbers can be chosen in ${}^{20}C_6$ ways.

$\therefore$ Total number of elementary events $= {}^{20}C_6 = 38760$

It is given that a persons wins the prize if six selected numbers match with the six numbers already fixed by the committee.

$\therefore$ Favourable number of ways = 1

Hence, required probability $= \dfrac{1}{38760}$

EXAMPLE 41 *A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits. If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?*

SOLUTION It is given that a PIN is a sequence of four symbols selected from 36 (26 letters and 10 digits) symbols. Therefore,

Total numbers of PINs $= 36 \times 36 \times 36 \times 36 = 36^4 = 1{,}679{,}616$

Total number of PINs with distinct symbols $= 36 \times 35 \times 34 \times 33 = 1{,}413{,}720$.

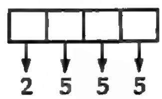
$\therefore$ The number of PINs that contain at least one repeated symbol $= 1{,}679{,}616 - 1{,}413{,}720$
$= 2{,}65{,}896$

Hence,

The probability that a randomly chosen PIN contains a repeated symbol $= \dfrac{2{,}65{,}896}{1{,}679{,}616}$
$= 0.1583$

EXAMPLE 42 *If 4-digit numbers greater than or equal to 5000 are randomly formed from the digits 0, 1, 3, 5 and 7, what is the probabilty of forming number divisible by 5 when*
(i) the digits may be repeated (ii) the repetition of digits is not allowed. [NCERT]

SOLUTION (i) Total number of 4-digit numbers formed from the digits 0, 1, 3, 5 and 7 and greater than or equal to 5000 is $2 \times 5 \times 5 \times 5 = 250$.



A number is divisible by 5, if units digit is 0 to 5. Therefore, number of 4 digit numbers formed from the digits 0, 1, 3, 5 and 7, divisible by 5 and greater than or equal to 5000 is $2 \times 5 \times 5 \times 2 = 100$

$\therefore$ Probability of forming a number divisible by $5 = \dfrac{100}{250} = \dfrac{2}{5}$

(ii) If repetition of digits is not allowed, then the total number of 4 digit numbers formed from the digits 0, 1, 3, 5 and 7 is $2 \times 4 \times 3 \times 2 = 48$.

Now,

Number of 4 digit numbers divisible by 5 having 0 at one's place $= 2 \times 3 \times 2 \times 1 = 12$

Number of 4 digit numbers divisible by 5 having 5 at one's place $= 1 \times 3 \times 2 \times 1 = 6$

$\therefore$ Number of 4 digit numbers with distinct digits and divisible by $5 = 12 + 6 = 18$

Hence, probability of forming a number divisible by $5 = \dfrac{18}{48} = \dfrac{3}{8}$

EXAMPLE 43 *A fair coin is tossed four times, and a person wins Re 1 for each head and lose Rs. 1.50 for each tail that turns up. From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.* [NCERT]

SOLUTION The sample space associated with the random experiment of tossing four coins is given by

$$S = \{HHHH, HHHT, HHTH, THHH, HTHH, HHTT, TTHH, HTHT, HTTH, THTH, THHT, TTHT, HTTT, THTT, TTTH, TTTT\}$$

Let Rs X be the amount won by the person in a single throw of four coins.

If the person gets 4 heads in a throw of four coins, then amount won $=$ ₹ $(4 \times 1) =$ ₹ 4 i.e. $X = 4$.

If the person gets 3 heads and one tail in a throw of four coins, then amount won
$=$ ₹ $(3 \times 1 - 1.50 \times 1) =$ ₹ 1.50 i.e., $X = 1.50$

If the person gets 2 heads and 2 tails in a throw of four coins, then amount won
$=$ ₹ $(2 \times 1 - 1.50 \times 2) = -$ ₹ 1 i.e. amount lost $= -1$

If the person gets 1 head and 3 tails in a throw of four coins, then amount won/lost
$=$ ₹ $(1 \times 1 - 3 \times 1.50) = -$ ₹ 3.50. So, $X = -3.50$

If the person gets all tails in a throw of 4 coins, then amount lost
$=$ ₹ $(0 - 4 \times 1.50) = -$ ₹ 6 i.e., $X = -6$

Now, $P(X = 4) =$ Probability of getting all heads $= \dfrac{1}{16}$

$P(X = 1.50)$ = Probability of getting 3 heads and 1 tail $= \frac{4}{16} = \frac{1}{4}$

$P(X = -1)$ = Probability of getting 2 heads and 2 tails $= \frac{6}{16} = \frac{3}{8}$

$P(X = -3.50)$ = Probability of getting 1 head and 3 tails $= \frac{4}{16} = \frac{1}{4}$

and, $P(X = -6)$ = Probability of getting all tails $= \frac{1}{16}$

EXAMPLE 44 *Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope contains exactly one letter. Find the probability that at least one letter is in its proper envelope.* [NCERT]

SOLUTION Three letters can be inserted in three envelopes in $3! = 6$ ways.

Let us now find the number of ways of inserting 3 letters in three envelopes so that no letter is put in proper envelope.

Number of ways of inserting 2 letters in 2 envelopes so that no letter is in proper envelope = 1

∴ Number of ways of inserting 3 letters in 3 envelopes so that no letter is in proper envelope

= Total number of ways of inserting 3 letters in 3 envelopes

– Number of ways in which one letter is in proper envelope and remaining two are in wrong envelopes

– Number of ways in which all are in proper envelopes

$= 3! - {}^3C_1 \times 1 - 1 = 6 - 3 - 1 = 2$

∴ Probability that at least one letter is in its proper envelope

$= 1 -$ Probability that no letter is in its proper envelope $= 1 - \frac{2}{6} = \frac{2}{3}$

EXAMPLE 45 *A word consists of 9 letters; 5 consonants and 4 vowels. Three letters are chosen at random. What is the probability that more than one vowel will be selected?*

SOLUTION Three letters can be chosen out of 9 letters in ${}^9C_3$ ways.

∴ Total number of elementary events $= {}^9C_3$

More than one vowels can be chosen in one of the following ways:

(i) 2 vowels and one consonant or, (ii) 3 vowels.

So, favourable number of elementary events $= {}^4C_2 \times {}^5C_1 + {}^4C_3$

Hence, required probability $= \frac{{}^4C_2 \times {}^5C_1 + {}^4C_3}{{}^9C_3} = \frac{17}{42}$

EXAMPLE 46 *A bag contains 50 tickets numbered 1, 2, 3, ..., 50 of which five are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$). Find the probability that $x_3 = 30$.*

SOLUTION Five tickets out of 50 can be drawn in ${}^{50}C_5$ ways.

∴ Total number of elementary events $= {}^{50}C_5$

Since $x_1 < x_2 < x_3 < x_4 < x_5$ and $x_3 = 30$. Therefore, $x_1, x_2 < 30$ i.e., x_1 and x_2 should come from tickets numbered 1 to 29 and this may happen in ${}^{29}C_2$ ways. Remaining two i.e., $x_4, x_5 > 30$, should come from 20 tickets numbered from 31 to 50 in ${}^{20}C_2$ ways.

∴ Favourable number of elementary events $= {}^{29}C_2 \times {}^{20}C_2$

Hence, required probability $= \frac{{}^{29}C_2 \times {}^{20}C_2}{{}^{50}C_5} = \frac{551}{15134}$

EXAMPLE 47 *A bag contains tickets numbered 1 to 30. Three tickets are drawn at random from the bag. What is the probability that the maximum number on the selected tickets exceeds 25?*

SOLUTION It is given that the maximum number on the selected tickets exceeds 25. This means that at least one of the selected tickets should bear a number that exceeds 25. Note that the negation of 'at least one' is none and in this case it will be easier for us to find the probability that none of selected tickets bear number exceeding 25.

Let A be the event that none of the selected tickets bear number exceeding 25. Then, $\overline{A}$ denotes the event that at least one of the selected tickets bears a number that exceeds 25.

$\therefore$ Required probability $= P(\overline{A}) = 1 - P(A)$

Now, we calculate $P(A)$.

The total number of ways of drawing three tickets out of 30 is ${}^{30}C_3$.

$\therefore$ Total number of elementary events $= {}^{30}C_3$

Since none of the selected tickets bear number exceeding 25. Therefore, three tickets are drawn from tickets bearing number 1 to 25. This can be done in ${}^{25}C_3$ ways.

$\therefore$ Favourable number of elementary events $= {}^{25}C_3$

So, $$P(A) = \frac{{}^{25}C_3}{{}^{30}C_3} = \frac{115}{203}$$

Hence, required probability $= P(\overline{A}) = 1 - \dfrac{115}{203} = \dfrac{88}{203}$

EXAMPLE 48 *Twelve balls are distributed among three boxes, find the probability that the first box will contain three balls.*

SOLUTION Since each ball can be put into any one of the three boxes. So, the total number of ways in which 12 balls can be put into three boxes in 3^{12}. Out of 12 balls, 3 balls can be chosen in ${}^{12}C_3$ ways. Put these three balls in the first box and put remaining 9 balls in the remaining two boxes, which can be done in 2^9 ways.

So, the total number of ways in 3 balls can be put in the first box and the remaining 9 in other two boxes is ${}^{12}C_3 \times 2^9$.

Hence, required probability $= \dfrac{{}^{12}C_3 \times 2^9}{3^{12}}$.

EXAMPLE 49 *Find the probability that the birth days of six different persons will fall in exactly two calender months.*

SOLUTION Since each person can have his birth day in any one of the 12 calender months. So, there are 12 options for each person.

$\therefore$ Total number of ways in which 6 persons can have their birth days

$$= 12 \times 12 \times 12 \times 12 \times 12 \times 12 = 12^6$$

Out of 12 months, 2 months can be chosen in ${}^{12}C_2$ ways.

Now, birth days of six persons can fall in these two months in 2^6 ways. Out of these 2^6 ways, there are two ways when all six birth days fall in one month. So, there are $(2^6 - 2)$ ways in which six birth days fall in the chosen 2 months.

$\therefore$ Number of ways in which six birth days fall in exactly two calender month $= {}^{12}C_2 \times (2^6 - 2)$

Hence, required probability $= \dfrac{{}^{12}C_2 \times (2^6 - 2)}{12^6} = \dfrac{341}{12^5}$

EXAMPLE 50 *Three dice are thrown simultaneously. Find the probability that:*
(i) *all of them show the same face.* (ii) *all show distinct faces.* (iii) *two of them show the same face.*

SOLUTION The toal number of elementary events associated to the random experiment of throwing three dice simultaneously is $6 \times 6 \times 6 = 6^3$

(i) All dice show the same face in one of the following mutually exclusive ways :

$$(1, 1, 1), (2, 2, 2), (3, 3, 3,), (4, 4, 4), (5, 5, 5), (6, 6, 6)$$

So, favourable number of elementary events = 6.

Hence, required probability $= \frac{6}{6^3} = \frac{1}{36}$

(ii) The total number of ways in which all dice show different faces is equal to the number of ways of arranging 6 distinct objects by taking three at a time i.e. ${}^6C_3 \times 3!$.

So, favourable number of elementary events $= {}^6C_3 \times 3!$

Hence, required probability $= \frac{{}^6C_3 \times 3!}{6^3} = \frac{5}{9}$

(iii) Select a number which occurs on two dice out of the six numbers 1, 2, 3, 4, 5, 6 marked on the six faces of a die. This can be done in ${}^6C_1$ ways. Now, select a number from the remaining 5 numbers which occurs on the remaining one die. This can be done in ${}^5C_1$ ways. Now, we have three numbers like 1, 1, 2 ; 2, 2, 5 etc. These three digits can be arranged in $\frac{3!}{2!}$ ways.

So, the favourable number of elementary events $= {}^6C_1 \times {}^5C_1 \times \frac{3!}{2!}$

Hence, required probability $= \frac{90}{6^3} = \frac{5}{12}$

EXAMPLE 51 *What is the probability that in a group of*
(i) *2 people, both will have the same birth-day?* (ii) *3 people, at least two will have the same birth-day? assuming that there are 365 days in a year and no one has his/her birth day on 29th February.*

SOLUTION (i) First person may have any one of the 365 days of the year as a birth day. Similarly, second person may have any one of 365 days of the year as a birth day.

So, the total number of ways in which two persons may have their birth days = $365 \times 365 = 365^2$

The number of ways in which two persons have the same birth-day = 365.

Hence, required probability $= \frac{365}{365^2} = \frac{1}{365}$

(ii) Let A be the event "At least two people have the same birth day". Then,

$\overline{A}$ = No two or more people have the same birth day = All the three persons have distinct birth-days.

$$\therefore \quad P(\overline{A}) = \frac{365 \times 364 \times 363}{365^3} = \frac{364 \times 363}{365^2}$$

Hence, required probability $= 1 - P(\overline{A}) = 1 - \frac{364 \times 363}{365^2}$

EXAMPLE 52 *If n biscuits are distributed among N beggars, find the chance that a particular beggar will get r (< n) biscuits.*

SOLUTION Since a biscuit can be given to any one of N beggars. Therefore, each biscuit can be distributed in N ways.

So, total number of ways of distributing n biscuits among N beggars $= \underbrace{N \times N \times \ldots \times N}_{n-\text{times}} = N^n$.

Now, r biscuits can be given to a particular beggar in ${}^nC_r$ ways and the remaining $(n-r)$ biscuits can be distributed to $(N-1)$ beggars in $(N-1)^{n-r}$ ways. Thus, the number of ways in which a particular beggar receives r biscuits is ${}^nC_r \times (N-1)^{n-r}$.

Hence, required probability $= \dfrac{{}^nC_r \times (N-1)^{n-r}}{N^n}$

EXAMPLE 53 *The letters of word 'SOCIETY' are placed at random in a row. What is the probability that three vowels come together?*

SOLUTION There are 7 letters in the word 'Society'. These 7 letters can be arranged in a row in 7! ways.

$\therefore$ Total number of elementary events = 7!

There are 3 vowels viz. O, I, E in word 'SOCIETY'. Considering these three vowels as one letter we have 5 letters which can be arranged in a row in 5! ways. But, three vowels O, I, E can be put together in 3! ways. Therefore, the total number of arrangements in which three vowels come together is $5! \times 3!$.

So, favourable number of elementary events $= 5! \times 3!$

Hence, required probability $= \dfrac{5! \times 3!}{7!} = \dfrac{1}{7}$

EXAMPLE 54 *Find the probability that in a random arrangement of the letters of the word 'UNIVERSITY' the two I's come together.*

SOLUTION The total number of words which can be formed by permuting the letters of the word 'UNIVERSITY' is $\dfrac{10!}{2!}$ as there are two I's.

$\therefore$ Total number of elementary events $= \dfrac{10!}{2!}$

Regarding $2I$'s as one letter, number of ways of arrangement in which both I's are together = 9!

$\therefore$ Favourable number of elementary events = 9!

Hence, required probability $= \dfrac{9!}{10!/2!} = \dfrac{1}{5}$

EXAMPLE 55 *Six new employees, two of whom are married to each other, are to be assigned six desks that are lined up in a row. If the assignment of employees to desks is made randomly, what is the probability that the married couple will have non-adjacent desks?* **[NCERT EXEMPLAR]**

SOLUTION Six employees can be seated in row in six desks in 6! ways. Married couple can occupy adjacent seats in the following 5 ways.

$1-2, 2-3, 3-4, 4-5, 5-6,$

Also, they can interchange their seats and the remaining 4 seats can be occupied by remaining 4 employees in 4! ways.

$\therefore$ Number of ways in which married couple will have adjacent seats $= 5 \times 2! \times 4!$

So, Number of ways in which married couple will have non-adjacent seats $= 6! - 5 \times 2! \times 4!$
$= 480.$

Hence, required probability $= \dfrac{480}{720} = \dfrac{2}{3}$

EXAMPLE 56 *In how many ways, can three girls and nine boys be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls should sit together in a back row on adjacent seats? Now, if all the seating arrangements are equally likely, what is the probability of 3 girls sitting together in a back row on adjacent seats?*

SOLUTION Each van has 7 seats. So, there are 14 numbered seats in two vans.

The total number of ways in which 3 girls and 9 boys can sit on these seats is $^{14}C_{12} \times 12!$

So, total number of seating arrangements $= {}^{14}C_{12} \times 12!$

In a van 3 girls can choose adjacent seats in the back row in two ways (1, 2, 3, or 2, 3, 4). So, the number of ways in which 3 girls can sit in the back row on adjacent seats is 2 (3!) ways. The number of ways in which 9 boys can sit on the remaining 11 seats is $^{11}C_9 \times 9!$ ways.

So, the number of ways in which 3 girls and 9 boys can sit in two vans

$$= 2(3!) \times {}^{11}C_9 \times 9! + 2(3!) \times {}^{11}C_9 \times 9!$$

Hence, required probability $= \dfrac{4 \times 3! \times {}^{11}C_9 \times 9!}{{}^{14}C_{12} \times 12!} = \dfrac{1}{91}$

EXAMPLE 57 *Five persons entered the lift cabin on the ground floor of an 8-floor house. Suppose that each of them independently and with equal probability can leave the cabin at any floor beginning with the first. Find out the probability of all five persons leaving at different floors.*

SOLUTION Besides ground floor, there are 7 floors. Since a person can leave the cabin at any of the seven floors, therefore there are 7 ways for a person to leave the lift cabin. There are five persons in the cabin and each can leave the cabin in 7 ways. Therefore,

The total number of ways in which 5 persons can leave the cabin $= 7 \times 7 \times 7 \times 7 \times 7 \times = 7^5$

$\therefore$ Total number of elementary events $= 7^5$

The total number of ways in which five persons can leave the lift cabin at different floors is same as the number of arrangements of 7 by taking 5 at a time i.e., $^7C_5 \times 5!$

$\therefore$ Favourable number of elementary events $= {}^7C_5 \times 5!$

Hence, required probability $= \dfrac{{}^7C_5 \times 5!}{7^5}$

EXAMPLE 58 *If n persons are seated on a round table, what is the probability that two named individuals will be neighbours?*

SOLUTION Total number of ways in which n persons can sit on a round table is $(n-1)!$.

$\therefore$ Total number of elementary events $= (n-1)!$

Considering two named individuals as one person there are $(n-1)$ persons who can sit on a round table in $(n-2)!$ ways. But, two named individual can be seated together in 2! ways.

$\therefore$ Favourable number of elementary events $= (n-2)! \times 2!$

So, required probability $= \dfrac{(n-2)! \times 2!}{(n-1)!} = \dfrac{2}{n-1}$

EXAMPLE 59 *There are 4 letters and 4 addressed envelopes. Find the probability that all the letters are not dispatched in right envelopes.*

SOLUTION Four letters can be put in four addressed envelopes in 4! ways.

$\therefore$ Total number of elementary events = 4!

All four letters can be put in correct envelopes in exactly one way.

$\therefore$ Probability that all four letters are put in correct envelopes $= \frac{1}{4!}$

Hence, required probability $= 1 - \frac{1}{4!} = \frac{23}{24}$

EXAMPLE 60 *Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary six faced die. Find the probability that the equation will have real roots.*

SOLUTION Since each of the coefficients a, b and c can take the values from 1 to 6.

$\therefore$ Total number of equations $= 6 \times 6 \times 6 = 216$.

The roots of the equation $ax^2 + bx + c = 0$ will be real if $b^2 - 4ac \geq 0$ i.e. $b^2 \geq 4ac$.

The favourable number of elementary events can be enumerated as follows:

ac	*a*	*c*	*4ac*	*b* (so that $b^2 \geq 4ac$)	*No. of ways*
1	1	1	4	2, 3, 4, 5, 6	$1 \times 5 = 5$
2	1 2	2 1	8	3, 4, 5, 6	$2 \times 4 = 8$
3	1 3	3 1	12	4, 5, 6	$2 \times 3 = 6$
4	1 4 2	4 1 2	16	4, 5, 6	$3 \times 3 = 9$
5	1 5	5 1	20	5, 6	$2 \times 2 = 4$
6	1 6 2 3	6 1 3 2	24	5, 6	$4 \times 2 = 8$
7	*ac* is not possible				0
8	2 4	4 2	32	6	$2 \times 1 = 2$
9	3	3	36	6	1
					Total $= 43$

Since $b^2 \geq 4ac$ and since the maximum value of b^2 is 36, therefore ac = 10, 11, 12 ... etc. is not possible.

$\therefore$ Total number of favourable elementary events = 43.

Hence, required probability $= \frac{43}{216}$.

EXAMPLE 61 *Two numbers b and c are chosen at random with replacement from the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9. Find the probability that $x^2 + bx + c > 0$ fro all $x \in R$.*

SOLUTION Since *b* and *c* both can assume value from 1 to 9. So, total numbers of ways of choosing *b* and *c* is $9 \times 9 = 81$.

Now, $x^2 + bx + c > 0$ for all $x \in R$

$\Rightarrow$ Disc < 0 *i.e.* $b^2 - 4c < 0$

The following table shows the possible values of *b* and *c* for which $b^2 - 4c < 0$

c	b	Total
1	1	1
2	1, 2	2
3	1, 2, 3	3
4	1, 2, 3	3
5	1, 2, 3, 4	4
6	1, 2, 3, 4	4
7	1, 2, 3, 4, 5	5
8	1, 2, 3, 4, 5	5
9	1, 2, 3, 4, 5	5
		32

So, favourable number of favourable elementary events = 32

Hence, required probability = 32/81.

EXAMPLE 62 *Three squares of Chess board are selected at random. Find the probability of getting 2 squares of one colour and other of a different colour.* [NCERT EXEMPLAR]

SOLUTION In a Chess board, there are 64 squares of which 32 are white and 32 are black. Out of 64 squares 3 square can be chosen in ${}^{64}C_3$. Since 2 of one colour and 1 ways of other colour can be 2W, 1B or 1W, 2B. Therefore, the number of ways of selecting 2 squares of one colour and one other colour is ${}^{32}C_2 \times {}^{32}C_1 + {}^{32}C_1 \times {}^{32}C_2 = 2\,({}^{32}C_2 \times {}^{32}C_1)$

$\therefore$ Favourable number of ways $= 2\,({}^{32}C_2 \times {}^{32}C_1)$

Hence, required probability $= \dfrac{2\,({}^{32}C_2 \times {}^{32}C_1)}{{}^{64}C_3} = \dfrac{16}{21}$.

EXERCISE 33.3

LEVEL-1

1. Which of the following cannot be valid assignment of probability for elementary events or outcomes of sample space $S = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$:

Elementary events:	w_1	w_2	w_3	w_4	w_5	w_6	w_7
(i)	0.1	0.01	0.05	0.03	0.01	0.2	0.6
(ii)	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
(iii)	0.7	0.6	0.5	0.4	0.3	0.2	0.1
(iv)	$\frac{1}{14}$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{6}{14}$	$\frac{15}{14}$

2. A die is thrown. Find the probability of getting:
   (i) a prime number (ii) 2 or 4 (iii) a multiple of 2 or 3.
3. In a simultaneous throw of a pair of dice, find the probability of getting:
   (i) 8 as the sum (ii) a doublet
   (iii) a doublet of prime numbers (iv) a doublet of odd numbers
   (v) a sum greater than 9 (vi) an even number on first
   (vii) an even number on one and a multiple of 3 on the other
   (viii) neither 9 nor 11 as the sum of the numbers on the faces
   (ix) a sum less than 6 (x) a sum less than 7
   (xi) a sum more than 7 (xii) neither a doublet nor a total of 10

(xiii) odd number on the first and 6 on the second

(xiv) a number greater than 4 on each die

(xv) a total of 9 or 11 (xvi) a total greater than 8.

4. In a single throw of three dice, find the probability of getting a total of 17 or 18.

5. Three coins are tossed together. Find the probability of getting:
   (i) exactly two heads (ii) at least two heads
   (iii) at least one head and one tail.

6. What is the probability that an ordinary year has 53 Sundays?

7. What is the probability that a leap year has 53 Sundays and 53 Mondays?

8. A bag contains 8 red and 5 white balls. Three balls are drawn at random. Find the probability that:
   (i) All the three balls are white. (ii) All the three balls are red.
   (iii) One ball is red and two balls are white. [NCERT EXEMPLAR]

9. In a single throw of three dice, find the probability of getting the same number on all the three dice.

10. Two unbiased dice are thrown. Find the probability that the total of the numbers on the dice is greater than 10.

11. A card is drawn at random from a pack of 52 cards. Find the probability that the card drawn is:
    (i) a black king (ii) either a black card or a king
    (iii) black and a king (iv) a jack, queen or a king
    (v) neither a heart nor a king (vi) spade or an ace
    (vii) neither an ace nor a king (viii) a diamond card
    (ix) not a diamond card (x) a black card
    (xi) not an ace (xii) not a black card.

12. In shuffling a pack of 52 playing cards, four are accidently dropped; find the chance that the missing cards should be one from each suit.

13. From a deck of 52 cards, four cards are drawn simultaneously, find the chance that they will be the four honours of the same suit.

14. Tickets numbered from 1 to 20 are mixed up together and then a ticket is drawn at random. What is the probability that the ticket has a number which is a multiple of 3 or 7?

15. A bag contains 6 red, 4 white and 8 blue balls. If three balls are drawn at random, find the probability that one is red, one is white and one is blue.

16. A bag contains 7 white, 5 black and 4 red balls. If two balls are drawn at random, find the probability that: (i) both the balls are white (ii) one ball is black and the other red (iii) both the balls are of the same colour.

17. A bag contains 6 red, 4 white and 8 blue balls. If three balls are drawn at random, find the probability that: (i) one is red and two are white (ii) two are blue and one is red (iii) one is red.

18. Five cards are drawn from a pack of 52 cards. What is the chance that these 5 will contain:
    (i) just one ace (ii) at least one ace?

19. The face cards are removed from a full pack. Out of the remaining 40 cards, 4 are drawn at random. What is the probability that they belong to different suits?

20. There are four men and six women on the city councils. If one council member is selected for a committee at random, how likely is that it is a women? [NCERT]

21. A box contains 100 bulbs, 20 of which are defective. 10 bulbs are selected for inspection. Find the probability that: (i) all 10 are defective (ii) all 10 are good (iii) at least one is defective (iv) none is defective

22. Find the probability that in a random arrangement of the letters of the word 'SOCIAL' vowels come together.

23. The letters of the word 'CLIFTON' are placed at random in a row. What is the chance that two vowels come together?

24. The letters of the word 'FORTUNATES' are arranged at random in a row. What is the chance that the two 'T' come together.

25. A committee of two persons is selected from two men and two women. What is the probability that the committee will have (i) no man ? (ii) one man? (iii) two men? [NCERT]

26. If odds in favour of an event be 2 : 3, find the probability of occurrence of this event.

27. If odds against an event be 7 : 9, find the probability of non-occurrence of this event.

28. Two balls are drawn at random from a bag containing 2 white, 3 red, 5 green and 4 black balls, one by one without, replacement. Find the probability that both the balls are of different colours.

29. Two unbiased dice are thrown. Find the probability that:
    (i) neither a doublet nor a total of 8 will appear
    (ii) the sum of the numbers obtained on the two dice is neither a multiple of 2 nor a multiple of 3

30. A bag contains 8 red, 3 white and 9 blue balls. If three balls are drawn at random, determine the probability that (i) all the three balls are blue balls (ii) all the balls are of different colours.

31. A bag contains 5 red, 6 white and 7 black balls. Two balls are drawn at random. What is the probability that both balls are red or both are black?

32. If a letter is chosen at random from the English alphabet, find the probability that the letter is (i) a vowel (ii) a consonant

33. In a lottery, a person chooses six different numbers at random from 1 to 20, and if these six numbers match with six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prize in the game? [NCERT]

34. 20 cards are numbered from 1 to 20. One card is drawn at random. What is the probability that the number on the cards is: (i) a multiple of 4? (ii) not a multiple of 4? (iii) odd? (iv) greater than 12? (v) divisible by 5? (vi) not a multiple of 6?

35. Two dice are thrown. Find the odds in favour of getting the sum
    (i) 4 (ii) 5 (iii) What are the odds against getting the sum 6?

36. What are the odds in favour of getting a spade if a card is drawn from a well-shuffled deck of cards? What are the odds in favour of getting a king?

37. A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn at random. From the box, what is the probability that: (i) all are blue? (ii) at least one is green?

38. A box contains 6 red marbles numbered 1 through 6 and 4 white marbles numbered from 12 through 15. Find the probability that a marble drawn is (i) white (ii) white and odd numbered (iii) even numbered (iv) red or even numbered.

39. A class consists of 10 boys and 8 girls. Three students are selected at random. What is the probability that the selected group has (i) all boys? (ii) all girls? (iii) 1 boy and 2 girls? (iv) at least one girl? (v) at most one girl?

40. Five cards are drawn from a well-shuffled pack of 52 cards. Find the probability that all the five cards are hearts.

41. A bag contains tickets numbered from 1 to 20. Two tickets are drawn. Find the probability that (i) both the tickets have prime numbers on them (ii) on one there is a prime number and on the other there is a multiple of 4.

42. An urn contains 7 white, 5 black and 3 red balls. Two balls are drawn at random. Find the probability that (i) both the balls are red (ii) one ball is red and the other is black (iii) one ball is white.

LEVEL-2

43. A and B throw a pair of dice. If A throws 9, find B's chance of throwing a higher number.

44. In a hand at Whist, what is the probability that four kings are held by a specified player?

45. Find the probability that in a random arrangement of the letters of the word 'UNIVERSITY', the two I's do not come together.

ANSWERS

1. (i), (ii) 2. (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ (iii) $\frac{2}{3}$ 3. (i) $\frac{5}{36}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{12}$ (iv) $\frac{1}{12}$ (v) $\frac{1}{6}$ (vi) $\frac{1}{2}$ (vii) $\frac{11}{36}$ (viii) $\frac{5}{6}$ (ix) $\frac{5}{18}$ (x) $\frac{5}{12}$ (xi) $\frac{5}{12}$ (xii) $\frac{7}{9}$ (xiii) $\frac{1}{12}$ (xiv) $\frac{1}{9}$ (xv) $\frac{1}{6}$ (xvi) $\frac{5}{18}$ 4. $\frac{1}{54}$ 5. (i) $\frac{3}{8}$ (ii) $\frac{1}{2}$ (iii) $\frac{3}{4}$

6. $\frac{1}{7}$ 7. $\frac{1}{7}$ 8. (i) $\frac{5}{143}$ (ii) $\frac{28}{143}$ (iii) $\frac{40}{143}$ 9. $\frac{1}{36}$ 10. $\frac{1}{12}$

11. (i) $\frac{1}{26}$ (ii) $\frac{7}{13}$ (iii) $\frac{1}{26}$ (iv) $\frac{3}{13}$ (v) $\frac{9}{13}$ (vi) $\frac{4}{13}$ (vii) $\frac{11}{13}$ (viii) $\frac{1}{4}$ (ix) $\frac{3}{4}$ (x) $\frac{1}{2}$ (xi) $\frac{12}{13}$ (xii) $\frac{1}{2}$

12. $\frac{2197}{20825}$ 13. $\frac{4}{270725}$ 14. $\frac{2}{5}$ 15. $\frac{4}{17}$

16. (i) $\frac{7}{40}$ (ii) $\frac{1}{6}$ (iii) $\frac{37}{120}$ 17. (i) $\frac{3}{68}$ (ii) $\frac{7}{34}$ (iii) $\frac{33}{68}$

18. (i) $\frac{3243}{10829}$ (ii) $\frac{18472}{54145}$ 19. $\frac{1000}{9139}$ 20. $\frac{3}{5}$ 21. (i) $\frac{{}^{20}C_{10}}{{}^{100}C_{10}}$ (ii) $\frac{{}^{80}C_{10}}{{}^{100}C_{10}}$ (iii) $1-\frac{{}^{80}C_{10}}{{}^{100}C_{10}}$ (iv) $\frac{{}^{80}C_{10}}{{}^{100}C_{10}}$ 22. $\frac{1}{5}$ 23. $\frac{2}{7}$

24. $\frac{1}{5}$ 25. (i) $\frac{1}{6}$ (ii) $\frac{2}{3}$ (iii) $\frac{1}{6}$ 26. $\frac{2}{5}$ 27. $\frac{7}{16}$ 28. 0.78 29. (i) $\frac{13}{18}$ (ii) $\frac{1}{3}$ 30. (i) $\frac{7}{95}$ (ii) $\frac{18}{95}$ 31. $\frac{31}{153}$ 32. (i) $\frac{5}{26}$ (ii) $\frac{21}{26}$

33. $\frac{1}{38760}$ 34. (i) $\frac{1}{4}$ (ii) $\frac{3}{4}$ (iii) $\frac{1}{2}$ (iv) $\frac{2}{5}$ (v) $\frac{1}{5}$ (vi) $\frac{17}{20}$ 35. (i) 1 : 11 (ii) 1 : 8 (iii) 31 : 5 36. (i) 1 : 3, 1 : 12

37. (i) $\frac{{}^{20}C_5 \times {}^{40}C_0}{{}^{60}C_5} = \frac{34}{11977}$ (ii) $\frac{4367}{4484}$ 38. (i) $\frac{2}{5}$ (ii) $\frac{1}{5}$ (iii) $\frac{1}{2}$

(iv) $\frac{4}{5}$ 39. (i) $\frac{5}{34}$ (ii) $\frac{7}{102}$ (iii) $\frac{35}{102}$ (iv) $\frac{29}{34}$ (v) $\frac{10}{17}$

40. $\frac{{}^{13}C_5}{{}^{52}C_5} = \frac{33}{66640}$ 41. (i) $\frac{14}{95}$ (ii) $\frac{4}{19}$ 42. (i) $\frac{1}{35}$ (ii) $\frac{1}{7}$ (iii) $\frac{8}{15}$

43. $\frac{1}{6}$ 44. $\frac{11}{4165}$ 45. $\frac{4}{5}$

HINTS TO NCERT & SELECTED PROBLEMS

12. Required probability $= \frac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4}$

13. Four honours means king, queen, jack and ace.

So required probability $= \frac{{}^4C_4 + {}^4C_4 + {}^4C_4 + {}^4C_4}{{}^{52}C_4}$

14. Total number of elementary events $= {}^{20}C_1 = 20.$

A multiple of 3 or 7 can be obtained as follows: 3, 6, 9, 12, 15, 18, 7, 14.

So, favourable number of elementary events $= {}^8C_1 = 8.$

15. Total number of elementary events $= {}^{18}C_3.$

Favourable number of elementary events $= {}^6C_1 \times {}^4C_1 \times {}^8C_1$

20. Out of 4 men and 6 women one person can be chosen in ${}^{10}C_1 = 10$ ways.

The number of ways of selecting 1 women out of 6 women $= {}^6C_1 = 6.$

$\therefore$ Required probability $= \frac{6}{10} = \frac{3}{5}$

25. A committee of two persons can be formed from two men and two women in ${}^4C_2 = 6$ ways.

(i) Number of committees having no man $= {}^2C_2 = 1$

$\therefore$ Probability that a committee has no man $= \frac{1}{6}$

(ii) Number of committees having one man $= {}^2C_1 \times {}^2C_1 = 4$

$\therefore$ Probability that the committee will have one man $= \frac{4}{6} = \frac{2}{3}$

$\therefore$ Probability that the committee has two men $= \frac{1}{6}$

33. Total number of ways of selecting six numbers from numbers 1 to 20 $= {}^{20}C_6$

$\therefore$ Total number of elementry events $= {}^{20}C_6 = 38760$

Favourable number of elementary events $= 1$

$\therefore$ Required probability $= \frac{1}{38760}$

44. Total number of elementary events $= {}^{52}C_{13}$

Favourable number of elementary events $= {}^{48}C_9 \times {}^4C_4$

33.8 ADDITION THEOREMS ON PROBABILITY

Uptill now we have been computing the probability of occurrence or non-occurrence of an event by using favourable and total number of elementary events. But it is not always convenient to compute favourable number of elementary events to a given event. In such cases, we express the given event as the union of two or more events and the probability of the given event is expressed in terms of the probabilities of these events. Theorems which express the probability

of an event in terms of the probabilities of those events whose union is the given event are known as addition theorems on probability. In this section, we shall discuss addition theorems for two or more events.

THEOREM 1 *(Addition Theorem for two events) If A and B are two events associated with a random experiment, then* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

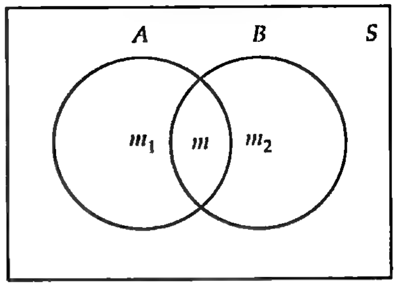


Fig. 33.3

PROOF Let S be the sample space associated with the given random experiment. Suppose the random experiment results in n mutually exclusive ways. Then, S contains n elementary events.

Let m_1, m_2 and m be the number of elementary events favourable to A, B and $A \cap B$ respectively. Then,

$$P(A) = \frac{m_1}{n},\ P(B) = \frac{m_2}{n} \text{ and } P(A \cap B) = \frac{m}{n}.$$

The number of elementary events favourable to A only is $m_1 - m$. Similarly, the number of elementary events favourable to B only is $m_2 - m$. Since m elementary events are favourable to both A and B. Therefore, the number of elementary events favourable to A or B or both i.e. $A \cup B$ is $m_1 - m + m_2 - m + m = m_1 + m_2 - m$.

So, $P(A \cup B) = \dfrac{m_1 + m_2 - m}{n} = \dfrac{m_1}{n} + \dfrac{m_2}{n} - \dfrac{m}{n}$

$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$ Q.E.D.

COROLLARY *If A and B are mutually exclusive events, then* $P(A \cap B) = 0$.

$\therefore \quad P(A \cup B) = P(A) + P(B)$

This is the addition theorem for mutually exclusive events.

THEOREM 2 *(Addition Theorem for three events) If A, B, C are three events associated with a random experiment, then*

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

PROOF Let $D = B \cup C$. Then,

$P(A \cup B \cup C) = P(A \cup D) = P(A) + P(D) - P(A \cap D)$ ...(i)

But, $A \cap D = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$\therefore \quad P(A \cap D) = P[(A \cap B) \cup (A \cap C)]$

$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$

$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$ ...(ii)

and, $P(D) = P(B \cup C) = P(B) + P(C) - P(B \cap C)$ ...(iii)

From (i), (ii) and (iii), we get

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$

$\Rightarrow P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Q.E.D.

COROLLARY *If A, B, C are mutually exclusive events, then*

$P(A \cap B) = P(B \cap C) = P(A \cap C) = P(A \cap B \cap C) = 0.$

$\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C)$

This is the addition theorem for three mutually exclusive events.

THEOREM 3 *Let A and B be two events associated to a random experiment. Then,*

(i) $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

(ii) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

(iii) $P((A \cap \overline{B}) \cup (\overline{A} \cap B)) = P(A) + P(B) - 2P(A \cap B)$

PROOF (i) Since $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive events such that

$$(A \cap B) \cup (\overline{A} \cap B) = B$$

$$\therefore \quad P(A \cap B) + P(\overline{A} \cap B) = P(B)$$

$$\Rightarrow \quad P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

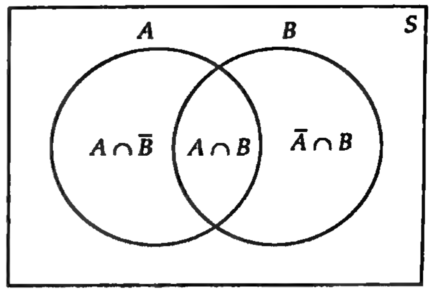


Fig. 33.4

(ii) Since $A \cap B$ and $A \cap \overline{B}$ are mutually exclusive events such that

$$(A \cap B) \cup (A \cap \overline{B}) = A$$

$$\therefore \quad P(A \cap B) + P(A \cup \overline{B}) = P(A)$$

$$\Rightarrow \quad P(A \cap \overline{B}) = P(A) - P(A \cap B).$$

(iii) Since $A \cap \overline{B}$ and $\overline{A} \cap B$ are mutually exclusive events. Therefore,

$$P\{(A \cap \overline{B}) \cup (\overline{A} \cap B)\} = P(A \cap \overline{B}) + P(\overline{A} \cap B)$$

$$= P(A) - P(A \cap B) + P(B) - P(A \cap B) \qquad \text{[Using (i) and (ii)]}$$

$$= P(A) + P(B) - 2P(A \cap B)$$

REMARK 1 $P(\overline{A} \cap B)$ is known as the probability of occurrence of B only. **Q.E.D.**

REMARK 2 $P(A \cap \overline{B})$ is known as the probability of occurrence of A only.

REMARK 3 $P\{(A \cap \overline{B}) \cup (\overline{A} \cap B)\}$ is known as the probability of occurrence of exactly one of two events A and B.

REMARK 4 *If A and B are two events associated to a random experiment such that $A \subset B$, then $\overline{A} \cap B \neq \phi$.*

$$\therefore \quad P(\overline{A} \cap B) \geq 0 \Rightarrow P(B) - P(A \cap B) \geq 0 \Rightarrow P(B) - P(A) \geq 0 \Rightarrow P(A) \leq P(B).$$

THEOREM 4 *For any two events A and B, prove that*

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B).$$

PROOF Since $A \cap B \subset A$. Therefore, we have

$$P(A \cap B) \leq P(A). \quad \text{...(i)}$$

Also, $A \subset A \cup B \Rightarrow P(A) \leq P(A \cup B)$ ...(ii)

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow \quad P(A \cup B) \leq P(A) + P(B) \quad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B).$$

Q.E.D.

THEOREM 5 *For any two events A and B, prove that the probability that exactly one of A, B occurs is given by $P(A) + P(B) - 2P(A \cup B) = P(A \cup B) - P(A \cap B)$.*

PROOF We have,

$$P(\text{Exactly one of } A, B \text{ occurs}) = P\{(A \cap \overline{B}) \cup (\overline{A} \cap B)\}$$

$$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$$

$$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - 2P(A \cap B)$$

$$= [P(A) + P(B) - P(A \cap B)] - P(A \cap B)$$

$$= P(A \cup B) - P(A \cap B).$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS BASED UPON FORMULAE**

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(ii) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

EXAMPLE 1 *Given* $P(A) = \frac{3}{5}$ *and* $P(B) = \frac{1}{5}$. *Find* $P(A \text{ or } B)$, *if A and B are mutually exclusive events.*

SOLUTION Since A and B are mutually exclusive events.

$\therefore \quad P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) = \frac{3}{5} + \frac{1}{5} = \frac{4}{5}$

EXAMPLE 2 *A and B are two mutually exclusive events of an experiment. If* $P(\text{'not } A\text{'}) = 0.65$, $P(A \cup B) = 0.65$ and $P(B) = p$, *find the value of p.*

SOLUTION By addition theorem for mutually exclusive events, we have

$$P(A \cup B) = P(A) + P(B)$$

$\Rightarrow \quad P(A \cup B) = 1 - P(\text{'not } A\text{'}) + P(B) \qquad [\because P(A) = 1 - P(\overline{A})]$

$\Rightarrow \quad 0.65 = 1 - 0.65 + p \Rightarrow p = 0.30$

EXAMPLE 3 *A and B are two non-mutually exclusive events. If* $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{5}$ *and* $P(A \cup B) = \frac{1}{2}$, *find the values of* $P(A \cap B)$ and $P(A \cap \overline{B})$.

SOLUTION We have, $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{1}{2}$

By addition theorem, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$\Rightarrow \quad \frac{1}{2} = \frac{1}{4} + \frac{2}{5} - P(A \cap B)$

$\Rightarrow \quad P(A \cap B) = \frac{1}{4} + \frac{2}{5} - \frac{1}{2} = \frac{3}{20}$

$\therefore \quad P(A \cap \overline{B}) = P(A) - P(A \cap B) = \frac{1}{4} - \frac{3}{20} = \frac{1}{10}$

EXAMPLE 4 *If E and F are two events such that* $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ *and* $P(E \text{ and } F) = \frac{1}{8}$, *find* *(i)* $P(E \text{ or } F)$ *(ii)* $P(\text{not } E \text{ and not } F)$.

SOLUTION We have,

$$P(E) = \frac{1}{4},\ P(F) = \frac{1}{2} \text{ and } P(E \cap F) = \frac{1}{8}$$

(i) $\quad P(E \text{ or } F) = P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8}$

(ii) $\quad P(\text{not } E \text{ and not } F) = P(\overline{E} \cap \overline{F})$

$= P(\overline{E \cup F})$

$= 1 - P(E \cup F) = 1 - \{P(E) + P(F) - P(E \cap F)\}$

$= 1 - \left\{\frac{1}{4} + \frac{1}{2} - \frac{1}{8}\right\} = 1 - \frac{5}{8} = \frac{3}{8}$

EXAMPLE 5 *The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then find* $P(\overline{A}) + P(\overline{B})$.

SOLUTION We have,

P (At least one of the events A and B occurs) = 0.6 i.e. $P(A \cup B) = 0.6$

and, P (A and B occur simultaneously) = 0.2 i.e. $P(A \cap B) = 0.2$

Now,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \Rightarrow \quad 0.6 &= P(A) + P(B) - 0.2 \\ \Rightarrow \quad 0.6 &= 1 - P(\bar{A}) + 1 - P(\bar{B}) - 0.2 \\ \Rightarrow \quad 0.6 &= 2 - 0.2 - [P(\bar{A}) + P(\bar{B})] \\ \Rightarrow \quad 0.6 &= 1.8 - [P(\bar{A}) + P(\bar{B})] \\ \Rightarrow \quad P(\bar{A}) + P(\bar{B}) &= 1.8 - 0.6 = 1.2 \end{aligned}$$

EXAMPLE 6 *Check whether the following probabilities P (A) and P (B) are consistently defined:*

(i) $P(A) = 0.5$, $P(B) = 0.7$, $P(A \cap B) = 0.6$ (ii) $P(A) = 0.5$, $P(B) = 0.4$, $P(A \cup B) = 0.8$

SOLUTION (i) We have, $P(A) = 0.5$, $P(B) = 0.7$, $P(A \cap B) = 0.6$

We know that $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$. But, for the given probabilities $P(A \cap B) \nleq P(A)$. So, given probabilities are not consistently defined.

(ii) We have,

$$P(A) = 0.5,\ P(B) = 0.4 \text{ and } P(A \cup B) = 0.8$$

$$\therefore \quad P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.5 + 0.4 - 0.8 = 0.1$$

Clearly, $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$. Hence, the given probabilities are consistently defined.

EXAMPLE 7 *Events E and F are such that P (not E or not F) = 0.25. State whether E and F are mutually exclusive.*

SOLUTION We have, P (not E or not F) = 0.25

i.e. $P(\bar{E} \cup \bar{F}) = 0.25 \Rightarrow P(\overline{E \cap F}) = 0.25 \Rightarrow 1 - P(E \cap F) = 0.25 \Rightarrow P(E \cap F) = 1 - 0.25 = 0.75 \neq 0$

Hence, E and F are not mutually exclusive.

EXAMPLE 8 *A, B, C are three mutually exclusive and exhaustive events associated with a random experiment. Find P(A), it being given that* $P(B) = \frac{3}{2} P(A)$ *and* $P(C) = \frac{1}{2} P(B)$.

SOLUTION Let $P(A) = p$. Then,

$$P(B) = \frac{3}{2} P(A) \Rightarrow P(B) = \frac{3}{2} p \text{ and } P(C) = \frac{1}{2} P(B) \Rightarrow P(C) = \frac{3}{4} p$$

Since A, B, C are mutually exclusive and exhaustive events associated with a random experiment.

$$\begin{aligned} \therefore \quad A \cup B \cup C &= S \\ \Rightarrow \quad P(A \cup B \cup C) &= P(S) \\ \Rightarrow \quad P(A \cup B \cup C) &= 1 && [\because P(S) = 1] \\ \Rightarrow \quad P(A) + P(B) + P(C) &= 1 && [\text{By addition Theorem}] \\ \Rightarrow \quad p + \frac{3}{2} p + \frac{3}{4} p = 1 \Rightarrow p &= \frac{4}{13} \Rightarrow P(A) = \frac{4}{13} \end{aligned}$$

EXAMPLE 9 ***Four candidates A, B, C, D have applied for the assignment to coach a school cricket team. If A is twice as likely to be selected as B, and B and C are given about the same chance of being selected, while C is twice as likely to be selected as D, what are the probabilities that (i) C will be selected? (ii) A will not be selected?*** **[NCERT EXEMPLAR]**

SOLUTION Let A_1, A_2, A_3 and A_4 be the events that candidates A, B, C and D respectively are selected as school cricket team coach. Then,

It is given that

$$P(A_3) = 2P(A_4), P(A_2) = P(A_3) \text{ and } P(A_1) = 2P(A_2)$$

$\Rightarrow \quad P(A_1) = 4P(A_4), P(A_2) = P(A_3) = 2P(A_4)$

Clearly, A_1, A_2, A_3 and A_4 are mutually exclusive and exhaustive events. Therefore,

$$A_1 \cup A_2 \cup A_3 \cup A_4 = S$$

$\Rightarrow \quad P(A_1 \cup A_2 \cup A_3 \cup A_4) = P(S)$

$\Rightarrow \quad P(A_1) + P(A_2) + P(A_3) + P(A_4) = 1$

$\Rightarrow \quad 4P(A_4) + 2P(A_4) + 2P(A_4) + P(A_4) = 1$

$\Rightarrow \quad 9P(A_4) = 1$

$\Rightarrow \quad P(A_4) = \frac{1}{9}.$

(i) Required probability $= P(A_3) = 2P(A_4) = \frac{2}{9}$

(ii) Required probability $= P(\overline{A}_1) = 1 - P(A_1) = 1 - 4P(A_4) = 1 - \frac{4}{9} = \frac{5}{9}$

EXAMPLE 10 *Probability that a truck stopped at a roadblock will have faulty brakes or badly worn tires are 0.23 and 0.24, respectively. Also, the probability is 0.38 that a truck stopped at the roadblock will have faulty brakes and or badly working tires. What is the probability that a truck stopped at this roadblock will have faulty brakes as well as badly worn tires?* [NCERT EXEMPLAR]

SOLUTION Let B be the event that a truck stopped at the roadblock will have faulty brakes and T be the event that it will have badly worn tires.

It is given that $P(B) = 0.23$, $P(T) = 0.24$ and $P(B \cup T) = 0.38$. We have to find $P(B \cap T)$.

We know that

$$P(B \cup T) = P(B) + P(T) - P(B \cap T) \quad \text{[By addition theorem]}$$

$\Rightarrow \quad P(B \cap T) = P(B) + P(T) - P(B \cup T) = 0.23 + 0.24 - 0.38 = 0.09$

EXAMPLE 11 *The probability of two events A and B are 0.25 and 0.50 respectively. The probability of their simultaneous occurrence is 0.14. Find the probability that neither A nor B occurs.*

SOLUTION We have, $P(A) = 0.25$, $P(B) = 0.50$ and $P(A \cap B) = 0.14$

$\therefore$ Required probability $= P(\overline{A} \cap \overline{B})$

$= P(\overline{A \cup B}) \quad [\because (\overline{A \cup B}) = \overline{A} \cap \overline{B}]$

$= 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - (0.25 + 0.50 - 0.14) = 0.39$

Type II **PROBLEMS BASED UPON ADDITION THEOREMS OF PROBABILITY BUT CAN BE SOLVED INDEPENDENTLY BY USING THE DEFINITION ONLY**

NOTE *Following problems will be solved by using addition theorems but these problems can be solved otherwise also. Students are advised to do these problems without using addition theorem.*

EXAMPLE 12 *Find the probability of getting an even number on the first die or a total of 8 in a single throw of two dice.*

SOLUTION Let S be the sample space associated with the experiment of throwing a pair of dice. Then, $n(S) = 36$.

$\therefore$ Total number of elementary events $= 36$

Let A and B be two events given by

A = Getting an even number on first die, B = Getting a total of 8.

Then, $A \cap B$ = Getting an even number on first die and a total of 8.

Clearly, $A = \{(2, 1), ..., (2, 6), (4, 1), (4, 2), ..., (4, 6), (6, 1), (6, 2), ..., (6, 6)\}$,

$B = \{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)\}$ and, $A \cap B = \{(2, 6), (6, 2), (4, 4)\}$.

$\therefore \quad P(A) = \frac{18}{36}, P(B) = \frac{5}{36}$ and $P(A \cap B) = \frac{3}{36}$

Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{18}{36} + \frac{5}{36} - \frac{3}{36} = \frac{20}{36} = \frac{5}{9}$

EXAMPLE 13 *A die is thrown twice. What is the probability that at least one of the two throws comes up with the number 4?*

SOLUTION Let S be the sample space associated with the random experiment of throwing a die twice. Then, $n(S) = 36$.

$\therefore$ Total number of elementary events = 36

Consider the events: A = First throw shows 4, B = Second throw shows 4

$\therefore \quad A \cap B$ = First and Second throw show 4 i.e. getting 4 in each throw.

Clearly,

$A = \{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)\}, B = \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4)\}$

and, $A \cap B = \{(4, 4)\}$

$\therefore \quad P(A) = \frac{6}{36}, P(B) = \frac{6}{36}$ and $P(A \cap B) = \frac{1}{36}$

$\therefore$ Required probability = Probability that at least one of the two throws shows 4.

$= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}$

EXAMPLE 14 *One number is chosen from numbers 1 to 200. Find the probability that it is divisible by 4 or 6?*

SOLUTION Let S be the sample space. Then, $n(S) = 200$.

$\therefore$ Total number of elementary events = 200

Let A be the event that the number selected is divisible by 4 and B be the event that the number selected is divisible by 6. Then,

$A = \{4, 8, 12, ..., 200\}, B = \{6, 12, ..., 198\}$ and $A \cap B = \{12, 24, ..., 192\}$

Clearly,

$n(A) = \frac{200}{4} = 50, \quad n(B) = \frac{198}{6} = 33$ and $n(A \cap B) = \frac{192}{12} = 16$

$\therefore \quad P(A) = \frac{50}{200}, \quad P(B) = \frac{33}{200}$ and $P(A \cap B) = \frac{16}{200}$

Required probability = P (a number is divisible by 4 or 6)

$= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{50}{200} + \frac{33}{200} - \frac{16}{200} = \frac{67}{200}$

EXAMPLE 15 *A card is drawn from a deck of 52 cards. Find the probability of getting a king or a heart or a red card.*

SOLUTION Consider the following events:

A = Getting a king, B = getting a heart card, C = Getting a red card.

Clearly,

$P(A) = \frac{{}^4C_1}{{}^{52}C_1} = \frac{4}{52}, \quad P(B) = \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{13}{52}, \quad P(C) = \frac{{}^{26}C_1}{{}^{52}C_1} = \frac{26}{52}$

$P(A \cap B) = P$ (Getting a king of heart) $= \frac{1}{52}$, $P(B \cap C) = P$ (Getting a heart card) $= \frac{13}{52}$

$P(C \cap A) = P$ (Getting a red king) $= \frac{2}{52}$, $P(A \cap B \cap C) = P$ (Getting a king of heart) $= \frac{1}{52}$

Required Probability $= P(A \cup B \cup C)$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{13}{52} - \frac{2}{52} + \frac{1}{52} = \frac{28}{52} = \frac{7}{13}$$

EXAMPLE 16 *A drawer contains 50 bolts and 150 nuts. Half of the bolts and half of the nuts are rusted. If one item is chosen at random, what is the probability that it is rusted or a bolt?*

SOLUTION Let A be the event that the item chosen is rusted and B be the event that the item chosen is a bolt. Clearly, there are 200 items in all, out of which 100 are rusted.

$\therefore$ $P(A) = \frac{100}{200}$, $P(B) = \frac{50}{200}$ and $P(A \cap B) = \frac{25}{200}$

Required probability $= P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B) = \frac{100}{200} + \frac{50}{200} - \frac{25}{200} = \frac{125}{200} = \frac{5}{8}$$

EXAMPLE 17 *Four cards are drawn at a time from a pack of 52 playing cards. Find the probability of getting all the four cards of the same suit.*

SOLUTION Since 4 cards can be drawn at a time from a pack of 52 cards in ${}^{52}C_4$ ways. Therefore,

Total number of elementary events $= {}^{52}C_4$

Consider the following events:

A = Getting all spade cards; B = Getting all club cards;

C = Getting all diamond cards, and D = Getting all heart cards.

Clearly, A, B, C and D are mutually exclusive events such that

$$P(A) = \frac{{}^{13}C_4}{{}^{52}C_4}, \; P(B) = \frac{{}^{13}C_4}{{}^{52}C_4}, \; P(C) = \frac{{}^{13}C_4}{{}^{52}C_4} \text{ and } P(D) = \frac{{}^{13}C_4}{{}^{52}C_4}$$

Required probability $= P(A \cup B \cup C \cup D)$

$= P(A) + P(B) + P(C) + P(D)$ [By addition Theorem]

$$= 4\left(\frac{{}^{13}C_4}{{}^{52}C_4}\right) = \frac{44}{4165}$$

EXAMPLE 18 *An integer is chosen at random from the numbers ranging from 1 to 50. What is the probability that the integer chosen is a multiple of 2 or 3 or 10?*

SOLUTION Out of 50 integers an integer can be chosen in ${}^{50}C_1$ ways.

$\therefore$ Total number of elementary events $= {}^{50}C_1 = 50$.

Consider the following events:

A = Getting a multiple of 2, B = Getting a multiple of 3 and, C = Getting a multiple of 10.

Clearly,

$A = \{2, 4, ..., 50\}$, $B = \{3, 6, ..., 48\}$, $C = \{10, 20, ..., 50\}$

$A \cap B = \{6, 12, ..., 48\}$, $B \cap C = \{30\}$, $A \cap C = \{10, 20, ..., 50\}$ and, $A \cap B \cap C = \{30\}$

$\therefore$ $P(A) = \frac{25}{50}$, $P(B) = \frac{16}{50}$, $P(C) = \frac{5}{50}$, $P(A \cap B) = \frac{8}{50}$, $P(B \cap C) = \frac{1}{50}$,

$P(A \cap C) = \frac{5}{50}$ and $P(A \cap B \cap C) = \frac{1}{50}$

Required probability $= P(A \cap B \cap C)$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap A) + P(A \cap B \cap C)$$

$$= \frac{25}{50} + \frac{16}{50} + \frac{5}{50} - \frac{8}{50} - \frac{1}{50} - \frac{5}{50} + \frac{1}{50} = \frac{33}{50}$$

EXAMPLE 19 *In an essay competition, the odds in favour of competitors P, Q, R, S are 1 : 2, 1 : 3, 1 : 4, and 1 : 5 respectively. Find the probability that one of them wins the competition.*

SOLUTION Let A, B, C, D be the events that the competitors P, Q, R and S respectively win the competition. Then,

$$P(A) = \frac{1}{3},\ P(B) = \frac{1}{4},\ P(C) = \frac{1}{5} \text{ and } P(D) = \frac{1}{6}$$

Since only one competitor can win the competition. Therefore, A, B, C, D are mutually exclusive events.

$\therefore$ Required probability $= P(A \cup B \cup C \cup D)$

$= P(A) + P(B) + P(C) + P(D)$ [By addition Theorem]

$$= \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{114}{120}$$

EXAMPLE 20 *(i) Two dice are thrown together. What is the probability that the sum of the numbers on the two faces is neither divisible by 3 nor by 4? (ii) What is the probability that the sum of the numbers on the two faces is divisible by 3 or 4?*

SOLUTION (i) Let S be the sample space associated with the experiment of throwing a pair of dice. Then, $n(S) = 36$.

$\therefore$ Total number of elementary events = 36

Consider the following events.

A = The sum of the numbers on two faces in divisible by 3

B = The sum of the numbers on two faces is divisible by 4.

Then, $A = \{(1, 2), (2, 1), (1, 5), (5, 1), (3, 3), (2, 4), (4, 2), (3, 6), (6, 3), (4, 5), (5, 4), (6, 6)\}$

$B = \{(2, 2), (1, 3), (3, 1), (2, 6), (6, 2), (4, 4), (3, 5), (5, 3), (6, 6)\}$ and, $A \cap B = \{(6, 6)\}$

$$\therefore \quad P(A) = \frac{12}{36} = \frac{1}{3},\ P(B) = \frac{9}{36} = \frac{1}{4} \text{ and } P(A \cap B) = \frac{1}{36}$$

(i) Required probability $= P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$$= 1 - \{P(A) + P(B) - P(A \cap B)\} = 1 - \left\{\frac{1}{3} + \frac{1}{4} - \frac{1}{36}\right\} = \frac{4}{9}$$

(ii) Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{36} = \frac{5}{9}$

EXAMPLE 21 *An urn contains twenty white slips of paper numbered from 1 through 20, ten red slips of paper numbered from 1 through 10, forty yellow slips of paper numbered from 1 through 40 and ten blue slips of paper numbered from 1 through 10. If these 80 slips of paper are thoroughly shuffled so that each slip has the same probability of being drawn. Find the probabilities of drawing a slip of paper that is (i) blue or white (ii) number 1, 2, 3, 4 or 5 (iii) red or yellow and numbered 1, 2, 3, or 4. (iv) numbered 5, 15, 25 or 35. (v) white and numbered higher than 12 or yellow and numbered higher than 26.*

[NCERT EXEMPLAR]

SOLUTION There are 80 slips of paper out of which one slip can be chosen in ${}^{80}C_1 = 80$ ways.

So, total number of elementary events = 80

(i) There are 10 blue and 20 white slips out of which one slip can be chosen in ${}^{30}C_1 = 30$ ways.

$\therefore$ Favourable number of ways = 30

Hence, P (Drawing a blue or white slip) $= \frac{30}{80} = \frac{3}{8}$.

(ii) Consider the following events:

W = Drawing a white slip numbered 1, 2, 3, 4 or 5,

R = Drawing a red slip numbered 1, 2, 3, 4 or 5,
Y = Drawing a yellow slip numbered 1, 2, 3, 4 or 5,
B = Drawing a blue slip numbered 1, 2, 3, 4 or 5

Clearly, these events are mutually exclusive.

Required probability $= P(W \cup R \cup Y \cup B)$

$$= P(W) + P(R) + P(Y) + P(B) = \frac{5}{80} + \frac{5}{80} + \frac{5}{80} + \frac{5}{80} = \frac{20}{80} = \frac{1}{4}$$

(iii) Consider the following events:

R = Drawing a red slip numbered 1, 2, 3 or 4,
Y = Drawing a yellow slip numbered 1, 2, 3, or 4

Clearly, R and Y are mutually exclusive events such that $P(R) = \frac{4}{80}$ and $P(Y) = \frac{4}{80}$.

Required probability $= P(R \cup Y) = P(R) + P(Y) = \frac{4}{80} + \frac{4}{80} = \frac{1}{10}$

(iv) Consider the following events:

A = Drawing a slip numbered 5, B = Drawing a slip numbered 15
C = Drawing a slip numbered 25, D = Drawing a slip numbered 35

We observe that A, B, C and D are mutually exclusive events such that

$P(A) = \frac{4}{80}$ [$\because$ There are 4 tickets, one of each colour numbered 5]

$P(B) = \frac{2}{80}$ [$\because$ There is one white and one yellow ticket each numbered 15]

$P(C) = \frac{1}{80}$ [$\because$ There is just one yellow ticket numbered 25]

and, $P(D) = \frac{1}{80}$ [$\because$ There is just one yellow ticket numbered 35]

Required probability $= (A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D) = \frac{4}{80} + \frac{2}{80} + \frac{1}{80} + \frac{1}{80} = \frac{1}{10}$

(v) Consider the following events:

A = Drawing a white slip numbered higher than 12
B = Drawing a yellow slip numbered higher than 26

We observe that A and B are mutually exclusive events such that $P(A) = \frac{8}{80}$ and $P(B) = \frac{14}{80}$.

Required probability $= P(A \cup B) = P(A) + P(B) = \frac{8}{80} + \frac{14}{80} = \frac{11}{40}$.

Type III PROBLEMS WHICH CAN BE SOLVED BY USING ADDITION THEOREMS ONLY

EXAMPLE 22 *Two cards are drawn from a pack of 52 cards. What is the probability that either both are red or both are kings?*

SOLUTION Out of 52 cards, two cards can be drawn in ${}^{52}C_2$ ways.

So, total number of elementary events $= {}^{52}C_2$.

Consider the following events:

A = Two cards drawn are red cards, B = Two cards drawn are kings.

Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [By addition Theorem] ...(i)

Let us now find $P(A), P(B)$ and $P(A \cap B)$.

There are 26 red cards, out of which 2 red cards can be drawn in ${}^{26}C_2$ ways.

$\therefore \quad P(A) = \frac{^{26}C_2}{^{52}C_2}$

Since there are 4 kings, out of which 2 kings can be drawn in $^4C_2$ ways.

$\therefore \quad P(B) = \frac{^{4}C_2}{^{52}C_2}$

There are 2 cards which are both red and kings.

$\therefore \quad P(A \cap B) =$ Probability of getting 2 cards which are both red and kings.

$=$ Probability of getting 2 red kings $= \frac{^{2}C_2}{^{52}C_2}$

Required probability $= P(A) + P(B) - P(A \cap B)$

$$= \frac{^{26}C_2}{^{52}C_2} + \frac{^{4}C_2}{^{52}C_2} - \frac{^{2}C_2}{^{52}C_2} = \frac{325}{1326} + \frac{1}{221} - \frac{1}{1326} = \frac{55}{221}$$

EXAMPLE 23 *A basket contains 20 apples and 10 oranges out of which 5 apples and 3 oranges are defective. If a person takes out 2 at random what is the probability that either both are apples or both are good?*

SOLUTION Out of 30 items, two can be selected in $^{30}C_2$ ways.

So, total number of elementary events $= {}^{30}C_2$.

Consider the following events:

$A =$ Getting two apples, $B =$ Getting two good items

There are 20 apples, out of which 2 can be drawn in $^{20}C_2$ ways.

$\therefore \quad P(A) = \frac{^{20}C_2}{^{30}C_2}$

There are 8 defective pieces and the remaining 22 are good. Out of 22 good pieces, two can be selected in $^{22}C_2$ ways.

$\therefore \quad P(B) = \frac{^{22}C_2}{^{30}C_2}$

Since there are 15 pieces which are good apples out of which 2 can be selected in $^{15}C_2$ ways.

$\therefore \quad P(A \cap B) =$ Probability of getting 2 pieces which are good apples $= \frac{^{15}C_2}{^{30}C_2}$

Required probability $= P(A) + P(B) - P(A \cap B) = \frac{^{20}C_2}{^{30}C_2} + \frac{^{22}C_2}{^{30}C_2} - \frac{^{15}C_2}{^{30}C_2} = \frac{316}{435}$

EXAMPLE 24 *A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If the die is rolled once, determine*

(i) $P(1)$ (ii) $P(1 \text{ or } 3)$ (iii) $P(\text{not } 3)$

SOLUTION Let A, B, C be three events defined by

$A =$ Getting a face with number '1', $B =$ Getting a face with number '2',
$C =$ Getting a face with number '3'

Then, $P(A) = \frac{2}{6} = \frac{1}{3}$, $P(B) = \frac{3}{6} = \frac{1}{2}$ and $P(C) = \frac{1}{6}$

(i) $P(2) = P(A) = \frac{1}{3}$

(ii) $P(1 \text{ or } 3) = P(A \cup C) = P(A) + P(C)$ [$\because$ A and C are mutually exclusive]

$\Rightarrow \quad P(1 \text{ or, } 3) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

(iii) $\quad P(\text{not } 3) = P(\bar{C}) = 1 - P(C) = 1 - \frac{1}{6} = \frac{5}{6}$

EXAMPLE 25 *The probability that a student will receive A, B, C or D grade are 0.40, 0.35, 0.15 and 0.10 respectively. Find the probability that a student will receive*

(i) *B or C grade* (ii) *at most C grade.*

SOLUTION Let E_1, E_2, E_3 and E_4 denote respectively the events that a student will receive A, B, C and D grades. Then,

$$P(E_1) = 0.40,\ P(E_2) = 0.35,\ P(E_3) = 0.15 \text{ and } P(E_4) = 0.10$$

(i) Required probability $= P(E_2 \cup E_3)$

$= P(E_2) + P(E_3)$ [$\because$ E_2 and E_3 are mutually exclusive events]

$= 0.35 + 0.15 = 0.50$

(ii) Required probability = Probability that the student receives C or D grade

$= P(E_3 \cup E_4)$

$= P(E_3) + P(E_4)$ [$\because$ E_3 and E_4 are mutually exclusive]

$= 0.15 + 0.10 = 0.25$

EXAMPLE 26 *The probability that a person will get an electric contract is $\frac{2}{5}$ and the probability that he will not get plumbing contract is $\frac{4}{7}$. If the probability of getting at least one contract is $\frac{2}{3}$, what is the probability that he will get both?*

SOLUTION Consider the following events:

A = Person gets an electric contract, B = Person gets plumbing contract

Clearly,

$$P(A) = \frac{2}{5},\ P(\bar{B}) = \frac{4}{7} \text{ and } P(A \cup B) = \frac{2}{3}$$

We have to find $P(A \cap B)$. By addition theorem, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad \frac{2}{3} = \frac{2}{5} + \left(1 - \frac{4}{7}\right) - P(A \cap B) \Rightarrow P(A \cap B) = \frac{2}{5} + \frac{3}{7} - \frac{2}{3} = \frac{17}{105}$$

EXAMPLE 27 *If a person visits his dentist, suppose the probability that he will have his teeth cleaned is 0.48, the probability that he will have cavity filled is 0.25, probability that he will have a tooth extracted is 0.20, the probability that he will have a teeth cleaned and cavity filled is 0.09, the probability that he will have his teeth cleaned and a tooth extracted is 0.12, the probability that he will have a cavity filled and tooth extracted is 0.07, and the probability that he will have his teeth cleaned, cavity filled, and tooth extracted is 0.03. What is the probability that a person visiting his dentist will have at least one of these things done to him?* **[NCERT EXEMPLAR]**

SOLUTION Consider the following events:

C = The person will have his teeth cleaned, F = The person will have cavity filled

E = The person will have a tooth extracted

It is given that $P(C) = 0.48, P(F) = 0.25, P(E) = 0.20,$

$$P(C \cap F) = 0.09,\ P(C \cap E) = 0.12,\ P(E \cap F) = 0.07 \text{ and } P(C \cap F \cap E) = 0.03.$$

$\therefore$ Required probability $= P(C \cup F \cup E)$

$= P(C) + P(F) + P(E) - P(C \cap F) - P(F \cap E) - P(C \cap E) + P(C \cap F \cap E)$

$= 0.48 + 0.25 + 0.20 - 0.09 - 0.12 - 0.07 + 0.03 = 0.68$

EXAMPLE 28 *The probability that a patient visiting a dentist will have a tooth extracted is 0.06, the probability that he will have a cavity filled is 0.2 and the probability that he will have a tooth extracted as well as cavity filled is 0.03. What is the probability of that a patient has either a tooth extracted or a cavity filled?*

SOLUTION Let A be the event that the patient will have his tooth extracted, B the event that he will have a cavity filled.

We have, $P(A) = 0.06$, $P(B) = 0.2$ and $P(A \cap B) = 0.03$

$\therefore$ Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.06 + 0.2 - 0.03 = 0.23$

EXAMPLE 29 *The probability that a person visiting a dentist will have his teeth cleaned is 0.44, the probability that he will have a cavity filled is 0.24. The probability that he will have his teeth cleaned or a cavity filled is 0.60. What is the probability that a person visiting a dentist will have his teeth cleaned and cavity filled?*

SOLUTION Let A be the event that the patient will have his teeth cleaned and B be the event that he will have cavity filled.

We have, $P(A) = 0.44$, $P(B) = 0.24$ and $P(A \cup B) = 0.60$

$\therefore$ Required probability $= P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.44 + 0.24 - 0.60 = 0.08$

EXAMPLE 30 *Probability that Hameed passes in Mathematics is $\frac{2}{3}$ and the probability that he passes in English is $\frac{4}{9}$. If the probability of passing both courses is $\frac{1}{4}$, what is the probability that Hameed will pass in at least one of these subjects?*

SOLUTION Let A be the event that Hameed passes in Mathematics and B be the event that he passes in English.

We have, $P(A) = \frac{2}{3}$, $P(B) = \frac{4}{9}$ and $P(A \cap B) = \frac{1}{4}$

$\therefore$ Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{3} + \frac{4}{9} - \frac{1}{4} = \frac{31}{36}$

EXAMPLE 31 *Find the probability of at most two tails or at least two heads in a toss of three coins.*

SOLUTION Consider the following events:

A = Getting at most two tails in a toss of three coins.

B = Getting at least two heads in a toss of three coins.

We have, $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$,

$A = \{HHH, HHT, HTH, THH, TTH, HTT, THT\}$, $B = \{HHT, HTH, THH, HHH\}$

$\therefore$ $P(A) = \frac{7}{8}$, $P(B) = \frac{4}{8} = \frac{1}{2}$ and $P(A \cap B) = \frac{4}{8} = \frac{1}{2}$

Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{7}{8} + \frac{1}{2} - \frac{1}{2} = \frac{7}{8}$

EXAMPLE 32 *In a town of 6000 people 1200 are over 50 years old and 2000 are female. It is known that 30% of the females are over 50 years. What is the probability that a random chosen individual from the town either female or over 50 years?*

SOLUTION Consider the following events :

A = A randomly chosen individual is a female

B = A randomly chosen individual is over 50 years old.

Clearly,

$$P(A) = \frac{2000}{6000} = \frac{1}{3},\ P(B) = \frac{1200}{6000} = \frac{1}{5}$$

and, $P(A \cap B) = P$ (An individual is a female over 50 years old) $= \frac{30\% \text{ of } 2000}{6000} = \frac{600}{6000} = \frac{1}{10}$

Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{5} - \frac{1}{10} = \frac{13}{30}$

EXAMPLE 33 *From the employees of a company, 5 persons are elected to represent them in the managing committee of the company. Particulars of the five persons are as follows:* [NCERT]

S. No.	*Person*	*Age (in years)*
1	*Male*	30
2	*Male*	33
3	*Female*	46
4	*Female*	28
5	*Male*	41

A person is selected at random from this group to act as a spokespersons. What is the probability that a spokespersons will be either male or over 35 years?

SOLUTION Consider the following events:

A = The spokespersons is a male, B = The spokespersons is over 35 years.

Clearly, $P(A) = \frac{3}{5}$, $P(B) = \frac{2}{5}$ and $P(A \cap B) = \frac{1}{5}$

$\therefore$ Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{5} + \frac{2}{5} - \frac{1}{5} = \frac{4}{5}$

EXAMPLE 34 *In class XI of a school, 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, find the probability that he will be studying Mathematics or Biology or both.*[NCERT]

SOLUTION Consider the following events:

M = A student studies Mathematics, B = A student studies Biology.

We have,

$$P(M) = \frac{40}{100},\ P(B) = \frac{30}{100} \text{ and } P(M \cap B) = \frac{10}{100}$$

Required probability $= P(M \cup B)$

$$= P(M) + P(B) - P(M \cap B) = \frac{40}{100} + \frac{30}{100} - \frac{10}{100} = \frac{60}{100} = \frac{3}{5}$$

EXAMPLE 35 *Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05 and that Ashima will qualify the examination is 0.10. The probability that both will qualify the examination is 0.02. Find the probability that:*

(i) *both Anil and Ashima will not qualify the exam.* [NCERT]
(ii) *at least one of them will not qualify the exam.*
(iii) *only one of them will qualify the exam.*

SOLUTION Let E and F denote the events that Anil and Ashima will qualify the examination. Then, $P(E) = 0.05$, $P(F) = 0.10$ and $P(E \cap F) = 0.02$

(i) Required probability $= P(\overline{E} \cap \overline{F})$

$$= P(\overline{E \cup F}) = 1 - P(E \cup F)$$
$$= 1 - \{P(E) + P(F) - P(E \cap F)\} = 1 - (0.05 + 0.10 - 0.02) = 0.87$$

(ii) Required probability $= P$ (At least one of them will not qualify the exam)

$= 1 - P$ (Both of them will qualify the exam)

$= 1 - P(E \cap F) = 1 - 0.02 = 0.98$

(iii) P (Only one of them will qualify the exam) $= P(E) + P(F) - 2P(E \cap F)$

$= 0.05 + 0.10 - 2 \times 0.02 = 0.15 - 0.04 = 0.11$

EXAMPLE 36 *In a class of 60 students 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that:*

(i) *the student opted for NCC or NSS* (ii) *the student has opted neither NCC nor NSS*

(iii) *the student has opted NSS but not NCC.* **[NCERT]**

SOLUTION Consider the following events:

A = A student opted NCC, B = A student opted NSS

We have,

$$P(A) = \frac{30}{60},\ P(B) = \frac{32}{60} \text{ and } P(A \cap B) = \frac{24}{60}$$

(i) Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{30}{60} + \frac{32}{60} - \frac{24}{60} = \frac{38}{60} = \frac{19}{30}$

(ii) Required probability $= P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cap B) = 1 - \frac{19}{30} = \frac{11}{30}$

(iii) Required probability $= P(\overline{A} \cap B) = P(B) - P(A \cap B) = \frac{32}{60} - \frac{24}{60} = \frac{8}{60} = \frac{2}{15}$

EXAMPLE 37 *One of the four persons John, Rita, Aslam or Gurpreet will be promoted next month. Consequently the sample space S consists of four elementary outcomes as given below.*

S = { John promoted, Rita promoted, Aslam promoted, Gurpreet promoted}

You are told that the chances of John's promotion is same as that of Gurpreet. Rita's chances of promotion are twice as likely as Johns. Aslam's chances are four times that of John.

(i) *Determine P (John promoted), P (Rita promoted), P (Aslam promoted), P (Gurpreet promoted)*

(ii) *If A = {John promoted or Gurpreet promoted }, find P(A)* **[NCERT EXEMPLAR]**

SOLUTION (i) Let P (John promoted) $= p$. Then, by hypothesis

P (Gurpreet promoted) $= p$, P (Rita promoted) $= 2p$ and, P (Aslam promoted) $= 4p$.

We have,

S = { John promoted, Rita promoted, Aslam promoted, Gurpreet promoted}

$\therefore$ P (John promoted) $+P$ (Rita promoted) $+P$ (Aslam promoted) $+P$ (Gurpreet promoted) $= 1$

$\Rightarrow p + 2p + 4p + p = 1$

$\Rightarrow p = \frac{1}{8}$

Hence, P(John promoted) $= \frac{1}{8}$, P (Rita promoted) $= 2p = \frac{2}{8} = \frac{1}{4}$,

P (Aslam promoted) $= 4p = \frac{4}{8} = \frac{1}{2}$ and, P (Gurpreet promoted) $= p = \frac{1}{8}$.

(ii) A = { John promoted or Gurpreet promoted }

$\therefore$ $P(A) = P$ (John promoted) $+ P$ (Gurpreet promoted) $= p + p = 2p = \frac{2}{8} = \frac{1}{4}$

LEVEL-2

EXAMPLE 38 *A, B, C are events such that* $P(A) = 0.3$, $P(B) = 0.4$, $P(C) = 0.8$, $P(A \cap B) = 0.08$, $P(A \cap C) = 0.28$, $P(A \cap B \cap C) = 0.09$. *If* $P(A \cup B \cup C) \geq 0.75$, *then show that* $P(B \cap C)$ *lies in the interval (0.23, 0.48).*

SOLUTION We know that the probability of occurrence of an event is always less than or equal to 1 and it is given that $P(A \cup B \cup C) \geq 0.75$

$\therefore \quad 0.75 \leq P(A \cup B \cup C) \leq 1$

$\Rightarrow \quad 0.75 \leq P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \leq 1$

$\Rightarrow \quad 0.75 \leq 0.3 + 0.4 + 0.8 - 0.08 - P(B \cap C) - 0.28 + 0.09 \leq 1$

$\Rightarrow \quad 0.75 \leq 1.59 - 0.36 - P(B \cap C) \leq 1$

$\Rightarrow \quad 0.75 \leq 1.23 - P(B \cap C) \leq 1$

$\Rightarrow \quad -0.48 \leq -P(B \cap C) \leq -0.23 \Rightarrow 0.23 \leq P(B \cap C) \leq 0.48$

EXAMPLE 39 *If A and B are any two events such that* $P(A \cup B) = \frac{1}{2}$ *and* $P(\overline{A}) = \frac{2}{3}$, *find* $P(\overline{A} \cap B)$.

SOLUTION Clearly, $\overline{A} \cap B$ and A are mutually exclusive events such that

$A \cup B = A \cup (\overline{A} \cap B)$

$\Rightarrow \quad P(A \cup B) = P(A) + P(\overline{A} \cap B)$

$\Rightarrow \quad \frac{1}{2} = 1 - P(\overline{A}) + P(\overline{A} \cap B)$

$\Rightarrow \quad \frac{1}{2} = 1 - \frac{2}{3} + P(\overline{A} \cap B)$

$\Rightarrow \quad P(\overline{A} \cap B) = \frac{1}{6}$

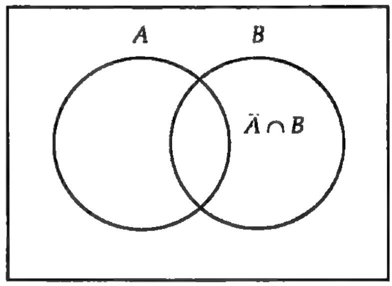


Fig. 33.5

EXAMPLE 40 *Figure 33.6 shows three events A, B and C and also the probabilities of the various intersections (for instance* $P(A \cap B) = 0.07$*) determine*

(i) $P(A)$ (ii) $P(B \cap \overline{C})$ (iii) $P(A \cup B)$

(iv) $P(A \cap \overline{B})$ (v) $P(B \cap C)$ (vi) *Probability of exactly one of the three events.*

[NCERT EXEMPLAR]

SOLUTION (i) We have, $P(A \cap \overline{B}) = 0.13$ and $P(A \cap B) = 0.07$

Since $A \cap \overline{B}$ and $A \cap B$ are mutually exclusive events such that

$A = (A \cap \overline{B}) \cup (A \cap B)$.

$\therefore \quad P(A) = P(A \cap \overline{B}) + P(A \cap B)$

$\Rightarrow \quad P(A) = 0.13 + 0.07 = 0.20$

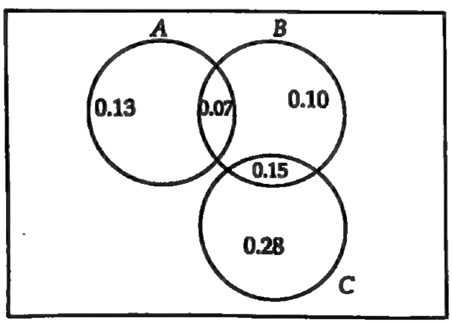


Fig. 33.6

(ii) It is evident from the Figure 33.6 that

$P(A \cap B) = 0.07, P(B \cap C) = 0.15$ and $P(\overline{A} \cap B \cap \overline{C}) = 0.10$

Also, $A \cap B$, $B \cap C$ and $\overline{A} \cap B \cap \overline{C}$ are mutually exclusive events such that

$B = (A \cap B) \cup (B \cap C) \cup (\overline{A} \cap B \cap \overline{C})$

$\therefore \quad P(B) = P(A \cap B) + P(B \cap C) + P(\bar{A} \cap B \cap \bar{C})$

$\Rightarrow \quad P(B) = 0.07 + 0.15 + 0.10 = 0.32$

Now, $\quad P(B \cap \bar{C}) = P(B) - P(B \cap C)$

$\Rightarrow \quad P(B \cap \bar{C}) = 0.32 - 0.15 = 0.17$

(iii) We have, $P(A) = 0.20$, $P(B) = 0.32$ and $P(A \cap B) = 0.07$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.20 + 0.32 - 0.07 = 0.45$

(iv) We have, $P(A) = 0.20$ and $P(A \cap B) = 0.07$

$\therefore \quad P(A \cap \bar{B}) = P(A) - P(A \cap B) = 0.20 - 0.07 = 0.13$

(v) It is evident from the Figure 33.6 that $P(B \cap C) = 0.15$.

(vi) Probability that exactly one of three events A, B and C occurs

$= P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap C) = 0.13 + 0.10 + 0.28 = 0.51$

EXAMPLE 41 *Let A, B, C be three events. If the probability of occuring exactly one event out of A and B is $1 - x$, out of B and C is $1 - 2x$, out of C and A is $1 - x$, and that of occuring three events simultaneously is x^2, then prove that the probability that atleast one out of A, B, C will occur is greater than $1/2$.*

SOLUTION We have,

$$P(A) + P(B) - 2P(A \cap B) = 1 - x \quad \text{...(i)}$$

$$P(B) + P(C) - 2P(B \cap C) = 1 - 2x \quad \text{...(ii)}$$

$$P(C) + P(A) - 2P(C \cap A) = 1 - x \quad \text{...(iii)}$$

and, $$P(A \cap B \cap C) = x^2 \quad \text{...(iv)}$$

Adding (i), (ii) and (iii), we get

$$P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = \frac{3 - 4x}{2} \quad \text{...(v)}$$

$\therefore$ Probability that at least one out of A, B, C will occur

$= P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

$= \dfrac{3 - 4x}{2} + x^2$ [Using (iv) and (v)]

$= x^2 - 2x + \dfrac{3}{2} = (x - 1)^2 + \dfrac{1}{2} > \dfrac{1}{2}$

EXAMPLE 42 *For the three events A, B and C, P (exactly one of the events A or B occurs) $= P$ (exactly one of the events B or C occurs) $= P$ (exactly one of the events C and A occurs) $= p$ and P (all the three events occur simultaneously) $= p^2$, where $0 < p < 1/2$. Then, find the probability of occurrence of at least one of the three events A, B, and C.*

SOLUTION It is given that

P (Exactly one of the events A or B occurs) $= p$, P (Exactly one of the events B or C occurs) $= p$

P (Exactly one of the events C or A occurs) $= p$

and, P (All the three events occur simultaneously) $= p^2$

i.e. $$P(A) + P(B) - 2P(A \cap B) = p \quad \text{...(i)}$$

$$P(B) + P(C) - 2P(B \cap C) = p \quad \text{...(ii)}$$

$$P(C) + P(A) - 2P(A \cap B) = p \quad \text{...(iii)}$$

and, $$P(A \cap B \cap C) = p^2 \quad \text{...(iv)}$$

Adding (i), (ii) and (iii), we get

$$P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)=\frac{3p}{2} \quad ...(v)$$

Required probability $= P(A\cup B\cup C)$

$$= P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$$

$$= \frac{3p}{2}+p^2 = \frac{3p+2p^2}{2}$$

EXAMPLE 43 *For a post three persons A, B and C appear in the interview. The probability of A being selected is twice that of B and the probability of B being selected is thrice that of C. What are the individual probabilities of A, B, C being selected?*

SOLUTION Let A_1, A_2 and A_3 be three events as defined below:

A_1 = Person A is selected, A_2 = Person B is selected, A_3 = Person C is selected.

We have,

$P(A_1)=2P(A_2)$ and $P(A_2)=3P(A_3) \Rightarrow P(A_1)=6P(A_3)$ and $P(A_2)=3P(A_3)$.

Since A_1, A_2, A_3 are mutually exclusive and exhaustive events.

$\therefore \quad A_1\cup A_2\cup A_3 = S$

$\Rightarrow \quad P(A_1\cup A_2\cup A_3) = P(S)$

$\Rightarrow \quad P(A_1)+P(A_2)+P(A_3) = 1$ [$\because$ A_1, A_2, A_3 are mutually exclusive]

$\Rightarrow \quad 6P(A_3)+3P(A_3)+P(A_3) = 1$

$\Rightarrow \quad 10\,P(A_3) = 1$

$\Rightarrow \quad P(A_3) = \frac{1}{10}$

$\therefore \quad P(A_1) = \frac{6}{10}$ and $P(A_2) = \frac{3}{10}$

EXAMPLE 44 *P and Q are two candidates seeking admission in I.I.T. The probability that P is selected is 0.5 and the probability that both P and Q are selected is at most 0.3. Prove that the probability of Q being selected is at most 0.8.*

SOLUTION Let A_1 and A_2 be two events defined as follows:

$A_1 = P$ is selected, $A_2 = Q$ is selected.

We have, $P(A_1) = 0.5$ and $P(A_1\cap A_2)\le 0.3$

Now, $P(A_1\cup A_2)\le 1$

$\Rightarrow \quad P(A_1)+P(A_2)-P(A_1\cap A_2)\le 1$

$\Rightarrow \quad 0.5+P(A_2)-P(A_1\cap A_2)\le 1$

$\Rightarrow \quad P(A_2)\le 0.5+P(A_1\cap A_2) \Rightarrow P(A_2)\le 0.5+0.3 \Rightarrow P(A_2)\le 0.8$

EXAMPLE 45 *A box contains 6 red, 4 white and 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is atleast one ball of each colour.*

SOLUTION We observe that at least one ball of each colour can be drawn in one of the following mutually exclusive ways :

(i) 1 red, 1 white and 2 black balls. (ii) 2 red, 1 white and 1 black balls.

(iii) 1 red, 2 white and 1 black balls.

Thus, if we define three events A, B and C as follows:

A = Drawing 1 red, 1 white and 2 black balls, B = Drawing 2 red, 1 white and 1 black balls

C = Drawing 1 red, 2 white and 1 black balls

We observe that A, B, C are mutually exclusive events.

$\therefore$ Required probability $= P(A \cup B \cup C)$

$= P(A) + P(B) + P(C)$ [By addition Theorem]

$$= \frac{{}^6C_1 \times {}^4C_1 \times {}^5C_2}{{}^{15}C_4} + \frac{{}^6C_2 \times {}^4C_1 \times {}^5C_1}{{}^{15}C_4} + \frac{{}^6C_1 \times {}^4C_2 \times {}^5C_1}{{}^{15}C_4}$$

$$= \frac{6 \times 4 \times 10 + 15 \times 4 \times 5 + 6 \times 6 \times 5}{{}^{15}C_4} = \frac{24 \times 720}{15 \times 14 \times 13 \times 12} = \frac{48}{91}$$

EXAMPLE 46 *A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find P(G), where G is the event that a number greater than 3 occurs on a single roll of the die.*

SOLUTION Let A_i denote the event "Getting number i on the upper face of the die", $i = 1, 2, 3, 4, 5, 6$. Clearly, $A_i; i = 1, 2, \dots, 6$ are mutually exclusive and exhaustive events.

It is given that $P(A_2) = P(A_4) = P(A_6) = p$ (*say*) and, $P(A_1) = P(A_3) = P(A_5) = 2p$.

Now, $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 = S$

$\Rightarrow$ $P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6) = P(S)$

$\Rightarrow$ $P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5) + P(A_6) = 1$

$\Rightarrow$ $2p + p + 2p + p + 2p + p = 1 \Rightarrow 9p = 1 \Rightarrow p = \frac{1}{9}$

Now, $G = A_4 \cup A_5 \cup A_6$

$\Rightarrow$ $P(G) = P(A_4 \cup A_5 \cup A_6)$

$\Rightarrow$ $P(G) = P(A_4) + P(A_5) + P(A_6)$ [$\because$ A_4, A_5, A_6 are mutually exclusive]

$\Rightarrow$ $P(G) = p + 2p + p = 4p = \frac{4}{9}$

EXAMPLE 47 *A team of medical students doing their internship have to assist during surgeries at a city hospital. The probabilities of surgeries rated as very complex, complex, routine, simple and very simple are respectively 0.15, 0.20, 0.31, 0.26, 0.08. Find the probabilities that a particular surgery will be rated (i) Complex or very complex (ii) neither very complex nor very simple (iii) routine or complex (iv) routine or simple.* [NCERT EXEMPLAR]

SOLUTION Consider the following events:

A = Surgery is very complex, B = Surgery is complex, C = Surgery is routine,

D = Surgery is simple, E = Surgery is very simple.

It is given that $P(A) = 0.15$ $P(B) = 0.20$, $P(C) = 0.31$, $P(D) = 0.26$ and $P(E) = 0.08$.

Clearly, A, B, C, D and E are mutually exclusive events.

(i) Required probability $= P(A \cup B) = P(A) + P(B) = 0.15 + 0.20 = 0.35$

(ii) Required probability $= P(\overline{A} \cap \overline{E})$

$= P(\overline{A \cup E})$

$= 1 - P(A \cup E) = 1 - \{P(A) + P(E)\} = 1 - (0.15 + 0.08) = 0.77$

(iii) Required probability $= P(C \cup B) = P(C) + P(B) = 0.31 + 0.20 = 0.51$

(iv) Required probability $= P(C \cup D) = P(C) + P(D) = 0.31 + 0.26 = 0.57$

EXERCISE 33.4

LEVEL-1

1. (a) If A and B be mutually exclusive events associated with a random experiment such that $P(A) = 0.4$ and $P(B) = 0.5$, then find:

(i) $P(A \cup B)$ (ii) $P(\overline{A} \cap \overline{B})$ (iii) $P(\overline{A} \cap B)$ (iv) $P(A \cap \overline{B})$.

(b) A and B are two events such that $P(A) = 0.54$, $P(B) = 0.69$ and $P(A \cap B) = 0.35$. Find (i) $P(A \cup B)$ (ii) $P(\bar{A} \cap \bar{B})$ (iii) $P(A \cap \bar{B})$ (iv) $P(B \cap \bar{A})$

(c) Fill in the blanks in the following table:

	$P(A)$	$P(B)$	$P(A \cap B)$	$P(A \cup B)$
(i)	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{15}$	......
(ii)	0.35	....	0.25	0.6
(iii)	0.5	0.35	.....	0.7

2. If A and B are two events associated with a random experiment such that $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cup B) = 0.5$, find $P(A \cap B)$.
3. If A and B are two events associated with a random experiment such that $P(A) = 0.5$, $P(B) = 0.3$ and $P(A \cap B) = 0.2$, find $P(A \cup B)$.
4. If A and B are two events associated with a random experiment such that $P(A \cup B) = 0.8$, $P(A \cap B) = 0.3$ and $P(\bar{A}) = 0.5$, find $P(B)$.
5. Given two mutually exclusive events A and B such that $P(A) = 1/2$ and $P(B) = 1/3$, find P (A or B).
6. There are three events A, B, C one of which must and only one can happen, the odds are 8 to 3 against A, 5 to 2 against B, find the odds against C.
7. One of the two events must happen. Given that the chance of one is two-third of the other, find the odds in favour of the other.

NOTE *Students are advised to do the following exercises by using addition theorems and also by using the definition only i.e. by calculating exhaustive number of cases and favourable number of cases.*

8. A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability of its being a spade or a king.
9. In a single throw of two dice, find the probability that neither a doublet nor a total of 9 will appear.
10. A natural number is chosen at random from amongst first 500. What is the probability that the number so chosen is divisible by 3 or 5?
11. A die is thrown twice. What is the probability that at least one of the two throws come up with the number 3?
12. A card is drawn from a deck of 52 cards. Find the probability of getting an ace or a spade card.
13. The probability that a student will pass the final examination in both English and Hindi is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75. What is the probability of passing the Hindi examination? [NCERT]
14. One number is chosen from numbers 1 to 100. Find the probability that it is divisible by 4 or 6?
15. From a well shuffled deck of 52 cards, 4 cards are drawn at random. What is the probability that all the drawn cards are of the same colour.
16. 100 students appeared for two examinations. 60 passed the first, 50 passed the second and 30 passed both. Find the probability that a student selected at random has passed at least one examination.
17. A box contains 10 white, 6 red and 10 black balls. A ball is drawn at random from the box. What is the probability that the ball drawn is either white or red?
18. In a race, the odds in favour of horses A, B, C, D are $1:3, 1:4, 1:5$ and $1:6$ respectively. Find probability that one of them wins the race.
19. The probability that a person will travel by plane is $3/5$ and that he will travel by train is $1/4$. What is the probability that he (she) will travel by plane or train?
20. Two cards are drawn from a well shuffled pack of 52 cards. Find the probability that either both are black or both are kings.

21. In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing at least one of them is 0.95. What is the probability of passing both? [NCERT]
22. A box contains 30 bolts and 40 nuts. Half of the bolts and half of the nuts are rusted. If two items are drawn at random, what is the probability that either both are rusted or both are bolts?
23. An integer is chosen at random from first 200 positive integers. Find the probability that the integer is divisible by 6 or 8.
24. Find the probability of getting 2 or 3 tails when a coin is tossed four times.
25. Suppose an integer from 1 through 1000 is chosen at random, find the probability that the integer is a multiple of 2 or a multiple of 9. [NCERT EXEMPLAR]
26. In a large metropolitan area, the probabilities are 0.87, 0.36, 0.30 that a family (randomly chosen for a sample survey) owns a colour television set, a black and white television set, or both kinds of sets. What is the probability that a family owns either any one or both kinds of sets? [NCERT EXEMPLAR]
27. If A and B are mutually exclusive events such that $P(A) = 0.35$ and $P(B) = 0.45$, find
    (i) $P(A \cup B)$ (ii) $P(A \cap B)$ (iii) $P(A \cap \overline{B})$ (iv) $P(\overline{A} \cap \overline{B})$ [NCERT EXEMPLAR]
28. A sample space consists of 9 elementary event $E_1, E_2, E_3, \dots, E_8, E_9$ whose probabilities are
    $P(E_1) = P(E_2) = 0.08,\ P(E_3) = P(E_4) = 0.1, P(E_6) = P(E_7) = 0.2, P(E_8) = P(E_9) = 0.07$
    Suppose $A = \{E_1, E_5, E_8\}$, $B = \{E_2, E_5, E_8, E_9\}$ [NCERT EXEMPLAR]
    (i) Compute $P(A)$, $P(B)$ and $P(A \cap B)$.
    (ii) Using the addition law of probability, find $P(A \cup B)$.
    (iii) List the composition of the event $A \cup B$, and calculate $P(A \cup B)$ by adding the probabilities of the elementary events.
    (iv) Calculate $P(\overline{B})$ from $P(B)$, also calculate $P(\overline{B})$ directly from the elementary events of $\overline{B}$.

ANSWERS

1. (a) (i) 0.9 (ii) 0.1 (iii) 0.5 (iv) 0.4 1. (b) (i) 0.88 (ii) 0.12 (iii) 0.19 (iv) 0.34

1. (c) (i) $\frac{7}{15}$ (ii) 0.5 (iii) 0.15 2. 0.2 3. 0.6 4. 0.6 5. 5/6

6. 43 : 34 7. 3 : 2 8. 4/13 9. 13/18 10. 233/500 11. $\frac{11}{36}$

12. $\frac{4}{13}$ 13. 0.65 14. $\frac{33}{100}$ 15. $\frac{92}{833}$ 16. $\frac{4}{5}$ 17. $\frac{8}{13}$

18. $\frac{171}{420}$ 19. $\frac{17}{20}$ 20. $\frac{55}{221}$ 21. 0.556 22. $\frac{185}{483}$ 23. $\frac{1}{4}$

24. $\frac{5}{8}$ 25. 0.556 26. 0.93 27. (i) 0.8 (ii) 0 (iii) 0.35 (iv) 0.2

28. (i) 0.25, 0.32, 0.17 (ii) 0.40 (iii) 0.40 (iv) 0.68

HINTS TO NCERT & SELECTED PROBLEMS

1. (ii) Use : $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$
   (iii) Use : $P(\overline{A} \cap B) = P(B) - P(A \cap B)$
   (iv) Use : $P(A \cap \overline{B}) = P(A) - P(A \cap B)$
6. It is given that A, B, C are mutually exclusive and exhaustive.
   $\therefore\ A \cup B \cup C = S \Rightarrow P(A \cup B \cup C) = P(S) \Rightarrow P(A) + P(B) + P(C) = 1$

7. Let A, B be two events. Then A, B are mutually exclusive and exhaustive.

$\therefore\quad A \cup B = S$

$\Rightarrow\quad P\ (A \cup B) = 1$

$\Rightarrow\quad P(A) + P(B) = 1$

$\Rightarrow\quad \frac{2}{3} P(B) + P(B) = 1 \qquad \left[\because\ P(A) = \frac{2}{3} P(B)\right]$

$\Rightarrow\quad P(B) = \frac{3}{5}$

Odds in favour of B are $P(B) : P(\overline{B})$ i.e. 3/5 : 2/5 or 3 : 2

9. Let A = Getting a doublet, B = Getting a total of 9. Then,

$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$

12. A = Getting an ace, B = Getting a spade card.

Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

13. Let E and H denote the events that the student will pass in English and Hindi examination respectively. Then, we have

$P(E \cap H) = 0.5,\ P(\overline{E} \cap \overline{H}) = 0.1$ and $P(E) = 0.75$

Now,

$P(\overline{E} \cap \overline{H}) = 0.1$

$\Rightarrow\quad P(\overline{E \cup H}) = 0.1$

$\Rightarrow\quad 1 - P(E \cup H) = 0.1$

$\Rightarrow\quad P(E \cup H) = 0.9$

$\Rightarrow\quad P(E) + P(H) - P(E \cap H) = 0.9 \Rightarrow 0.75 + P(H) - 0.5 = 0.9 \Rightarrow P(H) = 0.65.$

15. Let A = 4 cards drawn are red, B = 4 cards drawn are black. Then, A, B are mutually exclusive events.

So, required probability $= P(A \cup B) = P(A) + P(B)$

21. Let A and B denote the events that a randomly chosen student passes first and second examinations respectively. Then, $P(A) = 0.8$, $P(B) = 0.7$ and $P(A \cup B) = 0.95$

Required probability $= P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.8 + 0.7 - 0.95 = 0.55$

25. Consider the following events:

A = Integer chosen is a multiple of 2, B = Integer chosen is a multiple of 9.

We have,

$P(A) = \frac{500}{1000} = \frac{1}{2},\ P(B) = \frac{111}{1000}$ and $P(A \cap B) = \frac{55}{1000}$

Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{111}{1000} - \frac{55}{1000} = \frac{555}{1000}$

26. Consider the following events:

A = Family owns colours television set, B = Family owns black and white television set

It is given that $P(A) = 0.87$, $P(B) = 0.36$ and $P(A \cap B) = 0.30$

Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.87 + 0.36 - 0.30 = 0.93$

27. It is given that A and B are mutually exclusive events. Therefore, $P(A \cap B) = 0$.

(i) $P(A \cup B) = P(A) + P(B) = 0.35 + 0.45 = 0.8$

(ii) $P(A \cap B) = 0$

(iii) $P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) = 0.35$

(iv) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cap B}) = 1 - P(A \cup B) = 1 - \{P(A) + P(B)\} = 1 = 0.8 = 0.2$

28. We have, $P(E_1) + P(E_2) + P(E_3) + P(E_4) + P(E_5) + P(E_6) + P(E_7) + P(E_8) = 1 \Rightarrow P(E_5) = 0.1$

(i) $P(A) = P(E_1) + P(E_5) + P(E_8) = 0.08 + 0.1 + 0.07 = 0.25$

$$P(B) = P(E_2) + P(E_5) + P(E_8) + P(E_9) = 0.08 + 0.1 + 0.07 + 0.07 = 0.32$$
$$P(A \cap B) = P(E_5) + P(E_8) = 0.1 + 0.07 = 0.17$$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 025 + 0.32 - 0.17 = 0.40$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Three numbers are chosen at random from numbers 1 to 30. Write the probability that the chosen numbers are consecutive.
2. n (> 3) persons are sitting in a row. Two of them are selected. Write the probability that they are together.
3. A single letter is selected at random from the word 'PROBABILITY'. What is the probability that it is a vowel?
4. What is the probability that a leap year will have 53 Fridays or 53 Saturdays?
5. Three dice are thrown simultaneously. What is the probability of getting 15 as the sum?
6. If the letters of the word 'MISSISSIPPI' are written down at random in a row, what is the probability that four S's come together.
7. What is the probability that the 13th days of a randomly chosen month is Friday?
8. Three of the six vertices of a regular hexagon are chosen at random. What is the probability that the triangle with these vertices is equilateral.
9. If E and E_2 are independent evens, write the value of $P\left((E_1 \cup E_2) \cap (\overline{E} \cap \overline{E}_2)\right)$.
10. If A and B are two independent events such that $P(A \cap B) = \frac{1}{6}$ and $P(\overline{A} \cap \overline{B}) = \frac{1}{3}$, then write the values of $P(A)$ and $P(B)$.

ANSWERS

1. $\frac{144}{145}$ 2. $\frac{2}{n}$ 3. $\frac{4}{11}$ 4. $\frac{3}{7}$ 5. $\frac{13}{216}$ 6. $\frac{4}{165}$ 7. $\frac{1}{84}$ 8. $\frac{1}{10}$

9. 0 10. $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. One card is drawn from a pack of 52 cards. The probability that it is the card of a king or spade is
   (a) 1/26 (b) 3/26 (c) 4/13 (d) 3/13
2. Two dice are thrown together. The probability that at least one will show its digit greater than 3 is
   (a) 1/4 (b) 3/4 (c) 1/2 (d) 1/8
3. Two dice are thrown simultaneously. The probability of obtaining a total score of 5 is
   (a) 1/18 (b) 1/12 (c) 1/9 (d) none of these
4. Two dice are thrown simultaneously. The probability of obtaining total score of seven is
   (a) 5/36 (b) 6/36 (c) 7/36 (d) 8/36
5. The probability of getting a total of 10 in a single throw of two dice is
   (a) 1/9 (b) 1/12 (c) 1/6 (d) 5/36

6. A card is drawn at random from a pack of 100 cards numbered 1 to 100. The probability of drawing a number which is a square is
(a) 1/5 (b) 2/5 (c) 1/10 (d) none of these.

7. A bag contains 3 red, 4 white and 5 blue balls. All balls are different. Two balls are drawn at random. The probability that they are of different colour is
(a) 47/66 (b) 10/33 (c) 1/3 (d) 1

8. Two dice are thrown together. The probability that neither they show equal digits nor the sum of their digits is 9 will be
(a) 13/15 (b) 13/18 (c) 1/9 (d) 8/9

9. Four persons are selected at random out of 3 men, 2 women and 4 children. The probability that there are exactly 2 children in the selection is
(a) 11/21 (b) 9/21 (c) 10/21 (d) none of these

10. The probabilities of happening of two events A and B are 0.25 and 0.50 respectively. If the probability of happening of A and B together is 0.14, then probability that neither A nor B happens is
(a) 0.39 (b) 0.25 (c) 0.11 (d) none of these

11. A die is rolled, then the probability that a number 1 or 6 may appear is
(a) 2/3 (b) 5/6 (c) 1/3 (d) 1/2

12. Six boys and six girls sit in a row randomly. The probability that all girls sit together is
(a) 1/122 (b) 1/112 (c) 1/102 (d) 1/132

13. The probabilities of three mutually exclusive events A, B and C are given by 2/3, 1/4 and 1/6 respectively. The statement
(a) is true (b) is false
(c) nothing can be said (d) could be either

14. If $\frac{(1-3p)}{2}, \frac{(1+4p)}{3}, \frac{(1+p)}{6}$ are the probabilities of three mutually exclusive and exhaustive events, then the set of all values of p is
(a) (0,1) (b) (−1/4, 1/3) (c) (0,1/3) (d) $(0, \infty)$

15. A pack of cards contains 4 aces, 4 kings, 4 queens and 4 jacks. Two cards are drawn at random. The probability that at least one of them is an ace is
(a) 1/5 (b) 3/16 (c) 9/20 (d) 1/9

16. If three dice are throw simultaneously, then the probability of getting a score of 5 is
(a) 5/216 (b) 1/6 (c) 1/36 (d) none of these

17. One of the two events must occur. If the chance of one is 2/3 of the other, then odds in favour of the other are
(a) 1 : 3 (b) 3 : 1 (c) 2 : 3 (d) 3 : 2

18. The probability that a leap year will have 53 Fridays or 53 Saturdays is
(a) 2/7 (b) 3 / 7 (c) 4/7 (d) 1/7

19. A person write 4 letters and addresses 4 envelopes. If the letters are placed in the envelopes at random, then the probability that all letters are not placed in the right envelopes, is
(a) 1/4 (b) 11/24 (c) 15/24 (d) 23/24

20. A and B are two events such that $P(A) = 0.25$ and $P(B) = 050$. The probability of both happening together is 0.14. The probability of both A and B not happening is
(a) 0.39 (b) 0.25 (c) 0.11 (d) none of these

21. If the probability of A to fail in an examination is $\frac{1}{5}$ and that of B is $\frac{3}{10}$. Then, the probability that either A or B fails is
(a) 1/2 (b) 11/25 (c) 19/50 (d) none of these

22. A box contains 10 good articles and 6 defective articles. One item is drawn at random. The probability that it is either good or has a defect, is
(a) 64/64 (b) 49/64 (c) 40/64 (d) 24/64

23. Three integers are chosen at random from the first 20 integers. The probability that their product is even is
(a) 2/19 (b) 3/29 (c) 17/19 (d) 4/19

24. Out of 30 consecutive integers, 2 are chosen at random. The probability that their sum is odd, is
(a) 14/29 (b) 16/29 (c) 15/29 (d) 10/29

25. A bag contains 5 black balls, 4 white balls and 3 red balls. If a ball is selected randomwise, the probability that it is black or red ball is
(a) 1/3 (b) 1/4 (c) 5/12 (d) 2/3

26. Two dice are thrown simultaneously. The probability of getting a pair of aces is
(a) 1/36 (b) 1/3 (c) 1/6 (d) none of these

27. An urn contains 9 balls two of which are red, three blue and four black. Three balls are drawn at random. The probability that they are of the same colour is
(a) 5/84 (b) 3/9 (c) 3/7 (d) 7/17

28. Five persons entered the lift cabin on the ground floor of an 8 floor house. Suppose that each of them independently and with equal probability can leave the cabin at any floor beginning with the first, then the probability of all 5 persons leaving at different floor is
(a) $\frac{^7P_5}{7^5}$ (b) $\frac{7^5}{^7P_5}$ (c) $\frac{6}{^6P_5}$ (d) $\frac{^5P_5}{5^5}$

29. A box contains 10 good articles and 6 with defects. One item is drawn at random. The probability that it is either good or has a defect is
(a) 64/64 (b) 49/64 (c) 40/64 (d) 24/64

30. A box contains 6 nails and 10 nuts. Half of the nails and half of the nuts are rusted. If one item is chosen at random, the probability that it is rusted or is a nail is
(a) 3/16 (b) 5/16 (c) 11/16 (d) 14/16

31. If S is the sample space and $P(A) = \frac{1}{3} P(B)$ and $S = A \cup B$, where A and B are two mutually exclusive events, then $P(A) =$
(a) 1/4 (b) 1/2 (c) 3/4 (d) 3/8

32. One mapping is selected at random from all the mappings of the set $A = \{1, 2, 3, \ldots, n\}$ into itself. The probability that the mapping selected is one to one is
(a) $\frac{1}{n^n}$ (b) $\frac{1}{n!}$ (c) $\frac{(n-1)!}{n^{n-1}}$ (d) None of these

33. If A, B, C are three mutually exclusive and exhaustive events of an experiment such that $3P(A) = 2P(B) = P(C)$, then $P(A)$ is equal to
(a) $\frac{1}{11}$ (b) $\frac{2}{11}$ (c) $\frac{5}{11}$ (d) $\frac{6}{11}$

34. If A and B are mutually exclusive events then
(a) $P(A) \leq P(\bar{B})$ (b) $P(A) \geq P(\bar{B})$ (c) $P(A) < P(\bar{B})$ (d) None of these

35. If $P(A \cup B) = P(A \cap B)$ for any two events A and B, then
(a) $P(A) = P(B)$ (b) $P(A) > P(B)$ (c) $P(A) < P(B)$ (d) None of these

36. Three numbers are chosen from 1 to 20. The probability that they are not consecutive is
(a) $\frac{186}{190}$ (b) $\frac{187}{190}$ (c) $\frac{188}{190}$ (d) $\frac{18}{^{20}C_3}$

37. 6 boys and 6 girls sit in a row at random. The probability that all the girls sit together is
(a) $\frac{1}{432}$ (b) $\frac{12}{431}$ (c) $\frac{1}{132}$ (d) None of these

38. Without repetition of the numbers, four digit numbers are formed with the numbers 0, 2, 3, 5. The probability of such a number divisible by 5 is
(a) $\frac{1}{5}$ (b) $\frac{4}{5}$ (c) $\frac{1}{30}$ (d) $\frac{5}{9}$

39. If the probability for A to fail in an examination is 0.2 and that for B is 0.3, then the probability that either A or B fails is
(a) > 0.5 (b) 0.5 (c) ≤ 0.5 (d) 0

40. Three digit numbers are formed using the digits 0, 2, 4, 6, 8. A number is chosen at random out of these numbers what is the probability that this number has the same digits?
(a) $\frac{1}{16}$ (b) $\frac{16}{25}$ (c) $\frac{1}{645}$ (d) $\frac{1}{25}$

ANSWERS

1. (c)	2. (b)	3. (c)	4. (b)	5. (b)	6. (c)	7. (a)	8. (b)
9. (c)	10. (a)	11. (c)	12. (d)	13. (b)	14. (b)	15. (c)	16. (c)
17. (d)	18. (b)	19. (d)	20. (a)	21. (c)	22. (a)	23. (c)	24. (c)
25. (d)	26. (a)	27. (a)	28. (a)	29. (a)	30. (c)	31. (a)	32. (c)
33. (b)	34. (a)	35. (a)	36. (b)	37. (c)	38. (d)	39. (c)	40. (d)

SUMMARY

1. An experiment whose outcomes cannot be predicted or determined in advance is called a random experiment.
2. Each outcome of a random experiment is known as an elementary event.
3. The set of all possible outcomes (elementary events) of a random experiment is called the sample space associated with it.
4. A subset of the sample space associated with a random experiment is called an event.
5. An event is said to occur if any one of the elementary events belonging to it is an outcome.
6. An event associated with a random experiment is called a certain event if it always occurs whenever the experiment is performed.
   The sample space associated with a random experiment defines a certain event.
7. The null set of the sample space defines an impossible event.
8. An event associated with a random experiment is a compound event, if it is the disjoint union of two or more elementary events.

9. Two or more events associated with a random experiment are said to be mutually exclusive or incompatible events if the occurrence of any one of them prevents the occurrence of all others i.e. no two or more of them can occur simultaneously in the same trial.
If A and B are mutually exclusive events, then $A \cap B = \phi$.

10. Events $A_1, A_2, A_3, \ldots, A_n$ associated with a random experiment with sample space S are exhaustive if $A_1 \cup A_2 \cup \ldots \cup A_n = S$.

11. Let S be the sample space associated with a random experiment. A set of events $A_1, A_2, \ldots, A_n$ is said to form a set of mutually exclusive and exhaustive system of events if
(i) $A_1 \cup A_2 \cup \ldots \cup A_n = S$ (ii) $A_i \cap A_j = \phi$ for $i \neq j$

12. *Probability function:* Let $S = \{w_1, w_2, \ldots, w_n\}$ be the sample space associated with a random experiment. Then a function P which assigns every event $A \subset S$ to a unique non-negative real number $P(A)$ is called the probability function if the following axioms hold :
A-1 : $0 \le P(w_i) \le 1$ for all $w_i \in S$
A-2 : $P(S) = 1$ i.e. $P(w_1) + P(w_2) + \ldots + P(w_n) = 1$
A-3 : For any event $A \subset S$, $P(A) = \sum_{w_k \in A} P(w_k)$, the number $P(w_k)$ is called probability of elementary event w_k.

13. *Probability of an event:* If there are n elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of occurrence of A is defined as:
$$P(A) = \frac{m}{n} = \frac{\text{Favourable number of elementary events}}{\text{Total number of elementary events}}$$
The odds in favour of occurrence of the event A are defined by $m : (n - m)$
The odds against the occurrence of A are defined by $(n - m) : m$.
The probability of non-occurrence of A is given by $P(\overline{A}) = 1 - P(A)$.

14. If A and B are two events associated with a random experiment, then
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

15. If A, B, C are three events associated with a random experiment, then
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C$$

16. If A and B are two events associated with a random experiment, then
(i) $P(\overline{A} \cap B) = P(B) - P(A \cap B)$ i.e. probability of occurrence of B only $= P(B) - P(A \cap B)$
(ii) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$ i.e. probability of occurrence of A only $= P(A) - P(A \cap B)$
(iii) Probability of occurrence of exactly one of A and B is $P(A) + P(B) - 2P(A \cap B)$
$= P(A \cup B) - P(A \cap B)$

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